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Ricci flow of almost non-negatively curved three manifolds

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Abstract

In this manuscript we study the evolution of almost non-negatively curved (possibly singular) three dimensional metric spaces by Ricci flow. The non-negatively curved metric spaces which we consider arise as limits of smooth Riemannian manifolds $(M_i, {}^i g)$, $i \in \mathbb{N}$, whose Ricci curvature is bigger than $-\frac{1}{i}$, and whose diameter is less than d_0 (independent of i) and whose volume is bigger than $v_0 > 0$ (independent of i). We show for such spaces, that a solution to Ricci flow exists for a short time, and we examine the local and global behaviour of such solutions.

Contents

Chapter 1

Introduction and statement of results

1.1 Geometry and topology of spaces with almost non-negative curvature

A much studied question in geometry and topology is

Problem 1.1.1. *(General) If we know certain facts about the geometry of a Riemannian manifold, can we say something about its topology/differential structure?*

Here we use the word *geometry* to denote objects related to the metric tensor g. For example the curvature, the diameter, the volume, and the metric tensor itself belong to the geometry of a manifold.

An example of a positive answer to Problem 1.1 is easily obtained in two dimensions. The Gauss-Bonnet Theorem tells us that the possible topologies of a two dimensional closed manifold are determined by the integral of the scalar curvature over the manifold. In particular, the Gauss-Bonnet theorem implies the following.

Fact 1.1.2. *Let (M, g) be an arbitrary $2-$dimensional closed Riemannian manifold with scalar curvature $\mathrm{R}(g) \geq 0$. Then the manifold is diffeomorphic to one of:*

- *the torus $\mathbb{T}^2$,*
- *the Klein bottle K^2*
- *the sphere $\mathbb{S}^2$*
- *the projective plane $\mathbb{RP}^2$.*

Notice that the projective plane is the quotient of the sphere by the subgroup of isometries $\{\mathrm{Id}, -\mathrm{Id}\}$ of the sphere, where $\mathrm{Id}(x) = -x$, for $x \in S^2 \subset \mathbb{R}^3$, and the Klein bottle and torus are quotients of $\mathbb{R}^2$ by a group of fixed point free isometries. In the case that the curvature on a two dimensional Riemannian manifold is not too negative, the Gauss-Bonnet theorem still gives a good classification of the possible topologies/differential Structures:

Fact 1.1.3. *Let (M^2, g) be an arbitrary two dimensional closed Riemannian manifold. with* $\mathrm{vol}(M, g) = 1$. *If* $\mathrm{R}(g) > -4\pi$ *then the surface must be one of the examples listed in 1.1.2.*

We may write this as

Fact 1.1.4. *If a closed 2−dimensional Riemannian manifold (M^2, g) satisfies* $\mathrm{vol} \cdot \mathrm{R} > -2\pi$ *then M^2 is diffeomorphic to S^2 or a quotient of $\mathbb{R}^2$ or S^2 by a group of fixed point free isometries.*

In Theorem [13] and [14], Hamilton showed using the Ricci flow that

Theorem A. (Theorem 1.2 of [14]) *If M^3 is a closed three dimensional Riemannian manifold with non-negative Ricci curvature then M^3 is diffeomorphic to a quotien of S^3, $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$ by a group of fixed point free isometries acting properly discontinuously.*

It is interesting to note that in order to apply the theorem we only require information on the Ricci curvatures (not the sectional curvatures). We say that a smooth family of metrics $(M, g(t))_{t\in[0,T)}$ is a solution to the Ricci flow with initial value g_0, or is a Ricci flow of g_0 if

$$\begin{aligned} &\frac{\partial}{\partial t} g(t) = -2\mathrm{Ricci}(g(t)) \ \forall \ t \in [0, T), \\ &g(0) = g_0 \end{aligned} \tag{1.1.1}$$

In [14], Hamilton continued his study of the effect of the Ricci flow on manifolds with non-negative curvature. He showed there that a similar result to the one above holds also in four dimensions, if one assumes non-negative curvature operator (this is a lot more restrictive than having non-negative Ricci curvature or non-negative sectional curvature). More precisely, he proved

Theorem B. (Theorem 1.3 of [14]) *A compact four-manifold with non-negative curvature operator is diffeomorphic to a quotient of one of the spaces S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $S^3 \times \mathbb{R}^1$, $S^2 \times \mathbb{R}^2$ or $\mathbb{R}^4$ by a group of fixed point free isometries in the standard metric acting properly discontinuously.*

In higher dimensions, the sphere theorems of Rauch [28], Berger [3], Klingenberg [22] et al. tell us (see chapter 6 of [10] for a proof of this result)

Theorem C. (Theorem 6.1 of [10]) *If (M^n, g) is a complete, simply connected n-dimensional Riemannian manifold with all sectional curvatures larger than $\frac{1}{4}$ but not larger than 1, then M is homeomorphic to the sphere.*

In the two and three dimensional cases, non-negative Ricci curvature was enough to classify the possible topologies. At least in the two dimensional case, it is sufficient that, after scaling so that the volume is equal to one, the curvature is not too negative.

In three and four dimensions, there are also results allowing some negative curvature, as the following theorems illustrate. In Simon [29] it was shown that

Theorem D. (Theorem 1.2 of [29]) *For every $d_0, v_0 > 0$ there exists a constant $\varepsilon = \varepsilon(d_0, v_0) > 0$ such that if (M^n, g), $n = 2$ or 3 ($n = 4$), is a closed Riemannian manifold with*

$$\sup_M |\text{Riem}| \leq 1 \tag{1.1.2}$$

$$\text{vol} \geq v_0 > 0 \tag{1.1.3}$$

$$\text{diam } \leq d_0 < \infty \tag{1.1.4}$$

$$\text{Ricci} \geq -\varepsilon \quad (\mathcal{R} \geq -\varepsilon) \tag{1.1.5}$$

then M admits a smooth metric with non-negative Ricci curvature (curvature operator). Here $\mathcal{R}$ refers to the curvature operator of (M, g). Hence, for $n = 3$, M^3 must be one of the possibilities listed in Theorem A. For $n = 4$, M^4 is diffeomorphic to one of the possibilities listed in Theorem B.

Notice that after scaling so that $\sup |\text{Riem}| = 1$ (which implies condition (1.1.2)), (1.1.3) and (1.1.4) always hold for some constants d_0, v_0. Hence the condition (1.1.5) says that M is not too negatively Ricci (curvature operator) curved once we have scaled so that (1.1.2), (1.1.3), and (1.1.4) hold.

Before continuing this discussion, we would like to say a few words about the method of proof of this theorem, as this method will be a guide line for the discussion which follows.

Let d_0, v_0 be fixed and assume that the theorem is not true (fix $n = 3$). Then there exists a sequence of Riemannian manifolds $(M_i^3, {}^i g)$ satisfying (1.1.2), (1.1.3), and (1.1.4), and so that $\text{Ricci} \geq -\frac{1}{i}$, but so that M_i^3 *does not* admit a metric with $\text{Ricci} \geq 0$.

Definition 1.1.5. *Let $\mathcal{B}(n, d_0, -k_1^2, k_2^2, v_0)$ denote the space of smooth n-dimensional Riemannian manifolds, whose sectional curvatures are bounded from below by $-k_1^2$ bounded above by k_2^2, whose diameter is bounded above by d_0 and whose volume is bounded below by v_0.*

It is well known that the space $\mathcal{B}(n, d_0, -k_1^2, k_2^2, v_0)$ is precompact in the Gromov-Hausdorff space. That is, given a sequence of smooth n-dimensional Riemannian manifolds $(M_i^n, g_i)_{i \in \mathbb{N}} \in \mathcal{B}(n, d_0, -k_1^2, k_2^2, v_0)$ there exists a metric space (X, d_∞) and a subsequence of (M_i^n, g_i) (which we also call (M_i^n, g_i) for ease of reading) such that $(M_i^n, d(g_i)) \xrightarrow{i \to \infty} (X, d_\infty)$, in the Gromov-Hausdorff sense, where here $d(g)$ denotes the distance function (metric) $d(g) : M \times M \to \mathbb{R}_0^+$ arising from the Riemannian metric g (see Appendix A). The Gromov Hausdorff (space) distance between two metric spaces is defined in Appendix A. It is a very weak measure of how close two metric spaces are to being isometric to one another.

As $\mathcal{B}(n, d_0, -k_1^2, k_2^2, v_0)$ has very well controlled geometry, it turns out that in this case all possible limits X are compact n-dimensional manifolds, and that d_∞ is the distance metric arising from a $C^{1,\alpha}$ Riemannian metric g_∞. Furthermore, X is diffeomorphic to M_i for large enough i. See [26]. In [29] it was shown, using Ricci flow techniques, that it is possible to *smooth out* the limit metric g_∞ to obtain a smooth metric g so that (X, g) has non-negative Ricci curvature (a Ricci flow of g_∞ was constructed by taking a limit of the solutions arising from each g_i). This is a contradiction. Hence, such an ε must exist.

As we mentioned above, $\mathcal{B}(n, d_0, -k_1^2, k_2^2, v_0)$ has very well controlled geometry, and so its closure does not contain very irregular Riemannian manifolds.

In [12] Fukaya and Yamaguchi showed that

Theorem E. (Corollary 0.13 of [12]) *There exists an $\varepsilon > 0$ such that if (M^3, g) is a Riemannian manifold whose diameter is not larger than* 1, *and has* $\sec \geq -\varepsilon$, *then a finite covering of M is either*

- *homotopic to an S^3 or*
- *diffeomorphic to one of*
  - *(a)* T^3
  - *(b)* $S^1 \times S^2$
  - *(c)* Nil

Hence, using that the Poincaré Conjecture is correct (see Perelman's papers [24], [25]) (that is, a homotopy S^3 is homeomorphic to S^3), we have a good topological classification of 3-manifolds with $\sec \cdot \text{diam}^{\,2} \geq -\varepsilon$ and ε small enough.

In studying the problem of Fukaya and Yamaguchi everything becomes a lot more complicated than in the situation of [29] above. There are essentially two extra problems that can now occur when studying this problem: *collapsing* and *no upper curvature bound*. We explain this in the following.

Definition 1.1.6. *For $n \in \mathbb{N}$, $d_0 \in \mathbb{R}^+$ and $k \in \mathbb{R}$ let $\mathcal{S}(n, d_0, k)$ denote the space of smooth n-dimensional Riemannian manifolds of dimension n with diameter bounded by d_0 and sectional curvature not less than k.*

It is well known, see for example proposition 10.7.3, and Remark 10.7.5 in [5] that the space $\mathcal{S}(n, d_0, k)$ is precompact in the Gromov-Hausdorff space, just as $\mathcal{B}$ was. That is, given a sequence of smooth n-dimensional Riemannian manifolds $(M_i^n, g_i)_{i\in\mathbb{N}} \in \mathcal{S}(n, d_0, k)$, there exists a metric space (X, d_∞) and a subsequence of (M_i^n, g_i)(for ease of reading we will denote this subsequence also as (M_i^n, g_i)) such that $(M_i, d(g_i))$ converges to (X, d_∞) in the Gromov Hausdorff space.

It is however now possible that X is not diffeomorphic to the M_i's (for large i) and X could be a manifold (with or without boundary) of lower dimension. One may see this easily by considering the following example of shrinking torii

Example 1.1.7. *Let*

$$(M_i, g_i) = (S^1 \times \ldots \times S^1, f_1(i)d\alpha^2 \oplus f_2(i)d\alpha^2 \ldots \oplus f_n(i)d\alpha^2),$$

$i \in \mathbb{N}$ where $d\alpha^2$ is the standard metric on S^1, and $f_j(i) \in \mathbb{R}^+ \overset{i\to\infty}{\longrightarrow} 0$, for all $j \in \{1, \dots, n-m\}$, and $f_j(i) = 1$ for all $j \in \{n-m+1, \dots, n\}$ (that is, $n-m$ circles shrink to points as $i \to \infty$ and the others stay fixed). Then $(M_i^n, d(g_i))$ converges in the Gromov Hausdorff space to $(\mathbb{T}^m, d_{\mathbb{T}^m})$, the smooth standard $m-$dimensional torus.

Definition 1.1.8. *If*

$$\mathrm{vol}(M_i, g_i) \overset{i\to\infty}{\longrightarrow} 0$$

for a sequence of smooth Riemannian manifolds (M_i, g_i) then we say that the sequence is a collapsing sequence, or that the sequence collapses. If there exists a $v_0 > 0$ such that

$$\mathrm{vol}(M_i, g_i) \geq v_0 \ \forall \ i \in \mathbb{N},$$

then we say that the sequence is a non-collapsing sequence, or that the sequence does not collapse.

It is also possible that the limit space (X, d_∞) does not enjoy the regularity properties of the spaces occurring in the converging sequence, as one sees in the following example.

Example 1.1.9. *Let $(S_i^n, g_i)_{i\in\mathbb{N}}$ be a sequence of spheres with Riemannian metrics, where the metrics are chosen so that*

- *the sectional curvature is non-negative*
- *the manifolds are becoming cone like in a fixed compact region (topologically a closed disc) as $i \to \infty$, and stay smooth away from this region.*
- *the diameter is bounded above by $0 < d_0 < \infty$ and the volume bounded below by $v_0 > 0$ where d_0, v_0 are constants independent of $i \in \mathbb{N}$.*

Then $(S_i^n, d(g_i))$ converges in the Gromov Hausdorff space to (S^n, d), where d is a (non-standard) metric on the sphere, and there exists a Riemannian metric g which is smooth away from the tip, induces d, but cannot be extended in a C^0 way to the tip. It is not possible to find a C^0 Riemannian metric g which induces d.

So even when a sequence $(M_i^n, g_i)_{i\in\mathbb{N}} \in \mathcal{S}(n, d, k)$ is non-collapsed, it is possible that the limit space (X, d) may be *irregular* if the curvature of the manifolds in the sequence is not bounded from above.

Let us return now to the situation of Fukaya and Yamaguchi (Theorem E). Assume the theorem were false. Then there exists a sequence of (M_i, g_i) with

$$\begin{aligned} \sec(g_i) &\geq -\frac{1}{i} \\ \text{diam } (g_i) &\leq 1 \end{aligned} \tag{1.1.6}$$

but so that M_i is not one of the listed possibilities. If we rescale so that diam $= \frac{1}{i}$ we obtain a new sequence (which for convenience we will also call (M_i, g_i)) with

$$\sec(g_i) \geq -1$$

$$\text{diam}\ (g_i) \quad = \frac{1}{i} \tag{1.1.7}$$

but so that M_i is not one of the listed possibilities. That is, the new sequence is a collapsing sequence (in view of the Bishop-Gromov comparison principle) which collapses to a single point and has curvature bound from below by -1. In this sense, the theorem gives us information on collapsing sequences in $\mathcal{S}(n, 1, -1)$.

In [32], further results about collapsing sequences of manifolds in $\mathcal{S}(n, d_0, -1)$ were proved by Shiyoa and Yamaguchi. In particular, they examined the cases that the limit X of a collapsing sequence in $\mathcal{S}(3, d_0, -1)$ is a *one or two dimensional Alexandrov space with curvature bounded from below by* -1 (that is, the metric space in question satisfies certain comparison inequalities, similar to those satisfied by Riemannian manifolds with curvature bounded from below by -1: see Appendix A). They proved

Theorem F. (Theorems 0.2 - 0.7 of [32]) *For a given $d_0 < \infty$, there exists a positive constant $\varepsilon = \varepsilon(d_0) > 0$ satisfying the following: If*

$$\begin{aligned} (M, g) \quad &\in \mathcal{S}(3, d_0, -1), \\ \text{vol}(M, g) \quad &\leq \varepsilon \end{aligned} \tag{1.1.8}$$

then one of the following holds

- *M is a graph manifold, or*
- *a finite cover of M is a simply connected manifold.*

Hence, using that the Poincaré Conjecture is correct (see Perelman [24] , [25]), we obtain a good classification of three dimensional manifolds in $\mathcal{S}(3, d_0, -1)$ with vol $\leq \varepsilon$ for $\varepsilon = \varepsilon(d_0)$ small enough.

In [33] Shiyoa and Yamaguchi continued their study of such collapsing sequences. In their paper [33] they consider the case that the diameter of a collapsing sequence may go to infinity. They showed

Theorem G. (Theorem 1.1 of [33]) *There exist $0 < \varepsilon_1 < \infty$ and $d_1 < \infty$ such that if (M^3, g) satisfies* $\sec \geq -1$ *and* vol $\leq \varepsilon_1$ *then either*

- *[case 1] M is homeomorphic to a graph manifold or,*
- *[case 2] a finite cover of M has finite fundamental group, and* diam $(M, g) \leq d_1$.

Hence, once again, using that the Poincaré Conjecture is correct (see Perelman [24], [25]), we obtain a good classification of three dimensional manifolds M^3 with $\sec \cdot \text{vol}^{\frac{2}{3}} \geq -\varepsilon$ for ε small enough.

In this paper we consider the somewhat weaker space of *Riemannian manifolds with Ricci curvature bounded from below.*

Definition 1.1.10. *For $n \in \mathbb{N}$, $d_0 \in \mathbb{R}^+$ and $k \in \mathbb{R}$ let $\mathcal{M}(n, d_0, k)$ denote the space of smooth n-dimensional Riemannian manifolds of dimension n with diameter bounded above by d_0 and Ricci curvature not less than k.*

Clearly $\mathcal{S}(n, d_0, k) \subset \mathcal{M}(n, d_0, (n-1)k)$.

In this manuscript we will be chiefly concerned with metric spaces (M^3, d_∞) which arise as Gromov-Hausdorff limits of non-collapsing sequences of Riemannian manifolds $(M_i^3, g_i) \in \mathcal{M}(3, d_0, -\varepsilon(i))$ where $\varepsilon(i) \to 0$ as $i \to \infty$.

We show that a Ricci flow solution with initial data (M^3, d_∞) exists for a short time (see Theorem 5.2), and we examine the local and global behaviour of such solutions

Theorem 5.2 also contains the following classification theorem.

Theorem 1.1.11. *For all $0 < v_0 < \infty, 0 < d_0 < \infty$ there exists an $\varepsilon = \varepsilon(v_0, d_0) > 0$ such that if (M^3, g) is closed and satisfies*

$$\text{vol}(M, g) \geq v_0$$

and $(M, g) \in \mathcal{M}(3, d_0, -\varepsilon)$ then M is diffeomorphic to one of

(a) A quotient of S^3 by a finite group of fixed-point free isometries acting properly discontinuously

(b) A quotient of $S^2 \times \mathbb{R}$ by a finite group of fixed-point free isometries acting properly discontinuously ,

(c) A quotient of $\mathbb{R}^3$ by a finite group of fixed-point free isometries acting properly discontinuously

We may write this in a more scale invariant form:

Theorem 1.1.12. *Let d_0 be given. There exist $0 < \varepsilon_2 = \varepsilon_2(d_0) < \infty$ such that if (M^3, g) satisfies*

$$\begin{aligned} &\text{Ricci} \cdot \text{vol}^{\frac{2}{3}} \geq -\varepsilon_2 \\ &\text{diam}\ ^3 \leq d_0^3 \cdot \text{vol} \end{aligned} \tag{1.1.9}$$

then M is one of

(a) S^3 or a quotient thereof by a finite group of fixed-point free isometries acting properly discontinuously,

(b) A quotient of $S^2 \times \mathbb{R}$ by a finite group of fixed-point free isometries acting properly discontinuously,

(c) A quotient of $\mathbb{R}^3$ by a finite group of fixed-point free isometries acting properly discontinuously

Proof. Assume the theorem is false. Then there exists a sequence of Riemannian manifolds (M_i, g_i) with Ricci $\cdot$ vol$^{\frac{2}{3}} \geq -\varepsilon_i$ for a sequence $\varepsilon_i \in \mathbb{R}^+$ with $\varepsilon_i \stackrel{i\to\infty}{\longrightarrow} 0$ and so that diam $^3(g_i) \leq d_0^3 \mathrm{vol}(g_i)$. Scale all (M_i, g_i) so that $\mathrm{vol}(g_i) = 1$. We then have diam $(g_i) \leq d_0$ and Ricci $\geq -\varepsilon_i$ and so we may then apply the Theorem 1.1.11 to obtain a contradiction. □

As a byproduct, we note that if we combine Theorem G and Theorem 1.1.11 we obtain the same classification which is implied by using Theorem G, but we do not need to use that the Poincaré Conjecture is correct.

Theorem 1.1.13. *Let ε_1 and d_1 be as in Theorem G. There exist $0 < \varepsilon_3 < \infty$ such that if (M^3, g) satisfies* sec $\cdot$vol$^{\frac{2}{3}} \geq -\varepsilon_3$ *and*

- *[case 1]* diam $^3 \geq \frac{2(d_1)^3}{\varepsilon_1}$vol *then M is homeomorphic to a graph manifold*
- *[case 2]* diam $^3 \leq \frac{2(d_1)^3}{\varepsilon_1}$vol *then M is one of*
  - *(a) S^3 or a quotient thereof by a finite group of fixed-point free isometries acting properly discontinuously,*
  - *(b) A quotient of $S^2 \times \mathbb{R}$ by a finite group of fixed-point free isometries acting properly discontinuously ,*
  - *(c) A quotient of $\mathbb{R}^3$ by a finite group of fixed-point free isometries acting properly discontinuously*

Proof. Assume the theorem is false. Then there exists a sequence of Riemannian manifolds (M_i^3, g_i) with sec $\cdot$vol$^{\frac{2}{3}} \geq -\varepsilon_i$ for a sequence $\varepsilon_i \in \mathbb{R}^+$ with $\varepsilon_i \stackrel{i\to\infty}{\longrightarrow} 0$ and so that either (after taking a subsequence)

- case1 diam $^3(g_i) \geq \frac{2(d_1)^3}{\varepsilon_1}\mathrm{vol}(g_i)$ but M_i is not homeomorphic to a graph manifold, or
- case2 diam $^3(g_i) \leq \frac{2(d_1)^3}{\varepsilon_1}\mathrm{vol}(g_i)$ but M_i is not one of the possibilities listed in case 2 of the statement of this theorem.

In case 1, we scale all (M_i, g_i) so that $\mathrm{vol}(g_i) = \varepsilon_1$. Then diam $(g_i) \geq 2d_1$, and sec ≥ -1 for i sufficiently large, and so we may apply Theorem G to obtain a contradiction.

In case 2, let us scale all (M_i, g_i) so that $\mathrm{vol}(g_i) = 1$. We then have diam $(g_i) \leq \frac{2d_1}{(\varepsilon_1)^{\frac{1}{3}}}$, and sec $\geq -\varepsilon_i$ and so we may then apply the Theorem 1.1.11 to obtain a contradiction. □

1.2 Discussion of the Ricci flow of almost non-negatively curved metric spaces and statement of results

In this manuscript we will cheifly be concerned with metric spaces (M^3, d_∞) which arise as Gromov-Hausdorff limits of non-collapsing sequences of Riemannian manifolds $(M_i^3, g_i) \in \mathcal{M}_i(3, d_0, -\varepsilon(i))$ where $\varepsilon(i) \to 0$ as $i \to \infty$. In particular, we wish to flow such metric spaces (M^3, d_∞) by Ricci Flow. As we saw in the previous section (see Example 1.1.9) such limits can be quite irregular (not even C^0). Nevertheless, they will be Alexandrov Spaces of curvature bounded from below by 0, and so do carry some structure (see Appendix A). In order to flow (M^3, d_∞) we will flow each of the (M_i^3, g_i) and then take a Hamilton limit of the solutions (see [17]). The two main obstacles to this procedure are:

- It is possible that the solutions $(M_i, g_i(t))$ are defined only for $t \in [0, T_i)$ where $T_i \to 0$ as $i \to \infty$.

- In order to take this limit, we require that each of the solutions satisfy uniform bounds of the form

  $$\sup_{M_i} |\text{Riem}(g_i(t))| \leq c(t), \ \forall\, t \in (0, T),$$

  for some well defined common time interval $(0, T)$ ($c(t) \to \infty$ as $t \to 0$ would not be a problem here). Furthermore they should all satisfy a uniform lower bound on the injectivity radius of the form

  $$\text{inj}(M, g_i(t_0)) \geq \sigma_0 > 0$$

  for some $t_0 \in (0, T)$.

As a first step to solving these two problems, in Lemma 2.1.4 of Chapter 2 we see that a (three dimensional) smooth solution to the Ricci flow $(M, g(t))_{t\in[0,T)}$ cannot become singular at time T as long as Ricci ≥ -1, the diameter remains bounded (by say d_0) and the volume stays bounded away from zero (say it is bigger than v_0). Furthermore, a bound of the form

$$|\text{Riem}(g(t))| \leq \frac{c_0(d_0, v_0)}{t} \forall t \in [0, T) \cap [0, 1]$$

for such solutions is proved: that is, the curvature of such solutions is quickly smoothed out.

In Lemma 2.1.5 we present an application of the proof of 2.1.4.Notice that Proposition 11.4 of [24] for the three dimensional case implies the Lemma 2.1.5. Perelmans method of proof is somewhat different from that used in Lemma 2.1.5.

Chapter three is concerned with proving (for an arbitrary three dimensional solution to the Ricci flow) lower bounds for the Ricci curvature of the evolving metric, which depend on

- the bound from below for the Ricci curvature of the initial metric
- the scalar curvature of the evolving metric.

One of the major applications is: if (M, g_0) satisfies $\mathrm{Ricci}(g_0) \geq -\varepsilon_0$ (ε_0 small enough) and the solution satisfies $\mathrm{R}(g(t)) \leq \frac{c_0}{t}$, then

$$\mathrm{Ricci}(g(t)) \geq -2c_0\varepsilon_0 \forall t \in (0, T_*) \cap (0, T)$$

for some universal constant $T_* = T_* > 0$ ($(0, T)$ is the time interval for which the solution is defined). See Lemma 3.4.

In Chapter 4, we consider smooth solutions to the Ricci flow which satisfy

$$\mathrm{Ricci}(g(t)) \geq -c_0 \tag{1.2.1}$$
$$|\mathrm{Riem}(g(t))|t \leq c_0 \tag{1.2.2}$$
$$\mathrm{diam}\ (M, g_0) \leq d_0 \tag{1.2.3}$$

In Lemma 4.1, well known bounds on the evolving distance for a solution to the Ricci flow are proved for such solutions.

We combine this Lemma with some results on Gromov-Hausdroff convergence to show (Corollary 4.2) that such solutions can only lose volume at a controlled rate.

In Chapter 5 we show (using the a priori estimates from the previous chapters) that a solution to the Ricci flow of (M, d_∞) exists, where (M, d_∞) is the Gromov-Hausdorff limit of $(M_i, d(g_i))$ where the (M_i, g_i) satisfy

$$\begin{aligned}&\mathrm{Ricci}(g_i) \geq -\varepsilon(i)\\&\mathrm{vol}(M, g_i) \geq v_0\\&\mathrm{diam}\ (M, g_0) \leq d_0\end{aligned}$$

where $\varepsilon(i) \to 0$ as $i \to \infty$. It is also shown that the solution is smooth for all $t > 0$ (as long as the solution exists) and is in the class of those solutions considered in Chapter 4. That is, the solution satisfies 1.2.1, 1.2.2, 1.2.3, (for $t > 0$) and (due to Corollary 4.2) can only lose volume at a certain rate.

Chapter 6 is broken up into 3 sections. In the first section we show, using some ideas of Shi [34], that if an initial metric g_0 is C^0 close enough to a fixed smooth background metric h, (we assume that (M, h) is compact) then a Ricci flow solution exists for a well defined time interval, and the volume and diameter remain well controlled, and the curvature improves on this time interval. The theorem of this section is largely an application of Theorem 1.1 of [29].

In Section 2 of Chapter 6, we construct a family of smooth (initial) metrics $g_i(0)$ which pinch in time $\varepsilon(i)$ ($\varepsilon(i) \to 0$ as $i \to \infty$) when evolved by Ricci flow, and satisfy uniform estimates

$$\frac{1}{\delta} g_1(0) \leq g_i(0) \leq \delta g_1(0)$$

for all $i \in \mathbb{N}$ for some $\infty > \delta > 0$. That is, the geometry being bounded at a C^0 level at time zero is not enough to ensure that the solution exists for some well defined time interval.

In Section 3 of Chapter 6, we show that a bound of the form $g_0 \geq h$ for some fixed background metric h on M^n is enough to ensure that the volume of a solution to Ricci flow $(M^n, g(t))_{t\in[0,T)}$ $(g(0) = g_0)$ decreases at most at a finite rate. In particular, for a well controlled time interval (depending only on (M^n, h)) the volume of the solution remains bigger than $\delta(n, h) > 0$.

In Chapter 7 we obtain estimates which show that if a solution to Ricci flow is of the type coming from Chapter 5, then certain initial local behaviours are preserved by the flow.

In particular, in Section 2 of Chapter 7, we show that: if $(M, g(t))_{t\in[0,T)}$ is a solution to Ricci flow of the type coming from Chapter 5, and (Ω, g_0) is bounded on the C^0 and curvature level (see Definition 6.2.1) then for some well defined interior region $\Omega' \subset \Omega$ and some well defined time interval $(0, S)$, $(\Omega', g(t))$ is bounded on the C^0 and curvature level for $t \in (0, S)$. See Theorem 6.2.2.

In Section 3 of Chapter 7, we prove similar results to those of Section 2 for solutions to the heat flow.

Appendix A contains definitions, results and facts about Gromov-Hausdorff space, which we require in this manuscript.

In Appendix B we give a brief introduction to C-essential points and δ-like necks. See the Book of B.Chow and D.Knopf ([7]) for more information and results related to C-essential points and δ-like necks.

A proof of the (well known) Lemma 4.1 is contained in Appendix C.

Appendix D is a description of the Notation used in this manuscript.

Chapter 2

Bounding the blow up time using bounds on the geometry.

2.1 Controlling the blow up time of Ricci flow using the geometry

An important property of the Ricci flow is that:
if certain geometrical quantities are controlled (bounded) on a half open finite time interval $[0, T)$, then the solution does not become singular as $t \nearrow T$ and may be extended to a solution defined on the time interval $[0, T + \varepsilon)$ for some $\varepsilon > 0$. We are interested in the question:

Problem 2.1.1. *What elements of the geometry need to be controlled, in order to guarantee that a solution does not become singular?*

In [13], it was shown that for (M, g_0) a closed smooth Riemannian manifold, the Ricci flow equation

$$\begin{aligned} &\frac{\partial}{\partial t} g = -2\text{Ricci}(g) \\ &g(\cdot, 0) = g_0, \end{aligned} \tag{2.1.1}$$

always has a solution $(M, g(t))_{t\in[0,T)}$ for a short time. It was also shown that two such solutions defined on the same time interval must agree, if there initial values agree. Furthermore, for each smooth, closed (M, g_0) there exists a maximal time interval $[0, T_{Max})$ ($T_{Max} > 0$) for which, there exists a solution $(M, g(t))_{t\in[0,T_{Max})}$ to (2.1.1), and if $T_{Max} < \infty$ then there is no solution $(M, g(t))_{t\in[0,T_{Max}+\varepsilon)}$ to (2.1.1) (for any $\varepsilon > 0$). Such a solution $(M, g(t))_{t\in[0,T_{Max})}$ is called a *maximal solution*.

Definition 2.1.2. *(Maximal Solutions) Let $(M, g(t))_{t\in[0,T)}$ be a solution to Ricci flow. We say that the solution blows up at time T if*

$$\sup_{M\times[0,T)} |\text{Riem}| = \infty, \tag{2.1.2}$$

It was also shown in [13] that

Lemma 2.1.3. *Let* $(M, g(t)_{t\in[0,T)}$ *be a closed, smooth solution to Ricci flow, with* $g(0) = g_0$ *and* $T < \infty$, *with*

$$\sup_{M\times[0,T)} |\mathrm{Riem}| < \infty. \tag{2.1.3}$$

Then, for some $\varepsilon > 0$, *there exists a solution* $(M, g(t))_{t\in[0,T+\varepsilon)}$, *with* $g(0) = g_0$.

So we see that a bound on the supremum of the Riemannian curvature (that is, *control* of this geometrical quantity) on a finite time interval $[0, T)$ guarantees that this solution does not become singular as $t \nearrow T$. In the following lemma, we present other bounds on geometrical quantities which guarantee that a solution to the Ricci flow does not become singular as $t \nearrow T$.

Lemma 2.1.4. *Let* $(M^3, g(t))_{t\in[0,T)}$, $T \leq 1$ *be an arbitrary smooth solution to Ricci flow* (M^3 *closed*) *satisfying*

$$\begin{aligned} \mathrm{Ricci}(g) &\geq -1, \\ \mathrm{vol}(M, g) &\geq v_0 > 0 \\ \mathrm{diam}\ (g) &\leq d_0 < \infty \end{aligned} \tag{2.1.4}$$

for all $t \in [0, T)$. *Then there exists a* $c = c(d_0, v_0)$, *such that*

$$\mathrm{R}(g(t))t \leq c$$

for all $t \in [0, T)$. *In particular,* $(M^3, g(t))_{t\in[0,T)}$ *is not maximal.*

Corollary 2.1.5. *Let* $(M^3, g(t))_{t\in[0,T)}$ *be an arbitrary smooth solution to Ricci flow satisfying*

$$\begin{aligned} \mathrm{Ricci}(g) &\geq -1, \\ \mathrm{vol}(M, g) &\geq v_0 > 0 \\ \mathrm{diam}\ (g) &\leq d_0 < \infty \end{aligned} \tag{2.1.5}$$

for all $t \in [0, T)$. *Then there exists a* $c = c(d_0, v)$, *such that*

$$\mathrm{R}(g(t)) \leq c \max(\frac{1}{t}, 1)$$

for all $t \in [0, T)$. *In particular,* $(M^3, g(t))_{t\in[0,T)}$ *is not maximal.*

The proof of the corollary is a trivial iteration argument.

Proof. (of the corollary) Fix $t_0 \in [0, T)$. We wish to show that

$$\mathrm{R}(g(t_0)) \leq c \max(\frac{1}{t_0}, 1).$$

If $t_0 \leq \frac{1}{2}$ then we apply Lemma 2.1.4. If $\frac{(N+1)}{2} > t_0 \geq \frac{N}{2}$, ($N \in \mathbb{N}$) then we apply Lemma 2.1.4 to the solution $(M, g(\frac{(N-1)}{2} + t))_{t\in[\frac{1}{2},1)}$ of Ricci flow (notice that $\frac{(N-1)}{2} + t = t_0$ implies that $1 > t \geq \frac{1}{2}$). ∎

We now prove Lemma 2.1.4.

Proof. Assume to the contrary that there exist solutions $(M_i, {}^i g(t))_{t\in[0,T_i)}$, $T_i \le 1$ to Ricci flow such that

$$\sup_{(x,t)\in M_i\times(0,T_i)} {}^i\mathrm{R}(x,t)t \overset{i\to\infty}{\longrightarrow} \infty, \tag{2.1.6}$$

or there exists some $j \in \mathbb{N}$ with

$$\sup_{(x,t)\in M_j\times(0,T_j)} {}^j\mathrm{R}(x,t)t = \infty, \tag{2.1.7}$$

where ${}^i\mathrm{R} := \mathrm{R}({}^i g)$. It is then possible to choose points $(p_i, t_i) \in M_i \times [0, T_i)$ (or in $M_j \times [0, T_j)$: in this case we redefine $M_i = M_j$ and $T_i = T_j$ for all $i \in \mathbb{N}$ and hence we do not need to treat this case separately) such that

$$\mathrm{R}(p_i, t_i)t_i = \sup_{(x,t)\in M_i\times(0,t_i]} {}^i\mathrm{R}(x,t)t \overset{i\to\infty}{\longrightarrow} \infty. \tag{2.1.8}$$

Define

$${}^i\hat{g}(\cdot,\hat{t}) := c_i\, {}^i g(\cdot, t_i + \frac{\hat{t}}{c_i}), \tag{2.1.9}$$

where $c_i := {}^i\mathrm{R}(p_i, t_i)$. This solution to Ricci flow is defined for $0 \le t_i + \frac{\hat{t}}{c_i} < T_i$, that is, at least for $0 \ge \hat{t} > -t_i c_i$. Let $A_i := t_i c_i$. Then the solution ${}^i\hat{g}(\hat{t})$ is defined at least for $\hat{t} \in (-A_i, 0)$. By the choice of (p_i, t_i) we see that the solution is defined for $\hat{t} > -A_i = -t_i c_i = -t_i\, {}^i\mathrm{R}(p_i, t_i) \overset{i\to\infty}{\longrightarrow} -\infty$. Since $t_i \le T_i \le 1$, we also have

$$c_i \overset{i\to\infty}{\longrightarrow} \infty, \tag{2.1.10}$$

in view of the fact that

$$t_i c_i = t_i\, {}^i\mathrm{R}(p_i, t_i) \overset{i\to\infty}{\longrightarrow} \infty.$$

Furthermore, letting $s(\hat{t}, i) := t_i + \frac{\hat{t}}{c_i}$, where $-A_i < \hat{t} \le 0$ we have

$$\begin{aligned} {}^i\hat{\mathrm{R}}(\cdot,\hat{t}) &= \frac{1}{c_i}\, {}^i\mathrm{R}(\cdot, s(\hat{t}, i)) && (2.1.11)\\ &= \frac{{}^i\mathrm{R}(\cdot, s)}{{}^i\mathrm{R}(p_i, t_i)} \\ &= \frac{{}^i\mathrm{R}(\cdot, s)s}{{}^i\mathrm{R}(p_i, t_i)t_i}\frac{t_i}{s} \\ &\le \frac{t_i}{s} \\ &= \frac{t_i}{t_i + \frac{\hat{t}}{c_i}} \overset{i\to\infty}{\longrightarrow} 1. && (2.1.12) \end{aligned}$$

in view of the definition of (p_i, t_i), and $0 \leq s \leq t_i$ (follows from the definition of s and the fact that $\hat{t} \leq 0$), and (2.1.10). Due to the conditions (2.1.4) we see that there exist $l = l(v_0, d, n)$, and $\varepsilon = \varepsilon(v_0, d, n)$, such that

$$l \geq \frac{\mathrm{vol}(B_r(p), {}^i g(t))}{r^3} \geq \varepsilon \ \forall \ r \leq \mathrm{diam} \ (M_i, {}^i g(t)), \tag{2.1.13}$$

(in view of the Bishop Gromov comparison principle) which implies the same result for any rescaling of the manifolds. Notice that the conditions (2.1.4) imply that

$$\mathrm{diam} \ (M, {}^i g(t)) \geq d_1(n, v_0) > 0 \tag{2.1.14}$$

for some $\infty > d_1(n, v_0) > 0$. Otherwise $\mathrm{vol}(M, {}^i g(t)) \leq c(n) d_1^3 \omega_n$ (Bishop-Gromov comparison principle), and hence $\mathrm{vol}(M, {}^i g(t)) < v_0$ if d_1 is too small, which would be a contradiction. Hence, $\mathrm{diam} \ (M, {}^i\hat{g}(0)) \overset{i\to\infty}{\longrightarrow} \infty$, in view of the inequalities (2.1.14) and (2.1.10). Now using

$$l \geq \frac{\mathrm{vol}(B_r(p), {}^i\hat{g}(t))}{r^3} \geq \varepsilon_0, \forall r \leq \mathrm{diam} \ (M_i, {}^i\hat{g}(\hat{t})), \tag{2.1.15}$$

we obtain a bound on the injectivity radius from below, in view of the theorem of Cheeger/Gromov/Taylor, [9] (the theorem of Cheeger/Gromov/Taylor says that for a complete Riemannian manifold (M, g) with $|Riem| \leq 1$, we have

$$\mathrm{inj}(x, g) \geq r \frac{\mathrm{vol}(g, B_r(x))}{\mathrm{vol}(g, B_r(x)) + \omega_n \exp^{n-1}},$$

for all $r \leq \frac{\pi}{4}$. In particular, using that $\mathrm{diam} \ (M, g) \geq d_1 > 0$ and $\mathrm{Riem} \leq c$ (see [i] beolow) for the Riemannian manifolds in question, we obtain

$$\mathrm{inj}(x, g) \geq \varepsilon \frac{s^{n+1}}{l s^n + \omega_n \exp^{n-1}} \geq c^2(d_0, v_0, n) > 0 \tag{2.1.16}$$

for $s = \min((\omega_n \exp^{n-1})^{\frac{1}{n}}, \mathrm{diam} \ (M, g), \frac{\pi}{4}))$.

This allows us to take a pointed *Hamilton limit* (see [17]), which leads to a Ricci flow solution $(\Omega, o, g(t)_{t\in(-\infty,\omega)})$, with $R \leq R(o, 0) = 1$, and Ricci ≥ 0, $\omega > 0$ (at $t = 0$, as explained below, the full Riemannian curvature tensor of ${}^i\hat{g}(0)$is bounded by $c(3)$ and so clearly each solution lives at least to a time $\omega > 0$ independent of i). More Precisely:

- [i] The bound from below on the Ricci curvature, and the bound from above on the scalar curvature imply that the Ricci curvatures are bounded absolutely by the constant 5 for i big enough. In three dimensions, bounds from above and below on the Ricci curvatures imply bounds from above and below on the sectional curvatures and hence on the norm of the full Riemannian curvature tensor. This, together with the bound from below on the injectivity radius, allows us to a take a Hamilton limit of these Ricci flows.

- [ii] In fact the limit solution satisfies sec ≥ 0, which can be seen as follows: Each rescaled solution ${}^i\hat{g}$ is defined on $M_i \times [-A_i, \omega]$ where $A_i \stackrel{i\to\infty}{\longrightarrow} \infty$. They also each satisfy sec ≥ -2 and $|\text{Riem}| \leq c(n)$ for all $t \in (-S, 0)$ for any fixed S and all i big enough, in view of (2.1.12) and Ricci ≥ -1.

  Let us translate in time by S , so that these solutions are defined on $M_i \times [-A_i + S, S]$ and satisfy sec ≥ -2 and $|\text{Riem}| \leq c(n)$ on $(0, S)$ (for i big enough).Without loss of generality, we assume that sec ≥ -1. We then use the improved pinching result of Hamilton [18] (see also [21]):

  Theorem 2.1.6. *Let $g(t)$ be a solution to Ricci flow defined on $M \times [0, T)$, M closed. Assume at $t = 0$ that the eigenvalues $\alpha \geq \beta \geq \gamma$ of the curvature operator at each point are bounded below by $\gamma \geq -1$. The scalar curvature is their sum $\mathrm{R} = \alpha + \beta + \gamma$, and $X := -\gamma$. Then at all points and all times we have the pinching estimate*

  $$\mathrm{R} \geq X[\log X + \log(1 + t) - 3],$$

  whenever $X > 0$.

  Notice that this estimate is also valid for the translated limit solution (defined on $[0, S)$) as it is valid for each i and the scalar curvature and X converge as $i \to \infty$ to the corresponding quantities of the translated (by S) limit solution.

  Let $\delta > 0$ be any arbitrary small constant. Assume there exists $(x, t) \in \Omega \times [\frac{S}{2}, S)$ such that $X(x, t) \geq \delta$. Then we get

  $$\begin{aligned} \log(\delta) \leq \log X(x,t) &\leq \frac{\mathrm{R}(x,t)}{\delta} - \log(1+t) + 3 \\ &\leq \frac{c(n)}{\delta} - \log(1 + \frac{S}{2}) + 3 \end{aligned} \tag{2.1.17}$$

  which is a contradiction for S big enough. Hence our initial limit solution (without any translations in time) has $X(x, 0) \leq \delta$. As δ was arbitrary we get $X(\cdot, 0) \leq 0$. A similar argument shows $X \leq 0$ everywhere. That is, the limit space satisfies sec $\geq 0 \forall t \in (-\infty, 0)$

The volume ratio estimates

$$l \geq \frac{\text{vol}(B_r(p))}{r^3} \geq \varepsilon_0 \forall r > 0, \tag{2.1.18}$$

are also valid for (Ω, g), as these estimates are scale invariant, and diam $(\Omega, g) = \infty$. At this point we could apply Proposition 11.4 of [24] to obtain a contradiciton. We prefer however to introduce an alternative method to Perelman in order to obtain a contradiction (this method may be of independent interest). We now consider the following two cases.

(case 1) $\sup_{\Omega \times (-\infty, 0]} |t| \mathrm{R} = \infty$, (case 2) $\sup_{\Omega \times (-\infty,0]} |t| \mathrm{R} < \infty$.

(case 1) In the first case, in view of [7], chapter 8, section 6, we may assume w.l.o.g. that there exists a solution $(\Omega, o, g(t)_{t\in(-\infty,\infty)})$, with

$$\sup_{\Omega\times(-\infty,\infty)} |\mathrm{R}(t)| \le 1 = |\mathrm{R}(0,o)|. \tag{2.1.19}$$

Note: we must slightly modify the argument there, by replacing Riem with R wherever it appears. We also use the fact (as mentioned above) that $|\mathrm{Riem}| \le c(3)\mathrm{R}$ in the case that Ricci ≥ 0 (in dimension three) and that our scale invariant volume estimate (2.1.18) remains true for any rescalings of our solution: these two facts ensure that in the rescaling process of the argument in [7], chapter 8, section 6, an injectivity radius estimate is satisfied, and that the limit solution is well defined.

(case 1.1) the sectional curvature is everywhere positive.

(case 1.2) there exists $(p_0, t_0) \in \Omega \times (-\infty, \infty)$, and $v_{p_0}, w_{p_0} \in T_{p_0}\Omega$ with $\sec(p_0,t_0)(v_{p_0}, w_{p_0}) = 0$.

First we consider case 1.1

(case 1.1) This means Ω is diffeomorphic to $\mathbb{R}^3$ in view of the soul theorem (see [10], Chapter 8) and in particular, Ω is simply connected. We may then apply the gradient soliton theorem of Hamilton [15] which implies, in view of (2.1.19), that $(\Omega, g(t))_{t\in(-\infty,\infty)}$ is a gradient soliton. We may then, using the dimension reduction theorem of Hamilton, Theorem 22.3 of [16], take a Hamilton limit of rescalings of this solution, to obtain a new solution, $(\mathbb{R}\times N, dx^2 \oplus \gamma(t))_{t\in(-\infty,\infty)}$, or a quotient thereof by a group of isometries acting properly discontinuously, where dx^2 is the standard metric on $\mathbb{R}$, and $(N, \gamma(t))_{t\in(-\infty,\infty)}$ is a solution to the Ricci flow, N is a surface, and $\mathrm{R}(\cdot, t) > 0$, on N. In the case that we have a quotient of $(\mathbb{R} \times N, dx^2 \oplus \gamma(t))$ then we notice that $(\mathbb{R} \times N, dx^2 \oplus \gamma(t))$ still satisfies (2.1.18) (the bound from below follows as the Riemannian covering map $f : (\mathbb{R} \times N, dx^2 \oplus \gamma(t)) \to (\Omega, g(t))$ is a Riemannian-Submersion, and the bound from above follows in view of the Bishop-Gromov comparison principle) and so, without loss of generality, we may assume that we do not have a quotient. If N is compact, then $(\mathbb{R}\times N, dx^2 \oplus \gamma)$, does not satisfy the estimates (2.1.18), and so we obtain a contradiction. So w.l.o.g. we may assume that N is non-compact. Now we break this up into two cases:

(case 1.1.1) $\sup_{N\times(-\infty,\infty)} |t||\mathrm{R}(t)| = \infty$, and

(case 1.1.2) $\sup_{N\times(-\infty,\infty)} |t||\mathrm{R}(t)| < \infty$.

First we handle

(case 1.1.1) Once again, w.l.o.g ([7] chap.8, sec.6), we may assume

$$\sup_{N\times(-\infty,\infty)} \mathrm{R} \leq 1 = \mathrm{R}(o,0).$$

$\mathrm{R}(t) > 0$, and N non-compact implies N is diffeomorphic to $\mathbb{R}^2$, which is simply connected. We may then use the gradient soliton theorem of Hamilton, [15] , to obtain that (N,γ), is a gradient soliton, which implies (thm. 26.3, [16]), that (N,γ) is the cigar (Σ, cig). But $(\mathbb{R}\times\Sigma, dx^2 \oplus \mathrm{cig})$ do not satisfy the estimates (2.1.18), and so we obtain a contradiction.

(case 1.1.2) $\sup_{N\times(-\infty,\infty)} |t||\mathrm{R}(t)| < \infty$. Hamilton, Thm. 26.1 of [16] implies that $(N,\gamma) = (\mathbb{S}^2 \text{ or } \mathbb{R}^2, \gamma)/\Gamma$, where γ is the standard solution on S^2 or $\mathbb{R}^2$, and Γ is a finite group of isometries acting without fixed points on the standard S^2 or standard $\mathbb{R}^2$. $(\mathbb{R}^2,\gamma)$ cannot occur, since the surface should satisfy $\mathrm{R}(t) > 0$ everywhere (the standard $(\mathbb{R}^2,\gamma)$ is flat). But then N is compact, and $(\mathbb{R}\times N, dx^2\oplus\gamma)$, does not satisfy the estimates (2.1.18), and once again we obtain a contradiction.

(case 1.2) there exists $(p_0,t_0) \in \Omega\times(-\infty,\infty)$, and $v_{p_0}, w_{p_0} \in T_{p_0}\Omega$ with $\sec(p_0,t_0)(v_{p_0},w_{p_0}) = 0$. Then the maximum principle applied to the evolution equation of the curvature operator, implies that $(\Omega, o, g(t))_{t\in(-\infty,\infty)} = (\mathbb{R}\times N, dx^2\oplus\gamma(t))_{t\in(-\infty,\infty)}$, or a quotient thereof by a group of isometries(see [14], chapter 9) and $\sup_{N\times(-\infty,\infty)} \mathrm{R}(t) \leq 1 = \mathrm{R}(o,0)$. Without loss of generality, we may assume that we don't have a quotient, as explained in case 1.1. $\mathrm{R}(t) > 0$, implies N is diffeomorphic to $\mathbb{S}^2/\Gamma$ or $\mathbb{R}^2$. In the case that N is diffeomorphic to $\mathbb{S}^2/\Gamma$, we obtain a contradiction, as then (Ω, g) does not satisfy (2.1.18). So w.l.o.g. N is diffeomorphic to $\mathbb{R}^2$, in particular N is simply connected. We may use the gradient soliton theorem of Hamilton [15], to get that (N,γ) is a soliton and it must be the Cigar, in view of theorem 26.3 of Hamilton [16]. This leads to a contradiction as then (Ω, g) does not satisfy (2.1.18) (similarly for the covering case).

(Case 2) $B := \sup_{\Omega\times(-\infty,0]} |t||\mathrm{Riem}(t)| < \infty$.

(Case 2.1) The asymptotic scalar curvature ratio $A = \limsup_{s\to\infty} \mathrm{R}s^2 = \infty$. Then we use the dimension-reduction argument of Hamilton (see Lemma 22.2 of [16] and the argument directly after the proof of Lemma 22.2) to obtain a new solution $(N\times\mathbb{R}, \gamma\oplus dx^2)$ or a quotient thereof by a group of isometries where (N,γ) is a solution to Ricci flow defined on $(-\infty, T]$ $(T > 0)$ (note, our injectivity radius estimate is still valid in view of the volume ratio estimate (2.1.18) which survives into every limit). If N is compact then we obtain a contradiction to (2.1.18). So we may assume that N is non-compact. We then consider the cases $\sup_{N\times(-\infty,\infty)} |t||\mathrm{R}(t)| = \infty$, and $\sup_{N\times(-\infty,\infty)} |t||\mathrm{R}(t)| < \infty$. Then, using the exact same arguments as in Cases 1.1.1 and 1.1.2, we obtain a contradiction.

(Case 2.2) The asymptotic scalar curvature ratio $A = \limsup_{s\to\infty} \mathrm{R}s^2 < \infty$. Remember that the asymptotic scalar curvature ratio is a constant in time for ancient solutions which have bounded curvature at each time and non-negative curvature operator. A is also independent of which origin we choose: see theorem 19.1 [16].

Now we use another splitting argument of Hamilton (see Theorem 24.7 of [16] for the compact version of this argument).

(Case 2.2.1) There exists a $C > 0$, s.t., for all $\tau \in (-\infty, 0)$, for all $\delta \in (0,1)$, there exists $(x,t) \in \Omega \times (-\infty, \tau)$ such that (x,t) is a C-essential δ-necklike point (see Appendix B). Let $\{\delta_i\}_{i\in\mathbb{N}}$ be a positive sequence,$\delta_i \overset{i\to\infty}{\longrightarrow} 0$, and let (x_i,t_i) be chosen so that (x_i,t_i) is an C-essential δ_i-necklike point. Assume θ_i is a unit 2-form on $T_{x_i}\Omega$ with

$$|\mathrm{Riem}(x_i,t_i) - \mathrm{R}(x_i,t_i)(\theta_i \otimes \theta_i)| \le \delta_i |\mathrm{Riem}|(x_i,t_i).$$

Let ${}^{i}g(x,t) = \frac{1}{|t_i|} g(x, t_i + t|t_i|)$. Then

$$\begin{aligned} {}^{{}^{i}g}|{}^{i}\mathrm{Riem}(x,t)| &= |t_i|^{g}|\mathrm{Riem}(x, t_i + t|t_i|)| \\ &= |t_i|^{g}|\mathrm{Riem}(x, (t-1)|t_i|)| \\ &= \frac{|(t-1)|t_i||}{|1-t|} {}^{g}|\mathrm{Riem}(x,(t-1)|t_i|)| \\ &\le \frac{B}{|1-t|} \le 2B \end{aligned} \tag{2.1.20}$$

for $t \le \frac{1}{2}$. Notice that

$$t_i + \frac{1}{2}|t_i| = t_i - \frac{1}{2}t_i = \frac{1}{2}t_i < 0 \tag{2.1.21}$$

and so ${}^{i}g(t)$ is defined for (at least) $-\infty < t \le \frac{1}{2}$. Furthermore,

$${}^{{}^{i}g}|{}^{i}\mathrm{Riem}(x_i,0)| = |t_i|^{g}|\mathrm{Riem}(x_i,t_i)| \ge C > 0, \tag{2.1.22}$$

since (x_i,t_i) is C-essential. Set

$$\psi_i := \frac{1}{|t_i|}\theta_i.$$

ψ_i is then a unit two form on $T_{x_i}\Omega$ with respect to $g^i(x,0)$. Then

$${}^{{}^{i}g}|{}^{i}\mathrm{Riem}(x_i,0) - {}^{i}\mathrm{R}(x_i,0)(\psi_i \otimes \psi_i)| \le \delta_i B.$$

Now taking a Hamilton pointed limit (our injectivity radius estimate is still valid) we obtain a solution $(\tilde{\Omega}, \tilde{g})$, defined for $t \le \frac{1}{2}$ with

$${}^{\tilde{g}}|\tilde{\mathrm{Riem}}(o,0) - \tilde{\mathrm{R}}(o,0)(\psi \otimes \psi)| \le 0.$$

where ψ is a unit two form defined on T_oM, $\psi = \lim_{i\to\infty}\psi_i$. More precisely this ψ is obtained (in coordinates) as $\psi^{\alpha\beta}(o) := \lim_{i\to\infty}\frac{\partial F_i^\alpha}{\partial x^k}(o)\frac{\partial F_i^\beta}{\partial x^l}(o)\psi_i^{kl}(x_i)$, where $F_i : B_i(o) \to U_i \subset M_i$, $F_i(o) = x_i$ are the diffeomorphisms occurring in the Hamilton limit process.

Furthermore $\mathrm{R}(o,0) \geq C > 0$, (in view of (2.1.22)) which implies (in view of the strong maximum principle applied to the evolution equation for R) that $\mathrm{R} > 0$. Hence, due to the maximum principle, $(\tilde{\Omega},\tilde{g}) = (N\times\mathbb{R},\gamma\oplus dx^2)$, or a quotient thereof by a group of isometries, where (N,γ) is a solution to the Ricci flow (see Appendix B for a more detailed explanation of this fact). If N is compact we obtain a contradiction to the volume ratio estimates. If N is non-compact, then we argue exactly as in cases 1.1.1 and 1.1.2 to obtain a contradiction.

(Case 2.2.2) For all $C > 0$, there exists $\tau \in (-\infty,0)$, and $\delta \in (0,1)$, such that for all $(x,t) \in (\Omega\times(-\infty,\tau))$, (x,t) is not a C-essential δ-necklike point. Choose $C \leq \frac{1}{16}$, and let τ,δ be the τ,δ from the statement at the beginning of this case. Set

$$G := |t|^{\frac{\varepsilon}{2}}\frac{|\mathring{\mathrm{Riem}}|^2}{\mathrm{R}^{2-\varepsilon}},$$

with $\varepsilon \leq \eta(\delta) := \frac{\delta}{100(3-\delta)}$ (notice that this function is well defined,as $\mathrm{R} > 0$ everywhere). Then, as Chow and Knopf show in [7] (see the proof of theorem 9.19 there)

$$\frac{\partial}{\partial t}G \leq \Delta G + 2\frac{(1-\varepsilon)}{\mathrm{R}}\langle\nabla G,\nabla \mathrm{R}\rangle - \frac{\varepsilon}{2|t|}G, \qquad (2.1.23)$$

for all $t \leq \tau$. Let us examine G a little more carefully. For fixed $t < 0$ and a fixed x_0 we have the estimate

$$\begin{aligned}\lim_{d(x,x_0,t)\to\infty} G(x,t) &= \lim_{d(x,x_0,t)\to\infty}|t|^{\frac{\varepsilon}{2}}\frac{|\mathring{\mathrm{Riem}}(x,t)|^2}{\mathrm{R}^2(x,t)}\mathrm{R}^\varepsilon(x,t)\\ &\leq |t|^{\frac{\varepsilon}{2}}c(n)\lim_{d(x,x_0,t)\to\infty}\mathrm{R}^\varepsilon(x,t)\\ &= 0 \qquad (2.1.24)\end{aligned}$$

in view of the fact that the asymptotic scalar curvature ratio is less than infinity. Also, as Chow and Knopf point out, we have

$$G = |t|^\varepsilon \mathrm{R}^\varepsilon\frac{|\mathring{\mathrm{Riem}}|^2}{\mathrm{R}^2}\frac{1}{|t|^{\frac{\varepsilon}{2}}} \leq \frac{B^\varepsilon c(n)}{|t|^{\frac{\varepsilon}{2}}}, \qquad (2.1.25)$$

in view of the fact that $B := \sup_{\Omega\times(-\infty,\omega]}|t||\mathrm{R}(t)| < \infty$, and hence

$$\lim_{t\to-\infty}\sup_{x\in M} G(x,t) = 0. \qquad (2.1.26)$$

Let $\tau' < \tau - 2$ be a constant with $\sup_\Omega G(\cdot, t) < \varepsilon_0$ for all $t \le \tau'$. We know that

$$\sup_{M\times(-\infty,0]} |\text{Riem}| \le c(n) \tag{2.1.27}$$

and without loss of generality

$$\sup_{M\times[\tau',\tau]} |\nabla \text{Riem}|^2 + |\nabla^2 \text{Riem}|^2 \le c(n) \tag{2.1.28}$$

in view of the interior gradient estimates of Shi (see [16], chapter 13). We also know that for given $\varepsilon_1 > 0$ and $s \in [\tau', \tau]$ there exists an $r(s, \varepsilon_1) > 0$ such that

$$\sup_{\{x\in M: d^2(x,x_0,s)\ge r\}} |\text{Riem}|(x, s) \le \varepsilon_1, \tag{2.1.29}$$

in view of the fact that the asymptotic scalar curvature ratio is finite. Hence, for all $\varepsilon_2 > 0$ there exists a $\delta > 0$, such that

$$\sup_{x\in M, t\in(s,s+\delta): d^2(x,x_0,s)\ge r} |\text{Riem}|(x, t) \le \varepsilon_1 + \varepsilon_2,$$

in view of (2.1.27) and (2.1.28) and the evolution equation for $|\text{Riem}|^2$. In particular if $\sup_M G(\cdot, s) < \varepsilon_0$, then $\sup_{M\times(s,s+\delta)} G(\cdot, t) < \varepsilon_0$, for small enough δ (outside of a fixed large compact set K, $G < \varepsilon_0$ for all $t \in (s, s + \delta)$ and inside K we use the fact that G is smooth). That is, the set

$$Z := \{r : \sup_\Omega G(\cdot, t) < \varepsilon_0 \forall t \in [\tau', r)\}$$

is open. Hence either

$$\sup_\Omega G(\cdot, t) < \varepsilon_0$$

for all $t \in [\tau', \tau)$, or there is a first time $t_0 \in (\tau', \tau)$ such that $\sup_\Omega G(\cdot, t_0) = \varepsilon_0$. In the second case, we see (using equation (2.1.29) with $s = t_0$) that there must also be a point $x_0 \in M$ such that $G(x_0, t_0) = \varepsilon_0$. But this contradicts the maximum principle in view of (2.1.23).

This means that

$$\sup_\Omega G(\cdot, t) < \varepsilon_0,$$

for all $t \in (-\infty, \tau)$, and hence, since ε_0 was arbitrary,

$$G \equiv 0.$$

Hence $\Omega = S^3/\Gamma$, which is a contradiction to the fact that Ω is non-compact.

□

2.2 An application of the proof of Lemma 2.1.4.

In certain cases, the proof of Lemma 2.1.4 is applicable even if M is non-compact. For example, the theorem below is proved similarly to Lemma 2.1.4. This theorem was initially proved (using other methods) by Perelman Proposition 11.4 [24].

Theorem 2.2.1. *Let $(\Omega^3, g(t))_{t\in(-\infty,0]}$ be an ancient non-compact complete solution to Ricci flow, with (for some fixed origin $o \in M$)*

$$\sec \geq 0$$

$$\sup_{\Omega} |\mathrm{Riem}(g(t))| < \infty \quad \forall t \in (-\infty, 0)$$

$$\mathcal{V}(\tau) := \lim_{r\to\infty} \frac{\mathrm{vol}(B_r(o,\tau))}{r^n} \geq \mathcal{V}_0 > 0. \tag{2.2.1}$$

for some time τ, $\tau \in (-\infty, 0)$. Then $(\Omega^3, g(t))$ is flat for all $t \in (-\infty, 0)$.

Remark 2.2.2. *The limit in the statement of the theorem exists in view of the fact that $\frac{\mathrm{vol}(B_r(o,\tau))}{r^n}$ is non-increasing as r increases (in view of the Bishop-Gromov comparison principle).*

Proof. Assume that the asymptotic scalar curvature ratio $A = \limsup_{s\to\infty} \mathrm{R}s^2 = \infty$ (this is a constant independent of time).

Notice that for this solution, and any scaling of this solution which has bounded curvature in a ball of radius one around some origin o', we have a uniform bound on the injectivity radius from below at o' (and time s), in view of 2.2.1 and [9]: we have the estimate

$$\frac{\mathrm{vol}(B_r(o',s))}{r^n} \geq \mathcal{V}_0 > 0$$

for all $r > 0$ in view of 2.2.1 and the Bishop Gromov volume comparison principle. Furthermore $\frac{\mathrm{vol}(B_r(o',s))}{r^n} \leq \omega_n$ trivially using the Bishop Gromov volume comparison principle. We may then apply the result of [9] to obtain our estimate for the bound on the injectivity radius, exactly as we did in the argument of Lemma 2.1.4. Also, the estimates

$$\omega_n \geq \frac{\mathrm{vol}(B_r(o,s))}{r^n} \geq \mathcal{V}_0 > 0 \forall r \geq 0 \tag{2.2.2}$$

remain valid under scaling (as the inequality is scale invariant). Hence, we obtain a uniform bound from below on the injectivity radius estimate at o', for any scaling of this solution which has bounded curvature by some fixed constant c on a ball of radius one around o'

Translate in time so that $\tau = 0$. Then we use the dimension-reduction argument of Hamilton (see Lemma 22.2 of [16] and the argument directly after the proof of Lemma 22.2) to obtain a new solution $(N \times \mathbb{R}, \gamma \times dx^2)$ or a quotient thereof by group of isometries (without loss of generality, we may assume that we don't have a quotient, as explained in case 1.1 of the proof of Lemma 2.1.4). Notice that the dimension-reduction argument of Hamilton is applicable, in view of the bounds from below on the injectivity radius at the centres of the balls occurring in the argument (due to the argument at the beginning of this theorem). Also 2.2.2 remains true for the resulting solution, as 2.2.2 is scale invariant. Without loss of generality the solution is defined on $(N \times \mathbb{R}, \gamma \times d\sigma^2)$ for $t \in (-\infty, \omega]$ for some $\omega > 0$, in view of the short time existence result of Shi, [34].

Assume that $\mathrm{R}(0, o) \neq 0$ on $N \times (-\infty, \omega)$. Then, see Lemma 26.2 in [16], we have

$$A = \limsup_{s \to \infty} \mathrm{R}s^2 < \infty$$

is a constant independent of $t \in (-\infty, \omega)$ on N. This means that

$$\mathcal{V}(t) = \lim_{r \to \infty} \frac{\mathrm{vol}(B_r(o,t))}{r^n}$$

is a constant on N independent of time, and in particular

$$\omega_n \geq \frac{\mathrm{vol}(B_r(o,t))}{r^n} \geq \mathcal{V}_0 > 0 \forall r \geq 0 \quad \forall t \in (-\infty, \omega)$$

(see theorem 18.3 in [16]).

We then consider the following two cases:

(case 1) $\sup_{N \times (-\infty, \omega]} |t||\mathrm{R}(t)| = \infty$,

(case 2) $\sup_{N \times (-\infty, \omega]} |t||\mathrm{R}(t)| < \infty$.

exactly as in then proof of Lemma 2.1.4. Both cases lead to a contradiction.

In the case that $A = \limsup_{s \to \infty} \mathrm{R}s^2 < \infty$ then we also know that

$$\mathcal{V}(t) =:= \lim_{r \to \infty} \frac{\mathrm{vol}(B_r(o,t))}{r^n}$$

is a constant on Ω independent of time, and in particular

$$\omega_n \geq \frac{\mathrm{vol}(B_r(o,t))}{r^n} \geq \mathcal{V}_0 > 0 \forall r \geq 0 \quad \forall t \in (-\infty, 0).$$

Translate in time so that the solution is defined on $(-\infty, \omega)$, $\omega > 0$. We then consider the following two cases.

(case 1) $\sup_{\Omega \times (-\infty, 0])} |t||\mathrm{R}(t)| = \infty$,

(case 2) $\sup_{\Omega \times (-\infty, 0]} |t||\mathrm{R}(t)| < \infty$.

exactly as in then proof of Lemma 2.1.4. Both cases lead to a contradiction.

□

Chapter 3

Bounds on the Ricci curvature from below under Ricci flow

We prove quantitative estimates that tell us how quickly the Ricci curvature can decrease, if we assume at time zero that the Ricci curvature is not too negative. Both lemmas may be read independently of the rest of the results in this paper.

The first lemma is suited to the case that we have a sequence of solutions to Ricci flow $(M_i, {}^i g(t))_{t\in[0,T)}$ whose initial data satisfies

$$\mathrm{Ricci}({}^i g(0)) \geq -\varepsilon_i \mathrm{R}({}^i g(0)) {}^i g(0) - \varepsilon_i {}^i g(0), \tag{3.1}$$

where $\varepsilon_i \overset{i\to\infty}{\longrightarrow} 0$. One application of this lemma is: if a subsequence of subsets $(\Omega_i, {}^i g(t)), t \in [0,T)$ (Ω_i open) converges (in the sense of Hamilton, see [17]) to a smooth solution $(\Omega, g(t)), t \in (0,T)$, then the lemma tells us that the Ricci curvature of $(\Omega, g(t))$ is non-negative for all $t \in (0,T)$. This is very general, but does require that a limit solution exist.

The second lemma is suited to the case that we have a sequence of solutions to Ricci flow $(M_i, {}^i g(t))_{[0,T)}$ whose initial data satisfies

$$\mathrm{Ricci}({}^i g(0)) \geq -\varepsilon_i {}^i g(0), \tag{3.2}$$

where $\varepsilon_i \overset{i\to\infty}{\longrightarrow} 0$. Once again, one application of this lemma is: if a subsequence of subsets $(\Omega_i, {}^i g(t)), t \in (0,S)$ converges (in the sense of Hamilton, see [17]) to a smooth solution $(\Omega, g(t)), t \in (0,S)$, then the lemma tells us that the Ricci curvature of $(\Omega, g(t))$ is non-negative for all $t \in (0,S)$. Another useful application of the second lemma is: if a solution $(M, g(t)), t \in [0,T)$ satisfies

$$\begin{aligned} &|\mathrm{Riem}(g)| \leq \frac{c_0}{t} \\ &\mathrm{Ricci}(g(0)) \geq -\varepsilon g(0) \end{aligned} \tag{3.3}$$

then for a well controlled time interval the solution satisfies

$$\mathrm{Ricci}(g) \geq -c_0 \varepsilon g.$$

As we saw in Lemma 2.1.4, such a bound is relevant to the question of existence of solutions to the Ricci flow. We apply this lemma in the main Theorem 5.1 and the Application 5.2.

Lemma 3.3. *Let g_0 be a smooth metric on a 3-dimensional manifold M^3 which satisfies*

$$\mathrm{Ricci}(g_0) \geq -\frac{\varepsilon_0}{4}\mathrm{R}g_0 - \frac{\varepsilon_0}{4} \quad (\sec(g_0) \geq -\mathrm{R}\frac{\varepsilon_0}{4}g_0 - \frac{\varepsilon_0}{4}) \tag{3.4}$$

for some $0 < \varepsilon_0 < \frac{1}{100}$, and let $(M, g(\cdot,t))_{t\in[0,T)}$ be a solution to Ricci flow with $g(0) = g_0(\cdot)$. Then

$$\begin{array}{rl} \mathrm{Ricci}(g(t)) & \geq -\varepsilon_0(1+4t)g(t) - \varepsilon_0(1+4t)\mathrm{R}(g(t))g(t), \quad \forall \;\; t \in [0,T) \cap [0, \frac{1}{8}) \\ (\sec(g(t)) & \geq -\varepsilon_0(\frac{1}{2}+t)g(t) - \varepsilon_0(\frac{1}{2}+t)\mathrm{R}(g(t))g(t), \quad \forall \;\; t \in [0,T) \cap [0, \frac{1}{8})) \end{array}$$

Proof. Define $\varepsilon = \varepsilon(t) = \varepsilon_0(1+4t)$, and the tensor $L(t)$ by

$$L_{ij} := \mathrm{Ricci}_{ij} + \varepsilon \mathrm{R} g_{ij} + \varepsilon g_{ij}.$$

We shall often write ε for $\varepsilon(t)$ (not to be confused with ε_0). Notice that $\varepsilon_0 < \varepsilon(t) \leq 2\varepsilon_0$, for all $t \in [0, \frac{1}{8})$: we will use this freely. Then $L_i{}^j = (R_i{}^j + \varepsilon R \delta_i{}^j + \varepsilon \delta_i{}^j)$, and

$$\begin{aligned} (\frac{\partial}{\partial t}L)_{ij} &= (\frac{\partial}{\partial t}L_i^l)g_{jl} - 2L_i^l R_{jl} \\ &= g_{jl}\Big(\frac{\partial}{\partial t}(R_{ik}g^{kl}) + \varepsilon\frac{\partial}{\partial t}R\delta_i{}^l + 4\varepsilon_0 R\delta_i{}^l + 4\varepsilon_0\delta_i{}^l\Big) - 2L_i^l R_{jl} \\ &= g_{jl}\frac{\partial}{\partial t}(R_{ik}g^{kl}) + \varepsilon g_{ij}\frac{\partial}{\partial t}R + 4\varepsilon_0 R g_{ij} + 4\varepsilon_0 g_{ij} - 2L_i^l R_{jl} \\ &= g_{jl}\Big((\Delta \mathrm{Ricci})_i{}^l - Q_i{}^l + 2R_{ik}R_{sm}g^{km}g^{ls}\Big) \\ &\quad + \varepsilon g_{ij}\Big(\Delta R + 2|\mathrm{Ricci}|^2\Big) + 4\varepsilon_0 R g_{ij} + 4\varepsilon_0 g_{ij} - 2L_i^l R_{jl} \\ &= (\Delta L)_{ij} - Q_{ij} + 2R_{ik}R_{jm}g^{km} + 2\varepsilon|\mathrm{Ricci}|^2 g_{ij} \\ &\quad + 4\varepsilon_0 R g_{ij} + 4\varepsilon_0 g_{ij} - 2L_i^l R_{jl}, \end{aligned}$$

where Q is the tensor

$$\begin{array}{l} Q_{ij} := 6S_{ij} - 3RR_{ij} + (R^2 - 2S)g_{ij}, \\ S_{ij} := g^{kl}R_{ik}R_{jl} \end{array} \tag{3.5}$$

(see [13], theorem 8.4) Clearly $L_{ij}(0) > 0$. Define N_{ij} by

$$N_{ij} := -Q_{ij} + 2R_{im}R_{sj}g^{ms} + 2\varepsilon|\mathrm{Ricci}|^2 g_{ij} + 4\varepsilon_0 R g_{ij} + 4\varepsilon_0 g_{ij} - 2L_i^l R_{jl}.$$

We argue as in the proof of Hamilton's maximum principle, Theorem 9.1, [13].

We claim that $L_{ij}(g(t)) \geq 0$. Assume there exist a first time and point (p_0, t_0) and a direction w_{p_0} for which $L(w, w)(g(t))(p_0, t_0) = 0$. Choose coordinates about p_0 so that at (p_0, t_0) they are orthonormal, and so that Ricci is diagonal at (p_0, t_0). Clearly L is then also diagonal at (p_0, t_0). W.l.o.g.

$$\begin{aligned} R_{11} &= \lambda \\ R_{22} &= \mu \\ R_{33} &= \nu \end{aligned} \tag{3.6}$$

and $\lambda \leq \mu \leq \nu$, and so

$$L_{11} = \lambda + \varepsilon(t_0)R + \varepsilon(t_0) \leq L_{22} \leq L_{33},$$

and so $L_{11} = 0$, (otherwise $L(p_0, t_0) > 0$: a contradiction). In particular,

$$N_{11}(p_0, t_0) = (\mu - \nu)^2 + \lambda(\mu + \nu) + 2\varepsilon\lambda^2 + 2\varepsilon\mu^2 + 2\varepsilon\nu^2 + 4\varepsilon_0 R + 4\varepsilon_0, \tag{3.7}$$

in view of the definition of Q (see [13] Corollary 8.2, Theorems 8.3,8.4) and the fact that $L_{11} = 0$. Also, $L_{11} = 0 \Rightarrow \lambda = -\varepsilon R - \varepsilon$ at (p_0, t_0), and so, substituting this into (3.7), we get

$$\begin{aligned} N_{11}(p_0, t_0) &= (u - v)^2 + (-\varepsilon R - \varepsilon)(\mu + \nu) + 2\varepsilon(\lambda^2 + \mu^2 + \nu^2) + 4\varepsilon_0 R + 4\varepsilon_0 \\ &\geq \varepsilon(-(\lambda + \mu + \nu)(\mu + \nu) + 2\lambda^2 + 2\mu^2 + 2\nu^2) + 4\varepsilon_0 R + 4\varepsilon_0 - \varepsilon(\mu + \nu) \\ &= \varepsilon(-(\lambda + \mu + \nu)(\mu + \nu) + 2\lambda^2 + 2\mu^2 + 2\nu^2) + 4\varepsilon_0 R + 4\varepsilon_0 - \varepsilon R + \varepsilon\lambda \\ &\geq \varepsilon(\lambda - \lambda\mu - \lambda\nu + \mu^2 + \nu^2 + 2\lambda^2 - 2\mu\nu) + 4\varepsilon_0 R + 4\varepsilon_0 - \varepsilon R. \end{aligned}$$

To show $N_{11} > 0$, we consider a number of cases.

- **Case 1**. $\lambda \geq 0$. This combined with $L_{11} = 0$ implies that $R < 0$. A contradiction to the fact that $\lambda \geq 0$ and λ is the smallest eigenvalue of Ricci.
- **Case 2**. $\lambda \leq 0, R \geq 0$. This implies $\nu \geq 0$ and hence

  $$N_{11} \geq \varepsilon(\lambda - \lambda\mu + \mu^2 + \nu^2 + 2\lambda^2 - 2\mu\nu) + 4\varepsilon_0,$$

  in view of the fact that $\varepsilon R \leq 2\varepsilon_0 R$. In the case $\mu \geq 0$ we obtain

  $$N_{11} \geq \varepsilon(\lambda + \mu^2 + \nu^2 + 2\lambda^2 - 2\mu\nu) + 4\varepsilon_0 \geq -\varepsilon + 4\varepsilon_0 > 0,$$

  after an application of Young's inequality, and similarly in the case $\mu \leq 0$ we get

  $$N_{11} \geq \varepsilon(\lambda - \lambda\mu + \mu^2 + \nu^2 + 2\lambda^2) + 4\varepsilon_0 > 0.$$

- **Case 3**. $\lambda \leq 0, R \leq 0$. We know that $R(g_0) \geq -3\varepsilon_0$ will be preserved by Ricci flow, and hence $0 \geq R(g(t)) \geq -3\varepsilon_0$. We break case 3 up into three sub-cases (3.1,3.2,3.3).

- **Case 3.1** $\mu, \nu \le 0$. This with $R \ge -3\varepsilon_0$ implies that $|\lambda|, |\mu|, |\nu| \le 3\varepsilon_0$ and hence

$$N_{11} \ge -3\varepsilon\varepsilon_0 - 36\varepsilon\varepsilon_0^2 - 12\varepsilon_0^2 + 4\varepsilon_0 \ge -100\varepsilon_0^2 + 4\varepsilon_0 > 0,$$

  since $0 < \varepsilon_0 < \frac{1}{100}, \varepsilon < 2\varepsilon_0 < 1$.

- **Case 3.2** $\mu \le 0, \nu \ge 0$. Implies

$$N_{11} \ge \varepsilon(\lambda - \lambda\mu + \mu^2 + \nu^2 + 2\lambda^2) - 12\varepsilon_0^2 + 4\varepsilon_0 > 0,$$

  in view of Young's inequality, $\varepsilon_0 \le \frac{1}{100}$, and $0 < \varepsilon < 2\varepsilon_0$.

- **Case 3.3** $\mu \ge 0 (\Rightarrow \nu \ge 0)$. Then, similarly,

$$N_{11} \ge \varepsilon(\lambda + \mu^2 + \nu^2 + 2\lambda^2 - 2\mu\nu) - 12\varepsilon_0^2 + 4\varepsilon_0 > 0.$$

So in all cases $N_{11} > 0$. The rest of the proof is standard (see [13] Theorem9.1): extend $w(p_0, t_0) = \frac{\partial}{\partial x^1}(p_0, t_0)$ in space to a vector field $w(\cdot)$ in a small neighbourhood of p_0 so that ${}^{g(t_0)}\nabla w(\cdot)(p_0, t_0) = 0$, and let $w(\cdot, t) = w(\cdot)$. Then

$$0 \ge (\frac{\partial}{\partial t} L(w, w))(p_0, t_0) \ge (\Delta L(w, w))(p_0, t_0) + N(w, w) > 0,$$

which is a contradiction.

The case for the sectional curvatures is similar: >from [14], Sec. 5, we know that the reaction equations for the curvature operator are

$$\begin{aligned} \frac{\partial}{\partial t}\alpha &= \alpha^2 + \beta\gamma \\ \frac{\partial}{\partial t}\beta &= \beta^2 + \alpha\gamma \\ \frac{\partial}{\partial t}\gamma &= \gamma^2 + \alpha\beta. \end{aligned}$$

Note that:

$$\begin{aligned} R &= \alpha + \beta + \gamma, \\ |\mathrm{Ricci}|^2 &= ((\frac{\alpha+\beta}{2})^2 + (\frac{\alpha+\gamma}{2})^2 + (\frac{\beta+\gamma}{2})^2) \\ &= \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma). \end{aligned} \tag{3.8}$$

Similar to the Ricci case, we examine the function $\alpha + \varepsilon(t)R + \varepsilon(t)$ where $\alpha \le \beta \le \gamma$ are eigenvalues of the curvature operator, and $\varepsilon(t) = \varepsilon_0(\frac{1}{2} + t)$. In order to make the following inequalities more readable, we write ε in place of $\varepsilon(t)$: that is, $\varepsilon = \varepsilon_0(\frac{1}{2} + t)$.

$$\begin{aligned} \frac{\partial}{\partial t}(\alpha + \varepsilon R + \varepsilon) &= \varepsilon_0 + \varepsilon_0 R + \alpha^2 + \beta\gamma + 2\varepsilon|\mathrm{Ricci}|^2 \\ &= \varepsilon_0 + \varepsilon_0 R + \alpha^2 + \beta\gamma + \varepsilon(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma), \end{aligned}$$

and so in the case that $\beta,\gamma \geq 0$, or $\beta,\gamma \leq 0$, $\frac{\partial}{\partial t}(\alpha+\varepsilon R+\varepsilon) \geq \varepsilon_0(1+R) > 0$. So assume that $\alpha \leq \beta \leq 0$, and $\gamma \geq 0$. Combining these inequalities with $\varepsilon(t) \leq \varepsilon_0$, we see that

$$\begin{aligned}
\frac{\partial}{\partial t}(\alpha+\varepsilon R+\varepsilon) &\geq \varepsilon_0+\varepsilon_0 R+\alpha^2+\alpha\gamma+\varepsilon(\alpha^2+\beta^2+\gamma^2+\alpha\beta+\alpha\gamma+\beta\gamma)\\
&= \varepsilon_0+\varepsilon_0 R+\alpha^2+(\alpha+\varepsilon R+\varepsilon)\gamma\\
&\quad -\varepsilon R\gamma-\varepsilon\gamma+\varepsilon(\alpha^2+\beta^2+\gamma^2+\alpha\beta+\alpha\gamma+\beta\gamma))\\
&= \varepsilon_0+\varepsilon_0 R+\alpha^2+(\alpha+\varepsilon R+\varepsilon)\gamma-\varepsilon\gamma+\varepsilon(\alpha^2+\beta^2+\alpha\beta),\\
&\geq \alpha^2+(\alpha+\varepsilon R+\varepsilon)\gamma+\varepsilon_0(1+R-\gamma)+\varepsilon(\alpha^2+\beta^2),
\end{aligned}$$

which, using $\varepsilon(t) \geq \frac{\varepsilon_0}{2}$, is

$$\begin{aligned}
&\geq \alpha^2+(\alpha+\varepsilon R+\varepsilon)\gamma+\varepsilon_0(1+\alpha+\beta+\frac{\alpha^2}{2}+\frac{\beta^2}{2}),\\
&\geq \alpha^2+(\alpha+\varepsilon R+\varepsilon)\gamma,
\end{aligned}$$

in view of Young's inequality. At a point where $\alpha+\varepsilon R+\varepsilon = 0$, the last sum is strictly bigger than zero (if $\alpha = 0$, then, $R \geq 0$, and hence $\alpha+\varepsilon R+\varepsilon \geq \varepsilon > 0$: a contradiction). Then we argue as above.

□

The above lemma shows us that if the Ricci curvature at time zero is bigger than $-\varepsilon$ (ε small) then the Ricci curvature divided by the scalar curvature is at most $-c\varepsilon$ at points where the scalar curvature is bigger than one (for a short well defined time interval). It can of course happen that the Ricci curvature becomes very large and negative in a short time, if the scalar curvature is very large and positive in a short time.

Now we prove an improved version of the above theorem, which allows for some scaling in time. In particular, for the class of solutions where $|\text{Riem}|t \leq c_0$ it tells us that: if the Ricci curvature at time zero is bigger than $-\varepsilon$ (ε small) then the Ricci curvature is at most $-c\varepsilon$ for some short well defined time interval.

Lemma 3.4. *Let g_0 be a smooth metric on a 3-dimensional manifold M^3 which satisfies*

$$\begin{aligned}
&\text{Ricci}(g_0) \geq -\frac{\varepsilon_0}{4}g_0,\\
&(\sec(g_0) \geq -\frac{\varepsilon_0}{4}g_0)
\end{aligned} \tag{3.9}$$

for some $0 < \varepsilon_0 < \frac{1}{100}$, and let $(M, g(\cdot,t))_{t\in[0,T)}$ be a solution to Ricci flow with $g(0) = g_0(\cdot)$. Then

$$\begin{aligned}
\text{Ricci}(g(t)) &\geq -\varepsilon_0(1+kt)g(t)-\varepsilon_0(1+kt)t\text{R}(g(t))g(t), \quad \forall \ \ t\in[0,T)\cap[0,T')\\
(\sec(g(t)) &\geq -\varepsilon_0(\tfrac{1}{2}+kt)g(t)-\varepsilon_0(\tfrac{1}{2}+kt)t\text{R}(g(t))g(t), \quad \forall \ \ t\in[0,T)\cap[0,T'))
\end{aligned}$$

where $k = 100$ and $T' = T'(100) > 0$ is a universal constant.

Proof. The proof is similar to that above. Define $\varepsilon = \varepsilon(t) = \varepsilon_0(1 + kt)$, and the tensor $L(t)$ by

$$L_{ij} := \mathrm{Ricci}_{ij} + \varepsilon t \mathrm{R} g_{ij} + \varepsilon g_{ij}.$$

We shall often write ε for $\varepsilon(t)$ (not to be confused with ε_0). Notice that $\varepsilon_0 < \varepsilon(t) \leq 2\varepsilon_0$, for all $t \in [0, \frac{1}{k})$: we will use this freely. Then

$$L_i{}^j = (R_i{}^j + \varepsilon t R \delta_i{}^j + \varepsilon \delta_i{}^j),$$

and

$$\begin{aligned}
(\frac{\partial}{\partial t} L)_{ij} &= (\frac{\partial}{\partial t} L_i^l) g_{jl} - 2L_i^l R_{jl} \\
&= g_{jl}\Big(\frac{\partial}{\partial t}(R_{ik} g^{kl}) + \varepsilon \mathrm{R} \delta_i{}^l + \varepsilon t \frac{\partial}{\partial t} \mathrm{R} \delta_i{}^l + k\varepsilon_0 t \mathrm{R} \delta_i{}^l + k\varepsilon_0 \delta_i{}^l\Big) - 2L_i^l R_{jl} \\
&= g_{jl} \frac{\partial}{\partial t}(R_{ik} g^{kl}) + \varepsilon \mathrm{R} g_{ij} + \varepsilon t g_{ij} \frac{\partial}{\partial t} \mathrm{R} + k\varepsilon_0 t \mathrm{R} g_{ij} + k\varepsilon_0 g_{ij} - 2L_i^l R_{jl} \\
&= g_{jl}\Big((\Delta \mathrm{Ricci})_i{}^l - Q_i{}^l + 2R_{ik} R_{sm} g^{km} g^{ls}\Big) + \varepsilon \mathrm{R} g_{ij} \\
&\quad + \varepsilon t g_{ij}\Big(\Delta R + 2|\mathrm{Ricci}|^2\Big) + k\varepsilon_0 t \mathrm{R} g_{ij} + k\varepsilon_0 g_{ij} - 2L_i^l R_{jl} \\
&= (\Delta L)_{ij} - Q_{ij} + 2R_{ik} R_{jm} g^{km} + \varepsilon \mathrm{R} g_{ij} + 2\varepsilon t |\mathrm{Ricci}|^2 g_{ij} \\
&\quad + k\varepsilon_0 t R g_{ij} + k\varepsilon_0 g_{ij} - 2L_i^l R_{jl},
\end{aligned}$$

where Q is the tensor defined in Equation (3.5). Clearly $L_{ij}(0) > 0$. Define N_{ij} by

$$N_{ij} := -Q_{ij} + 2R_{ik} R_{jm} g^{km} + \varepsilon \mathrm{R} g_{ij} + 2\varepsilon t |\mathrm{Ricci}|^2 g_{ij} + k\varepsilon_0 t R g_{ij} + k\varepsilon_0 g_{ij} - 2L_i^l R_{jl}.$$

We argue as in the proof of Hamilton's maximum principle, Theorem 9.1, [13].

We claim that $L_{ij}(g(t)) > 0$ for all $t \in [0, T)$. Assume there exist a first time and point (p_0, t_0) and a direction w_{p_0} for which $L(w, w)(g(t))(p_0, t_0) = 0$. Choose coordinates about p_0 so that at (p_0, t_0) they are orthonormal, and so that Ricci is diagonal at (p_0, t_0). Clearly L is then also diagonal at (p_0, t_0). W.l.o.g.

$$\begin{aligned}
R_{11} &= \lambda, \\
R_{22} &= \mu, \\
R_{33} &= \nu,
\end{aligned} \tag{3.10}$$

and

$$\lambda \leq \mu \leq \nu,$$

and so

$$L_{11} = \lambda + \varepsilon(t_0) t_0 R + \varepsilon(t_0) \leq L_{22} \leq L_{33},$$

and so $L_{11} = 0$, (otherwise $L(p_0, t_0) > 0$: a contradiction). In particular,

$$N_{11}(p_0, t_0) = \quad (\mu - \nu)^2 + \lambda(\mu + \nu) + 2\varepsilon t \lambda^2 + 2\varepsilon t \mu^2 + 2\varepsilon t \nu^2$$

$$+\varepsilon \mathrm{R} g_{ij} + k\varepsilon_0 t R g_{ij} + k\varepsilon_0 g_{ij} \tag{3.11}$$

in view of the definition of Q (see [13] Corollary 8.2, Theorems 8.3,8.4) and the fact that $L_{11} = 0$. We will show that $N_{11}(p_0, t_0) > 0$. $L_{11} = 0 \Rightarrow \lambda = -\varepsilon t_0 R - \varepsilon$ at (p_0, t_0), and so, substituting this into (3.7), we get

$$\begin{aligned}
N_{11}(p_0, t_0) &= (u-v)^2 + (-\varepsilon t_0 R - \varepsilon)(\mu + \nu) + 2\varepsilon t_0(\lambda^2 + \mu^2 + \nu^2) \\
&\quad + \varepsilon \mathrm{R} + k\varepsilon_0 t \mathrm{R} g_{ij} + k\varepsilon_0 \\
&\geq \varepsilon t_0(-(\lambda + \mu + \nu)(\mu + \nu) + 2\lambda^2 + 2\mu^2 + 2\nu^2) - \varepsilon(\mu + \nu) \\
&\quad + \varepsilon \mathrm{R} + k\varepsilon_0 t_0 \mathrm{R} + k\varepsilon_0 \\
&= \varepsilon t_0(-(\lambda + \mu + \nu)(\mu + \nu) + 2\lambda^2 + 2\mu^2 + 2\nu^2) \\
&\quad + ((-\varepsilon^2 t_0 + k\varepsilon_0 t_0)\mathrm{R} - \varepsilon^2 + k\varepsilon_0) \\
&\geq \varepsilon t_0(-\lambda\mu - \lambda\nu + \mu^2 + \nu^2 + 2\lambda^2 - 2\mu\nu) \\
&\quad + ((-\varepsilon^2 t_0 + k\varepsilon_0 t_0)\mathrm{R} - \varepsilon^2 + k\varepsilon_0)
\end{aligned}$$

where here we have used once again that

$$\lambda(x_0, t_0) = -\varepsilon(t_0) t_0 R(x_0, t_0) - \varepsilon(t_0).$$

If $\mathrm{R}(x_0, t_0) \leq 0$, then using the fact that $\mathrm{R} \geq -3\varepsilon_0$ is preserved by the flow, we see that

$$(-\varepsilon^2(t_0) t_0 + k\varepsilon_0(t_0) t_0)\mathrm{R}(x_0, t_0) - \varepsilon^2 + k\varepsilon_0 \geq \frac{k}{2}\varepsilon_0.$$

Furthermore,

- [i] $\lambda = -\varepsilon R - \varepsilon \leq \varepsilon$ (since $R \geq -3\varepsilon_0$) and $\lambda = -\varepsilon R - \varepsilon \geq -\varepsilon$, that is $|\lambda| \leq \varepsilon$.
- [ii] Similarly $|\mu + \nu| = |R - \lambda| \leq 4\varepsilon$.

Hence

$$\varepsilon t_0(-\lambda(\mu + \nu) + \mu^2 + \nu^2 + 2\lambda^2 - 2\mu\nu) \geq -50\varepsilon_0^2,$$

and so $N_{11}(p_0, t_0) > 0$. Hence we must only consider the case $R(p_0, t_0) \geq 0$.

- **Case 1**. $\lambda \geq 0$. This combined with $L_{11} = 0$ implies that $R(p_0, t_0) < 0$. A contradiction.
- **Case 2**. $\lambda \leq 0, \mu \geq 0, \nu \geq 0$.

  In this case we trivially obtain $N_{11} > 0$.
- **Case 3**. $\lambda \leq 0, \mu \leq 0, \nu \geq 0$. Implies

  $$N_{11} > \varepsilon t_0(-\lambda\mu + \mu^2 + \nu^2 + 2\lambda^2) \geq 0,$$

  in view of Young's inequality.

So in all cases $N_{11} > 0$. The rest of the proof is standard (see [13] Theorem9.1): extend $w(p_0, t_0) = \frac{\partial}{\partial x^1}(p_0, t_0)$ in space to a vector field $w(\cdot)$ in a small neighbourhood of p_0 so that ${}^{g(t_0)}\nabla w(\cdot)(p_0, t_0) = 0$, and let $w(\cdot, t) = w(\cdot)$. Then

$$0 \geq (\frac{\partial}{\partial t} L(w, w))(p_0, t_0) \geq (\Delta L(w, w))(p_0, t_0) + N(w, w) > 0,$$

which is a contradiction.

The case for the sectional curvatures is similar: $>$from [14], Sec. 5, we know that the reaction equations for the curvature operator are

$$\begin{aligned} \frac{\partial}{\partial t}\alpha &= \alpha^2 + \beta\gamma \\ \frac{\partial}{\partial t}\beta &= \beta^2 + \alpha\gamma \\ \frac{\partial}{\partial t}\gamma &= \gamma^2 + \alpha\beta. \end{aligned}$$

In what follows, we use the formulae (3.8) freely.

Similar to the Ricci case, we examine the function $\alpha + \varepsilon(t)tR + \varepsilon(t)$ where $\alpha \leq \beta \leq \gamma$ are eigenvalues of the curvature operator, and $\varepsilon(t) = \varepsilon_0(\frac{1}{2} + kt)$. In order to make the following inequalities more readable, we write ε in place of $\varepsilon(t)$: that is, $\varepsilon = \varepsilon_0(\frac{1}{2} + kt)$. We assume $t \leq \frac{1}{2k}$ so that $\varepsilon_0 \frac{1}{2} \leq \varepsilon(t) \leq \varepsilon_0$.

$$\begin{aligned} \frac{\partial}{\partial t}(\alpha + \varepsilon tR + \varepsilon) &= \varepsilon R + k\varepsilon_0 tR + k\varepsilon_0 + \alpha^2 + \beta\gamma + 2\varepsilon t|\mathrm{Ricci}|^2 \\ &= \varepsilon R + k\varepsilon_0 tR + k\varepsilon_0 + \alpha^2 + \beta\gamma \\ &\quad + \varepsilon t(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma), \end{aligned}$$

and so in the case that $\beta, \gamma \geq 0$, or $\beta, \gamma \leq 0$,

$$\begin{aligned} \frac{\partial}{\partial t}(\alpha + \varepsilon R + \varepsilon) &\geq \varepsilon R + k\varepsilon_0 tR + k\varepsilon_0 \\ &\geq -3\varepsilon_0^2 - 3\varepsilon_0^2 + k\varepsilon_0 > 0 \end{aligned} \tag{3.12}$$

So assume that $\alpha \leq \beta \leq 0$, and $\gamma \geq 0$. Combining these inequalities with $\varepsilon(t) \leq \varepsilon_0$, we see that

$$\begin{aligned} \frac{\partial}{\partial t}(\alpha + \varepsilon tR + \varepsilon) &\geq \varepsilon R + k\varepsilon_0 tR + k\varepsilon_0 + \alpha\gamma \\ &\quad + \varepsilon t(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma) \\ &= \varepsilon R + k\varepsilon_0 tR + k\varepsilon_0 + (\alpha + \varepsilon tR + \varepsilon)\gamma \\ &\quad - \varepsilon tR\gamma - \varepsilon\gamma + \varepsilon t(\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \alpha\gamma + \beta\gamma) \\ &= \varepsilon R - \varepsilon\gamma + k\varepsilon_0 tR + k\varepsilon_0 + (\alpha + \varepsilon tR + \varepsilon)\gamma \\ &\quad + \varepsilon t(\alpha^2 + \beta^2 + \alpha\beta) \\ &= \varepsilon(\alpha + \beta) + k\varepsilon_0 tR + k\varepsilon_0 + (\alpha + \varepsilon tR + \varepsilon)\gamma \\ &\quad + \varepsilon t(\alpha^2 + \beta^2 + \alpha\beta) \end{aligned}$$

$$\geq \quad (2\varepsilon\alpha + k\varepsilon_0 tR + k\varepsilon_0) + \varepsilon t(\alpha^2 + \beta^2 + \alpha\beta)$$

at a point where $\alpha + \varepsilon tR + \varepsilon = 0$. Using $\alpha + \varepsilon tR + \varepsilon = 0$ again, we get

$$\begin{aligned} 2\varepsilon\alpha + k\varepsilon_0 tR + k\varepsilon_0 &= 2\varepsilon(-\varepsilon tR - \varepsilon) + k\varepsilon_0 tR + k\varepsilon_0 \\ &= Rt(-2\varepsilon^2 + k\varepsilon_0) + k\varepsilon_0 - 2\varepsilon^2 \\ &> \frac{k}{2}\varepsilon_0, \end{aligned}$$

since $R \geq -3\varepsilon_0$ is preserved by the flow, and $t \leq \frac{1}{k}$. Hence

$$\frac{\partial}{\partial t}(\alpha + \varepsilon tR + \varepsilon) \quad \geq \quad \frac{k}{2}\varepsilon_0 + \varepsilon t(\alpha^2 + \beta^2 + \alpha\beta) > 0,$$

at a point where $\alpha + \varepsilon tR + \varepsilon = 0$. Then we argue as above.

□

So although the Ricci curvature can become very large and negative under the Ricci flow, it can only do so at a controlled rate. In particular, as we mentioned before this lemma, if the curvature satisfies $|\mathrm{Riem}|t \leq c_0$ for all $t \in [0, T)$ (in addition to the initial conditions) then Ricci $\geq -c_1(c_0)\varepsilon_0$, is true on some well defined time interval $[0, T')$ (in dimensions two and three).

Chapter 4

Bounding the diameter and volume in terms of the curvature

The results of this chapter hold for all dimensions.

Lemma 4.1. *Let $(M^n, g(t))_{t\in[0,T)}$ be a solution to Ricci flow with*

$$\begin{aligned} &\mathrm{Ricci}(g(t)) \geq -c_0 \\ &|\mathrm{Riem}(g(t))|t \leq c_0 \\ &\mathrm{diam}\ (M, g_0) \leq d_0 \end{aligned} \tag{4.1}$$

Then

$$d(p,q,0) - c_1(t, d_0, c_0, n) \geq d(p,q,t) \geq d(p,q,0) - c_2(n,c_0)\sqrt{t} \tag{4.2}$$

for all $t \in [0,T)$, where

$$c_1(t, d_0, c_0, n) \to 0$$

as $t \to 0$.

In particular if ${}^i g_0$ is a sequence of smooth metrics on manifolds M_i with

$$\begin{aligned} &\mathrm{diam}\ (M_i, {}^i g_0) \leq d_0 \\ &\mathrm{d}_{GH}((M_i, d({}^i g_0)), (X, d_X)) \stackrel{i\to\infty}{\longrightarrow} 0 \end{aligned} \tag{4.3}$$

and $(M_i, {}^i g(t))_{t\in[0,T_i)}$ are solutions to Ricci flow with

$$\begin{aligned} &{}^i g(0) = {}^i g_0 \\ &\sec({}^i g(t)) \geq -c_0 \quad (\mathrm{Ricci}({}^i g(t)) \geq -c_0) \\ &|\mathrm{Riem}({}^i g(t))|t \leq c_0\ \forall t \in [0, T_i), \end{aligned} \tag{4.4}$$

then

$$\mathrm{d}_{GH}((M_i, d({}^i g(t_i))), (X, d_X)) \stackrel{i\to\infty}{\longrightarrow} 0$$

for any sequence $t_i \in [0, T_i), i \in \mathbb{N}$ where $t_i \stackrel{i\to\infty}{\longrightarrow} 0$.

Proof. The first inequality

$$d(p,q,t) \geq d(p,q,0) - c_1(n,c_0)\sqrt{t}$$

is proved in [16], theorem 17.2 (with a slight modification of the proof: see Appendix C). The second inequality follows easily from [16], Lemma 17.3: see Appendix C.

The second statement is a consequence of the first result, and the triangle inequality which is valid for the Gromov Hausdorff distance:

$$\begin{aligned}
&\mathrm{d}_{GH}((M_i, d({}^i g(t_i))), (X, d_X)) \\
&\leq \mathrm{d}_{GH}((M_i, d({}^i g(t_i))), (M_i, d({}^i g_0))) + \mathrm{d}_{GH}((M_i, d({}^i g_0)), (X, d_X)) \\
&\leq c(t_i) + \mathrm{d}_{GH}((M_i, d({}^i g_0)), (X, d_X)) \overset{i\to\infty}{\longrightarrow} 0. \qquad (4.5)
\end{aligned}$$

Here we have used the characterisation of Gromov Hausdorff distance given in A.9, and the fact that the identity map $I : (M_i, d({}^i g(t_i))) \to (M_i, d({}^i g_0))$, is an $c(t_i)$ -Hausdorff approximation, where $c(t) \to 0$ as $t \to 0$: see Appendix A, Definition A.8 and Lemma A.9. □

Corollary 4.2. *Let $(M^n, {}^i g(t))_{t\in[0,T)}$ be an arbitrary solution to Ricci flow (with $g(\cdot,0) = g_0$) satisfying the conditions 4.1 and assume that there exists $v_0 > 0$ such that*

$$\mathrm{vol}(M, g_0) \geq v_0 > 0. \qquad (4.6)$$

Then there exists an $S = S(d_0, v_0, n) > 0$ such that

$$\mathrm{vol}(M, g(t)) \geq \frac{3v_0}{4} \quad \forall \quad t \in [0,T) \cap [0,S)$$

Proof. If this were not the case, then there exist solutions $(M_i^n, {}^i g(t))_{t\in[0,T_i)}$ satisfying the stated conditions and there exist $t_i \in [0, T_i)$, $t_i \overset{i\to\infty}{\longrightarrow} 0$ such that $\mathrm{vol}(M_i, {}^i g(t_i)) = \frac{3v_0}{4}$. But then

$$\mathrm{d}_{GH}((M_i, d({}^i g(t_i))), (X, d_X)) \overset{i\to\infty}{\longrightarrow} 0$$

from the lemma above. According to [6], thm 10.8 for the case that $\sec({}^i g(t)) \geq -c_0$ (for the Ricci case we use theorem 5.4 of [8] Cheeger-Colding) we also have

$$v_0 \leq \mathrm{vol}(M_i, {}^i g_0) = \mathcal{H}^n(M_i, d({}^i g_0)) \overset{i\to\infty}{\longrightarrow} \mathcal{H}^n(X, d_X)$$

which implies $\mathcal{H}^n(X, d_X) \geq v_0$. Here $\mathcal{H}^n(X, d_X)$ is the n-dimensional Hausdorff mass of X with respect to the metric d_X. Similarly we have

$$\frac{3v_0}{4} = \mathcal{H}^n(M_i, d({}^i g(t_i))) \overset{i\to\infty}{\longrightarrow} \mathcal{H}^n(X, d_X)$$

Implies $\mathcal{H}^n(X, d_X) = \frac{3v_0}{4}$. A contradiction. □

Chapter 5

Non-collapsed compact three manifolds of almost non-negative curvature.

The results of this chapter are only valid for dimensions two and three-

Theorem 5.1. *Let M be a closed three (or two) manifold satisfying*

$$\begin{array}{l} \text{diam } (M, g_0) \le d_0 \\ \text{Ricci}(g_0) \quad (\sec(g_0)) \ge -\varepsilon g_0 \\ \text{vol}(M, g_0) \ge v_0 > 0, \end{array}$$

where $\varepsilon \le \frac{1}{10c^2}$ and $c = c(v_0, d_0) \ge 1$ is the constant from Lemma 2.1.4 Then there exists an $S = S(d_0, v_0) > 0$ and $K = K(d_0, v_0)$ such that the maximal solution $(M, g(t))_{t \in [0,T)}$ to Ricci-flow satisfies $T \ge S$, and

$$\sup_M |\text{Riem}(g(t))| \le \frac{K}{t},$$

for all $t \in (0, S)$.

Proof. Let $[0, T')$ be the maximal time interval for which

$$\begin{aligned} \text{vol}(M, g(t)) &> \frac{v_0}{2}, \\ \text{Ricci}(g(t)) &\ge -1 \\ \text{diam } (g(t)) &\le 5d_0. \end{aligned}$$

If $T' \le 1$, then the diameter condition will not be violated as long as the other conditions are not violated (as one easily sees by examining the evolution equation for distance under Ricci flow). So we assume w.l.o.g $T' \le 1$ and the diameter condition is not violated. >From Lemma 2.1.4, we know that there exists a $c =$

$c(d_0, v_0)$ such that $R(t) \leq \frac{c}{t}$, for all $t \in [0, T')$. Using Lemma 3.4 we see that there exists a $T'' = T''(c) > 0$ such that Ricci $\geq -\frac{1}{2}$ for all $t \in [0, T''] \cap [0, T')$. So the Ricci curvature condition is not violated on $[0, T''] \cap [0, T')$. Furthermore, in view of Corollary 4.2 there exists a $T''' = T'''(v_0, d_0, c)$, such that $\mathrm{vol}(M, g(t)) > \frac{3v_0}{4}$ for all $t \in [0, T'''] \cap [0, T''] \cap [0, T']$ Hence $T' \geq \min(T''(c), T'''(v_0, d_0)) > 0$, as required. The estimate for the curvature and the existence of S then follow from Lemma 2.1.4. □

Theorem 5.2. *Let $(M_i, {}^i g_0)$ be a sequence of closed three (or two) manifolds satisfying*

$$\begin{aligned} &\mathrm{diam}\ (M_i, {}^i g_0) \leq d_0 \\ &\mathrm{Ricci}({}^i g_0)(\sec({}^i g_0)) \geq -\varepsilon(i){}^i g_0 \\ &\mathrm{vol}(M, {}^i g_0) \geq v_0 > 0, \end{aligned}$$

where $\varepsilon(i) \to 0$, as $i \to \infty$.

Then there exists an $S = S(v_0, d_0) > 0$ and $K = K(v_0, d_0)$ such that the maximal solutions $(M_i, {}^i g(t))_{t\in[0,T_i)}$ to Ricci-flow satisfy $T_i \geq S$, and

$$\sup_M |\mathrm{Riem}({}^i g(t))| \leq \frac{K}{t},$$

for all $t \in (0, S)$. In particular the Hamilton limit solution $(M, g(t))_{t\in(0,S)} = \lim_{i\to\infty}(M_i, {}^i g(t))_{t\in(0,S)}$ (see [17]) exists and satisfies

$$\begin{aligned} &\sup_M |\mathrm{Riem}(g(t))| \leq \frac{K}{t} \\ &\mathrm{Ricci}(g(t)) \geq 0 \quad (\sec(g(t)) \geq 0), \end{aligned} \tag{5.1}$$

for all $t \in (0, S)$ and $(M, g(t))$ is closed. Furthermore

$$\mathrm{d}_{GH}((M, d(g(t))), (M, d_\infty)) \to 0 \tag{5.2}$$

as $t \to 0$ where $(M, d_\infty) = \lim_{i\to\infty}(M_i, d({}^i g_0))$ (the Gromov-Hausdorff limit). Hence, if $M = M^3$, then M^3 is diffeomorphic to to a quotient of one of S^3, $S^2 \times \mathbb{R}$ or $\mathbb{R}^3$ by a finite group of fixed point free isometries acting properly discontinuously.

Proof. We apply the previous theorem. Then notice that lemma 3.3 (or lemma 3.4) implies that $\mathrm{Ricci}(g(t)) \geq 0$ $(\sec(g(t)) \geq 0)$ for this limit solution, for all $t \in (0, S)$. To prove that $\mathrm{d}_{GH}((M, d(g(t))), (M, d_\infty)) \to 0$ use the triangle inequality as in the proof of Lemma 4.1:

$$\begin{aligned} &\mathrm{d}_{GH}((M, d(g(t))), (M, d_\infty)) \\ &\leq \mathrm{d}_{GH}((M, d(g(t))), (M_i, d({}^i g(t))) + \mathrm{d}_{GH}((M_i, d({}^i g(t))), (M, d_\infty)) \\ &\leq \mathrm{d}_{GH}((M, d(g(t))), (M_i, d({}^i g(t))) + \mathrm{d}_{GH}((M_i, d({}^i g(t))), (M_i, d({}^i g_0))) \\ &+\mathrm{d}_{GH}((M_i, d({}^i g_0)), (M, d_\infty)) \\ &\leq \mathrm{d}_{GH}((M, d(g(t))), (M_i, d({}^i g(t))) + c(t) + \mathrm{d}_{GH}((M_i, d({}^i g_0)), (M, d_\infty)) \\ &\overset{i\to\infty}{\longrightarrow} c(t), \end{aligned} \tag{5.3}$$

for all $t > 0$, where $c(t) \to 0$ as $t \to 0$: here we have used **??**, and the characterisation of Gromov-Hausdorff distance given in A.9 to obtain $c(t)$. □

Chapter 6

Local results for Ricci flow and reaction diffusion equations in general

We have seen, in theorem 5.1, that for certain initial conditions, it is possible to obtain a lower bound on the time for which a solution to Ricci flow exists, and also to obtain control over the curvature, volume and diameter for some well defined time interval. This control on the curvature (curvature behaves like $\frac{c}{t}$) is global, and suffices to prove the main theorem 5.1. We are, however, also interested in the local behaviour of such solutions. For example,

Problem 6.0.1. *For the class of solutions whose curvature is bounded by $\frac{c}{t}$ on some time interval $[0, T)$: if a region is diffeomorphic to a Euclidean ball and has bounded geometry at the curvature and C^0 level at time zero, does some smaller region (contained in the initial region) remain bounded geometrically at the curvature and C^0 level for a well controlled time interval?*

In the paper [24], Perelman proved such a local result (see [24] Section 10) in the setting that one has a solution to Ricci flow on a compact manifold which has bounded curvature at each time, and for which the absolute value of the Riemannian curvature on $B(g_0)(x_0, r_0)$ is bounded by $\frac{1}{r_0}$, and for which $B(g_0)(x_0, r_0)$ is close in some C^0 sense to the Euclidean ball of radius r_0 (see Theorem 10.3 of [24] for an exact statement of the theorem). There, the behaviour like $\frac{c}{t}$ is not assumed, but is in fact proved in a a seperate theorem, Theorem 10.1 of [24]. In Theorem 10.1 he proves that the curvature at a point x_0 behaves like $\frac{c}{t}$ (for $t \in [0, \delta^2 r^2]$, $\delta = \delta(n) > 0$)under the even weaker assumptions that the scalar curvature on $B(g_0)(x_0, r_0)$ is bounded from below by $-r_0^2$, and $B(g_0)(x_0, r_0)$ is close in some C^0 sense to the Euclidean ball of radius r_0: see theorem 10.1 of [24] for an exact statement of the theorem.

Another interesting question is:

Problem 6.0.2. *For the class of solutions whose curvature is bounded by $\frac{c}{t}$ on some time interval $[0, T)$: if a region is diffeomorphic to a Euclidean ball and has bounded* *geometry at the curvature and C^0 level at time zero, and has positive curvature (Scalar? Ricci($n = 3$)? Sectional?), does the curvature remain positive on a smaller region (contained in the initial region) for some well defined controlled time interval or can it almost immediately become negative somewhere?*

We show in the following, that a global bound like $\frac{c}{t}$ for the curvature on some time interval, is enough to get a lot of local control under Ricci flow.

Most of the results of this chapter are valid for all dimensions.

First, for completeness, we prove two standard technical lemmas which we will require in the proofs that follow.

6.1 Local results for Ricci flow depending on a bound on the Riemannian curvature tensor

Lemma 6.1.1. *Let $({}^I B_r(0), g(t))_{t\in[0,T)}$, $T \leq 1$ be a smooth solution to Ricci flow satisfying*

$$\sup_{{}^I B_r(0)\times[0,T)} {}^g|\mathrm{Riem}(g)| \leq K \tag{6.1.1}$$

and the initial conditions

$$\begin{aligned} &\sup_{x\in {}^I B_r(0)} {}^I|{}^I\Gamma - {}^{g_0}\Gamma|(x) \leq K \\ &\frac{1}{K} I \leq g_0 \leq KI \text{ on } {}^I B_r(0) \end{aligned} \tag{6.1.2}$$

where ${}^I B_r(0)$ is the open Euclidean ball of radius r, and I is the standard metric on this ball ($K \geq 1$). Then

$$\sup_{x\in {}^I B_{\frac{r}{2}}(0), t\in[0,T)} {}^I|{}^I\Gamma - {}^g\Gamma|(x) \leq c_0(K, n, \frac{1}{r}), \tag{6.1.3}$$

$$\frac{1}{c_0} I \leq g(t) \leq c_0 I \text{ on } {}^I B_{\frac{r}{2}}(0)\ \forall\, t \in [0,T) \tag{6.1.4}$$

for some constant $c_0(K, n, \frac{1}{r})$ and this constant is monotone non-increasing in the third variable.

Proof. This lemma can easily be proved using standard parabolic techniques. Notice that it is unimportant which metric we use for $|T|^2 = g^{ij}g^{kl}T_{ik}T_{jl}$, ($T$ an arbitrary tensor) as all metrics are equivalent:

$$\frac{1}{c} I \leq g(t) \leq cI, \tag{6.1.5}$$

($c = c(K,n) > 0$) in view of conditions (6.1.1) and the initial conditions (6.1.2). Using the interior estimates of W.-X.Shi (see [16], theorem 13.1) we obtain

$$\sup_{{}^{I}B_{\frac{r}{2}}(0)\times[0,T)} |{}^{g}\nabla \mathrm{Riem}| \leq \frac{c_0(n,\frac{1}{r},K)}{\sqrt{t}}. \tag{6.1.6}$$

In [16] (page 20) it also shown that $\frac{\partial}{\partial t}|{}^{g}\Gamma - {}^{g_0}\Gamma| \leq c(|{}^{g}\nabla \mathrm{Riem}| + |\mathrm{Riem}||{}^{g}\Gamma - {}^{g_0}\Gamma|)$, which combined with the above inequality implies that

$$\frac{\partial}{\partial t}|({}^{g}\Gamma - {}^{g_0}\Gamma)| \leq \frac{c}{\sqrt{t}} + c|{}^{g}\Gamma - {}^{g_0}\Gamma|, \tag{6.1.7}$$

for all $(x,t) \in {}^{I}B_{\frac{r}{2}}(0) \times [0,T)$, in view of (6.1.1), which implies the result, in view of (6.1.2),(6.1.5) and the fact that $T \leq 1$. □

Lemma 6.1.2. *Let $(U, g(t))_{t\in[0,T)}$, $T \leq 1$ be a smooth solution to Ricci flow with*

$$\sup_{U\times[0,T)} {}^{g}|\mathrm{Riem}(g)| \leq K, \tag{6.1.8}$$

$$\mathrm{inj}(x) \geq i_0 > 0 \quad \forall \quad x \in {}^{g_0}B_{r_0}(p) \tag{6.1.9}$$

$$\tag{6.1.10}$$

for some ${}^{g_0}B_{r_0}(p) \subset U$. Then there are coordinates $\phi : V_p \to {}^{I}B_r(0)$, where $r = r(i_0, r_0, K, n) > 0$, and $p \in V_p$, such that in these coordinates

$$\sup_{x_0\in {}^{I}B_r(0), t\in[0,T)} {}^{I}|{}^{I}\Gamma - {}^{g}\Gamma|(x,t) \leq c_0(K, i_0, n, \frac{1}{r_0}), \tag{6.1.11}$$

$$\frac{1}{c_0} I \leq g(t) \leq c_0 I \ \forall \ t \in [0,T) \tag{6.1.12}$$

$$\tag{6.1.13}$$

where the constant $c_0(K, i_0, n, \frac{1}{r_0})$ is non-decreasing in the last variable.

Proof. >From [19], Theorem 1.2, there exists a coordinate map $\phi : V \to {}^{I}B_r(0)$ (so called harmonic coordinates) such that the initial metric satisfies (6.1.2) in ${}^{I}B_r(0)$. We then apply the above lemma. □

6.2 Local results for solutions to the Ricci flow which satisfy $|\mathrm{Riem}(g(t))| \leq \frac{c}{t}$.

First, we define more precisely what we mean by: *the geometry is bounded on the curvature and C^0 level.*

Definition 6.2.1. *The Euclidean n-ball of radius $r > 0$ $U = {}^I B_r(0)$ with a Riemannian metric g_0 is geometrically bounded by c if*

$$\sup_U {}^{g_0}|\mathrm{Riem}(g_0)| \leq \frac{c}{r} \tag{6.2.1}$$

$$cI \leq g_0 \leq \frac{1}{c} I \tag{6.2.2}$$

where I is the standard metric on ${}^I B_r(0)$.

Clearly the set is topologically the same as a ball (per definition). Hence a Euclidean ball with a metric g_0 is *bounded geometrically* if the given Riemannian metric g_0 can be compared to the standard metric I on the curvature and C^0 level.

In the following lemma we show that if U is a Euclidean ball of radius one, and (U, g_0) is bounded geometrically by some constant c, and $(\bar{U}, g(t))_{t\in[0,T)}$ is a smooth solution to the Ricci flow whose curvature is bounded above by $\frac{c}{t}$, then, for a well controlled time interval, a Euclidean ball of radius $r(c, n) > 0$ (contained in the initial ball with the same middle point) with the metric $g(t)$ is bounded geometrically by $4c$.

Lemma 6.2.2. *Let $(U = {}^I B_1(0), g_0)$ be geometrically bounded by c_0. Define*

$${}^I\mathrm{dist}(x) = \mathrm{dist}(I)(\partial U, x),$$

where the distance is taken with respect to the Euclidean metric I. Let $({}^I B_{1+\delta}(0), g(t))_{t\in[0,T)}$ $(\delta > 0)$ be any smooth solution to the Ricci flow satisfying $g(0) = g_0$ and

$$t\,{}^{g(t)}|\mathrm{Riem}(x,t)| \qquad \leq c_0, \tag{6.2.3}$$

$$\tag{6.2.4}$$

for all $x \in {}^I B_{1+\delta}(0)$ and for all $t \in [0, T)$. Then there exists an $N = N(c_0, n)$, such that

$${}^{g(t)}|\mathrm{Riem}(x,t)|\,{}^I\mathrm{dist}^2(x) < N,$$

for all $x \in U$ with ${}^I\mathrm{dist}^2(x) \geq Nt$, and $t \leq T$. In particular, $({}^I B_{\frac{1}{4N}}(0), g(t))$ is bounded geometrically by $e^1 c_0$ for all $t \leq \min(\frac{1}{4N}, T)$.

Proof. For the proof, $|\mathrm{Riem}(h)|$ will always refer to ${}^h|\mathrm{Riem}(h)|$. Choose N big, and assume that the theorem does not hold. Then there must be a first time $t_0 \in [0, \min(T, \frac{1}{4N}))$ and point $x_0 \in U$ where the theorem does not hold.

$$\begin{aligned}
&{}^I\mathrm{dist}^2(x_0) \geq N t_0,\\
&x_0 \in U,\\
&|\mathrm{Riem}(x_0, t_0)|\,{}^I\mathrm{dist}^2(x_0) = N,
\end{aligned}$$

and

$$|\mathrm{Riem}(x,t)|{}^{I}\mathrm{dist}^{2}(x) \le N \ \forall\ (x,t) \in U \times [0,t_0] \text{ with} {}^{I}\mathrm{dist}^{2}(x) \ge Nt \tag{6.2.5}$$

Let us rescale: $\tilde{g}(x,s) := ag(x,\frac{s}{a})$, for $s \in [0, at_0]$, and $\tilde{I} = aI$. We define: ${}^{\tilde{I}}\mathrm{dist}^{2}(x) := \mathrm{dist}^2(\tilde{I})(\partial U, x)$ (that is, the distance from the boundary of U to the point x with respect to the metric $\tilde{I}$) which gives us ${}^{\tilde{I}}\mathrm{dist}^{2}(x) = a{}^{I}\mathrm{dist}^{2}(x)$. Set

$$a := \frac{N}{{}^{I}\mathrm{dist}^{2}(x_0)},$$

(a is then bigger than N) and for $t \in [0, t_0]$, let $\tilde{t} := at$. Then ${}^{\tilde{I}}\mathrm{dist}^{2}(x_0) = N$, and hence $N = |\mathrm{Riem}(x_0,t_0)|{}^{I}\mathrm{dist}^{2}(x_0) = |\tilde{\mathrm{Riem}}(x_0,\tilde{t}_0)|{}^{\tilde{I}}\mathrm{dist}^{2}(x_0) = |\tilde{\mathrm{Riem}}(x_0,\tilde{t}_0)|N$, which implies that $|\tilde{\mathrm{Riem}}(x_0,\tilde{t}_0)| = 1$. Furthermore, $\tilde{t_0} = \frac{t_0 N}{{}^{I}\mathrm{dist}^{2}(x_0)} \le 1$, since ${}^{I}\mathrm{dist}^{2}(x_0) \ge Nt_0$. We also have

$$|\tilde{\mathrm{Riem}}(x,\tilde{t})|{}^{\tilde{I}}\mathrm{dist}^{2}(x) = |\mathrm{Riem}(x,t)|{}^{I}\mathrm{dist}^{2}(x) \le N \tag{6.2.6}$$

for all $(x,\tilde{t}) \in U \times [0,\tilde{t}_0]$ which satisfy ${}^{\tilde{I}}\mathrm{dist}^{2}(x) \ge N\tilde{t}$ in view of the facts

(i) ${}^{I}\tilde{\mathrm{dist}}^{2}(x) \ge N\tilde{t} \iff {}^{I}\mathrm{dist}^{2}(x) \ge Nt$ and

(ii) $\tilde{t} \le \tilde{t}_0 \iff t \le t_0$ and

(iii) equation (6.2.5) holds).

Now we consider two cases:

- case 1: ${}^{\tilde{I}}\mathrm{dist}^{2}(x_0) \ge 2N\tilde{t}_0(\iff \tilde{t}_0 \le \frac{1}{2})$
- case 2: ${}^{\tilde{I}}\mathrm{dist}^{2}(x_0) < 2N\tilde{t}_0(\iff \tilde{t}_0 > \frac{1}{2})$

Assume case one holds. Then for y satisfying ${}^{\tilde{I}}\mathrm{dist}^{2}(y) \ge \frac{N}{2}$ we see that (use $\tilde{t}_0 \le \frac{1}{2}$)

$${}^{\tilde{I}}\mathrm{dist}^{2}(y) \ge N\tilde{t}_0 \ge N\tilde{t}$$

for all $\tilde{t} \le \tilde{t}_0$, and hence

$$|\tilde{\mathrm{Riem}}(y,\tilde{t})|{}^{\tilde{I}}\mathrm{dist}^{2}(y) \le N,$$

in view of (6.2.6). Hence,

$$|\tilde{\mathrm{Riem}}(y,\tilde{t})| \ \le \ 2 \ \forall\ \tilde{t} \le \tilde{t}_0, \tag{6.2.7}$$

in view of the assumption on y (notice that ${}^{\tilde{I}}\mathrm{dist}^{2}(x_0) = N \ge \frac{N}{2}$ and so $y = x_0$ is valid in (6.2.7): that is, $|\tilde{\mathrm{Riem}}(x_0,\tilde{t})| \le 2$ for all $\tilde{t} \le \tilde{t}_0$). That is,

$$|\tilde{\mathrm{Riem}}(y,\tilde{t})| \le 2 \ \forall\ \tilde{t} \le \tilde{t}_0, \quad \forall y \in \tilde{B}_{\frac{N}{4}}(x_0).$$

Using the fact that

$$\frac{I}{c_0} \leq g_0 \leq c_0 I$$

we obtain that

$$\frac{\tilde{I}}{c_0} \leq \tilde{g}_0 \leq c_0 \tilde{I}$$

and

$$\sup_{y \in \tilde{B}_{\frac{N}{4}}(x_0)} |\mathrm{Riem}(\tilde{g}_0)|(y) \leq \frac{c_0}{N},$$

since $\tilde{g}_0 = a g_0$, and $a = \frac{N}{{}^I\mathrm{dist}^2(x_0)} \geq N$ and $(U, g_0) = ({}^I B_1(0), g_0)$ is geometrically bounded by c_0. We also have $|\tilde{\mathrm{Riem}}(x_0, \tilde{t}_0)| = 1$. This contradicts Lemma 6.2.3, if $N = N(c_0, n)$ is chosen large enough.

So assume case two holds. This is equivalent to $\frac{1}{2} < \tilde{t}_0$. Then, for $\tilde{t} \leq \frac{1}{2}$, we still have the estimate

$$|\tilde{\mathrm{Riem}}(y, \tilde{t})| \leq 2, \quad \forall y \text{ with } {}^{\tilde{I}}\mathrm{dist}^2(y) \geq \frac{N}{2}$$

(this may be seen as follows:

$${}^{\tilde{I}}\mathrm{dist}^2(y) \geq \frac{N}{2} \geq N\tilde{t},$$

since $\tilde{t} \leq \frac{1}{2} < \tilde{t}_0$, and so, using (6.2.6), we obtain the estimate).

For $\tilde{t}_0 \geq \tilde{t} > \frac{1}{2}$, we have

$$|\tilde{\mathrm{Riem}}(y, \tilde{t})| \leq \frac{c_0}{\tilde{t}} \leq 2c_0,$$

in view of the assumption (6.2.3). W.l.o.g. $c_0 \geq 1$. Hence

$$|\tilde{\mathrm{Riem}}(y, \tilde{t})| \leq 2c_0, \forall t \in [0, \tilde{t}_0), \quad \forall\, y \in \tilde{B}_{\frac{N}{4}}(x_0).$$

Furthermore, $|\tilde{\mathrm{Riem}}(x_0, \tilde{t}_0)| = 1$. Once again, this leads to a contradiction for $N = N(c_0, n)$ chosen large enough, in view of Lemma 6.2.3. (as in case 1, we have

$$\begin{aligned} &\frac{\tilde{I}}{c_0} \leq \tilde{g}_0 \leq c_0 \tilde{I}, \\ &\sup_{\tilde{B}_{\frac{N}{4}}(x_0)} |Riem(\tilde{g}_0)| \leq \frac{c_0}{N}, \end{aligned} \tag{6.2.8}$$

and so we may apply the lemma). □

The following lemma, which is used to help prove the locality theorem above, is a standard result from the theory of parabolic equations

Lemma 6.2.3. *Let* $(M, g(t))_{t\in[0,T)}$, *be a complete smooth solution to Ricci flow and* (U, x_0, I), $U \subset M$ *be isometric to to the (open) Euclidean ball* $B_N(0)$*of radius* N, *(*$x_0 \sim 0$*) and centre* 0. *Assume that:*

$$\frac{I}{c_0} \leq g_0 \leq c_0 I$$
$$\sup_U |\mathrm{Riem}(\cdot,t)| \leq c_1 \ \forall \ t \in [0,T),$$
$$\sup_U |\mathrm{Riem}|(\cdot,0) \leq \varepsilon.$$

Then for all $\sigma^2 \geq \varepsilon^2$ *there exists* $a = a(c_0, c_1, n)$ *and* $N_* = N_*(\sigma, c_0, c_1, n) > 0$ *such that if* $N \geq N_*$ *then*

$$|\mathrm{Riem}|^2(x_0,t) \leq \sigma^2 \exp^{at} \ \ \forall \ t \in [0,T) \tag{6.2.9}$$

In particular, if $\varepsilon \leq \exp^{-a}\delta$ *for some* $\delta > 0$ *then (choose* $\sigma = \exp^{-a}\delta$*: then* N_* *is a constant depending only on* c_0, c_1, n, δ *) there is a* $N_* = N_*(c_0, c_1, n, \delta) > 0$ *such that*

$$|\mathrm{Riem}(x_0,t_0)| \leq \delta^2, \ \ \forall \ t_0 \in [0,T) \cap [0,1)$$

if $N \geq N_*$.

Proof. Clearly, in view of the conditions in the assumption, and the equation of evolution for the metric, we have

$$\frac{1}{c_2(c_0,c_1,n)} I \leq g(t) \leq c_2(c_0,c_1,n) I \ \forall \ t \in [0,T)$$

Set

$$f(\cdot,t) := |\mathrm{Riem}|^2(\cdot,t) - \sigma^2(1+\rho^2)\exp^{at},$$

where a is to be chosen. and $\rho(x,t) := \mathrm{dist}(g(t))(x,x_0)$. Then

$$\begin{aligned}
\frac{\partial}{\partial t} f \leq & \quad {}^g\Delta f + 4|\mathrm{Riem}|^3 - a\sigma^2(1+\rho^2)\exp^{at} - 2\sigma^2(\frac{\partial}{\partial t}\rho)\rho\exp^{at} + \sigma^2\exp^{at}\,{}^g\Delta(\rho^2) \\
\leq & \quad {}^g\Delta f + 4c_1|\mathrm{Riem}|^2 - a\sigma^2(1+\rho^2)\exp^{at} \\
& \quad + 4\sigma^2(n-1)c_1\rho^2\exp^{at} + c_1\sigma^2\exp^{at} c(c_1,n)\rho \\
\leq & \quad {}^g\Delta f + 4c_1|\mathrm{Riem}|^2 - \frac{a}{2}\sigma^2(1+\rho^2)\exp^{at}
\end{aligned}$$

for all $x \in B_{\frac{N}{2c_2^2}}(0)$ for appropriately chosen $a = a(n, c_1)$, where here we have used the Hessian comparison principle in order to estimate ${}^g\Delta(\rho^2)$, and the fact that all distance minimising geodesics (in terms of $g(t)$) between 0 and points in ${}^IB_{\frac{N}{2c_2^2}}(0)$ must lie in ${}^IB_N(0)$ (this last fact may be seen as follows: the length of a ray from 0 to $p \in \partial B_{\frac{N}{2c_2^2}}(0)$ is trivially bounded from above by $c_2\frac{N}{2c_2^2} = \frac{N}{2c_2}$. For any curve starting from 0 which reaches the boundary of $B_N(0)$, the length is bounded from below by $\frac{1}{c_2}N$). Using the definition of f, we get

$$\frac{\partial}{\partial t}f \;\leq\; {}^{g}\Delta f + 4c_1 f + 4c_1\sigma^2(1+\rho^2)\exp^{at} - \frac{a}{2}\sigma^2(1+\rho^2)\exp^{at} < 4c_1 f,$$

for appropriately chosen $a = a(n, c_1)$. Now for $\sigma^2 \geq \varepsilon^2$ we have $f(\cdot, 0) < 0$ on $B_{\frac{N}{2c_2^2}}(0)$. Now choose $N = N(\sigma, c_0, c_1, n)$ so large that $f(\cdot, t) < 0$ on $\partial B_{\frac{N}{2c_2^2}}(0)$ for all $t \in [0, T)$. The maximum principle then implies that f is less than zero for all $t \in [0, T)$, for all $x \in B_{\frac{N}{2c_2^2}}(0)$. Note that although ρ^2 is not smooth everywhere, using a trick of Calabi, we may still draw the same conclusion: see the proof of a Theorem 7.1 in [29] (essentially we define a new function $\tilde{\rho}(x,t) = \rho(x,q,t) + \rho(q,x_0,t)$ for some appropriately chosen q so that $\tilde{\rho}$ is smooth in a small neighbourhood of (p_0, t_0) where (p_0, t_0) is the first time and point where $f(p_0, t_0) = 0$. Thn we define:

$$\tilde{f}(\cdot, t) := |\mathrm{Riem}|^2(\cdot, t) - \sigma^2(1+\tilde{\rho}^2)\exp^{at},$$

and argue with $\tilde{f}$: due to the triangle inequality we have $\tilde{f} \leq f$ and hence $\tilde{f} < 0$ for $t \leq t_0$. Furthermore $\tilde{f}(x_0, t_0) = 0$, as q lies on a shortest geodesic between x_0 and p_0 at time t_0. Hence we may apply the maximum principle and still obtain a contradiction. Hence $f < 0$. □

So we see that if a local region is relatively well controlled at time zero, and the curvature behaves globally like $\frac{c}{t}$ near time zero, then we can show that a (well defined) smaller region remains well behaved for a a well defined time interval.

If a region remains geometrically bounded, and the curvature at time zero is positive on this region, then one might ask whether the curvature remains positive on some smaller region. A partial answer to this question is given in the following lemma

Lemma 6.2.4. *Let $(U, g(t))_{t\in[0,T)}$, $T \leq 1$ be a solution to Ricci flow where U is diffeomorphic to a Euclidean ball. Assume that*

$$\begin{aligned} &|\mathrm{Riem}(g(t))| + |\Gamma(g(t)) - \Gamma(I)| \leq K \;\forall t \in [0,T) \text{ on } U \\ &\frac{1}{K}I \leq g(t) \leq KI \;\forall\, t \in [0,T) \text{ on } U \end{aligned} \tag{6.2.10}$$

$(K \geq 1)$ where (U, I) is the Euclidean unit ball with centre zero. Assume furthermore that $n = 3$ and

$$\begin{aligned} \mathrm{Ricci}(g_0) &\geq \varepsilon_0 \\ \mathrm{Ricci}(g(t)) &\geq -K\varepsilon_0 \;\; \forall t \in [0,T). \end{aligned} \tag{6.2.11}$$

Then there exists $S = S(K, n) > 0$, such that (in the case that $n = 3$)

$$\mathrm{Ricci}(g)(0,t) \;\geq\; K\varepsilon_0 g(0,t)$$

for all $t \in [0, \min(T, S))$.

Remark 6.2.5. *Notice that if a region* $({}^{I}B_2, g_0)$ *is geometrically bounded by some constant* c *and* $({}^{I}B_2, g(t))_{t\in[0,T)}$ *is a solution to Ricci flow satisfying* $\sup_{{}^{I}B_2} |\mathrm{Riem}(g(t))|t \le c$ *for all* $t \in [0,T)$ *and furthermore*

$$\sup_{x\in {}^{I}B_2} {}^{I}|{}^{I}\Gamma - {}^{g_0}\Gamma|(x) \le c,$$

then using Lemma 6.1.1 and Lemma 6.2.2, we see that the conditions (6.2.10) are satisfied for some constant K *for* $t \le T'(n,K,c)$.

Proof. Let $n = 3$. Set

$$L(x,t) := \mathrm{Ricci}(x,t) - K\varepsilon_0 \exp^{-kt} g(x,t) + 3K\varepsilon_0|x|^2 g(x,t).$$

Then $L(\cdot,t) > 0$ on the boundary of U for all $t \le S$, $S = S(n,k)$ small enough.

$$\begin{aligned}
\frac{\partial}{\partial t}L_{ij} = \quad & \frac{\partial}{\partial t}L_i{}^s g_{js} + L_i{}^s \frac{\partial}{\partial t} g_{js} \\
\ge \quad & (\Delta L_i{}^s + \mathrm{Ricci}_i{}^s\mathrm{R} - \mathrm{Ricci}_i{}^p\mathrm{Ricci}_p{}^s + kK\varepsilon_0 \exp^{-kt} \delta_i{}^s)g_{js} \\
& + L_i{}^s \frac{\partial}{\partial t} g_{js} - 3K\varepsilon_0\Delta(|x|^2)g_{ij} \\
\ge \quad & (\Delta L_i{}^s + \mathrm{Ricci}_i{}^s\mathrm{R} - \mathrm{Ricci}_{ip}\mathrm{Ricci}^{ps} + kK\varepsilon_0 \exp^{-kt} \delta_i{}^s)g_{js} \\
& + L_i{}^s \frac{\partial}{\partial t} g_{js} - 2a_{K,n}\varepsilon_0 g_{ij} \\
= \quad & \Delta L_{ij} + \mathrm{Ricci}_{ij}\mathrm{R} - \mathrm{Ricci}_{is}\mathrm{Ricci}_j^s + kK\varepsilon_0 \exp^{-kt} g_{ij} \\
& + L_i{}^s \frac{\partial}{\partial t} g_{js} - a_{K,n}\varepsilon_0 g_{ij} \\
= \quad & \Delta L_{ij} + \mathrm{R}L_{ij} + K\varepsilon_0\mathrm{R}\exp^{-kt} g_{ij}(x,t) - 3K\mathrm{R}\varepsilon_0|x|^2 g_{ij}(x,t) \\
& - L_{is}\mathrm{Ricci}_j^s + (-K\varepsilon_0 \exp^{-kt} g_{is} + 3K\varepsilon_0|x|^2 g_{is})\mathrm{Ricci}_j^s \\
& + kK\varepsilon_0 \exp^{-kt} g_{ij} + L_i{}^s \frac{\partial}{\partial t} g_{js} - a_{K,n}\varepsilon_0 g_{ij} \\
\ge \quad & \Delta L_{ij} + \mathrm{R}L_{ij} - L_{is}\mathrm{Ricci}_j^s + L_i{}^s \frac{\partial}{\partial t} g_{js} - b(n,K)\varepsilon_0 g_{ij} + kK\varepsilon_0 \exp^{-kt} g_{ij} \\
> \quad & \Delta L_{ij} + \mathrm{R}L_{ij} - L_{is}\mathrm{Ricci}_j^s + L_i{}^s \frac{\partial}{\partial t} g_{js}
\end{aligned}$$

for $k = k(n,K)$ big enough, and $t \in [0,S)$, $S = S(n,K) > 0$ small enough. At a first time and point where there is a direction in which L is zero we obtain a contradiction. Hence $L > 0$ on U for all $t \in [0,S)$. This implies the result.

□

The condition 6.2.11 is not very satisfactory, as it will not automatically hold on a region with bounded geometry, even if it is satisfied at time zero (we always expect influence from outside the region to instantaneously effect the region). We may however combine this lemma with Theorem 3.4 on Ricci curvature to obtain a relatively satisfactory result in dimensions two and three.

Lemma 6.2.6. *Let $(M^n, g(t))_{t\in[0,T}$ be a closed solution to Ricci flow $n = 2$ or 3, and $U \subset M$ be diffeomorphic to a Euclidean ball. Assume that*

$$\begin{aligned} &|\mathrm{Riem}(g(t))| + |\Gamma(g(t)) - \Gamma(I)| \leq K \quad \text{on } U \ \forall t \in [0,T) \\ &\frac{1}{K} I \leq g(t) \leq K I \quad \text{on } U \ \forall t \in [0,T) \end{aligned} \tag{6.2.12}$$

$(K \geq 1)$ where (U, I) is a Euclidean unit ball with centre zero. Then there exists $c'(K) > 0, k = k(K) > 0, S = S(K) > 0$ such that if

$$\begin{aligned} \mathrm{Ricci}(g_0) &\geq \varepsilon_0 \quad \text{on } U \\ \mathrm{Ricci}(g_0) &\geq -K\varepsilon_0 \quad \text{on } M \end{aligned} \tag{6.2.13}$$

and $c'(K)\varepsilon_0 \leq \frac{1}{100}$, $K \geq 1$, then

$$\mathrm{Ricci}(g)(0,t) \geq k\varepsilon_0 g(0,t)$$

for all $t \in [0, \min(T, S))$.

Proof. We know from Lemma 3.4

$$\mathrm{Ricci}(g(t)) \geq -c\varepsilon_0 t R - c\varepsilon_0$$

on M for all $t \in [0, T')$ for some $T' = T'(K) > 0$ for some $c = c(K) < \infty$. This gives us that

$$\mathrm{Ricci}(g(t)) \geq -c'\varepsilon_0$$

on U for all $t \in [0, T')$ for constants $0 < T'(K), c'(K) < \infty$, in view of the condition (6.2.12). Hence, the conditions of Lemma 6.2.4 are satisfied, and we may apply that lemma to obtain the result. □

Combining this with Lemma 6.1.1 and Lemma 6.2.2 we obtain:

Lemma 6.2.7. *Let $(M^n, g(t))_{t\in[0,T}$ be a closed solution to Ricci flow $n = 2$ or 3, and $U \subset M$ be diffeomorphic to a Euclidean ball ${}^{I}B_r(0)$. Assume that $({}^{I}B_r(0), g_0)$ has geometry bounded by K and*

$${}^{I}|\Gamma(g_0) - \Gamma(I)| \leq \frac{K}{r} \quad \text{on } U$$

$(K \geq 1)$. Then there exists $c'(K) > 0, k = k(K) > 0, S = S(K) > 0$ such that if

$$\begin{aligned} \mathrm{Ricci}(g_0) &\geq \frac{\varepsilon_0}{r^2} \quad \text{on } U \\ \mathrm{Ricci}(g_0) &\geq -\frac{K}{r^2}\varepsilon_0 \quad \text{on } M \end{aligned} \tag{6.2.14}$$

and $c'(K)\varepsilon_0 \leq \frac{1}{100}$ then

$$\mathrm{Ricci}(g)(0,t) \geq k\frac{\varepsilon_0}{r^2} g(0,t)$$

for all $t \in [0, \min(T, Sr^2))$.

Proof. Rescale g_0 and I so that $r = 1$ and then apply Lemmata 6.2.2, 6.1.1 and then use Lemma 6.2.6. □

6.3 Local results for solutions f to the heat equation which satisfy a bound of the form $f \leq \frac{1}{t}$.

In fact, the important bound which leads to the locality result (Lemma 6.2.2), is the bound of the form $|\text{Riem}| \leq \frac{c}{t}$. The argument was a scaling argument which used the parabolic maximum principle, and didn't really have anything to do with Ricci-flow. We illustrate this somewhat more precisely, by showing that a similar result holds for the heat flow.

Lemma 6.3.1. *Let f be a smooth solution to the heat flow on $B_2(0) \times [0,1]$ with*

$$ft|_{B_1(0)\times[0,1]} \leq 1 \quad \forall t \in [0,1], \tag{6.3.1}$$

$$f(x,t) \geq 0, \quad \forall x \in B_2(0), t \in [0,1] \tag{6.3.2}$$

$$\sup_{\bar{B}_1(0)} f(x,0)(1-|x|^2)^2 \leq 1 \tag{6.3.3}$$

Then

$$\sup_{\bar{B}_1(0)} f(x,t)((1-|x|^2)^2 - 50nt) \quad \leq 50n$$

for all $t \in [0,1]$.

Proof. Set

$$l(x,t) := f(x,t)((1-|x|^2)^2 - Mt) - M,$$

where $M = 50n$. Then $l \in C^\infty(\bar{B}_{\frac{3}{2}}(0) \times [0,1])$, and $l(x,0) \leq -50$, and $l(\cdot,t) \leq -50n < 0$ on $\partial(B_1(0))$ for all $t \in [0,1]$. Hence, if there is a time and point $(x,t) \in \bar{B}_1(0) \times [0,1]$ where $l(x,t) \geq 0$, then due to compactness, there must be a first time t_0 and point $x_0 \in B_1(0)$, where $l(x_0,t_0) = 0$ (possibly there is more than one point x_0, but there exists at least one point x_0). Assume that (x_0,t_0) is such a point, and (x_0,t_0) satisfies $(1-|x_0|^2)^2 - Mt_0 \leq \frac{M}{2}t_0$. Then

$$\begin{aligned} l(x_0,t_0) \quad &= f(x_0,t_0)((1-|x_0|^2)^2 - Mt_0) - M \\ &\leq f(x_0,t_0)\frac{M}{2}t_0 - M \\ &\leq -\frac{M}{2} < 0, \end{aligned} \tag{6.3.4}$$

(in view of the conditions (6.3.1) and (6.3.2)) which contradicts the fact that $l(x_0,t_0) = 0$. So we may assume, without loss of generality, that

$$(1-|x_0|^2)^2 - Mt_0 \geq \frac{M}{2}t_0 \tag{6.3.5}$$

Notice that this implies

$$\frac{2}{3}(1-|x_0|^2)^2 \geq Mt_0$$

which further implies that

$$(1-|x_0|^2)^2 - Mt_0 \geq \frac{1}{3}(1-|x_0|)^2 \tag{6.3.6}$$

We calculate the evolution equation for l for such an (x_0, t_0).

$$\begin{aligned}
\frac{\partial}{\partial t} l = \quad & \Delta(l) - Mf - f\Delta((1-|x|^2)^2) - 2\nabla_i f \nabla_i (1-|x|^2)^2 \\
= \quad & \Delta(l) - Mf - f(8|x|^2 - 4n(1-|x|^2)) - 2\nabla_i f \nabla_i (1-|x|^2)^2 \\
\leq \quad & \Delta(l) - Mf + 4nf - 2\nabla_i f \nabla_i (1-|x|^2)^2 \\
\leq \quad & \Delta(l) - \frac{M}{2} f - 2\nabla_i f \nabla_i (1-|x|^2)^2 \\
= \quad & \Delta(l) - \frac{M}{2} f - \frac{2\nabla_i (f((1-|x|^2)^2 - Mt))\nabla_i (1-|x|^2)^2}{((1-|x|^2)^2 - Mt)} \\
& + 2f \frac{\nabla_i (1-|x|^2)^2 \nabla_i (1-|x|^2)^2}{((1-|x|^2)^2 - Mt)} \\
= \quad & \Delta(l) - \frac{M}{2} f - \frac{2\nabla_i (f((1-|x|^2)^2 - Mt))\nabla_i (1-|x|^2)^2}{((1-|x|^2)^2 - Mt)} \\
& + 8f|x|^2 \frac{(1-|x|^2)^2}{((1-|x|^2)^2 - Mt)}
\end{aligned} \tag{6.3.7}$$

and hence, in view of the inequality (6.3.6),

$$\begin{aligned}
\frac{\partial}{\partial t} l \quad & \leq \Delta(l) - \frac{M}{2} f - \frac{2\nabla_i l \nabla_i (1-|x|^2)^2}{((1-|x|^2)^2 - Mt)} + 24f|x|^2 \\
& \leq \Delta(l) - \frac{2\nabla_i l \nabla_i (1-|x|^2)^2}{((1-|x|^2)^2 - Mt)} + 24f - \frac{M}{2} f \\
& < \Delta(l) - \frac{2\nabla_i l \nabla_i (1-|x|^2)^2}{((1-|x|^2)^2 - Mt)}.
\end{aligned} \tag{6.3.8}$$

As $l(x_0, t_0)$ is a local maximum, we obtain a contradiction (note that $((1-|x_0|^2)^2 - Mt_0) > 0$ due to the assumption (6.3.5)). □

In fact a scaled version of this lemma is true whenever we have at most polynomial growth of f in t, as the following lemma shows.

Lemma 6.3.2. *Let f be a smooth solution to the heat flow on $B_2(0) \times [0,1]$ with*

$$\begin{aligned}
& ft^p|_{B_1(0)\times[0,1]} \leq 1 \\
& f(x,t) \geq 0, \quad \forall x \in B_2(0), t \in [0,1] \qquad (6.3.9) \\
& \sup_{x\in \bar{B}_1(0)} (f(x,0)+1)^{\frac{1}{p}} (1-|x|^2)^2 \leq 2 \qquad (6.3.10)
\end{aligned}$$

$(p \geq 1)$. Then

$$\sup_{x\in \bar{B}_1(0)} (f(x,t)+1)^{\frac{1}{p}} ((1-|x|^2)^2 - M(n,p)t) \quad \leq c(n,p) \quad \forall t \in [0,1].$$

Proof. If

$$ft^p \leq 1$$

then

$$(1+f)t^p \leq 2 \ \forall \ t \leq 1,$$

and hence

$$(1+f)^{\frac{1}{p}} t \leq 2.$$

Setting

$$\tilde{f} := (1+f)^{\frac{1}{p}},$$

we have

$$\begin{aligned}
&\tilde{f} t \leq 2, \\
&\tilde{f}(x,0)(1-|x|^2)^2 \leq 2. \qquad (6.3.11)
\end{aligned}$$

Using the fact that f solves the heat equation and $\tilde{f} \geq 1$, we calculate

$$\frac{\partial}{\partial t}\tilde{f} = \Delta(\tilde{f}) - \frac{1}{p}(\frac{1}{p} - 1)\frac{|\nabla \tilde{f}|^2}{\tilde{f}}. \qquad (6.3.12)$$

Define

$$\tilde{l} := \tilde{f}(x,t)((1-|x|^2)^2 - Mt) - M,$$

where $M = M(n,p)$ is a constant to be chosen. Let (x_0, t_0) be a first time and point where $\tilde{l} = 0$. Arguing as above, we obtain

$$\frac{\partial}{\partial t}\tilde{l} \ < \ \Delta(\tilde{l}) - 2(\nabla_i \tilde{l})V_i - \frac{M}{2}\tilde{f} - \frac{1}{p}(\frac{1}{p} - 1)((1-|x|^2)^2 - Mt)\frac{|\nabla \tilde{f}|^2}{\tilde{f}} \quad (6.3.13)$$

where

$$V_i = \frac{\nabla_i (1-|x|^2)^2}{((1-|x|^2)^2 - Mt)}.$$

Arguing as in the previous lemma, we see that we may assume that $((1-|x|^2)^2 - Mt) \geq \frac{1}{3}(1-|x|^2)^2$, and so V_i can be considered to be smooth in a small neighbourhood of (x_0, t_0). But the last term in the above inequality can also be estimated, as the following calculation shows.

$$\begin{aligned}
\frac{c(p)}{\tilde{f}}((1-|x|^2)^2 - Mt)\nabla_i \tilde{f} \nabla_i \tilde{f} = \quad & \frac{c(p)}{\tilde{f}}\nabla_i(((1-|x|^2)^2 - Mt)\tilde{f})\nabla_i \tilde{f} \\
& -c(p)\nabla_i((1-|x|^2)^2 - Mt)\nabla_i \tilde{f}. \quad (6.3.14)
\end{aligned}$$

Remembering that we may assume that $((1-|x|^2)^2 - Mt) \geq \frac{1}{3}(1-|x|^2)^2$, we get

$$\begin{aligned}
& -c(p)\nabla_i((1-|x|^2)^2 - Mt)\nabla_i \tilde{f} \\
= \quad & -\frac{c(p)}{((1-|x|^2)^2 - Mt)}\nabla_i((1-|x|^2)^2 - Mt)\nabla_i(\tilde{f}((1-|x|^2)^2 - Mt))
\end{aligned}$$

$$
\begin{aligned}
& +c(p)\frac{\tilde{f}}{((1-|x|^2)^2-Mt)}\nabla_i(1-|x|^2)^2\nabla_i(1-|x|^2)^2 \\
\leq \quad & -\frac{3c(p)}{((1-|x|^2)^2}\nabla_i((1-|x|^2)^2-Mt)\nabla_i(\tilde{f}((1-|x|^2)^2-Mt)) \\
& +32c(p)\tilde{f} \qquad (6.3.15)
\end{aligned}
$$

Substituting this inequality into (6.3.14), we get

$$
\frac{c(p)}{\tilde{f}}((1-|x|^2)^2-Mt)\nabla_i\tilde{f}\nabla_i\tilde{f} \leq W_i\nabla_i\tilde{l} + 12c(p)\tilde{f},
$$

is a smooth vector field which is defined in a small neighbourhood (space and time) of the point (x_0, t_0). Substituting this inequality into equation (6.3.13) we get

$$
\frac{\partial}{\partial t}(\tilde{l}) \quad < \quad \Delta(\tilde{l}) + \nabla_i(\tilde{l})Z_i. \qquad (6.3.16)
$$

at the point (x_0, t_0), where Z is a smooth vector field which is defined in a small neighbourhood (space and time) of the point (x_0, t_0). This gives us a contradiction, as (x_0, t_0) is a local maximum for $\tilde{l}$.

□

Appendix A

Gromov Hausdorff space and Alexandrov spaces

Definition A.1. *Let (Z, d) be a metric space, $p \in Z$, $r > 0$.*

$$B_r(p) := \{x \in Z : d(x, p) < r\}.$$

For two non-empty subsets $A, B \subset Z$

$$\mathrm{dist}(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

$$B_r(A) := \{x \in Z : \mathrm{dist}(x, A) < r\}.$$

Definition A.2. *For subsets $X, Y \subset (Z, d)$ we define the Hausdorff distance between X and Y by*

$$d_H(X, Y) := \inf\{\varepsilon > 0 : X \subset B_\varepsilon(Y) \text{ and } Y \subset B_\varepsilon(X)\}.$$

Then, (see [5] Prop.7.3.3)

Proposition A.3.

- *d_H is a semi-metric on 2^Z (the set of all subsets of Z),*
- *$d_H(A, \bar{A}) = 0$ for all $A \subset Z$, where $\bar{A}$ is the closure of A (in (Z, d))*
- *If A and B are closed subsets of (Z, d) and $d_H(A, B) = 0$ then $A = B$.*

Definition A.4. *For a subset $X \subset Z$, (Z, d) a metric space, we define $d|_X$ to be the metric on X defined by*

$$d|_X(a, b) = d(a, b).$$

We then define the Gromov-Hausdorff distance between two abstract metric spaces (X, d_X) and (Y, d_Y) as follows:

Definition A.5. *$d_{GH}((X, d_X), (Y, d_Y))$ is the infimum over all $r > 0$ such that there exists a metric space (Z, d) and maps $f : X \to Z$, $X' := f(X)$, and $g : Y \to Z$, $Y' := g(Y)$ such that $f : (X, d_X) \to (X', d|_{X'})$ and $g : (Y, d_Y) \to (Y', d_{Y'})$ are isometries and $d_H(X', Y') < r$.*

Fact A.6. d_{GH} *satisfies the triangle inequality,i.e.,*

$$d_{GH}((X_1,d_1),(X_3,d_3)) \le d_{GH}((X_1,d_1),(X_2,d_2)) + d_{GH}((X_2,d_2),(X_3,d_3))$$

for all metric spaces $(X_1,d_1),(X_2,d_2),(X_3,d_3)$.

Proof. See [5] Prop.7.3.16. □

Definition A.7. *An* ν*-Hausdorff approximation* $f : X \to Y$ *for metric spaces* (X,d_X) *and* (Y,d_Y) *is a map which satisfies*

$$\begin{aligned} &|d_Y(f(x),f(x')) - d_X(x,x')| \le \nu \\ &B_\nu(f(X)) = Y \end{aligned} \tag{A.1}$$

Definition A.8. $\mathrm{Happrox}((X,d_X),(Y,d_Y))$ *is the infimum of* ν *such that there exists a* ν*-Hausdorff approximation* $f : X \to Y$.

We prove the following simple well known lemma

Lemma A.9.

$$\mathrm{Happrox}((X,d_X),(Y,d_Y)) \le 2d_{GH}((X,d_X),(Y,d_Y) \le 4\mathrm{Happrox}((X,d_X),(Y,d_Y)).$$

Proof. See Corollary 7.3.28 of [5]. □

Now we state the compactness result of Gromov.

Proposition A.10. $\mathcal{M}(n,k,d_0)$ *is precompact in Gromov-Hausdorff space.*

Proof. See [5] Remark 10.7.5. □

Clearly $\mathcal{S}(n,k,d_0) \subset \mathcal{M}(n,(n-1)k,d_0)$ and so it is also precompact in Gromov-Hausdorff space.

In [6] (Theorem 10.8), the following fact about the convergence of Hausdorff measure was shown.

Theorem A.11. *Let* $(M_i,g_i) \in \mathcal{S}(n,k,d_0)$*,*$i \in \mathbb{N}$ *be a sequence of smooth Riemannian manifolds with* $\mathrm{vol}(M_i,g_i) \ge v_0 > 0$*, for all* $i \in \mathbb{N}$ *and*

$$(M_i,d(g_i)) \overset{i\to\infty}{\longrightarrow} (X,d_X)$$

in Gromov Hausdorff space. Then

$$\mathrm{vol}(M_i,g_i) = \mathcal{H}_i(M_i) \overset{i\to\infty}{\longrightarrow} \mathcal{H}(M),$$

where $\mathcal{H}_i : M_i \to \mathbb{R}_0^+$ *is* n*-dimensional Hausdorff measure with respect to* $d(g_i)$ *and* $\mathcal{H} : X \to \mathbb{R}_0^+$, *is* n*-dimensional Hausdorff measure with respect to* d_X.

Proof. See for example Theorem 10.10.10 in [5]. □

In [8] (Theorem 5.4) the same result was proved for $\mathcal{M}(n, k, d_0)$.

Theorem A.12. *Let* $(M_i, g_i) \in \mathcal{M}(n, k, d_0)$,$i \in \mathbb{N}$ *be a sequence of smooth Riemannian manifolds with* $\operatorname{vol}(M_i, g_i) \geq v_0 > 0$ *for all* $i \in \mathbb{N}$ *, and*

$$(M_i, d(g_i)) \overset{i\to\infty}{\longrightarrow} (X, d_X)$$

in Gromov Hausdorff space. Then

$$\operatorname{vol}(M_i, g_i) = \mathcal{H}_i(M_i) \overset{i\to\infty}{\longrightarrow} \mathcal{H}(M),$$

where $\mathcal{H}_i : M_i \to \mathbb{R}_0^+$ *is n-dimensional Hausdorff measure with respect to* $d(g_i)$ *and* $\mathcal{H} : X \to \mathbb{R}_0^+$, *is n-dimensional Hausdorff measure with respect to* d_X.

Proof. See Theorem 5.4 of [8]. □

The spaces that arise in this chapter, are obtained as limits of spaces whose curvature is bounded >from below. A.D. Alexandrov studied such spaces extensively, and there is a large field of literature which is souly concerned with such spaces. Here we give one possible definition of the class of spaces with curvature bounded from below (see [6] and [5] for further possible definitions and properties of such spaces).

Definition A.13. *The complete metric space* (M, d) *is called an intrinsic metric space if for any* $x, y \in M$, $\delta > 0$ *there is a finite sequence of points* $z_0 = x, z_1, \ldots, z_k = y$, *such that* $d(z_i, z_{i+1}) \leq \delta \forall i \in \{0, \ldots, k-1\}$ *and*

$$\sum_{i=0}^{k-1} d(z_i, z_{i+1}) \leq d(x, y) + \delta.$$

Definition A.14. *The length* $L_d(\gamma)$ *of a continuous curve* $\gamma : [a, b] \to M$ *is the supremum of the sums*

$$\sum_{i=0}^{k-1} d(\gamma(y_i), \gamma(y_{i+1}))$$

over all partitions $a = y_0 < y_1 < \ldots y_k = b$, *of* $[a, b]$ *(notice that this length could be infinite).*

Definition A.15. *A geodesic is a continuous curve* $\gamma : [a, b] \to M$ *whose length is equal to* $d(\gamma(a), \gamma(b))$ *(that is, the distance between the endpoints of the curve).*

Definition A.16. *A collection of three points* $p, q, r \in M$ *and three geodesics* pq, pr, qr *is called a triangle in* M *and denoted by* $\triangle(p, q, r)$.

Fact A.17. *Let* (M_k, d_k) *denote the complete simply connected two dimensional Riemannian manifold of sectional curvature* k. *Let us fix the real number* k *and let* $\triangle(p, q, r)$ *be given. For* $k < 0$ *there exists a unique (up to an rigid motion) triangle* $\triangle(\tilde{p}, \tilde{q}, \tilde{r})$ *in the metric space* (M_k, d_k) *with* $d(p, q) = d_k(\tilde{p}, \tilde{q}), d(p, r) = d_k(\tilde{p}, \tilde{r}), d(r, q) = d_k(r, q)$. *For* $k > 0$ *we require that the perimeter of* $\triangle(p, q, r)$ *be less than* $\frac{2\pi}{\sqrt{k}}$ *in order that the triangle exist.*

Definition A.18. *A complete, locally compact space M with intrinsic metric d is called an Alexandrov space with curvature $\geq k$ if in some neighbourhood U_x of each point x, for any triangle $\triangle(p,q,r)$ with vertices in U_x and any point s on the geodesic qr, the inequality $d(p,s) \geq d_k(\tilde{p},\tilde{s})$ is satisfied, where $\triangle(\tilde{p},\tilde{q},\tilde{r})$ is the triangle from A.17, and $\tilde{s}$ is the point in $\tilde{q}\tilde{r}$ satisfying $d(q,s) = d_k(\tilde{q},\tilde{s})$ and $d(r,s) = d_k(\tilde{r},\tilde{s})$.*

Many reuslts which are valid for smooth Riemannian manifolds with curvature bounded from below by k are also valid for Alexandrov spaces with curvature $\geq k$. For example, Theorem 3.6 of [6] says that for (X,d) an Alexandrov space with curvature $\geq k$ we have

$$\text{diam }(X,d) \leq \frac{\pi}{\sqrt{k}}.$$

For other properties of Alexandrov spaces with curvature $\geq k$ see [6] or the book [5].

Appendix B

C-essential points and δ-like necks

Definition B.1. *Let $(M, g(t))_{t\in(-\infty,T)}$, $T \in \mathbb{R} \cup \{\infty\}$, be a solution to Ricci flow. We say that $(x, t) \in M \times (-\infty, T)$ is a C-essential point if*

$$|\mathrm{Riem}(x,t)||t| \geq C.$$

Definition B.2. *We say that $(x, t) \in M \times (-\infty, T)$ is a δ-necklike point if there exists a unit 2-form θ at (x, t) such that*

$$|\mathrm{Riem} - R(\theta \otimes \theta)| \leq \delta |\mathrm{Riem}|.$$

δ-necklike points often occur in the process of taking a limit around a sequence of times and points which are becoming singular. If $\delta = 0$, then the inequality reads

$$|\mathrm{Riem}(x,t) - R(x,t)(\theta \otimes \theta)| = 0.$$

In three dimensions this tells us that the manifold splits. This can be seen with the help of some algebraic lemmas.

Lemma B.3. *Let $\omega \in \Omega^2(\mathbb{R}^3)$. Then it is possible to write*

$$\omega = X \wedge V,$$

for two orthogonal vectors X and V.

Remark B.4. *Here we identify one forms with vectors using*

$$adx^1 + bdx^2 + cdx^3 \equiv (a, b, c).$$

Proof. Assume

$$\omega = adx^1 \wedge dx^2 + bdx^1 \wedge dx^3 + cdx^2 \wedge dx^3 \tag{B.1}$$

Without loss of generality $b \neq 0$. Then, we may write:

$$\omega = (dx^1 + \frac{c}{b}dx^2) \wedge (adx^2 + bdx^3) \tag{B.2}$$

So $\omega = X \wedge Y$. Now let X, Z, W be an orthogonal basis all of length $|X|$. Then

$$Y = a_1 X + a_2 Z + a_3 W.$$

This implies

$$\begin{aligned} \omega &= X \wedge (a_1 X + a_2 Z + a_3 W) \\ &= X \wedge (a_2 Z + a_3 W) \end{aligned} \tag{B.3}$$

as required ($V = a_2 Z + a_3 W$). □

Hence we may write the θ occurring above as

$$\theta = X \wedge V.$$

Hence

$$\mathrm{Riem}(x, t) = cX \wedge V \otimes X \wedge V,$$

with

$$\{X, V, Z\}$$

an orthonormal basis for $\mathbb{R}^3$.

The set $\{X \wedge V, X \wedge Z, V \wedge Z\}$ then forms an orthonormal basis and the curvature operator $\mathcal{R}$ can be written with respect to this basis as

$$\begin{pmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the manifold splits (if the solution is complete with bounded curvature and non-negative curvature operator) in view of the arguments in chapter 9 of [14].

Appendix C

Estimates on the distance function for Riemannian manifolds evolving by Ricci flow

For completeness, we prove some results which are implied or proved in [16] and stated in [4] as editors note 24 from the same paper in that book. The lemma we wish to prove is

Lemma C.1. *Let $(M^n, g(t))_{t\in[0,T)}$ be a solution to Ricci flow with*

$$\begin{aligned} &\mathrm{Ricci}(g(t)) \geq -c_0 \\ &|\mathrm{Riem}(g(t))|t \leq c_0 \\ &\mathrm{diam}\ (M, g_0) \leq d_0 \end{aligned} \tag{C.1}$$

Then

$$d(p,q,0) - c_1(t, d_0, c_0, n) \geq d(p,q,t) \geq d(p,q,0) - c_1(n, c_0)\sqrt{t} \tag{C.2}$$

for all $t \in [0,T)$, where

$$c_1(t, d_0, c_0, n) \to 0$$

as $t \to 0$.

Proof. The first inequality

$$d(p,q,t) \geq d(p,q,0) - c_1(n, c_0)\sqrt{t}$$

is proved in [16], theorem 17.2 after making a slight modification of the proof. If we examine the proof there (as pointed out in [4] as editors note 24 of the same book), we see that in fact what is proved is:

$$d(P,Q,t) \geq d(P,Q,0) - C\int_0^t \sqrt{M(t)}$$

where $\sqrt{M(t)}$ is any integrable function which satisfies

$$\sup_M |\text{Riem}(\cdot, t)| \leq M(t).$$

In particular, in our case we may set

$$M(t) = \frac{c}{t}$$

which then implies the first inequality. The second inequality is also a simple consequence of results obtained in [16]. Lemma 17.3 tells us that

$$\frac{\partial}{\partial t} d(P, Q, t) \leq -\inf_{\gamma \in \Gamma} \int_\gamma \text{Ricci}(T, T) ds$$

where the inf is taken over the compact set Γ of all geodesics from P to Q realising the distance as a minimal length, T is the unit vector field tangent to γ. Then in our case Ricci $\geq -c_0$ implies

$$\frac{\partial}{\partial t} d(P, Q, t) \leq c_0 d(P, Q, t).$$

This implies that

$$d(P, Q, t) \leq \exp^{c_0 t} d(P, Q, 0),$$

and as a consequence

$$\text{diam } (M, g(t)) \leq d_0 \exp^{ct}.$$

Hence

$$\begin{aligned} d(P, Q, t) \leq \exp^{c_0 t} d(P, Q, 0) \qquad &= d(P, Q, 0) + (\exp^{c_0 t} - 1) d(P, Q, 0) \\ &\leq d(P, Q, 0) + (\exp^{c_0 t} - 1) d_0 \exp^{ct}, \end{aligned} \tag{C.3}$$

which implies the result.

□

Appendix D

Notation

$\mathbb{R}^+$ is the set of positive real numbers.
$\mathbb{R}^+_0$ is the set of non-negative real numbers.
For a tensor T on M, we write ${}^g|T|^2$ to represent the norm of T with respect to the metric g on M. For example if T is a $\binom{0}{2}$ tensor, then

$$ {}^g|T|^2 = g^{ij}g^{kl}T_{ik}T_{jl}. $$

${}^h\nabla T$ refers to the covariant derivative with respect to h of T.
${}^h\text{Riem}$ or $\text{Riem}(h)$ refers to the Riemannian curvature tensor with respect to h on M.
${}^h\text{Ricci}$ or $\text{Ricci}(h)$ or ${}^hR_{ij}$ refers to the Ricci curvature of h on M.
${}^h\text{R}$ or $\text{R}(h)$ refers to the scalar curvature of h on M.
$\sec(p)(v, w)$ is the sectional curvature of the plane spanned by the linearly independent vectors v, w at p.
$\sec \geq k$ means that the sectional curvature of every plane at every point is bounded from below by k.
$\mathcal{R}$ denotes the curvature operator.
$\mathcal{R} \geq c$ means that the eigenvalues of the curvature operator are bigger than or equal to c at every point on the manifold.
$\Gamma(h)^k_{ij}$ or ${}^h\Gamma^k_{ij}$ refer to the Christoffel symbols of the metric h in the coordinates $\{x^k\}$,

$$ {}^h\Gamma^k_{ij} = \frac{1}{2}h^{kl}(\frac{\partial h_{il}}{\partial x^j} + \frac{\partial h_{jl}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^l}). $$

For a diffeomorphism $F : M \to N$ we will sometimes consider dF, a 1-form along F, defined by

$$ dF(x) := \frac{\partial F^\alpha}{\partial x^k} dx^k(x) \frac{\partial}{\partial y^\alpha}|_{(F(x))}. $$

For a general 1-form ω along F, $\omega = \omega^\alpha_i(x)dx^i(x) \otimes \frac{\partial}{\partial y^\alpha}|_{(F(x))}$, we define the norm of ω with respect to l (a metric on M) and γ (a metric on N) by

$$ {}^{l,\gamma}|\omega|^2(x) = l^{ij}(x)\gamma_{\alpha\beta}(F(x))\omega^\alpha_i(x)\omega^\beta_j(x). $$

For example,

$$^{l,\gamma}|dF|^2(x) = l^{ij}(x)\gamma_{\alpha\beta}(F(x))\frac{\partial F^\alpha}{\partial x^i}(x)\frac{\partial F^\beta}{\partial x^j}(x).$$

We define $^{g,h}\nabla dF$, a $\binom{0}{2}$ tensor along F, by

$$(^{g,h}\nabla dF)^\alpha_{ij} := \Big(\frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}(g)\frac{\partial F^\alpha}{\partial x^k} + \Gamma^\alpha_{\beta\sigma}(h)\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\sigma}{\partial x^j}\Big).$$

For a general $\binom{0}{2}$ tensor ψ along F, $\psi = \psi^\alpha_{ij}(x)dx^i(x) \otimes dx^j(x) \otimes \frac{\partial}{\partial y^\alpha}|_{(F(x))}$, we define the norm of ψ with respect to l (a metric on M) and γ (a metric on N) by

$$^{l,\gamma}|\psi|^2 = \gamma_{\alpha\beta}(F(x))l^{ks}(x)l^{ij}(x)\eta^\alpha_{ik}(x)\eta^\beta_{js}(x).$$

For example

$$\begin{aligned} ^{l,\gamma}|^{g,h}\nabla dF|^2 \quad = \quad & \gamma_{\alpha\beta}(F(x))l^{ks}(x)l^{ij}(x)\Big(\frac{\partial^2 F^\alpha}{\partial x^i \partial x^k} - \Gamma^r_{ik}(g)\frac{\partial F^\alpha}{\partial x^r} + \Gamma^\alpha_{\eta\sigma}(h)\frac{\partial F^\eta}{\partial x^i}\frac{\partial F^\sigma}{\partial x^k}\Big) \\ & \Big(\frac{\partial^2 F^\beta}{\partial x^j \partial x^s} - \Gamma^r_{js}(g)\frac{\partial F^\beta}{\partial x^r} + \Gamma^\beta_{\phi\rho}(h)\frac{\partial F^\phi}{\partial x^j}\frac{\partial F^\rho}{\partial x^s}\Big). \end{aligned}$$

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