Deforming Lipschitz metrics into smooth metrics while keeping their curvature operator non-negative

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Abstract

In this paper we study the evolution of Lipschitz continuous Riemannian metrics on smooth manifolds, by the dual Ricci-Harmonic map flow. This flow is equivalent (up to a diffeomorphism) to the Ricci flow. We show that a solution $g(\cdot,t)_{t\in(0,T)}$ exists for a short time $[0,T)$, $(T>0)$ and that the solution is smooth on $(0,T)$. Furthermore if the curvature operator (in dimension three Ricci curvature, or sectional curvature) of the initial Lipschitz metric $g_0(\cdot)$ is non-negative in a weak sense (see definition 1.2) then the curvature operator (in dimension three Ricci curvature, or sectional curvature) of $g(\cdot,t)$ is non-negative in the usual smooth sense for all $t\in(0,T)$.

§ 1. Introduction and statement of results

In this paper we are concerned with the Ricci flow of Lipschitz Riemannian metrics.

In the study of smooth Riemmanina manifolds one often considers Riemannian metrics whose tensor $g=\{g_{ij}\}$ is C^2. This allows one to define the Riemannian curvature tensor which is then continuous. Given a C^∞ Riemannian metric g_0 on a compact manifold M, we can always find a $T>0$ and a 1-parameter family of C^∞ Riemannian metrics $\{g(t)\}_{t\in[0,T]}$ on M, denoted $(M,g(t))$, such that

$$\begin{aligned}\frac{\partial}{\partial t}g(t) &= -2\mathrm{Ricci}(g(t)),\ \text{for all } t\in[0,T]\\ g(0) &= g_0,\end{aligned}\tag{1.1}$$

where g is $C^\infty(M\times[0,T])$ (C^∞ on the manifold $(M\times[0,T])$ with the induced structure), and $\mathrm{Ricci}(g(t))$ is the Ricci curvature of the Riemannian manifold $(M,g(t))$. Notice that (1.1) makes no sense if g is not twice differentiable in space for all $t\in[0,T]$. The family $(M,g(t))_{t\in[0,T]}$ is called a solution to the Ricci flow with initial value g_0. Ricci flow was invented, and used by R.Hamilton [Ha 1] to prove that every compact three manifold which admits a C^∞ Riemannian metric g_0 with $\mathrm{Ricci}(g_0)>0$ also admits a metric g_∞ of constant positive sectional curvature [Ha 1].

This paper is a further investigation of the Ricci flow of non-smooth metrics which we started in the paper [Si].

In this paper we shall consider the dual Ricci-Harmonic Map flow (see section 6. [Ha 3]) as we did in [Si]. This leads to a more general version of the Ricci DeTurck flow, considered initially by DeTurck in [DeT]. In the paper [Bem] the authors use Ricci flow to smooth out C^2 metrics by introducing harmonic co-ordinates at appropriately chosen times.

For a short introduction to the dual Ricci-Harmonic heat and Ricci-DeTurck flow, see [Si].

In Shi's paper [Sh], the Ricci-DeTurck flow was written term by term to obtain the evolution equation for solutions to (1.3) in co-ordinate form. We present here the evolution equation, in co-ordinate form, for metrics which solve (1.3) for an arbitrary smooth fixed background metric h. For the rest of the paper we shall be chiefly concerned with solutions of (1.3) and not solutions of Ricci flow. For this reason we will use the notation $g(t), t \in [0,T]$ to refer to a solution of (1.3). Let $g(t), t \in [0,T]$ be a solution to (1.3). Then $g(t)$, $t \in [0,T]$ solves the evolution equation

$$\begin{aligned} \frac{\partial}{\partial t} g_{ab} =& g^{cd}\tilde\nabla_c\tilde\nabla_d g_{ab} - g^{cd} g_{ap}\tilde g^{pq}\tilde R_{bcqd} - g^{cd} g_{bp}\tilde g^{pq}\tilde R_{acqd} \\ &+ \frac{1}{2} g^{cd} g^{pq}(\tilde\nabla_a g_{pc}\cdot\tilde\nabla_b g_{qd} + 2\tilde\nabla_c g_{ap}\cdot\tilde\nabla_q g_{bd} \\ &- 2\tilde\nabla_c g_{ap}\cdot\tilde\nabla_d g_{bq} - 4\tilde\nabla_a g_{pc}\cdot\tilde\nabla_d g_{bq}), \\ g(0) =& g_0, \end{aligned} \tag{1.5}$$

where $\tilde R_{abcd} = \mathrm{Riem}(h)_{abcd}$ and $\tilde\nabla$ is the co-variant derivative with respect to h. Note that if h is not twice differentiable, then (1.5) makes no sense, since then $\tilde R_{abcd} = \mathrm{Riem}(h)_{abcd}$ is not defined. If we choose $h = g_0$, that is we wish to examine the Ricci DeTurck flow, and g_0 is not twice differentiable, then we cannot make sense of the above equation. For this reason we will always choose a smooth h not equal to g_0 (but close to g_0 in some to be specified C^0 sense) when examining (1.5). In [Si] , it is shown that for initial metrics $g_0(\cdot)$ which are C^0, the system (1.5) has a short time solution (see statement of theorem A below). We shall call such solutions, solutions to the 'h flow'.

Definition 1.1 . *Let M be a complete manifold and g a C^0 metric, and $1 \leq \delta < \infty$ a given constant. A metric h is said to be a δ fair background metric for g, or 'δ fair to g', if h is C^∞ and there exists a constant k_0 with*

$$\sup_{x\in M} {}^h|\mathrm{Riem}(h)(x)| = k_0 < \infty, \tag{1.6}$$

and

$$\frac{1}{\delta} h(p) \leq g(p) \leq \delta h(p) \text{ for all } p \in M. \tag{1.7}$$

Remark 1. *By the result of Shi [Sh], if g is Riemannian metric and h a smooth Riemannian metric satisfying (1.6) and (1.7) then there exists a smooth metric h' which is $\delta + \epsilon$ fair to g, and*

$$\sup_{x\in M} {}^h|\overset{h}{\nabla} Riem(h)(x)| = k_j < \infty,$$

where ${}^h\nabla^j$ *is the jth covariant derivative with respect to h.*

Remark 2. *Let M be a compact manifold, and g a C^0 metric on M. Then for every $0 < \epsilon < 1$ there exists a metric $h(\epsilon)$, for which $h(\epsilon)$ is $1+\epsilon$ fair to g.*

Proof (of Remark 2): We may use de Rham regularisation [deR], or a locally finite partition of unity and Sobolev averaging (see section on mollifiers in [GT]) to obtain a C^∞ metric h which is C^0 as close as we like to g. A bound on the curvature follows from the compactness of M. $\diamondsuit$

The following existence result is proved in [Si 1].

Theorem A. *There exists an $\epsilon(n)$ with the following properties. Let g_0 be a complete metric and h a complete metric on M which is $1+\frac{\epsilon(n)}{2}$ fair to g_0. There exists a $T = T(n, k_0)$ and a family of metrics $g(t), t \in (0, T]$ in $C^\infty(M \times (0, T])$ which solves h flow for $t \in (0, T]$, h is $(1+\epsilon)$ fair to $g(\cdot, t)$, for $t \in (0, T]$ and satisfies*

$$\lim_{t\to 0} \sup_{x\in\Omega'} {}^h|g(\cdot, t) - g_0(\cdot)| = 0,$$

$$\sup_{x\in M} |{}^h\nabla^i g|^2 \leq \frac{c_i(n, k_0, \ldots, k_i)}{t^i}, \text{ for all } t \in (0, T], i \in \{1, 2, \ldots\},$$

where Ω' is any open set satisfying $\Omega' \subset\subset \Omega$, where Ω is any open set on which g_0 is continuous (see Theorem 5.2).

If the initial metric g_0 is Lipschitz, then we are able to show that certain geometric inequalities will be preserved by this flow. We shall use the notation $\mathcal{R}(g)$ for the curvature operator of a Riemannian manifold with metric g.

Definition 1.2. *Let M^n be an n-dimensional manifold, and g be a locally Lipschitz complete Riemannian metric on M. We say that $\mathcal{R}(g) \geq 0$ ($\mathrm{Ricci}(g) \geq 0$), if there exists a family $\{{}^\alpha g\}_{\alpha\in\{1,2,\ldots\}}$ of smooth metrics on M which satisfy $\mathcal{R}({}^\alpha g) \geq -\frac{1}{\alpha}$, ($\mathrm{Ricci}({}^\alpha g) \geq -\frac{1}{\alpha}$) and $\lim_{\alpha\to\infty} \sup_M {}^g|{}^\alpha g - g| = 0$, and ${}^g|\Gamma({}^\alpha g) - \Gamma(g)| \leq c_0$ for all $\alpha \in \{1, 2, \ldots\}$, where c_0 is some constant which does not depend on α, and $\Gamma(g)$ refers to the Christoffel symbols of g. If M is non-compact, we further require that*

$$\sup_M |\mathrm{Riem}({}^\alpha g)| < \infty,$$

for some sufficiently large α.

Remark: In [Si] the evolution of Lipschitz metrics on three dimensional manifolds whose Ricci curvature is non-negative is examined : see theorem 6.5. The extra condition (above) required for the non-compact case was not included in the definition (definition 6.4) in [Si], but is however required in the proof there (for the non-compact case) of theorem 6.5. Here the proof is presented in somewhat more detail than in [Si].

We prove the following theoerem.

Theorem 1.3. *Let M^n be a manifold, and g_0 be a complete locally Lipschitz metric on M which satisfies $\mathcal{R}(g_0) \geq 0$, $(\mathrm{Ricci}(g_0) \geq 0)$ in the weak sense of Definition 1.1, with Lipschitz constant c_0. Then the solution $g(x,t), t \in (0,T]$ to h flow of g_0 exists (for some smooth metric h, and $T = T(n, c_0, h) > 0$) and satisfies $\mathcal{R}(g(x,t)) \geq 0$ for all $t \in (0,T]$ in the usual smooth Riemannian sense (see Theorem 6.5). Furthermore, there exists a constant $c = c(n, c_0, h)$.* $\sup_M {}^g|\mathrm{Riem}(g(t))| < \frac{c(n,c_0,h)}{t}$ $\forall t \in (0,T]$

§ 2. A priori estimates.

Remark 2.0. *Let h be a $1+\epsilon$ fair complete metric for g_0 on a manifold M such that $\sup_M |\overset{h}{\nabla} g_0| \leq c_0$. W.l.o.g we may assume that*

$$\sup_M |\overset{h}{\nabla}{}^j \mathrm{Riem}(h)| = k_j < \infty. \qquad (*)$$

We will always assume this for the rest of this paper. If not then we flow h by Ricci flow as in [Sh] for a very short time to obtain a new metric $\hat{h}$ which satisfies (*). Note that (Theorem 5.1 , [Sh]) $\sup_M |\overset{h}{\nabla} \hat{h}|^2 \leq c(k_0, n)$, where $k_0 = \sup_M |\mathrm{Riem}(h)|^2 < \infty$. But then clearly $\sup_M {}^g|\overset{\hat{h}}{\nabla} g_0 - \overset{h}{\nabla} g_0| \leq c(c_0, n, k_0)$, and so $\sup_M {}^h|\overset{\hat{h}}{\nabla} g_0| \leq c(c_0, n, k_0)$. This proves the remark.

Lemma 2.1 . *Let g_0 be a complete smooth metric on M, and h be $1+\frac{\epsilon(n)}{2}$ fair background metric for g_0 for which $\sup_M |\overset{h}{\nabla} g_0| \leq c_0$ also holds. Let $g(t)$,$t \in [0,T]$ be a solution to $h-flow$ where T is as in theorem A (such a solution always exists: see theorem A above). Then*

$$\sup_M |\overset{h}{\nabla} g(\cdot,t)| \leq c(c_0,n,h) \forall t \in [0,T], \qquad (2.1)$$

and

$$\sup_M |\overset{h}{\nabla}{}^2 g(\cdot,t)| \leq \frac{c(c_0,n,h)}{\sqrt{t}} \forall t \in [0,T], \qquad (2.2)$$

which clearly implies

$$\sup_M {}^g|\mathrm{Riem}(g,t)| \leq \frac{c(c_0,n,h)}{\sqrt{t}}. \qquad (2.3)$$

Proof :
i) From the remark above, we may assume, without loss of generality, that

$$\sup_M |\overset{h}{\nabla}{}^j \mathrm{Riem}(h)| = k_j < \infty. \qquad (*)$$

ii) Proof of the first estimate.

Let $\psi(x,t) = (\phi(x,t) + a(n))|\overset{h}{\nabla} g(x,t)|^2$, where $\phi : M \times [0,T] \to$ R, is the function $\phi(x,t) = g^{ij}(x,t)h_{ij}(x)$ first examined by Shi in [Sh] , Lemma 2.2 (for $h = g_0$). One may calculate, as in [Sh] Lemma 2.2, that $\frac{\partial}{\partial t}\phi \leq h^{ij}\overset{h}{\nabla}_i\overset{h}{\nabla}_j\phi + c(k_0,n)\phi^2$, where here c is a constant which depends only on the supremum of the curvature of h, and the dimension of M^n (see [Si] Lemma 2.1/ [Sh] , Lemma 2.2 for details). The fact that ϕ satisfies a good evolution equation (notice now first order gradient times appear on the right hand side!), as was first noticed by Shi, is crucial in the development of this theory.

In Lemma 2.4 of [Si] it is calculated that

$$\frac{\partial}{\partial t}\psi \leq g^{ij}\overset{h}{\nabla}_i\overset{h}{\nabla}_j\psi - \frac{1}{2}\psi^2 + c_0(n,k_0,k_1),$$

where $k_1 = \sup_M |\text{Riem}(h)|^2$ and k_0 is as in the statement of the theorem. Let η be the cut-off function defined in Lemma 4.1 of [Si], whose support is contained in a ball $B_{2r}(y_0)$ of radius $2r$ and which is equal to one on $B_r(y_0)$. Then calculating as in that lemma, we get

$$\frac{\partial}{\partial t}\psi\eta \leq g^{ij}\overset{h}{\nabla}_i\overset{h}{\nabla}_j\eta\psi - \frac{1}{16}\eta\psi^2 + c_1(n,k_0,k_1,r)(1+\psi),$$

where we have used (as in the proof of Lemma 4.1 in [Si], upto equation 4.5) the properties (d1) - (d5) of the cut-off function η. Assume that (x_0,t_0) is a point in the domain $B_r(y_0) \times (0,T]$ where $\psi\eta$ obtains its maximum. Then

$$0 \leq \frac{\partial}{\partial t}\psi\eta(x_0,t_0) \leq -\frac{1}{16}\eta\psi^2 + c_1(n,k_0,k_1,r)(1+\psi),$$

which, when multiplied by $\eta(x_0)$, gives us

$$0 \leq -\frac{1}{16}(\eta\psi)^2 + c_1(n,k_0,k_1,r)(\eta + \psi\eta),$$

which implies that

$$\psi\eta(\frac{1}{16}(\eta\psi) - c_1) \leq c_1,$$

in view of the fact that $\eta \leq 1$. That is

$$\sup_{x\in B_r(y_0),t\in[0,T]} \psi(x,t) \leq c_2(n,r,k_0,k_1,\sup_M |\overset{h}{\nabla} g(x,0)|^2),$$

where here we have used that $\psi(x,0) \leq c(n)|\overset{h}{\nabla} g(x,0)|^2$. Hence $\sup_{x\in M}\psi \leq c_2(n,r,k_0,k_1)$, which implies the first estimate, in view of the fact that $(\phi(x,t) + a(n)) \geq a(n)$.
iii) Proof of the second and third estimate. The next estimate is obtained similarly. We let $w = t(a + |\overset{h}{\nabla} g|^2)|\overset{h}{\nabla}^2 g|^2$ as in [Si] lemma 4.2, where a is a constant $a = a(c_0,n,h) > 1$. We see that

$$\frac{\partial}{\partial t}w \leq g^{ij}\overset{h}{\nabla}\overset{h}{\nabla} w - (c_0,h,n)w^2 + c(c_0,h,n)w + c(c_0,h,n).$$

As in the first estimate, we may use a cut off function (see [Si] Lemma 4.2 for details) to obtain $w \leq c(c_0, h, n)$, which gives us the required estimate.

$\diamondsuit$

§ 3. Proof of theorem 1.3.

Before we begin the proof of theorem 1.3 we make the neccesary definitions and we introduce some notation. We define $\Lambda^2(T^*_xM) = T^*_xM \wedge T^*_xM$ and $\Lambda^2(M)$ to be the vector bundle over M with fibre $\Lambda^2(T^*_xM)$. $\mathcal{O}^2(M)$ will denote the space of smooth symmetric bi-linear forms on the vector bundle $\Lambda^2(M)$ (a smooth bi-linear form B on a vector bundle E over M means here a family $(B(p))_{p\in M}$ where $B(p) : E_p \times E_p \to \mathrm{R}$ is bi-linear (E_p is the fibre of E over p) , and for smooth sections v, w of E, we require that $B(v, w)$ be a smooth function on M, where $B(v, w)(p) = B(p)(v(p), w(p)))$. The curvature operator $\mathcal{R}(g) : \Lambda^2(M) \to \mathrm{R}$ of a Riemannian manifold (M, g) is then an element of $\mathcal{O}^2(M)$, and is defined by

$$\mathcal{R}(\phi, \psi) = \mathrm{R}_{abcd}\phi^{ab}\psi^{cd},$$

where R_{abcd} is the Riemannian curvature tensor of (M, g).

In [Ha 2], Hamilton uses time dependent isomorphisms $u(t) : (TM, g_0) \to (TM, g(t))$ to examine the evolution of the curvature operator under Ricci flow. In particular if $(M, g_{ij}(t)$ is a solution to the Ricci flow, then the pull back of the curvature operator is

$$\mathcal{R}(t)(\phi, \psi) = \mathrm{R}(t)_{abcd}\phi^{ab}\psi^{cd},$$

where $\mathrm{R}(t)_{abcd} = \mathrm{Riem}(g(t))_{ijkl}u^i_a u^j_b u^k_c u^l_a$, and the pull back of the metric is $g_{ab} = u^i_a(t)u^j_b(t)g_{ij}(t)$, and the isomorphisms $u(t)$ are chosen so that g_{ab} has zero time derivative, and hence g_{ab} is independent of t. That is

$$\frac{\partial}{\partial t}u^i_a = g^{ij}R_{jk}u^k_a.$$

Let $(\phi^\alpha)_{\alpha\in\{1\ldots p(n)\}}$ be a basis for $\Lambda^2(M)$ at some point p in M. Then for an operator N in $\mathcal{O}^2(M)$, we write $N_{\alpha\beta}(p)$ for $N(\phi^\alpha, \phi^\beta)(p)$. The evolution of $\mathcal{R}$ is then derived in [Ha 2] to be

$$\frac{\partial}{\partial t}\mathcal{R} = \Delta\mathcal{R} + \mathcal{R}^2 + \mathcal{R}\#\mathcal{R},$$

where $\mathcal{R}^2$ is the square of the curvature operator, $\#$ is the operator given by $T\#N_{\alpha\beta} = c^{\gamma\eta}_\alpha c^{\delta\theta}_\beta T_{\gamma\delta}N_{\eta\theta}$, and $c_{\alpha\gamma\eta}$ are the structure constants given by $c^{\alpha\beta\eta} = \langle\phi^\alpha, [\phi^\beta, \phi^\eta]\rangle$, where for two forms ϕ, ψ the inner product $G = \langle,\rangle$ is defined by

$$G(\phi, \psi) = \langle\phi, \psi\rangle = g^{ab}g^{cd}\phi_{ac}\psi_{bd},$$

and

$$[\phi,\psi]_{ab} = g^{cd}\phi_{ac}\psi_{bd} - g^{cd}\phi_{bc}\psi_{ad}.$$

Note that the laplacian appearing in the above evolution equation comes from the naturally occuring connection on (TM, g_{ab}): in particular $\nabla G = 0$ (see [Ha 2] for details). Let us choose an orthonormal basis $(\phi^\alpha)_{\alpha\in\{1\ldots p(n)\}}$ for the two forms in the neighbourhood of a point p. Assume also that we choose co-ordinates on M so that $g_{ab} = \delta_{ab}$ at p. Then $1 = \langle\phi^\alpha,\phi^\alpha\rangle = \sum_{a,b}(\phi^\alpha_{ab})^2$, which shows us that

$$|c^{\alpha\beta\eta}| \leq c(n), \tag{3.1}$$

at that point p for this orthonormal basis at p. Assume that we choose the basis at this point p so that $\mathcal{R}_{\alpha\beta}$ is diagonal $\mathcal{R}_{\alpha\beta} = \lambda_\alpha\delta_{\alpha\beta}$. Without loss of generallity $\lambda_1 \leq \lambda_2 \leq \ldots \lambda_{\frac{n(n-1)}{2}}$

Then the evolution equation for $\mathcal{R}$ at this point and time is

$$\frac{\partial}{\partial t}\mathcal{R}_{\alpha\alpha} = \qquad \Delta\mathcal{R}_{\alpha\alpha} + {\lambda_\alpha}^2 + \sum_{\beta\gamma} c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma$$

For the $h-flow$, the equation is, as calculated in [Si] Theorem 6.2,

$$\frac{\partial}{\partial t}\mathcal{R}_{\alpha\alpha} = {\lambda_\alpha}^2 + \overset{g}{\nabla}_k\mathcal{R}_{\alpha\alpha}V^k + \mathcal{R} * \nabla V + \sum_{\beta\gamma} c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma,$$

where $*$ means a tensor product obtained by taking traces with repsect to g. This tensor product actually has the property that $\mathcal{R} * \nabla V \geq -c(n)|\lambda_\alpha|^{g}|\nabla V|$, as one sees in [Si] thm.6.2.

If $g(\cdot,t)$ is a solution to h flow as in Lemma 2.1, then this implies

$$\frac{\partial}{\partial t}\mathcal{R}_{\alpha\alpha} \geq {\lambda_\alpha}^2 + (\overset{g}{\nabla}_k\mathcal{R}_{\alpha\alpha})V^k + \frac{-c(n,c_0,h)}{\sqrt{t}}{}^{g}|\mathcal{R}_{\alpha\alpha}| + \sum_{\beta\gamma} c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma,$$

in view of (2.1), (2.2) and the fact that ${}^{h}|\overset{h}{\nabla} V| \leq c(n)({}^{h}|\overset{h}{\nabla}{}^2 g| + {}^{h}|\overset{h}{\nabla} g|^2)$. The following lemma will be helpful in examining the evolution of the curvature operator of a Lipschitz metric.

Lemma 3.1. *Let g_0 be a complete smooth metric on M, and h be $1 + \frac{\epsilon(n)}{2}$ fair background metric for g_0 for which $\sup_M {}^{h}|\overset{h}{\nabla} g_0| \leq c_0$ also holds. Let $g(t)$,$t \in [0,T]$ be a solution to $h-flow$ where T is as in theorem A (such a solution always exists: see theorem A above). Assume further that*

$$\mathcal{R}(g_0) \geq -\delta,$$

where $|\delta| \leq 1$. Then there exists a $T_0 = T_0(T,n,c_0) > 0$ such that

$$\mathcal{R} \geq -2\delta \forall t \in [0,T_0).$$

Proof

Step i). A lower bound for $\mathcal{R}(g_0)$.

w.l.o.g $|\overset{h}{\nabla} g|^2 + |\overset{h}{\nabla}{}^2 g|^2 t \leq c(n, c_0, h)$ (see Lemma 2.1). In the rest of this paper, all constants that only depend on n, c_0, h will be denoted by $c(n, c_0, h)$ and we will agree that, for example, $2c^2(n, c_0, h) + 5 = c(n, c_0, h)$. Let η be our standard cutoff function, for a ball of radius one (see Lemma 2.1 above). Then the above evolution equation gives us

$$\begin{aligned}\frac{\partial}{\partial t}\eta\mathcal{R}_{\alpha\alpha} \geq &\Delta(\eta\mathcal{R})_{\alpha\alpha} + \eta\sum_{\beta\gamma} c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma \\ &+ \overset{g}{\nabla}_k(\eta\mathcal{R}_{\alpha\alpha})V^k - \mathcal{R}_{\alpha\alpha}(\nabla_k\eta)V^k \\ &\quad - \mathcal{R}_{\alpha\alpha}\overset{g}{\Delta}\eta - 2\langle\overset{g}{\nabla}\eta, \overset{g}{\nabla}R_{\alpha\alpha}\rangle - \frac{c(n, h, c_0)}{\sqrt{t}}|R_{\alpha\alpha}|.\end{aligned}$$

Now use the fact that

$$c(n, h, c_0) \geq {}^g|\overset{h}{\nabla}\overset{h}{\nabla}\eta| \geq {}^g|\overset{g}{\nabla}\overset{g}{\nabla}\eta| - {}^g|\overset{h}{\nabla}\overset{h}{\nabla}\eta - \overset{g}{\nabla}\overset{g}{\nabla}\eta|,$$

to get

$${}^g|\overset{g}{\nabla}\overset{g}{\nabla}\eta| \leq c(n, h, c_0) + {}^g|(\Gamma(g) - \Gamma(h)) * \overset{g}{\nabla}\eta| \leq c(c_0, n, h),$$

in view of the fact that $\Gamma(g(t)) - \Gamma(h)$ and $\overset{h}{\nabla}\eta$ are uniformly bounded (Lemma 2.1). Also,

$$\begin{aligned}\langle\overset{g}{\nabla}\eta, \overset{g}{\nabla}R_{\alpha\alpha}\rangle &= \frac{\langle\overset{g}{\nabla}\eta, \overset{g}{\nabla}R_{\alpha\alpha}\eta\rangle}{\eta} - \frac{{}^g|\overset{g}{\nabla}\eta|^2}{\eta}R_{\alpha\alpha} \\ &\leq \frac{\langle\overset{g}{\nabla}\eta, \overset{g}{\nabla}R_{\alpha\alpha}\eta\rangle}{\eta} + \frac{c(n, h, c_0)}{\sqrt{t}},\end{aligned}$$

in view of the estimates $(d_1 - d_4)$ in [Si], Lemma 4.1, p 1048. Similarly $-\mathcal{R}_{\alpha\alpha}(\nabla_k\eta)V^k \geq \frac{-c(n,h,c_0)}{\sqrt{t}}$, and hence we get

$$\begin{aligned}\frac{\partial}{\partial t}(\eta\mathcal{R}_{\alpha\alpha}) > &\Delta(\eta\mathcal{R})_{\alpha\alpha} + \eta\sum_{\beta\gamma} c^2_{\alpha\beta\gamma}\eta\lambda_\beta\lambda_\gamma \\ &- \frac{2}{\eta}\langle\nabla\eta, \nabla(\eta R_{\alpha\alpha})\rangle - \frac{c(n, h, c_0)}{\sqrt{t}}(1 + |\eta\mathcal{R}_{\alpha\alpha}|)\end{aligned}$$

Let $b : [0, T) \to \mathrm{R}^+$ be a function dependent on time which is differentiable for all $t > 0$, and which satisfies $1 \leq b \leq 2$, $\forall t \in [0, T)$. Then the O.D.E. satisfies

$$\begin{aligned}&\frac{\partial}{\partial t}(\eta\mathcal{R}_{\alpha\alpha}(t) + b(t)G_{\alpha\alpha}(t)) \\ &= \frac{\partial}{\partial t}(\eta\mathcal{R}_{\alpha\alpha}(t)) + b(t)\frac{\partial}{\partial t}G_{\alpha\alpha}(t) + (\frac{\partial}{\partial t}b(t))G_{\alpha\alpha} \\ &\geq \eta\sum_{\beta\gamma} c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma - \frac{c(n, h, c_0)}{\sqrt{t}}(1 + |\eta\mathcal{R}_{\alpha\alpha}|) + (\frac{\partial}{\partial t}b(t))G_{\alpha\alpha} \\ &\geq \eta\sum_{\beta\gamma} c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma - \frac{c(n, h, c_0)}{\sqrt{t}}(1 + |\eta\mathcal{R}_{\alpha\alpha} + b(t)G_{\alpha\alpha}|) + (\frac{\partial}{\partial t}b(t))G_{\alpha\alpha},\end{aligned}$$

in view of 2.3 and the fact that $b \leq 2$. Let $b(t) = (1 + 2c_4\sqrt{t})$, where $c_4 = c_4(n, h, c_0)$, will be chosen later. Then for all $t \leq \frac{1}{c_4^2}$ we get $b(t) \leq 2$, and hence

$$\begin{aligned}
&\frac{\partial}{\partial t}(\eta \mathcal{R}_{\alpha\alpha}(t) + b(t)G_{\alpha\alpha}(t)) \\
&> \eta \sum_{\beta\gamma} c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma - \frac{c(n,h,c_0)}{\sqrt{t}} - \frac{c(n,c_0,h)}{\sqrt{t}} {}^{g}|\eta\mathcal{R}_{\alpha\alpha} + b(t)G_{\alpha\alpha}(t)| + \frac{c_4}{\sqrt{t}},
\end{aligned}$$

where here we have used that $b \leq 2$. This gives us

$$\begin{aligned}
&\frac{\partial}{\partial t}(\eta \mathcal{R}_{\alpha\alpha}(t) + b(t)G_{\alpha\alpha}(t)) \\
&> \eta \sum_{\beta\gamma} c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma - \frac{c(n,c_0,h)}{\sqrt{t}} {}^{g}|\eta\mathcal{R}_{\alpha\alpha}(t) + b(t)G_{\alpha\alpha}(t)| + \frac{c_4}{2\sqrt{t}},
\end{aligned} \tag{3.2}$$

if we choose $\frac{c_4}{2} \geq c(n, h, c_0)$.

From the assumptions of the lemma, and the definition of b.

$$(\mathcal{R} + bG)(\cdot, 0) > 0,$$

and hence $\eta\mathcal{R}(\cdot, 0) + b(0)G(\cdot, 0) > 0$. Let $T_0 = \min(\frac{1}{c_4^2}, T)$. If there is a first time $t_0 \in (0, T_0)$ and point p_0 and direction ϕ^α where this is no longer true, then clearly $p_0 \in {}^{h}B_2(x_0)$, and the above equation will lead to a contradiction: first note that

$$\eta\mathcal{R}(x_0, t_0) + b(t_0)G(x_0, t_0) \geq 0, \tag{3.3}$$

Also, $\eta\mathcal{R}_{\alpha\alpha}(x_0, t_0) + b(t_0)G_{\alpha\alpha}(x_0, t_0) = 0$ by assumption. Hence we must merely examine the sum appearing in (3.2). We examine this sum term by term. For β, γ given, we either have both $\lambda_\beta, \lambda_\gamma \leq 0$ or at least one of $\lambda_\beta, \lambda_\gamma$ is positive. In the first case, the term $\eta c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma$ is non-negative. In the second case, w.l.o.g $\lambda_\beta > 0$, and the term $\eta c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma$ at (p_0, t_0) may be estimated by

$$\eta c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma = c^2_{\alpha\beta\gamma}\lambda_\beta(\eta\lambda_\gamma + bG_{\gamma\gamma}) - c^2_{\alpha\beta\gamma}\lambda_\beta bG_{\gamma\gamma} \geq -\frac{c(n,h,c_0)}{\sqrt{t_0}},$$

in view of (3.3) and Lemma 2.1. Hence, the total sum may be estimated by

$$\eta \sum_{\beta\gamma} c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma \geq -\frac{c(n,h,c_0)}{\sqrt{t_0}}.$$

Substituting this into (3.2) we get at (p_0, t_0)

$$\frac{\partial}{\partial t}(\eta\mathcal{R} + bG)_{(p_0,t_0)}(\phi^\alpha, \phi^\alpha) > \Delta(\eta\mathcal{R} + bG)_{(p_0,t_0)}(\phi^\alpha, \phi^\alpha),$$

when we choose $\frac{c_4}{2} > c(n,h,c_0)$, which leads to a contradiction when we argue as in the compact maximum principle of Hamilton [Ha 1]. Hence $\mathcal{R} \geq -2$ for all $t \in [0,T_0)$.
Step ii) Compactification of the problem.
Note that $g(t)$ and g_0 and h are equivalent due to our setup (see Lemma 2.1). From step i), we know that $\mathcal{R}(g) \geq -2$. We construct the following help tensor as in [Si], theorem 7.3... Let $N(x,t) = \mathcal{R}(x,t) + \delta \mathrm{e}^{(1+s\sqrt{t})(1+\rho^2(x,t))} G(x,t)$, where $\rho(x,t) = dist(g(t))(x_0,x)$, for some fixed x_0, and $s = s(n,c_0,h)$ is a constant to be chosen later. Then as on page 1068 [Si], we calculate that for $t \leq \frac{1}{s^2}$

$$\begin{aligned} \Delta N &\leq \Delta \mathcal{R} + \delta \mathrm{e}^{b(x,t)}(c(n,h,c_0)^{g}|\mathrm{Riem}|(1+\rho^2))G \\ &\leq \Delta \mathcal{R} + \delta \mathrm{e}^{b(x,t)}(\frac{c(n,h,c_0)}{\sqrt{t}}(1+\rho^2))G, \end{aligned}$$

in view of (3.3). Also,

$$\begin{aligned} \frac{\partial}{\partial t}\rho(x,t) &= \int_\gamma \frac{\frac{\partial}{\partial t} g_{TT}}{\sqrt{g_{TT}}} ds \\ &= \int_\gamma \frac{-2\mathrm{Ricci}_{TT}}{\sqrt{g_{TT}}} ds \\ &\geq \int_\gamma -\frac{c(n,h,k_0)}{\sqrt{t}} \sqrt{g_{TT}} \\ &= -\frac{c(n,h,k_0)}{\sqrt{t}} \rho(x,t). \end{aligned}$$

Hence

$$\begin{aligned} (\frac{\partial}{\partial t} - \Delta)N &\geq \delta(\frac{s - c(n,h,c_0))}{\sqrt{t}})(1+\rho^2)\mathrm{e}^{b(x,t)}G + \sum_{\beta\gamma} c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma \\ &> \delta\frac{s}{2\sqrt{t}}\mathrm{e}^{b(x,t)}G + \sum_{\beta\gamma} c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma, \end{aligned} \tag{3.4}$$

if we choose $\frac{s}{2} > c(n,h,c_0)$.

At time zero, we clearly have $N > 0$. Either this remains true for all time, or there is some first time $t_0 \in [0,T_0)$ and point $p_0 \in M$ and some direction ϕ^α where this is no longer true: this follows from part i), since $\mathcal{R}(g) \geq -2 \Rightarrow N(x) > 0$ for all x with $dist(x_0,x) \geq D$, where $D = D(c_0,n,h)$ is some sufficiently large constant (that is we have compactified the problem). Assume there is some first time t_0 and point p_0 and direction ϕ^α where this is no longer true. Using the same trick as in part i), we estimate the sum $\sum_{\beta\gamma} c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma$ term by term to get $\sum_{\beta\gamma} c^2_{\alpha\beta\gamma}\lambda_\beta\lambda_\gamma \geq -\frac{c(n,h_0,c_0)}{\sqrt{t}}\delta \mathrm{e}^b$, at (p_0,t_0). Hence substituting this into (3.4) we see that $\frac{\partial}{\partial t}N_(p_0,t_0)(\phi^\alpha,\phi^\alpha) > 0$, if we choose $s > 2c(n,h_0,c_0)$, which, arguing as in the proof of the compact maximum

principle of [Ha 1], leads to a contradiction. Hence $N(x) > 0$. Hence $\mathcal{R}(x_0, \cdot) \geq -2\delta$ for all $t \in [0, T_0]$.
Note: if p_0 lies in the cut locus of x_0 then we must modify our argument a little, since *dist* is then not necessarily smooth at p_0. However this causes no problems: we use the trick of Calabi as explained in [Si], page 1068 in the proof of the non-compact maximum principle.

We are now in a position to prove our theorem.

(proof of theorem 1.3) Let $\overset{\alpha}{g}_0$, be the approximating metrics for g_0 obtained in definition 1.2, and let $h = \overset{l}{g}_0$ for some fixed $l \in \mathbf{N}$ chosen large so that ${}^h|h - g_0| \leq \epsilon(n)$. Once again we assume w.l.o.g that $k_j = \sup_M {}^h|{}^h\nabla^j \mathrm{Riem}(h)|^2 < \infty$. Let $\overset{\alpha}{g}(x,t), t \in [0,T]$ denote the corresponding solutions to the $hflow$ (see theorem A) , and $g(x,t), t \in (0,T]$ the limit (as $\alpha \to \infty$) solution. Note that each $\overset{\alpha}{g}_0$ satisfies $\sup_M {}^h|{}^h\nabla(\overset{\alpha}{g}_0)| \leq c_0$, independent of α , and hence we see, using lemma 3.1 that

$$\sup_{M \times (0,T_0)} \mathcal{R}(g_\alpha) \geq -\frac{2}{\alpha},$$

where $T_0 = c(T, n, h, c_0)$. Letting $\alpha \to \infty$, we obtain a smooth solution $g(x,t)$, $t \in (0,T)$ to the $hflow$ with $\mathcal{R}(g_\alpha) \geq 0$, and ${}^g|\mathrm{Riem}(g(t))| \leq \frac{c(c_0,n,h)}{\sqrt{t}}$ as required.

For completeness we mention the following results which are proved using the same techniques as above. Let M^4 be a four manifold and $\mathcal{I}$ denote the isotropic curvature on this manifold (see [Ha 5]).

Theorem 6.7. *Let g_0 be a Lipschitz metric on a real four manifold M^4, such that $\mathcal{I}(g_0) \geq 0$ in the sense of Definition 1.2 with Lipschitz constant c_0. Then there exists a metric h which is $1+\epsilon(4)$ fair to g_0, a $T(n,h,c_0) > 0$, and a family of smooth Riemannian metrics $g(x,t), t \in (0,T]$ such that $g(x,t), t \in (0,T]$ solves h flow, $g(\cdot,t) \to g_0(\cdot)$ uniformly on compact subsets of M as $t \searrow 0$, and*

$$0 \leq \mathcal{I}(g(x,t)) \text{ for all } t \in (0,T],$$

$${}^g|\mathrm{Riem}(g(t))| \leq \frac{c}{\sqrt{t}} \text{ for all } t \in (0,T].$$

Proof : In four dimensions one can decompose the real two forms Λ^2 into the direct sum of Λ^2_+ and Λ^2_-. Then the curvature operator defined on $\Lambda^2 \otimes \Lambda^2$ decomposes as a block matrix

$$\mathcal{R} = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}.$$

The manifold has non-negative isotropic curvature if and only if $a_1 + a_2 \geq 0$ and $c_1 + c_2 \geq 0$, where a_1 and a_2 are the two smallest eigenvalues of A and c_1, c_2 are the two smallest eigenvalues of C. The ordinary differential equation for the evolution of a (c is the same) under Ricci flow is (see [Ha 5], proof of Theorem 1.2)

$$\frac{\partial}{\partial t}(a_1 + a_2) \geq a_1^2 + a_2^2 + 2(a_1 + a_2)a_3 + b_1^2 + b_2^2 \ \forall t \in [0, T], \ \forall x \in M,$$

where b_1 and b_2 are the two smallest eigenvalues of B (we ignore the lapcaian term for the moment). We consider the function $f(x,t) = a_1(x,t) + a_2(x,t)$ and note that it satisfies the O.D.E.

$$\frac{\partial}{\partial t} f \geq 2a_3 f,$$

where

$$\sup_{M \times [0,T]} |a_3| \leq \frac{c}{\sqrt{t}}.$$

We then argue exactly as above to show that the limit solution $g(x,t)$ of h-flow satisfies $(a_1 + a_2) \geq 0$. Similarly we obtain $c_1 + c_2 \geq 0$. The other estimates are obtained in exactly the same way as in theorem 1.3 ♢

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