A MULTIGRID METHOD FOR INCOMPRESSIBLE FLOW PROBLEMS USING QUASI DIVERGENCE FREE FUNCTIONS

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Abstract. We consider a finite element method of higher order on a quadrilateral or triangular, hexahedral or tetrahedral mesh for solving the Stokes or Navier–Stokes problem by means of discontinuous elements for the pressure and suitable conforming elements for the velocity such that the global and the local inf-sup condition are satisfied. Our goal is the construction of a multigrid solver for the corresponding algebraic system of equations. The idea of this solver is to switch inside of the multigrid method to a new velocity basis which leads to a reduced system with a much lower number of unknowns as well as a very small number of couplings between them. We call this new basis quasi divergence free since most of the basis functions are dicretely divergence free which implies that they do not have any coupling to the pressure. The quasi divergence free basis functions can be constructed locally during the assembling process of the stiffness matrix. We create a multigrid method for solving the reduced problem efficiently. Since most of the velocity basis functions are completely decoupled from the pressure we can construct a smoother with low computational costs. The efficiency of the new multigrid method compared with other known multigrid solvers is demonstrated by numerical experiments for the Stokes problem. It is shown how the ideas for the construction of a quasi divergence free basis can be extended from the Stokes equations to the incompressible Navier-Stokes equations in the stationary and non-stationary case.

Key words. higher order finite elements, quasi divergence free basis, incompressible flow, Stokes equations, Navier-Stokes equations, multigrid

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1. Introduction. Many processes in science and engineering include fluid flow coupled with other physical phenomena. For the numerical simulation of such complex applications, it is very important to have a fast solver for flow problems. In the last years, higher order finite element methods turned out to be very efficient numerical methods concerning the number of unknowns in relation to the achieved accuracy. In this paper, we consider the numerical solution of an incompressible flow problem by means of a higher order finite element method and the construction of a multigrid solver for the corresponding algebraic system of equations.

For simplicity, we take as model problem the stationary d-dimensional Stokes equations ($d = 2$ or 3) with Dirichlet boundary conditions, i.e., we want to find a velocity $u : \Omega \to \mathbb{R}^d$ and a pressure $p : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\operatorname{div} \mathbb{T}(u,p) &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned} \tag{1.1}$$

where $\mathbb{T}(u,p) \in \mathbb{R}^{d\times d}$ denotes the stress tensor defined via the deformation tensor $\mathbb{D}(u) \in \mathbb{R}^{d\times d}$ for the velocity $u = (u_1, \ldots, u_d)$ in the following way

$$\mathbb{T}(u,p) := -p\mathbb{I} + 2\nu\mathbb{D}(u), \quad \mathbb{I} := (\delta_{i,j}), \quad \mathbb{D}(u) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Furthermore, $\Omega \subset \mathbb{R}^d$ is a connected, bounded domain with a polygonal ($d = 2$) or polyhedral ($d = 3$) boundary $\Gamma := \partial\Omega$, ν the viscosity, $f : \Omega \to \mathbb{R}^d$ a given body force

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and $g : \Gamma \to \mathbb{R}^d$ the prescribed boundary data satisfying the solvability condition $\int_\Gamma g \cdot n \, ds = 0$, where n denotes the outer normal unit vector on the boundary Γ.

We use a higher order conforming finite element method on triangles and/or quadrilaterals ($d = 2$) and tetrahedra or hexahedra ($d = 3$), respectively. It is essential for our method to have a discontinuos pressure approximation and a finite element pair for velocity and pressure that satisfies the local (elementwise) and the global inf-sup condition. Let us emphasize that a discontinuous pressure is not a disadvantage. By means of a simple post-processing procedure one can compute a continuous piecewise polynomial pressure function which maintains the optimal approximation order of the discontinuous pressure solution [15]. In order to get a method of order $r \geq 2$ in the energy norm, we approximate the pressure by discontinuous piecewise P_{r-1}-elements. On quadrilaterals and hexahedra a continuous piecewise Q_r-approximation for the velocity is used. For triangles and tetrahedra we approximate the velocity by continuous functions where the restriction to each element is contained in a suitable enrichment of the polynomial space P_r. Details are given in [4, 5]. For the sake of simplicity, we restrict ourselves to a fixed polynomial degree r for all elements. However, our method can be generalized easily to the case where the degree can vary from element to element.

Our goal is the construction of an efficient multigrid solver for the algebraic system of equations corresponding to the discrete problem. A problem of existing multigrid solvers for higher order finite element discretizations of the Stokes or Navier–Stokes equations is that they use a block Gauss-Seidel smoother (Vanka smoother) with a relatively large block size. Typically this block size is the number of all degrees of freedom for velocity and pressure that belong to one element. That means the block size becomes large for high order elements, in particular, in the 3D case [9, 10, 12].

The new idea of this solver is to switch inside of the multigrid method to a new basis of a suitable velocity subspace which leads at the end to a highly reduced number of velocity and pressure unknowns as well as to a very small number of couplings between them. We call this new basis a *quasi divergence free basis* or, for convinience, a *QDF basis* since most of the basis functions are dicretely divergence free. This implies that they do not have any coupling to the pressure. The QDF basis functions can be constructed locally during the assembling process of the stiffness matrix. The discrete Stokes problem is equivalent to a subspace formulation called *QDF formulation* in the following. This problem is further decomposed by an orthogonal splitting into two subproblems. The first one is a problem for the element bubble part u_b of the velocity solution $u_h = u_b + \widetilde{u}_s$. It can be solved independently from the remaining part $\widetilde{u}_s$ in an elementwise way during the assembling process of the stiffness matrix. The second problem of the decomposition is a *reduced Stokes problem* for the velocity part $\widetilde{u}_s$ and the piecewise constant part $\overline{p}_h$ of the pressure solution $p_h = \overline{p}_h + \widetilde{p}_h$. It has a much smaller dimension than the original discrete Stokes problem. The remaining pressure part $\widetilde{p}_h$ can be computed at the end by an elementwise post-processing procedure.

In this paper, we create a multigrid method for solving the QDF formulation of the Stokes problem efficiently. The most important ingredient of this method is the construction of a suitable QDF velocity basis where almost all basis functions are discretely divergence free. Only those basis functions which are associated with so-called *flux nodes* are not discretely divergence free. The number of flux nodes is equal to the number of the $(d-1)$-dimensional faces of the grid, i.e., it is relatively small compared with the total number of velocity unknowns in the reduced Stokes problem. Since each discretely divergence free basis function is completely decoupled from the

pressure, this, in priciple, allows to apply a point Gauss-Seidel smoothing to all non-flux nodes. Thus, switching inside of our multigrid method to the QDF velocity basis allows us to construct a smoothing procedure with very low computational costs compared to existing multigrid methods from the literature. For a bounded polynomial degree r, the computational overhead to work with the QDF basis is only proportional to the number of unknowns. The required memory for an efficient implementation of the QDF basis is less than the memory that will become free from the fact that a lot of non-zero matrix entries of the original finite element method are no longer used for our new multigrid method.

The idea to use a discretely divergence free basis for the velocity space is not new. For the two dimensional situation, we refer in case of continuous pressure elements to [21], [22] and in case of a discontinuous pressure approximation to [6, 7], [17], [4], [3], [18, 19], [20] for elements up to second order and to [16] for higher order finite element approximations. Efficient multigrid solvers based on a discretely divergence free basis of the velocity space are constructed in [3], [18, 19]. However, the main drawback of this approach is that the construction of such a basis in the 3D case is very complicated [8]. Therefore, it seems to be better to renounce a complete discretely divergence free basis and to work instead with a QDF basis which might be the right compromise.

The paper is organized as follows. We introduce some notation and define the finite element spaces in Section 2. The construction of the quasi divergence free basis of the velocity space is presented in Section 3. In Section 4 we show how the saddle point problem created from the original basis is decomposed into local subproblems on each element and a global system with a highly reduced dimension. The aspects of an efficient implementation of our new method are discussed in Section 5. In Section 6 we describe the main components of our multigrid solver. The performance of the new multigrid method compared with other multigrid solvers known from the literature is tested in Section 7 by means of a numerical example for the Stokes problem. Finally, in Section 8, we comment on generalizations of our approach to the stationary and non-stationary Navier–Stokes equations.

2. Preliminaries and Notation. For a domain $G \subset \mathbb{R}^d$, the $L^2(G)$ inner product and norm are denoted by $(\cdot,\cdot)_G$ and $\|\cdot\|_{0,G}$, respectively, for scalar functions as well as for vector- or tensor-valued functions. For example, for tensor-valued functions $A, B : G \to \mathbb{R}^{d\times d}$, we define the inner product as

$$(A,B)_G := \int_G \sum_{i,j=1}^{d} A_{i,j}(x)B_{i,j}(x)dx\,.$$

For a sufficiently smooth $(d-1)$-dimensional manifold $\gamma \subset \overline{\Omega}$, we denote by $\langle\cdot,\cdot\rangle_\gamma$ the inner product in $L^2(\gamma)$ or in $(L^2(\gamma))^d$, respectively. For a domain $G \subset \mathbb{R}^d$, the Sobolev spaces $H^k(G) = W^{k,2}(G)$ are defined in the usual way with the standard norm and semi-norm $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$, respectively.

We define the spaces X, X_0 and M, M_0 for velocity and pressure, respectively, as

$$\begin{aligned} X &:= (H^1(\Omega))^d\,, \qquad & X_0 &:= (H^1_0(\Omega))^d\,,\\ M &:= L^2(\Omega), & M_0 &:= L^2_0(\Omega) := \{q \in L^2(G) : (q,1)_G = 0\}. \end{aligned}$$

Let $\widetilde{g} \in X$ be an fixed extension of the boundary function $g : \Gamma \to \mathbb{R}^d$ of (1.1) into the domain Ω. Then, a weak formulation of problem (1.1) reads:

Find $(u,p) \in (\widetilde{g} + X_0) \times M_0$ *such that*

$$\begin{array}{rcll} a(u,v) \,+\, b(v,p) & = & (f,v)_\Omega & \forall\, v \in X_0, \\ b(u,q) & = & 0 & \forall\, q \in M_0, \end{array} \tag{2.1}$$

where

$$a(u,v) := 2\nu\ (\mathbb{D}(u), \mathbb{D}(v))_\Omega \qquad \text{and} \qquad b(v,p) := -(\operatorname{div} v, p)_\Omega\,. \tag{2.2}$$

For the discretization of this problem, we decompose the domain $\Omega \subset \mathbb{R}^d$ into elements $K \in \mathcal{T}_h$ which are allowed to be triangles and/or quadrilaterals in the case $d = 2$ and either tetrahedra or hexahedra in the case $d = 3$. We assume that the grid $\mathcal{T}_h$ is regular and shape-regular in the usual sense. The diameter of an element K is denoted by h_K and the global mesh-size of the grid $\mathcal{T}_h$ is defined as $h := \max_{K\in\mathcal{T}_h} h_K$. For an element $K \in \mathcal{T}_h$, we denote by $\mathcal{E}(K)$ the set of all $(d-1)$-dimensional faces of K, by $\mathcal{E}_h := \bigcup_{K\in\mathcal{T}_h} \mathcal{E}(K)$ the set of all faces of the grid $\mathcal{T}_h$, by $\mathcal{E}_h(\Gamma)$ the set of all boundary faces $E \subset \Gamma$ and by $\mathcal{E}_h^0 := \mathcal{E}_h \setminus \mathcal{E}_h(\Gamma)$ the set of all inner faces. Note that a $(d-1)$-dimensional face $E \in \mathcal{E}_h$ can belong to at most two different elements. For a given inner face $E \in \mathcal{E}_h^0$, let $K(E)$ and $K'(E)$ be those two elements of the grid $\mathcal{T}_h$ which have E as their common face. For a boundary face $E \in \mathcal{E}_h(\Gamma)$, let $K(E)$ denote that element such that $E \in \mathcal{E}(K(E))$. We will denote by n_E^K the unit normal vector of a face $E \in \mathcal{E}(K)$ which is directed outward with respect to the element K. To each face $E \in \mathcal{E}_h$ we assign the unit normal vector $n_E := n_E^{K(E)}$.

We denote by $\mathbb{P}_m(G)$ the space of all polynomials on the domain $G \subset \mathbb{R}^d$ with total degree less than or equal to m and by $\mathbb{Q}_m(G)$ the space of those polynomials where the maximum power in each coordinate is less than or equal to m. For an element K, let $F_K : \widehat{K} \to K$ be the one-to-one mapping between the reference element $\widehat{K}$ and K. For sufficient conditions for the bijectivity of non-affine reference mappings F_K we refer to [14].

It is essential for our approach that there are no continuity requirements for the pressure approximation. The finite element spaces M_h and $M_{h,0}$ for the pressure with $M_h \subset M$ and $M_{h,0} \subset M_0$ are defined as

$$M_h := \{q \in L^2(\Omega) :\ q\big|_K \circ F_K \in \mathbb{P}_{r-1}(\widehat{K}) \quad \forall K \in \mathcal{T}_h\}\,, \qquad M_{h,0} := M_h \cap M_0\,. \tag{2.3}$$

The velocity spaces X and X_0 will be approximated by the finite element spaces $X_h \subset X$ and $X_{h,0} \subset X_0$ of order $r \geq 2$ which are defined as

$$X_h := (S_h)^d\,, \qquad X_{h,0} := X_h \cap X_0 \tag{2.4}$$

with the scalar finite element space

$$S_h := \{\varphi \in H^1(\Omega) :\ \varphi\big|_K \circ F_K \in \mathbb{U}_r(\widehat{K}) \quad \forall K \in \mathcal{T}_h\}. \tag{2.5}$$

The space $\mathbb{U}_r(\widehat{K})$ depends on the shape of element K. We assume that it is chosen such that the local inf-sup condition on K is fulfilled. This means that there is a constant $\beta > 0$ which is independent on K such that

$$\inf_{q\in\widetilde{M}_h(K)} \sup_{v\in X_{h,0}(K)} \frac{(\operatorname{div} v, q)_K}{|v|_{1,K}\|q\|_{0,K}} \geq \beta \tag{2.6}$$

holds true for all $K \in \mathcal{T}_h$ where

$$\begin{array}{rcl} \widetilde{M_h}(K) & := & \{q \in M_{h,0} : q = 0 \text{ on } \Omega \setminus K\}, \\ X_{h,0}(K) & := & \{v \in X_h : v = 0 \text{ on } \Omega \setminus K\} \end{array} \tag{2.7}$$

are the local pressure and velocity spaces associated with the element K, respectively. As an example, the choice $\mathbb{U}_r(\widehat{K}) = \mathbb{Q}_r(\widehat{K})$ on quadrilaterals or hexahedra guarantees that assumption (2.6) is satisfied, see [14]. A possible choice for $\mathbb{U}_r(\widehat{K})$ on triangles and tetrahedra is a suitable bubble enrichment of $\mathbb{P}_r(\widehat{K})$. For details see [4, 5].

We use the *mapped* version of the local spaces where their definition is done via the reference transformation for both the velocity and the pressure. Thus, for a non-affine reference mapping, the pressure will be approximated by functions which may be no polynomials. Note that also the *unmapped* version of the local space is possible where only the velocity is defined using the reference transformation while the pressure will be approximated by piecewise polynomials from $\mathbb{P}_{r-1}(K)$ [5]. Our method will work also for the unmapped pressure definition. In the case of non-affine reference transformations the mapped and unmapped version differ but on families of regularly refined meshes we obtain the same optimal error estimates for both versions, see [1, 13].

Let $\widetilde{g}_h \in X_h$ be a suitable approximative extension of the boundary function g in (1.1) with

$$\langle \widetilde{g}_h, n \rangle_\Gamma = \langle g, n \rangle_\Gamma = 0. \tag{2.8}$$

Details will be given later on. Then, the discrete Stokes problem reads:

Find $(u_h, p_h) \in (\widetilde{g}_h + X_{h,0}) \times M_{h,0}$ *such that*

$$\begin{array}{rcll} a(u_h, v_h) + b(v_h, p_h) & = & (f, v_h)_\Omega & \forall\, v_h \in X_{h,0} \\ b(u_h, q_h) & = & 0 & \forall\, q_h \in M_{h,0}\,. \end{array} \tag{2.9}$$

Together with (2.6) we assume that the pair $(X_{h,0}, M_{h,0})$ satisfies also the global inf-sup-condition uniformly with respect to the mesh-size h. For example, in the case of quadrilaterals or hexahedra, the choice $\mathbb{U}_r(\widehat{K}) = \mathbb{Q}_r(\widehat{K})$ ensures this condition, see [14] for mapped and [5] for unmapped pressure elements. Thus, since the bilinear form $a(\cdot,\cdot)$ is elliptic due to Korn's inequality, the existence of a unique and stable solution (u_h, p_h) of problem (2.9) is guaranteed, see [5].

Finally, let us introduce some general notation. We denote by $\mathrm{card}(J)$ the number of elements of a finite set J. The dual space of a given space X will be denoted by X'. For a set $G \subset \mathbb{R}^m$, we denote by $\mathrm{int}(G)$ and $\overline{G}$ the interior and closure of G, respectively.

3. Construction of a Quasi Divergence Free Basis. Let φ_j, $j \in J$, denote the usual vector-valued Lagrange basis functions of the finite element space X_h, i.e.,

$$X_h = \mathrm{span}\{\varphi_j : j \in J\} \tag{3.1}$$

where the indices $j \in J$ will be called *nodes* in the sequel. Each node $j \in J$ is associated with a nodal point $a_j \in \overline{\Omega}$ of the grid $\mathcal{T}_h$. Let $J(\Gamma)$ and J_0 defined as

$$J(\Gamma) := \{j \in J : a_j \in \Gamma\}\,, \qquad J_0 := J \setminus J(\Gamma)\,, \tag{3.2}$$

denote the set of the boundary nodes and the set of the interior nodes, respectively, such that $X_{h,0} = \mathrm{span}\{\varphi_j : j \in J_0\}$.

Our construction of QDF basis functions ψ_j, $j \in \widetilde{J}$, of $\widetilde{X}_h$ is based on two projection operators which are applied to a subset of the basis functions φ_j of X_h. The first operator P_b maps into the space of the element bubble functions and the second operator P_f maps into a space of specially chosen face bubble functions.

Let us start with the definition of QDF functions and discretely divergence free functions. To this end, we need the discrete pressure subspace $\widetilde{M}_h \subset M_{h,0}$ defined by

$$\widetilde{M}_h := \{q \in M_{h,0} : \ q\big|_K \in L^2_0(K) \ \ \forall K \in \mathcal{T}_h\}\,. \tag{3.3}$$

DEFINITION 3.1. *A vector-valued function $v_h \in X_h$ is called a* **quasi divergence free function** *or a* **QDF function** *if it is divergence free with respect to the discrete pressure subspace $\widetilde{M}_h \subset M_{h,0}$, i.e.,*

$$b(v_h, q_h) = 0 \qquad \forall q_h \in \widetilde{M}_h\,. \tag{3.4}$$

The spaces

$$\widetilde{X}_h := \{v_h \in X_h : \ b(v_h, q_h) = 0 \ \ \forall q_h \in \widetilde{M}_h\} \qquad \textit{and} \qquad \widetilde{X}_{h,0} := \widetilde{X}_h \cap X_0 \tag{3.5}$$

are called the **subspaces of the QDF functions** *of X_h and $X_{h,0}$, respectively. Analogously, the spaces*

$$V_h := \{v_h \in X_h : \ b(v_h, q_h) = 0 \ \ \forall q_h \in M_h\} \qquad \textit{and} \qquad V_{h,0} := V_h \cap X_0 \tag{3.6}$$

are called the **subspaces of the discretely divergence free functions** *of X_h and $X_{h,0}$, respectively.*

In order to clarify the relation between QDF functions and discretely divergence free functions it is useful to introduce the spaces of the piecewise constant pressure functions

$$\overline{M}_h := \{q \in L^2(\Omega) : \ q\big|_K \in \mathbb{P}_0(K) \ \ \forall K \in \mathcal{T}_h\} \qquad \text{and} \qquad \overline{M}_{h,0} := \overline{M}_h \cap M_0\,. \tag{3.7}$$

Now, from $\widetilde{M}_h \subset M_{h,0} \subset M_h$ it follows that $V_h \subset \widetilde{X}_h$. For the pressure spaces we have the L^2-orthogonal decompositions

$$M_h = \overline{M}_h \oplus \widetilde{M}_h\,, \qquad M_{h,0} = \overline{M}_{h,0} \oplus \widetilde{M}_h \qquad \text{and} \qquad \widetilde{M}_h = \bigoplus_{K \in \mathcal{T}_h} \widetilde{M}_h(K). \tag{3.8}$$

This implies that a function $v_h \in X_h$ belongs to the discretely divergence free subspace V_h if and only if $v_h \in \widetilde{X}_h$ and

$$b(v_h, q_h) = 0 \qquad \forall q_h \in \overline{M}_h\,. \tag{3.9}$$

It is easy to see that the condition (3.9) is equivalent to the property that, for each element $K \in \mathcal{T}_h$, the mean value of the normal flux of v_h across the element boundary ∂K is zero, i.e.,

$$\sum_{E \in \mathcal{E}(K)} \left\langle v_h, n_E^K \right\rangle_E = 0 \qquad \forall K \in \mathcal{T}_h\,. \tag{3.10}$$

Using the fact that $b(v_h, 1) = 0$ for all functions $v_h \in X_{h,0}$, we get that

$$\text{“}\, v_h \in V_{h,0} \,\text{”} \qquad \text{if and only if} \qquad \text{“}\, v_h \in \widetilde{X}_{h,0} \ \text{and (3.10) is satisfied”}. \tag{3.11}$$

REMARK 3.2. *It is important to emphasize that an essential assumption of our approach is the fact that the pressure space is discontinuous which implies the property (3.8) and the above mentioned relation between the QDF functions and the discretely divergence free functions. Furthermore, the question whether a function $v_h \in X_h$ is a QDF function can be checked elementwise. Therefore, the construction of a QDF basis of $\widetilde{X}_h$ can be done in an elementwise fashion.*

3.1. Correction by element bubble functions. For a given element $K \in \mathcal{T}_h$, let

$$J(K) := \{j \in J : K \subset \mathrm{supp}(\varphi_j)\} \tag{3.12}$$

be the set of the local velocity nodes associated with K. Then, the local velocity space $X_h(K) \subset X_h$ related to element K is defined as

$$X_h(K) := \mathrm{span}\{\varphi_j : j \in J(K)\}. \tag{3.13}$$

For each $K \in \mathcal{T}_h$, we split the set of nodes $J(K)$ into the set $J_b(K)$ of *element bubble nodes*, where the associated basis functions φ_j are so-called *element bubble functions*, and the remaining set $J_s(K)$ of so-called *skeleton nodes* living on the *skeleton* of the grid $\mathcal{T}_h$, i.e.,

$$J_b(K) := \left\{j \in J(K) : \mathrm{supp}(\varphi_j) \subset \overline{K}\right\} \qquad \text{and} \qquad J_s(K) := J(K) \setminus J_b(K). \tag{3.14}$$

The associated local velocity spaces are defined as

$$X_b(K) := \mathrm{span}\{\varphi_j : \ j \in J_b(K)\} \qquad \text{and} \qquad X_s(K) := \mathrm{span}\{\varphi_j : \ j \in J_s(K)\} \tag{3.15}$$

such that $X_h(K) = X_b(K) \oplus X_s(K)$. Note that, due to the global continuity of the basis functions φ_j on Ω, we have $v_h\big|_K \in (H_0^1(K))^d$ for all element bubble functions $v_h \in X_b(K)$. Moreover, the space $X_b(K)$ coincides with $X_{h,0}(K)$ in (2.7). The global set of nodes can be splitted as $J = J_b \cup J_s$ with

$$J_b := \bigcup_{K \in \mathcal{T}_h} J_b(K) \qquad \text{and} \qquad J_s := J \setminus J_b = \bigcup_{K \in \mathcal{T}_h} J_s(K) \tag{3.16}$$

and the global velocity space is decomposed into $X_h = X_b \oplus X_s$ where

$$X_b := \mathrm{span}\{\varphi_j : \ j \in J_b\} \qquad \text{and} \qquad X_s := \mathrm{span}\{\varphi_j : \ j \in J_s\}. \tag{3.17}$$

Note that, due to the local support of the basis functions φ_j, $j \in J_b$, we have the following decomposition

$$X_b = \bigoplus_{K \in \mathcal{T}_h} X_b(K). \tag{3.18}$$

which is orthogonal in L^2 and H^1.

In the following, we will construct basis functions $\widetilde{\varphi}_j$, $j \in \widetilde{J}$, of the subspace of QDF functions $\widetilde{X}_h$. The idea for the construction of $\widetilde{\varphi}_j$ is to subtract from the original basis function φ_j a correction consisting of element bubble functions. To this end, we define the operators

$$P_b : X \to X_b \qquad \text{and} \qquad Q_b : X \to \widetilde{M}_h$$

as follows. For a given function $w \in X = (H^1(\Omega))^d$, the pair $(P_b w, Q_b w)$ is defined to be the solution of the problem:

Find $(P_b w, Q_b w) \in X_b \times \widetilde{M}_h$ *such that*

$$\begin{array}{lcll} a(P_b w, v_h) + b(v_h, Q_b w) & = & a(w, v_h) & \forall\, v_h \in X_b \\ b(P_b w, q_h) & = & b(w, q_h) & \forall\, q_h \in \widetilde{M}_h\,. \end{array} \tag{3.19}$$

Using the orthogonal splittings (3.18) and (3.8), we see that problem (3.19) is equivalent to $\mathrm{card}(\mathcal{T}_h)$ many local problems, i.e., $(P_b w, Q_b w)$ can be written as a sum

$$(P_b w, Q_b w) = \sum_{K \in \mathcal{T}_h} (P_b^K w, Q_b^K w)$$

with local operators $P_b^K : X \to X_b(K)$ and $Q_b^K : X \to \widetilde{M}_h(K)$ associated with the element $K \in \mathcal{T}_h$ such that, for a given $w \in X$, the pair $(P_b^K w, Q_b^K w)$ is defined as the solution of the following local problem:

Find $(P_b^K w, Q_b^K w) \in X_b(K) \times \widetilde{M}_h(K)$ *such that*

$$\begin{array}{lcll} a(P_b^K w, v_h) + b(v_h, Q_b^K w) & = & a(w, v_h) & \forall\, v_h \in X_b(K) \\ b(P_b^K w, q_h) & = & b(w, q_h) & \forall\, q_h \in \widetilde{M}_h(K)\,. \end{array} \tag{3.20}$$

Since $X_b(K)$ coincides with the space $X_{h,0}(K)$ from (2.7), the assumption (2.6) implies the inf-sup condition for problem (3.20). Thus, from the standard FEM-theory for the Stokes problem, we get that for each $K \in \mathcal{T}_h$ the local problem (3.20) has a unique solution which continuously depends on the local norm $\|w\|_{1,K}$. Hence, the global problem (3.19) has also a unique solution which depends continuously on the global norm $\|w\|_{1,\Omega}$.

Now, using the operator P_b, we define the following functions $\widetilde{\varphi}_j$ for the index set J_s of all skeleton nodes

$$\widetilde{\varphi}_j := (I - P_b)\varphi_j \qquad \forall j \in J_s, \tag{3.21}$$

where I denotes the identity operator. Since $(P_b \varphi_j)\big|_{\partial K} = 0$ for all $K \in \mathcal{T}_h$, we get that $\widetilde{\varphi}_j$ coincides with φ_j on the skeleton of the grid $\mathcal{T}_h$, i.e.,

$$\widetilde{\varphi}_j(x) := \varphi_j(x) \qquad \forall x \in \bigcup_{K \in \mathcal{T}_h} \partial K, \quad \forall j \in J_s. \tag{3.22}$$

Therefore, all the $\widetilde{\varphi}_j$ with $j \in J_s$ are linearly independent functions in X_h. Moreover, from the second equation in (3.19), it follows that $\widetilde{\varphi}_j \in \widetilde{X}_h$. Thus, the functions $\widetilde{\varphi}_j$, $j \in J_s$, are the basis functions of a subspace $\widetilde{X}_s \subset \widetilde{X}_h$ defined by

$$\widetilde{X}_s := \mathrm{span}\{\widetilde{\varphi}_j :\ j \in J_s\} \tag{3.23}$$

and the operator $I - P_b : X_s \to \widetilde{X}_s$ is an isomorphism.

However, for $j \in J_b$, the solution of problem (3.19) with $w = \varphi_j$ is $(P_b \varphi_j, Q_b \varphi_j) = (\varphi_j, 0) \in X_b \times \widetilde{M}_h$ such that $\widetilde{\varphi}_j$ defined by (3.21) would be zero. Therefore, in general, we have to construct for each element $K \in \mathcal{T}_h$ an additional set of basis functions $\widetilde{\varphi}_j$, $j \in \widetilde{J}_b(K)$, of the subspace $\widetilde{X}_b(K)$ of the quasi divergence free element bubble functions defined as

$$\widetilde{X}_b(K) := \{v_h \in X_b(K) :\ b(v_h, q_h) = 0 \ \ \forall q_h \in \widetilde{M}_h(K)\}. \tag{3.24}$$

Here "in general" means the case when the polynomial degree r of the velocity space is larger than 2. For example, in the case of quadrilateral elements with Q_2-elements for the velocity and discontinuous P_1-elements for the pressure we have for all elements K that $\dim(\widetilde{X}_b(K)) = \dim(X_b(K)) - \dim(\widetilde{M}_h(K)) = 0$, i.e., no basis functions have to be constructed. For a given element $K \in \mathcal{T}_h$, the problem to determine basis functions $\widetilde{\varphi}_j$ such that

$$\widetilde{X}_b(K) = \operatorname{span}\{\widetilde{\varphi}_j : \ j \in \widetilde{J}_b(K)\} \tag{3.25}$$

is equivalent to an algebraic problem of a small fixed size. This problem consists in the computation of basis vectors of the kernel of the matrix $B^K \in \mathbb{R}^{\dim(\widetilde{M}_h(K))\times\dim(X_b(K))}$ defined as

$$B^K_{i,j} := b(\varphi_j, \chi_i) \qquad \forall i \in \widetilde{I}(K),\ j \in J_b(K) \tag{3.26}$$

where the χ_i, $i \in \widetilde{I}(K)$, denote the basis functions of the local pressure space $\widetilde{M}_h(K)$. Basis vectors of $\ker(B^K)$ could be obtained, for instance, by means of a singular value decomposition of the matrix B^K. In the following, let us assume that the basis functions $\widetilde{\varphi}_j$, $j \in \widetilde{J}_b(K)$, are available for all elements $K \in \mathcal{T}_h$ in the sense that coefficients $c^K_{j,i} \in \mathbb{R}$ are known such that

$$\widetilde{\varphi}_j = \sum_{i\in J_b(K)} c^K_{j,i}\varphi_i \qquad \forall j \in \widetilde{J}_b(K)\,. \tag{3.27}$$

Then, the global space of the quasi divergence free element bubble functions $\widetilde{X}_b$ defined as

$$\widetilde{X}_b := \{v_h \in X_b : \ b(v_h, q_h) = 0 \ \ \forall q_h \in \widetilde{M}_h\} \tag{3.28}$$

can be represented by means of basis functions $\widetilde{\varphi}_j$ as follows

$$\widetilde{X}_b = \operatorname{span}\{\widetilde{\varphi}_j : \ j \in \widetilde{J}_b\} \qquad \text{where} \qquad \widetilde{J}_b := \bigcup_{K\in\mathcal{T}_h} \widetilde{J}_b(K). \tag{3.29}$$

Since all functions $v_h \in \widetilde{X}_b$ vanish on the element boundaries, the condition (3.10) is satisfied which implies that

$$\widetilde{X}_b = V_b := V_h \cap X_b \qquad \text{and} \qquad \widetilde{X}_b(K) = V_b(K) := V_h \cap X_b(K) \quad \forall K \in \mathcal{T}_h\,, \tag{3.30}$$

i.e., all quasi divergence free element bubble functions are discretely divergence free.

Now we are in the position to give the basis representation of the subspace of the QDF functions $\widetilde{X}_h$:

$$\widetilde{X}_h = \operatorname{span}\{\widetilde{\varphi}_j : \ j \in \widetilde{J}\} \qquad \text{where} \qquad \widetilde{J} := J_s \cup \widetilde{J}_b. \tag{3.31}$$

Using the space $\widetilde{X}_s$ defined in (3.23), we can decompose $\widetilde{X}_h$ in the direct sum

$$\widetilde{X}_h = \widetilde{X}_s \oplus V_b \tag{3.32}$$

where $V_b = V_h \cap X_b$ is the space of the discretely divergence free element bubble functions. In fact, by construction it is clear that the space $\widetilde{X}_s \oplus V_b$ is a subspace of $\widetilde{X}_h$ and if we consider the dimensions we obtain

$$\begin{aligned}\dim(\widetilde{X}_s \oplus V_b) &= \operatorname{card}(J_s) + \sum_{K\in\mathcal{T}_h}\Big(\dim(X_b(K)) - \dim(\widetilde{M}_h(K))\Big)\\ &= \dim(X_s) + \dim(X_b) - \sum_{K\in\mathcal{T}_h}\dim(\widetilde{M}_h(K)) = \dim(\widetilde{X}_h).\end{aligned}$$

Similarly we can decompose X_h in the direct sum

$$X_h = \widetilde{X}_s \oplus X_b \,. \tag{3.33}$$

In fact, an arbitrary function $v_h \in X_h$ has a representation as $v_h = v_s + \widetilde{v}_b$ with $v_s \in X_s$ and $\widetilde{v}_b \in X_b$. Using $\widetilde{v}_s := (I - P_b)v_s \in \widetilde{X}_s$ we get the representation $v_h = \widetilde{v}_s + v_b$ with $v_b := \widetilde{v}_b + P_b v_s \in X_b$. This shows that $X_h \subset \widetilde{X}_s \oplus X_b$. By construction we have $\widetilde{X}_s \oplus X_b \subset X_h$.

3.2. Correction by face bubble functions. The aim of this subsection is to perform a further decomposition of the subspace $\widetilde{X}_s$ in the form

$$\widetilde{X}_s = V_s \oplus \widetilde{X}_f$$

where $V_s \subset V_h$ is a subspace of discretely divergence free functions which should be as large as possible. An advantage of using V_h-subspaces is that velocity components lying in those subspaces do not have any couplings with the pressure in the discrete Stokes problem (2.9).

The property that a QDF basis function $\psi_j \in \widetilde{X}_s$ belongs to V_h is equivalent to the condition that ψ_j satisfies the condition (3.10). The idea to fulfill (3.10) for ψ_j is to start from the already known QDF basis function $\widetilde{\varphi}_j$ and to modify it in such a way that on each $(d-1)$-dimensional face $E \in \mathcal{E}_h$ the normal flux $\langle\, \psi_j, n_E \,\rangle_E$ becomes zero. To this end we modify the normal flux of $\widetilde{\varphi}_j$ across E by subtracting a multiple of a properly chosen face bubble function associated with the face E. This does not change the normal fluxes for all other faces.

In the following, we will describe how to choose these face bubble functions. Recall that each node $j \in J$ is associated with a nodal point $a_j \in \overline{\Omega}$. Then, for each face $E \in \mathcal{E}_h$, the set

$$J_0(E) := \{j \in J_s : \; a_j \in \mathrm{int}(E)\} \tag{3.34}$$

denotes the set of all skeleton nodes that correspond to the interior of the $(d-1)$-dimensional face E. Among all the nodes $j \in J_0(E)$ we choose a special *flux node of* E, denoted by $j_f(E)$, that realizes the maximum absolute normal flux of the corresponding basis functions, i.e.

$$|\,\langle\, \varphi_{j_f(E)}, n_E \,\rangle_E\,| = \max_{j \in J_0(E)} |\,\langle\, \varphi_j, n_E \,\rangle_E\,|. \tag{3.35}$$

Note that, due to the fact that $P_b \varphi_j\big|_E = 0$ for all $j \in J_0(E)$, the analogous equation holds true for the QDF basis functions $\widetilde{\varphi}_{j_f(E)}$ and $\widetilde{\varphi}_j$, respectively. To simplify the notation we introduce the abbreviations

$$\varphi_E := \varphi_{j_f(E)}, \qquad \widetilde{\varphi}_E := \widetilde{\varphi}_{j_f(E)} \qquad \forall E \in \mathcal{E}_h. \tag{3.36}$$

The set $J_f(K)$ of all flux nodes of an element $K \in \mathcal{T}_h$ and the set $\widetilde{J}_s(K)$ of the remaining skeleton nodes are defined by

$$J_f(K) := \{j_f(E) : \; E \in \mathcal{E}(K)\} \qquad \text{and} \qquad \widetilde{J}_s(K) := J_s(K) \setminus J_f(K). \tag{3.37}$$

Let J_f and $\widetilde{J}_s$ denote the corresponding global sets of nodes given by

$$J_f := \bigcup_{K \in \mathcal{T}_h} J_f(K) \qquad \text{and} \qquad \widetilde{J}_s := \bigcup_{K \in \mathcal{T}_h} \widetilde{J}_s(K) = J_s \setminus J_f \tag{3.38}$$

and $\widetilde{X}_f$ the subspace of $\widetilde{X}_s$ associated with the flux nodes, i.e.,

$$\widetilde{X}_f := \operatorname{span}\{\widetilde{\varphi}_j : \ j \in J_f\}. \tag{3.39}$$

Then, we introduce the operator $P_f : X \to \widetilde{X}_f$ such that for a vector-valued function $w \in X = (H^1(\Omega))^d$ the discrete function $P_f w \in \widetilde{X}_f$ is defined as

$$P_f w := \sum_{E \in \mathcal{E}_h} \frac{\langle\, w, n_E \,\rangle_E}{\langle\, \widetilde{\varphi}_E, n_E \,\rangle_E}\, \widetilde{\varphi}_E. \tag{3.40}$$

It is easy to see that this operator has the property

$$\langle\, P_f w, n_E \,\rangle_E = \langle\, w, n_E \,\rangle_E \qquad \forall E \in \mathcal{E}_h, \quad \forall w \in (H^1(\Omega))^d. \tag{3.41}$$

Now, starting from the basis functions $\widetilde{\varphi}_j$ of $\widetilde{X}_h$ with $j \in \widetilde{J} = (\widetilde{J}_s \cup J_f) \cup \widetilde{J}_b$, we define new basis functions ψ_j of $\widetilde{X}_h$ as

$$\psi_j := \begin{cases} \widetilde{\varphi}_j - P_f \widetilde{\varphi}_j, & j \in \widetilde{J}_s, \\ \widetilde{\varphi}_j, & j \in J_f \cup \widetilde{J}_b. \end{cases} \tag{3.42}$$

Note that the function $\psi_j = \widetilde{\varphi}_j - P_f \widetilde{\varphi}_j$ would be zero for $j \in J_f$ and it would be equal to $\widetilde{\varphi}_j$ for $j \in \widetilde{J}_b$. One easily verifies that the functions $\psi_j \in \widetilde{X}_s$ with $j \in \widetilde{J}_s$ are linearly independent. Moreover, the property (3.41) implies that $\psi_j \in V_h$ for all $j \in \widetilde{J}_s$. Thus, we get the desired decomposition $\widetilde{X}_s = V_s \oplus \widetilde{X}_f$ with the space

$$V_s := \operatorname{span}\{\psi_j : \ j \in \widetilde{J}_s\} \subset V_h.$$

Using (3.32) with $V_b = \operatorname{span}\{\widetilde{\varphi}_j; \ j \in \widetilde{J}_b\} = \operatorname{span}\{\psi_j; \ j \in \widetilde{J}_b\}$ we finally get the decomposition of the subspace $\widetilde{X}_h$ of the quasi divergence free functions

$$\widetilde{X}_h = \widetilde{X}_s \oplus V_b \qquad \text{with} \qquad \widetilde{X}_s = V_s \oplus \widetilde{X}_f = \operatorname{span}\{\psi_j : \ j \in J_s = \widetilde{J}_s \cup J_f\} \tag{3.43}$$

where both spaces V_s and V_b are subspaces of the space V_h of the discretely divergence free functions. That means that all the basis functions ψ_j with $j \in \widetilde{J}_s \cup \widetilde{J}_b$ do not have any couplings with the pressure in the Stokes problem (2.9).

4. Decomposition of the Saddle Point Problem. In this section, we show how the discrete Stokes problem (2.9) can be splitted into two subproblems using the decomposition (3.33) of the velocity space $X_h = \widetilde{X}_s \oplus X_b$. At first we explain how the discrete extension $\widetilde{g}_h \in X_h$ of the boundary function $g : \Gamma \to \mathbb{R}^d$ is chosen. Let

$$g_h = \sum_{j \in J(\Gamma)} g_j \varphi_j \in X_h \qquad \text{with} \qquad g_j \in \mathbb{R}$$

be an interpolate of g using only boundary nodes $j \in J(\Gamma)$ and satisfying the solvability condition $\langle\, g_h, n \,\rangle_\Gamma = \langle\, g, n \,\rangle_\Gamma = 0$. Then we choose

$$\widetilde{g}_h := \sum_{j \in J(\Gamma)} g_j \widetilde{\varphi}_j = (I - P_b) g_h \in \widetilde{X}_s \tag{4.1}$$

where $\widetilde{g}_h \in \widetilde{X}_s$ follows from the fact that $J(\Gamma) \subset J_s$. It is easy to see that the discrete solution u_h of (2.9) is a solution of the following *QDF formulation* of the discrete Stokes problem:

Find $(u_h, \overline{p}_h) \in (\widetilde{g}_h + \widetilde{X}_{h,0}) \times \overline{M}_{h,0}$ *such that*

$$\begin{array}{lcll} a(u_h, \widetilde{v}_h) + b(\widetilde{v}_h, \overline{p}_h) & = & (f, \widetilde{v}_h)_\Omega & \forall\, \widetilde{v}_h \in \widetilde{X}_{h,0} \\ b(u_h, \overline{q}_h) & = & 0 & \forall\, \overline{q}_h \in \overline{M}_{h,0}\,. \end{array} \tag{4.2}$$

Now, the splitting (3.32) of $\widetilde{X}_h$ as the direct sum $\widetilde{X}_h = \widetilde{X}_s \oplus V_b$ implies that the discrete solution $u_h \in \widetilde{g}_h + \widetilde{X}_{h,0}$ can be uniquely represented in the form

$$u_h = \widetilde{u}_s + u_b = (\widetilde{g}_h + \widetilde{u}_{s,0}) + u_b \tag{4.3}$$

with $\widetilde{u}_s \in \widetilde{X}_s$, $u_b \in V_b$ and $\widetilde{u}_{s,0} \in \widetilde{X}_{s,0} := \widetilde{X}_s \cap X_0$. From the definition of the operator $P_b : X \to X_b$ in (3.19), it follows by taking the test function $v_h = v_b \in V_b$ and using $b(v_b, p_h) = 0$ that

$$a(v_s - P_b v_s, v_b) = 0 \qquad \forall v_s \in X_s\,,\ v_b \in V_b\,.$$

Since the operator $I - P_b : X_s \to \widetilde{X}_s$ is an isomorphism we get the property

$$a(\widetilde{v}_s, v_b) = 0 \qquad \forall \widetilde{v}_s \in \widetilde{X}_s\,,\ v_b \in V_b \tag{4.4}$$

which says that the subspaces $\widetilde{X}_s$ and V_b are orthogonal with respect to the energy scalar product $a(\cdot,\cdot)$. This property is important for the decomposition of the Stokes problem (2.9) into smaller subproblems which will be described in the following.

If we insert the splitting $u_h = \widetilde{u}_s + u_b$ into the first equation of (2.9) and take the test function $v_h = v_b \in V_b \subset X_{h,0}$ we get, by means of $b(v_b, p_h) = 0$ and (4.4), the following problem for u_b:

Find $u_b \in V_b$ *such that*

$$a(u_b, v_b) = (f, v_b)_\Omega \qquad \forall v_b \in V_b\,. \tag{4.5}$$

That means that the element bubble part u_b of the solution u_h can be computed independently from the remaining part $\widetilde{u}_s$. Moreover, this computation can be done separately on each element $K \in \mathcal{T}_h$ inside of the assembling process of the stiffness matrix. If the basis functions $\psi_j = \widetilde{\varphi}_j$, $j \in \widetilde{J}_b(K)$, of V_b are available, the part of the u_b-problem (4.5) on element K can be calculated by solving the equivalent $\widetilde{n}_b(K) \times \widetilde{n}_b(K)$ linear system of equations where $\widetilde{n}_b(K) := \dim(X_b(K)) - \dim(\widetilde{M}_h(K))$. In the special case that the space X_h corresponds to conforming $\mathbb{Q}_2$-elements and M_h corresponds to discontinuous $\mathbb{P}_1$ elements, nothing is to do for solving (4.5) since $V_b = \{0\}$ and $u_b = 0$. If the basis functions ψ_j of V_b are not available, we can compute the part $u_b^K \in X_b(K)$ on an element K of $u_b = \sum_{K \in \mathcal{T}_h} u_b^K \in X_b$ corresponding to the splitting (3.18) by solving the mixed local problem associated to (4.5):

Find $(u_b^K, \theta^K) \in X_b(K) \times \widetilde{M}_h(K)$ *such that*

$$\begin{array}{lcll} a(u_b^K, v_h) + b(v_h, \theta^K) & = & (f, v_h)_\Omega & \forall\, v_h \in X_b(K) \\ b(u_b^K, q_h) & = & 0 & \forall\, q_h \in \widetilde{M}_h(K)\,. \end{array} \tag{4.6}$$

Now, we want to derive the problem for the part $\widetilde{u}_s$ of the solution u_h. Due to the decomposition (3.8) of the pressure space $M_{h,0}$ the pressure solution p_h can be uniquely splitted in the form $p_h = \overline{p}_h + \widetilde{p}_h$ with $\overline{p}_h \in \overline{M}_{h,0}$ and $\widetilde{p}_h \in \widetilde{M}_h$. If we take in (2.9) the test functions $v_h = \widetilde{v}_s \in \widetilde{X}_{s,0} = \widetilde{X}_s \cap X_0 \subset X_{h,0}$ and $q_h = \overline{q}_h \in \overline{M}_{h,0} \subset M_{h,0}$

and if we use the properties $b(\widetilde{v}_s, \widetilde{p}_h) = 0$, $b(u_b, \overline{q}_h) = 0$ and $a(u_b, \widetilde{v}_s) = 0$, where the symmetry of the bilinear form $a(\cdot,\cdot)$ has been exploited, we get the following *reduced Stokes problem* for $\widetilde{u}_s$ and $\overline{p}_h$:

Find $(\widetilde{u}_s, \overline{p}_h) \in (\widetilde{g}_h + \widetilde{X}_{s,0}) \times \overline{M}_{h,0}$ *such that*

$$\begin{array}{lcll} a(\widetilde{u}_s, \widetilde{v}_s) + b(\widetilde{v}_s, \overline{p}_h) & = & (f, \widetilde{v}_s)_\Omega & \forall\, \widetilde{v}_s \in \widetilde{X}_{s,0} \\ b(\widetilde{u}_s, \overline{q}_h) & = & 0 & \forall\, \overline{q}_h \in \overline{M}_{h,0}\,. \end{array} \tag{4.7}$$

THEOREM 4.1. *The problems (4.6) and (4.7) have unique solutions* $(u_b^K, \theta^K) \in X_b(K) \times \widetilde{M}_h(K)$ *for each* $K \in \mathcal{T}_h$ *and* $(\widetilde{u}_s, \overline{p}_h) \in (\widetilde{g}_h + \widetilde{X}_{s,0}) \times \overline{M}_{h,0}$*, respectively. Using these solutions, the velocity solution* $u_h \in \widetilde{g}_h + \widetilde{X}_{h,0}$ *of both the discrete Stokes problem (2.9) and the QDF formulation* (4.2) *has the representation*

$$u_h = \widetilde{u}_s + \sum_{K \in \mathcal{T}_h} u_b^K \in \widetilde{g}_h + X_{h,0}\,. \tag{4.8}$$

The discrete pressure solution $p_h \in M_{h,0}$ *of (2.9) has the representation* $p_h = \overline{p}_h + \widetilde{p}_h$ *where* $\overline{p}_h \in \overline{M}_{h,0}$ *is the pressure solution of the reduced Stokes problem (4.7) and* $\widetilde{p}_h = \sum_{K \in \mathcal{T}_h} \widetilde{p}_h^K \in \widetilde{M}_h$ *is composed of the elementwise solutions* $\widetilde{p}_h^K \in \widetilde{M}_h(K)$ *of the following local pressure problems:*

Find $\widetilde{p}_h^K \in \widetilde{M}_h(K)$ such that

$$b(v_b, \widetilde{p}_h^K) = (f, v_b)_K - a(u_h, v_b) \qquad \forall\, v_b \in X_b(K)\,. \tag{4.9}$$

For each $K \in \mathcal{T}_h$*, problem (4.9) has a unique solution* $\widetilde{p}_h^K \in \widetilde{M}_h(K)$*. Moreover, the pair* $(u_h, \overline{p}_h)$ *is the unique solution of the QDF formulation* (4.2) *of the discrete Stokes problem.*

Proof. Due to the assumption (2.6), the spaces $(X_b(K), \widetilde{M}_h(K))$ fulfill the inf-sup condition. Moreover, due to the zero boundary conditions in $X_b(K)$ and Korn's inequality, the bilinear form $a(\cdot,\cdot)$ is coercive on $X_b(K) \times X_b(K)$. Therefore, problem (4.6) has a unique solution (u_b^K, θ^K) for each $K \in \mathcal{T}_h$.

The bilinear form $a(\cdot,\cdot)$ is also coercive on $\widetilde{X}_{s,0} \times \widetilde{X}_{s,0}$. Furthermore, we can show that the pair of spaces $(\widetilde{X}_{s,0}, \overline{M}_{h,0})$ satisfies the inf-sup condition. We will only present the main idea of its proof because the details are beyond the scope of this paper. It is easy to see that the restriction of the operator P_f in (3.40) to the space $X_0 = (H_0^1(\Omega))^d$ is a mapping $P_f : X_0 \to \widetilde{X}_f \cap X_0$. Since $\widetilde{X}_f \cap X_0 \subset \widetilde{X}_{s,0}$ the operator P_f maps from X_0 into the space $\widetilde{X}_{s,0}$. By means of the property (3.41) of P_f one easily verifies that

$$b(P_f v, q_h) = b(v, q_h) \qquad \forall v \in X_0\,,\ q_h \in \overline{M}_{h,0}\,.$$

Moreover, we can show the stability estimate

$$\|P_f v\|_{1,\Omega} \le C \|v\|_{1,\Omega} \qquad \forall v \in X_0\,.$$

Then, the inf-sup condition for the spaces $(\widetilde{X}_{s,0}, \overline{M}_{h,0})$ follows from Fortin's Theorem [5, Theorem II.1.1]. This implies that the reduced Stokes problem (4.7) has a unique solution $(\widetilde{u}_s, \overline{p}_h)$.

Now, let $\widetilde{u}_s$ and $u_b = \sum_{K \in \mathcal{T}_h} u_b^K$ be the unique solutions of (4.7) and (4.5), respectively. Further, let $\overline{p}_h$ and $\widetilde{p}_h = \sum_{K \in \mathcal{T}_h} \widetilde{p}_h^K$ be the unique solutions of (4.7) and (4.9), respectively. Note that the unique solvability of (4.9) follows from the assumption

(2.6) that the local spaces $X_b(K) = X_{h,0}(K)$ and $\widetilde{M}_h(K)$ fulfill the inf-sup condition. In the following, we will show that $u_h := \widetilde{u}_s + u_b$ and $p_h := \overline{p}_h + \widetilde{p}_h$ are the solutions of the original Stokes problem (2.9). Let v_b be an arbitrary element of V_b. Then, from $a(\widetilde{u}_s, v_b) = 0$, see (4.4), $b(v_b, p_h) = 0$ and (4.5) we get

$$a(u_h, v_b) + b(v_b, p_h) = (f, v_b)_\Omega \qquad \forall\, v_b \in V_b\,. \tag{4.10}$$

Using $a(u_b, \widetilde{v}_s) = a(\widetilde{v}_s, u_b) = 0$, $b(\widetilde{v}_s, \widetilde{p}_h) = 0$ and the first equation of (4.7) we obtain

$$a(u_h, \widetilde{v}_s) + b(\widetilde{v}_s, p_h) = (f, \widetilde{v}_s)_\Omega \qquad \forall\, \widetilde{v}_s \in \widetilde{X}_{s,0}\,. \tag{4.11}$$

The decomposition $\widetilde{X}_h = \widetilde{X}_s \oplus V_b$ and the equations (4.10) and (4.11) yield

$$a(u_h, \widetilde{v}_h) + b(\widetilde{v}_h, p_h) = (f, \widetilde{v}_h)_\Omega \qquad \forall\, \widetilde{v}_h \in \widetilde{X}_{h,0}\,. \tag{4.12}$$

The equations (4.9) and $b(v_b, \overline{p}_h) = 0$ imply that

$$a(u_h, v_b) + b(v_b, p_h) = (f, v_b)_\Omega \qquad \forall\, v_b \in X_b(K)\,. \tag{4.13}$$

Due to the decomposition (3.18) of X_b, the equation (4.13) also holds for an arbitrary $v_b \in X_b$. Thus, it follows that u_h and p_h satisfy the first equation of the Stokes problem (2.9). From the second equation of (4.7) we get with $b(u_b, \overline{q}_h) = 0$ the equation

$$b(u_h, \overline{q}_h) = 0 \qquad \forall \overline{q}_h \in \overline{M}_{h,0}\,.$$

By definition we have $\widetilde{u}_s \in \widetilde{X}_s \subset \widetilde{X}_h$ and $u_b \in V_b \subset \widetilde{X}_h$ which implies $u_h \in \widetilde{X}_h$, i.e.

$$b(u_h, \widetilde{q}_h) = 0 \qquad \forall \widetilde{q}_h \in \widetilde{M}_h\,.$$

Thus, from the decomposition $M_{h,0} = \overline{M}_{h,0} \oplus \widetilde{M}_h$ it follows that u_h fulfills also the second equation of the Stokes problem (2.9). This proves the representations of u_h and p_h since problem (2.9) has a unique solution.
If we take in (2.9) the test functions $v_h \in \widetilde{X}_{h,0}$ and $q_h \in \overline{M}_{h,0}$ we easily see that $(u_h, \overline{p}_h)$ is a solution of (4.2). The inf-sup condtion for the pair $(\widetilde{X}_{h,0}, \overline{M}_{h,0})$ follows immediately from the inf-sup condtion for $(\widetilde{X}_{s,0}, \overline{M}_{h,0})$. This implies that the problem (4.2) has a unique solution which completes the proof of Theorem 4.1. □

In the following, we analyze the structure of the QDF formulation (4.2) of the Stokes problem by exploiting the fact that the space $\widetilde{X}_h$ can be decomposed in the direct sum $\widetilde{X}_h = V_b \oplus V_s \oplus \widetilde{X}_f$, see (3.43). The solution u_h of (4.2) can be decomposed as

$$u_h = \widetilde{g}_h + u_{h,0} \qquad \text{with} \qquad u_{h,0} \in \widetilde{X}_{h,0} = \operatorname{span}\{\psi_j : j \in \widetilde{J}_0\}, \quad \widetilde{J}_0 := \widetilde{J} \setminus J(\Gamma).$$

where $\widetilde{J} = \widetilde{J}_b \cup J_s$, see (3.31). By the disjoint decomposition (3.38) of the index set $J_s = \widetilde{J}_s \cup J_f$ the following disjoint decomposition of the index set $\widetilde{J}_0$ is induced:

$$\widetilde{J}_0 = \widetilde{J}_b \cup \widetilde{J}_{s,0} \cup J_{f,0} \qquad \text{with} \qquad \widetilde{J}_{s,0} := \widetilde{J}_s \setminus J(\Gamma) \quad J_{f,0} := J_f \setminus J(\Gamma).$$

Using this decomposition, the coefficient vector $\underline{u}^{h,0} = (u_j^{h,0})$ of $u_{h,0} = \sum_{j \in \widetilde{J}_0} u_j^{h,0} \psi_j$ can be partitioned into the two block vectors

$$\underline{u}^{h,0} = \begin{pmatrix} \underline{u}^{n,0} \\ \underline{u}^{f,0} \end{pmatrix} \qquad \text{with} \qquad \underline{u}^{n,0} := (u_j^{h,0})_{j \in \widetilde{J}_b \cup \widetilde{J}_{s,0}}, \quad \underline{u}^{f,0} := (u_j^{h,0})_{j \in J_{f,0}}.$$

Let $\overline{I}$ be the index set of the elements $K \in \mathcal{T}_h$ such that $\mathcal{T}_h = \{K_i : i \in \overline{I}\}$. Then, the piecewise constant pressure solution $\overline{p}_h$ of (4.7) corresponds to the coefficient vector $\underline{p}$ with

$$\underline{p} := (p_i)_{i\in\overline{I}} \in \mathbb{R}^{\mathrm{card}(\overline{I})} \qquad \text{where} \qquad \overline{p}_h = \sum_{i\in\overline{I}} p_i \chi_0^{K_i}\,.$$

Here, $\chi_0^{K_i}$ denotes the function in $\overline{M}_h$ which is 1 on K_i and 0 otherwise. Due to the fact that $\psi_j \in V_h$ for all non-flux nodes $j \in \widetilde{J}_b \cup \widetilde{J}_{s,0}$, the QDF formulation (4.2) corresponds to a block system of the following structure:

$$\begin{pmatrix} A^{nn} & A^{nf} & 0 \\ A^{fn} & A^{ff} & (B^f)^T \\ 0 & B^f & 0 \end{pmatrix} \begin{pmatrix} \underline{u}^{n,0} \\ \underline{u}^{f,0} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{r}^{n,0} \\ \underline{r}^{f,0} \\ \underline{s} \end{pmatrix}. \tag{4.14}$$

We see that only the QDF basis functions ψ_j for flux nodes $j \in J_{f,0}$ are coupled with the piecewise constant pressure. In contrast to the original basis functions, this is an important computational advantage for the construction of cheap block Gauss-Seidel smoothers since the structure of (4.14) allows, in principle, to apply a point Gauss-Seidel smoother for all non-flux nodes $j \in \widetilde{J}_0 \setminus J_{f,0}$. Moreover, the number of flux nodes is equal to the number of faces $E \in \mathcal{E}_h$ which is small compared with the total number of velocity unknowns in (4.14), in particular, in the case of higher order finite elements. That means that in the algebraic system (4.14) generated by means of the QDF basis functions most of the velocity unknowns are decoupled from the preesure unknowns.

5. Implementation Aspects.

5.1. Generation of the QDF basis functions. Within a numerical code it is not necessary to have explicit formulas for the modified QDF basis functions ψ_j. It suffices to know how one can obtain these functions by linear combinations of standard Lagrange basis functions.

First we will have a look at the element bubble correction operator P_b which is defined locally by (3.20). Using the standard bases $\{\varphi_j : j \in J(K)\}$ and $\{\chi_i : i \in \widetilde{I}(K)\}$ for the local velocity and pressure spaces $X_h(K)$ and $\widetilde{M}_h(K)$, respectively, problem (3.20) for $w = \varphi_j$, $j \in J_s(K)$, is equivalent to the following system of linear equations

$$\begin{pmatrix} A^K & (B^K)^T \\ B^K & 0 \end{pmatrix} \begin{pmatrix} \underline{c}_j^K \\ \underline{d}_j^K \end{pmatrix} = \begin{pmatrix} \underline{r}_j^K \\ \underline{s}_j^K \end{pmatrix} \tag{5.1}$$

with

$$\begin{aligned} (A^K)_{i,k} &:= a(\varphi_k, \varphi_i), \quad i,k \in J_b(K), \qquad B^K \text{ defined by (3.26)}\,, \\ (\underline{r}_j)_i &:= a(\varphi_j, \varphi_i), \quad i \in J_b(K), \qquad (\underline{s}_j)_i := b(\varphi_j, \chi_i), \quad i \in \widetilde{I}(K). \end{aligned}$$

We see that everything which we need to generate the above system is computed during the usual local assembling process of element stiffness matrices. From the solution vector $\underline{c}_j^K = (c_{j,i}^K) \in \mathbb{R}^{\mathrm{card}(J_b(K))}$ of the linear system (5.1) we get the following representation of the local part $P_b^K\varphi_j$ of the projection $P_b\varphi_j$ on element K:

$$P_b^K\varphi_j = \sum_{i\in J_b(K)} c_{j,i}^K \varphi_i, \qquad j \in J_s(K). \tag{5.2}$$

In order to be able to reconstruct the modified local functions $\widetilde{\varphi}_j\big|_K = \varphi_j\big|_K - P_b^K \varphi_j$, $j \in J_s(K)$, we store the coefficients $c_{j,i}^K$. The required storage for this is less than the memory in the original finite element method for storing the matrix entries $a(\varphi_j, \varphi_i)$ with $j \in J_s(K)$, $i \in J_b(K)$, which are no longer needed in the new method using the QDF basis.

Now we will describe how the action of the operator P_f defined in (3.40) can be implemented. For a basis function φ_j, which corresponds to the nodal point a_j, we have that $\varphi_j|_E \equiv \widetilde{\varphi}_j|_E \equiv 0$ for all faces $E \in \mathcal{E}_h$ with $a_j \notin \overline{E}$. Hence, the sum in (3.40) for $w = \widetilde{\varphi}_j$ reduces to a small sum over those faces E with

$$E \in \mathcal{E}_h(j) := \{E \in \mathcal{E}_h : a_j \in \overline{E}\}\,. \tag{5.3}$$

Then, the function $\psi_j = \widetilde{\varphi}_j - P_f \widetilde{\varphi}_j$, $j \in J_s(K) \setminus J_f(K)$, defined in (3.42) can be written as

$$\psi_j = \widetilde{\varphi}_j - \sum_{E \in \mathcal{E}_h(j)} \alpha_{j,E}\, \widetilde{\varphi}_{j_f(E)} \quad \text{with} \quad \alpha_{j,E} := \frac{\langle \varphi_j, n_E \rangle_E}{\langle \varphi_E, n_E \rangle_E} \tag{5.4}$$

where we have used that φ_j and $\widetilde{\varphi}_j$ coincide on all $E \in \mathcal{E}_h$. For each face $E \in \mathcal{E}_h$, we store the coefficients $\alpha_{j,E}$ for all nodes j belonging to the closure of E, i.e. for all $j \in J(E)$ with

$$J(E) := \{j \in J : a_j \in \overline{E}\}\,, \tag{5.5}$$

where $\alpha_{j,E}$ is defined by (5.4) if $j \in J_s \setminus J_f$ and by

$$\alpha_{j,E} := \langle \varphi_j, n_E \rangle_E \qquad \text{if} \qquad j = j_f(E) \in J(E) \cap J_f\,.$$

This is useful since for $j = j_f(E) \in J_f$ we have $\psi_j = \widetilde{\varphi}_j$ and

$$\alpha_{j,E} = \langle \varphi_j, n^{K(E)} \rangle_{\partial K(E)} = \langle \psi_j, n^{K(E)} \rangle_{\partial K(E)} = \begin{cases} -b(\psi_j, \chi_0^K) & \text{if} \quad K = K(E) \\ +b(\psi_j, \chi_0^K) & \text{if} \quad K = K'(E) \end{cases}$$

where $K(E)$ and $K'(E)$ denote the two elements associated with the face E (see Section 2) and $\chi_0^K \in \overline{M}_h$ is the function which is 1 on K and 0 on $\Omega \setminus K$. In the algebraic system, which is equivalent to the reduced Stokes problem (4.7), we need, for each velocity node $j \in J_s$, the matrix entries $b(\psi_j, \chi_0^K)$ for all $K \in \mathcal{T}_h$. Note that it is a common practice that the functions $\chi_0^K \in \overline{M}_h \setminus \overline{M}_{h,0}$, $K \in \mathcal{T}_h$, are used as a basis of the larger pressure space $\overline{M}_h$ and that the condition $\overline{p}_h \in \overline{M}_{h,0} = \overline{M}_h \cap L_0^2(\Omega)$ is realized within the discrete solver by subtracting a suitable global constant from the pressure approximation. Now, using that $\psi_j \in V_h$ for all $j \in \widetilde{J}_s = J_s \setminus J_f$, we get

$$b(\psi_j, \chi_0^K) = 0 \qquad \forall j \in J_s \setminus J_f\,,\ K \in \mathcal{T}_h\,,$$

i.e., the matrix entry $b(\psi_j, \chi_0^K)$ can be non-zero only if $j \in J_f(K)$. Therefore, if we store the coefficients $\alpha_{j,E} = \pm b(\psi_j, \chi_0^K)$ for $j = j_f(E) \in J_f(K)$, $E \in \mathcal{E}(K)$, then it is no longer necessary to store any matrix entries associated with the bilinear form $b(\cdot,\cdot)$. The memory required for storing the $\alpha_{j,E}$ is approximately equal to the memory that becomes free from the matrix entries of the bilinear form $b(\cdot,\cdot)$.

Now we consider the linear algebraic system which corresponds to the reduced Stokes problem (4.7) by using the new QDF basis $\{\psi_j : j \in J_s\}$ for $\widetilde{X}_s$ and the basis $\{\chi_0^K : K \in \mathcal{T}_h\}$ for $\overline{M}_h$. Using the stored coefficients $\alpha_{j,E}$ and $c_{j,i}^K$, this system is

easily generated by means of linear combinations of rows and columns of the element stiffness matrices of the original finite element method. For example, the entries of the global stiffness matrix $\overline{A} = (\overline{A}_{i,j})$ with $\overline{A}_{i,j} := a(\psi_j, \psi_i)$ are assembled in an elementwise fashion as

$$\overline{A}_{i,j} = \sum_{K \in \mathcal{T}_h} \overline{A}_{i,j}^K \qquad \text{with} \qquad \overline{A}_{i,j}^K := a_K(\psi_j, \psi_i) := 2\nu(\mathbb{D}(\psi_j), \mathbb{D}(\psi_i))_K \,,$$

where the modified element matrix entries $\overline{A}_{i,j}^K$ are computed from the original element matrix entries $A_{i,j}^K := a_K(\varphi_j, \varphi_i)$ in the following way. Using the representation (5.4) of the QDF basis functions ψ_j restricted to an element K, we have for non-flux nodes $i, j \in J_s \setminus J_f$

$$\begin{aligned}\overline{A}_{i,j}^K =& a_K(\widetilde{\varphi}_j, \widetilde{\varphi}_i) - \sum_{E \in \mathcal{E}_h^K(j)} \alpha_{j,E}\, a_K(\widetilde{\varphi}_{j_f(E)}, \widetilde{\varphi}_i) - \sum_{F \in \mathcal{E}_h^K(i)} \alpha_{i,F}\, a_K(\widetilde{\varphi}_j, \widetilde{\varphi}_{j_f(F)}) \\ &+ \sum_{E \in \mathcal{E}_h^K(j)} \sum_{F \in \mathcal{E}_h^K(i)} \alpha_{j,E}\, \alpha_{i,F}\, a_K(\widetilde{\varphi}_{j_f(E)}, \widetilde{\varphi}_{j_f(F)})\end{aligned}$$

where $\mathcal{E}_h^K(j) := \{E \in \mathcal{E}(K) : a_j \in \overline{E}\}$ denotes the set of those faces of element K that are associated with the nodal point a_j. In a similar way, one can compute in a previous step the needed intermediate matrix entries $\widetilde{A}_{i,j}^K := a_K(\widetilde{\varphi}_j, \widetilde{\varphi}_i)$ by linear combinations of the entries $A_{i,j}^K = a_K(\varphi_j, \varphi_i)$ of the original element stiffness matrix using the stored coefficients $c_{j,i}^K$.

REMARK 5.1. *The memory for storing the non-zero matrix entries $\overline{A}_{i,j}$ is the same as for storing the non-zero entries $A_{i,j}$, $i, j \in J_s$, of the original global stiffness matrix which are no longer needed for solving the reduced Stokes problem* (4.7). *Moreover, the additional work to compute the element matrix entries $\overline{A}_{i,j}^K$ from the original entries $A_{i,j}^K$ is only proportional to the number of velocity unknowns due to the local support of the QDF basis functions.*

5.2. Computation of $\widetilde{p}_h$. In the following, we want to discuss how the pressure part $\widetilde{p}_h \in \widetilde{M}_h$ of the pressure solution $p_h = \overline{p}_h + \widetilde{p}_h$ given in Theorem 4.1 can be determined in practice if the parts $\widetilde{u}_s$ and $u_b = \sum_{K \in \mathcal{T}_h} u_b^K$ of the velocity solution $u_h = \widetilde{u}_s + u_b$ have been already computed. From Theorem 4.1 we know that $\widetilde{p}_h$ can be decomposed elementwise as $\widetilde{p}_h = \sum_{K \in \mathcal{T}_h} \widetilde{p}_h^K$ where $\widetilde{p}_h^K \in \widetilde{M}_h(K)$ is the unique solution of the local problem (4.9). There are two ways to solve this problem. The first one is to solve for the coefficient vector $\underline{\widetilde{p}}^K$ of $\widetilde{p}_h^K$ the linear algebraic system that corresponds to (4.9). The structure of this system is :

$$\left(B^K\right)^T \underline{\widetilde{p}}^K = \underline{z}^K \tag{5.6}$$

where

$$\begin{aligned}&\left(B^K\right)^T \in \mathbb{R}^{\dim(X_b(K)) \times \dim(\widetilde{M}_h(K))}, \quad \underline{\widetilde{p}}^K \in \mathbb{R}^{\dim(\widetilde{M}_h(K))}, \\ &\underline{z}^K = (z_j^K) \in \mathbb{R}^{\dim(X_b(K))} \quad \text{with} \qquad z_j^K := (f, \varphi_j)_K - a_K(u_h, \varphi_j) \qquad \forall j \in J_b(K).\end{aligned}$$

For the definition of the matrix B^K see (3.26). Since (4.9) has a unique solution (see Theorem 4.1) and (5.6) is equivalent to (4.9) it follows that $\ker((B^K)^T) = \{0\}$.

Therefore, the symmetric matrix $B^K(B^K)^T$ is regular and $\underline{\widetilde{p}}^K$ can be computed by solving the linear system

$$B^K(B^K)^T\underline{\widetilde{p}}^K = B^K\underline{z}^K\,. \tag{5.7}$$

The disadvantage of this approach is that the local element matrices, which are needed to generate the systems (5.7) for all elements $K \in \mathcal{T}_h$, have to be assembled again since they are not stored inside of our new multigrid method for the reduced Stokes problem (4.7).

The second approach for computing $\widetilde{p}_h^K \in \widetilde{M}_h(K)$ uses the local pressure solutions $\theta^K \in \widetilde{M}_h(K)$ of the local Stokes problem (4.6) and $\theta_j^K := Q_b^K\varphi_j \in \widetilde{M}_h(K)$, $j \in J_s(K)$, of problem (3.20) with $w = \varphi_j$. The algebraic systems equivalent to these local Stokes problems have to be solved anyway by means of a direct solver in order to generate the QDF basis functions for the velocity. Thus, the coefficient vectors of θ^K and θ_j^K are available for free. For our second approach, we assume that these coefficient vectors will be stored. The required storage for this is less than the memory that becomes free from the matrix entries $b(\varphi_j, \chi_i)$, $j \in J_s(K)$, $i \in \widetilde{I}(K)$, and the right hand side components $-b(\widetilde{g}_h, \chi_i)$, $i \in \widetilde{I}(K)$, which are no longer needed for our new method relying on QDF basis functions. Now, let us assume that the local representation of the computed solution $\widetilde{u}_s \in \widetilde{X}_s$ of the reduced Stokes problem (4.7) on an element $K \in \mathcal{T}_h$ is given by means of coefficients $\widetilde{u}_j \in \mathbb{R}$ as follows

$$\widetilde{u}_s(x) = \sum_{j\in J_s(K)} \widetilde{u}_j\psi_j(x) \qquad \forall\, x \in K\,.$$

Applying the definition (3.42) of the ψ_j and (5.4), we can represent $\widetilde{u}_s$ in the form

$$\widetilde{u}_s(x) = \sum_{j\in J_s(K)} u_j\widetilde{\varphi}_j(x) \qquad \forall\, x \in K$$

using the basis functions $\widetilde{\varphi}_j \in \widetilde{X}_s$ defined by (3.21) and coefficients $u_j \in \mathbb{R}$ defined as $u_j := \widetilde{u}_j$ if $j \in J_s(K)\setminus J_f(K)$ and $u_j := \widetilde{u}_j - \sum_{i\in J(E)\setminus\{j\}} \alpha_{i,E}\widetilde{u}_i$ if $j = j_f(E) \in J_f(K)$. From the definition of $\widetilde{\varphi}_j\big|_K = \varphi_j\big|_K - P_b^K\varphi_j$ and $\theta_j^K := Q_b^K\varphi_j$ in (3.20), it follows that

$$b(v_b, \theta_j^K) = a(\widetilde{\varphi}_j, v_b) \qquad \forall v_b \in X_b(K).$$

Using the definition of θ^K in (4.6) we obtain that the pressure function $\widetilde{p}_h^K$ defined by

$$\widetilde{p}_h^K := \theta^K - \sum_{j\in J_s(K)} u_j\theta_j^K \tag{5.8}$$

is the unique solution of (4.9). This shows that we can calculate $\widetilde{p}_h^K$ by the cheap computation of the coefficients u_j from the coefficients $\widetilde{u}_j$ of the QDF basis representation of $\widetilde{u}_s$ if we invest the memory to store the coefficients that represent the pressure functions θ^K and θ_j^K.

5.3. Transformation of basis representations. In this subsection, we will describe two operators which allow to switch from the basis representation of a function in $\widetilde{X}_h = \operatorname{span}\{\psi_j : j \in \widetilde{J}\}$ to the one in $X_h = \operatorname{span}\{\varphi_j : j \in J\}$ and vice versa. Let $N := \operatorname{card}(J)$ and $\widetilde{N} := \operatorname{card}(\widetilde{J})$. By the finite element isomorphism a function

$v_h \in \widetilde{X}_h \subset X_h$ is uniquely associated with a coefficient vector $\underline{\widetilde{v}} = (\widetilde{v}_j) \in \mathbb{R}^{\widetilde{N}}$ such that $v_h = \sum_{j\in\widetilde{J}} \widetilde{v}_j \psi_j$. Thus, using (3.42) and (5.4), we have

$$v_h = \sum_{j\in\widetilde{J}_s} \widetilde{v}_j \big(\widetilde{\varphi}_j - P_f\varphi_j\big) + \sum_{j\in\widetilde{J}_b\cup J_f} \widetilde{v}_j\widetilde{\varphi}_j = \sum_{j\in\widetilde{J}} v_j^*\widetilde{\varphi}_j$$

with coefficients v_j^* defined by

$$v_j^* := \begin{cases} \widetilde{v}_j & \text{if} \quad j \in \widetilde{J}_s \cup \widetilde{J}_b, \\ \widetilde{v}_j - \displaystyle\sum_{i\in J(E)\setminus\{j\}} \alpha_{i,E}\widetilde{v}_i & \text{if} \quad j = j_f(E) \in J_f. \end{cases} \tag{5.9}$$

Now, from $\widetilde{J} = J_s \cup \widetilde{J}_b$ and the representation (3.27) of $\widetilde{\varphi}_j$ for $j \in \widetilde{J}_b$ we get

$$v_h = \sum_{j\in\widetilde{J}} v_j^*\widetilde{\varphi}_j = \sum_{j\in J_s} v_j^*\big(\varphi_j - P_b\varphi_j\big) + \sum_{K\in\mathcal{T}_h} \sum_{j\in\widetilde{J}_b(K)} v_j^* \sum_{i\in J_b(K)} c_{j,i}^K\varphi_i.$$

Using the local representation (5.2) of $P_b\varphi_j$ for $j \in J_s$, we obtain the following basis representation of $v_h = \sum_{j\in\widetilde{J}} \widetilde{v}_j\psi_j \in \widetilde{X}_h$ by means of the original basis $\{\varphi_j : j \in J\}$ of X_h:

$$v_h = \sum_{j\in J} v_j\varphi_j \qquad \text{with} \qquad v_j := \begin{cases} v_j^* & \text{if} \quad j \in J_s, \\ \displaystyle\sum_{i\in\widetilde{J}_b(K)} c_{i,j}^K v_i^* - \sum_{i\in J_s(K)} c_{i,j}^K v_i^* & \text{if} \quad j \in J_b(K). \end{cases} \tag{5.10}$$

Obviously, the mapping that assigns the coefficient vector $\underline{v}$ of the original basis representation to the coefficient vector $\underline{\widetilde{v}}$ of the QDF basis representation of a function $v_h \in \widetilde{X}_h$ is linear, i.e., there is a linear operator $\Pi : \mathbb{R}^{\widetilde{N}} \to \mathbb{R}^N$ such that

$$v_h = \sum_{j\in\widetilde{J}} (\underline{\widetilde{v}})_j\psi_j = \sum_{j\in J} (\Pi\underline{\widetilde{v}})_j\varphi_j \ \in \ \widetilde{X}_h \qquad \forall\, \underline{\widetilde{v}} \in \mathbb{R}^{\widetilde{N}}. \tag{5.11}$$

At the end of this section we want to construct for a given function $v_h = \sum_{j\in J} v_j\varphi_j \in X_h$ a suitable QDF approximation $\widetilde{v}_h = \sum_{j\in\widetilde{J}} \widetilde{v}_j\psi_j \in \widetilde{X}_h \subset X_h$ which will be used later on during the prolongation in the multigrid method. Using the decomposition $X_h = X_s \oplus X_b$ we can write any function $v_h \in X_h$ uniquely in the form

$$v_h = v_s + v_b, \qquad \text{with} \qquad v_s = \sum_{j\in J_s} v_j\varphi_j \in X_s, \quad v_b = \sum_{j\in J_b} v_j\varphi_j \in X_b.$$

Since the projector $I - P_b$ maps from X_s into the space $\widetilde{X}_s \subset \widetilde{X}_h$, a natural approximation of the part v_s of v_h is the function $\widetilde{v}_s$ with

$$\widetilde{v}_s := (I - P_b)v_s = \sum_{j\in J_s} v_j\widetilde{\varphi}_j \in \widetilde{X}_s.$$

Therefore, we have

$$v_h = \widetilde{v}_s + w_b \qquad \text{with} \qquad w_b := v_b + P_b v_s \ \in \ X_b,$$

which is, due to the decomposition $X_h = \widetilde{X}_s \oplus X_b$ (see (3.33)), the unique splitting of v_h into a function $\widetilde{v}_s \in \widetilde{X}_s$ and a function $w_b \in X_b$. Thus, the decomposition $\widetilde{X}_h = \widetilde{X}_s \oplus V_b$ (see (3.32)) suggests to define the approximation $\widetilde{v}_h \in \widetilde{X}_h$ of v_h as

$$\widetilde{v}_h := \widetilde{v}_s + \widetilde{w}_b \qquad \text{with} \qquad \widetilde{w}_b := L_b w_b \in V_b\,, \tag{5.12}$$

where $L_b : X_b \to V_b$ is a suitable projection operator. Using the elementwise decomposition of X_b (see (3.18)), we have

$$w_b = \sum_{K\in\mathcal{T}_h} w_b^K \qquad \text{with} \qquad w_b^K = \sum_{j\in J_b(K)} v_j\varphi_j + \sum_{i\in J_s(K)} v_i P_b^K\varphi_i = \sum_{j\in J_b(K)} w_j^K\varphi_j$$

where

$$w_j^K := v_j \,+ \sum_{i\in J_s(K)} v_i c_{i,j}^K \qquad \forall\, j \in J_b(K)\,. \tag{5.13}$$

We define L_b locally as

$$L_b w_b = \sum_{K\in\mathcal{T}_h} L_b^K w_b^K$$

where $L_b^K : X_b(K) \to V_b(K)$ is a suitable projection operator as, for example, the L^2 projection. The local coefficient vector $\underline{\widetilde{w}}^K := (\widetilde{w}_j^K)$ of the projection $\widetilde{w}_b^K = L_b^K w_b^K$ can be obtained from the coefficient vector $\underline{w}^K := (w_j^K)_{j\in J_b(K)}$ by matrix-vector multiplication with a small local matrix, say $D^K = (D_{i,j}^K) \in \mathbb{R}^{\mathrm{card}(\widetilde{J}_b(K))\times\mathrm{card}(J_b(K))}$, i.e.

$$\widetilde{w}_b^K = \sum_{j\in\widetilde{J}_b(K)} \widetilde{w}_j^K\psi_j \qquad \text{with} \qquad \widetilde{w}_j^K = \sum_{i\in J_b(K)} D_{j,i}^K w_i^K\,.$$

In an efficient implementation, the matrices D^K should be computed and stored for each element $K \in \mathcal{T}_h$ during the assembling process of the element stiffness matrices before starting the multigrid solver. The required memory is less than the one for storing the matrix entries of the bubble functions $a(\varphi_j, \varphi_i)$, $i, j \in J_b(K)$, in the original finite element method which is no longer needed in our new method.

Finally, a straightforward calculation yields the following basis representation for the approximation $\widetilde{v}_h \in \widetilde{X}_h$ of a given function $v_h = \sum_{j\in J} v_j\varphi_j \in X_h$:

$$\widetilde{v}_h = \sum_{j\in\widetilde{J}} \widetilde{v}_j\psi_j \qquad \text{with} \qquad \widetilde{v}_j := \begin{cases} v_j & \text{if} \quad j \in J_s \setminus J_f, \\ \displaystyle\sum_{i\in J_b(K)} D_{j,i}^K w_i^K & \text{if} \quad j \in \widetilde{J}_b(K), \\ \displaystyle v_j + \sum_{i\in J(E)\setminus\{j\}} v_i\,\alpha_{i,E} & \text{if} \quad j = j_f(E) \in J_f, \end{cases} \tag{5.14}$$

where the w_i^K are defined by (5.13).

Obviously, the resulting coefficient vector $\underline{\widetilde{v}} = (\widetilde{v}_j) \in \mathbb{R}^{\widetilde{N}}$ of $\widetilde{v}_h \in \widetilde{X}_h$ in the QDF basis depends linearly on the coefficient vector $\underline{v} = (v_j) \in \mathbb{R}^N$ of $v_h \in X_h$. Hence, there is a linear operator $\widetilde{\Pi} : \mathbb{R}^N \to \mathbb{R}^{\widetilde{N}}$ such that

$$\widetilde{v}_h = \sum_{j\in\widetilde{J}} (\widetilde{\Pi}\underline{v})_j\psi_j \;\in\; \widetilde{X}_h, \qquad \forall\, \underline{v} \in \mathbb{R}^N, \tag{5.15}$$

is the projection of v_h into the subspace $\widetilde{X}_h$. It is easy to verify that a given $v_h \in \widetilde{X}_h$ implies $\widetilde{v}_h = v_h$.

As a summary we can say that both operators Π and $\widetilde{\Pi}$, respectively, as well as all other tools that are necessary to work with the QDF basis can be implemented efficiently with a computational work which is only proportional to the number of unknowns. This fact is important to ensure optimal computational costs for our proposed multigrid solver. The required memory to store the additionally needed coefficients $c_{j,i}^K$, $\alpha_{j,E}$ and $D_{i,j}^K$ is less than the memory that will become free from the fact that a lot of non-zero matrix entries of the original finite element method are no longer used in the new multigrid solver based on the QDF basis functions.

6. Multigrid Method. In the following, we will describe the main ingredients of our multigrid method, namely the smoother and the transfer operators. Here, the multigrid cycles are designed to solve the corresponding QDF formulations of type (4.2) on the different grids.

6.1. Smoother. We apply a block Gauss-Seidel smoother to the corresponding problem of type (4.2) on each multigrid level ℓ. One smoothing step consists essentially of a loop over all elements where some block solves are performed for each element $K \in \mathcal{T}_h$. For $r > 2$, one of these blocks is associated with the set $\widetilde{J}_b(K)$. Due to the orthogonal decomposition (4.4), the corresponding block solve already yields the local bubble part u_b^K. Hence, this block solve has to be performed only once for each QDF formulation of type (4.2) on a newly entered coarser level ℓ when the right-hand side functional has changed due to defect correction.

In order to simplify the notation we omit the level ℓ and introduce, for a given element $K \in \mathcal{T}_h$ and a nodal point $a_m \in \partial K$, the set $J^K(a_m)$ of all non-flux nodes of K which are connected to a_m, i.e.

$$J^K(a_m) := \{j \in J(K) \setminus J_f(K) : a_j = a_m\}.$$

A block Gauss-Seidel smoother can be regarded as a successive subspace correction method. Therefore, we define, for an arbitrary index set $I \subset \widetilde{J}$, the velocity subspace

$$\widetilde{X}_I := \operatorname{span}\{\psi_j : j \in I\} \cap \widetilde{X}_{s,0}$$

and the local pressure subspaces

$$\overline{M}_K := \operatorname{span}\{\chi_0^K\} \subset \overline{M}_h \qquad \forall K \in \mathcal{T}_h.$$

Let $(u^{\text{old}}, p^{\text{old}}) \in \widetilde{X}_h \times M_{h,0}$ be an approximation of the solution $(u_h, \overline{p}_h)$ of the QDF formulation of type (4.2) on some grid level ℓ with a right-hand side functional $l(\cdot)$ which is defined by defect restriction on coarser levels and by $l(w) := (f, w)_\Omega$ on the finest level. Then, one smoothing step $(u^{\text{old}}, p^{\text{old}}) \mapsto (u, p)$ is defined as follows:

$(u, p) := (u^{\text{old}}, p^{\text{old}})$
for $K \in \mathcal{T}_h$ **do**
(i) **if** the right-hand side $l(\cdot)$ has changed **then**
Find $v \in \widetilde{X}_{\widetilde{J}_b(K)}$ such that

$$a(v, w) = l(w) - a(u, w) \qquad \forall w \in \widetilde{X}_{\widetilde{J}_b(K)} \tag{6.1}$$

update $u := u + v$
end if

(ii) Find $(v,q) \in \widetilde{X}_{J_f(K)} \times \overline{M}_K$ such that

$$\begin{aligned} a(v,w) + b(w,q) &= l(w) - a(u,w) - b(w,p) \qquad \forall w \in \widetilde{X}_{J_f(K)}, \\ b(v,\mu) &= -b(u,\mu) \qquad \forall \mu \in \overline{M}_K, \end{aligned} \tag{6.2}$$

update $(u,p) := (u,p) + (v,q)$

(iii) **for** all nodal points $a_m \in \partial K$ **do**

Find $v \in \widetilde{X}_{J^K(a_m)}$ such that

$$a(v,w) = l(w) - a(u,w) \qquad \forall w \in \widetilde{X}_{J^K(a_m)} \tag{6.3}$$

update $u := u + v$

end do

end do

L_0^2 projection $p := p - |\Omega|^{-1}(p,1)_\Omega$

The size of system (6.1) depends on the used element pair for velocity and pressure. For the pair Q_2/P_1^{disc}, we have $\widetilde{J}_b(K) = \emptyset$. Hence, no system has to be solved in this case. Also on the finest grid level the systems (6.1) will not be solved since the local parts u_b^K, $K \in \mathcal{T}_h$, have been already calculated during the generation of the stiffness matrices. In (6.2) we solve a system corresponding to all flux nodes $j \in J_f(K)$ and the one node for the pressure constant. This is a saddle-point problem of the dimension $\mathrm{card}\big(\mathcal{E}(K)\big) + 1$. For all nodal points $a_m \in \partial K$, we have $\widetilde{X}_{J^K(a_m)} \subset V_h$ due to $\widetilde{X}_s = V_s \oplus \widetilde{X}_f$, i.e., in the QDF formulation of type (4.2), the functions of the subspace $\widetilde{X}_{J^K(a_m)}$ are decoupled from the pressure. Therefore, the system (6.3) does not contain the pressure. The size of the linear system associated with (6.3) is equal to $\mathrm{card}\,\big(J^K(a_m)\big)$. It is either $(d-1)$, when a_m is connected to a flux node in the sense that a face $E \in \mathcal{E}(K)$ exists with $a_m = a_{j_f(E)}$, or it is d otherwise. For the special case of the Q_2-element in 2D, the different blocks of degrees of freedom used for solving the block systems (6.2) and (6.3) are shown in Figure 6.1. Since all

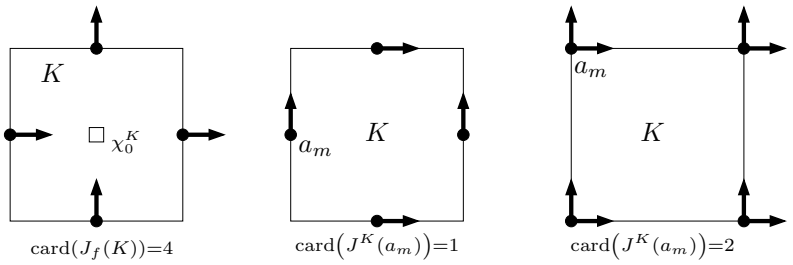

FIG. 6.1. *Blocks of degrees of freedom for the Q_2-element in 2D. Left: one 5×5-block from* (6.2)*, middle: four 1×1-blocks from* (6.3)*, right: four 2×2-blocks from* (6.3)*.*

local systems are of small size they can be solved very efficiently by means of a direct solver.

6.2. Prolongation and Restriction. In the following, we describe the transfer operators between the fine and coarse grid spaces during the multigrid method. At first, we will introduce some notations that use the corresponding grid level ℓ instead of the discretization parameter h. Let the final grid $\mathcal{T}_h$ be the finest grid $\mathcal{T}^{(L)} := \mathcal{T}_{h_L}$

of a sequence of uniformly refined grids $\mathcal{T}^{(\ell)} := \mathcal{T}_{h_\ell}$, $\ell = 0, 1, \ldots, L$, and let the corresponding spaces, basis functions, sets of nodes and dimensions at level ℓ be denoted as

$$X^{(\ell)} := X_{h_\ell} = \operatorname{span}\{\varphi_j^{(\ell)} : j \in J^{(\ell)}\}, \quad N^{(\ell)} := \operatorname{card}(J^{(\ell)}),$$
$$\widetilde{X}^{(\ell)} := \widetilde{X}_{h_\ell} = \operatorname{span}\{\psi_j^{(\ell)} : j \in \widetilde{J}^{(\ell)}\}, \quad \widetilde{N}^{(\ell)} := \operatorname{card}(\widetilde{J}^{(\ell)}).$$

Furthermore, let the operators Π and $\widetilde{\Pi}$ of Section 5 in the situation $X_h = X^{(\ell)}$ and $\widetilde{X}_h = \widetilde{X}^{(\ell)}$, $\ell = 0, \ldots, L$, be denoted by

$$\Pi^{(\ell)} : \mathbb{R}^{\widetilde{N}^{(\ell)}} \to \mathbb{R}^{N^{(\ell)}} \qquad \text{and} \qquad \widetilde{\Pi}^{(\ell)} : \mathbb{R}^{N^{(\ell)}} \to \mathbb{R}^{\widetilde{N}^{(\ell)}},$$

and let us assume that the local projection operator $L_b^K : X_b(K) \to V_b(K)$, used in the definition of the operator $\widetilde{\Pi}$ (see Section 5), is defined by $L_b^K w := 0$. This is necessary since no prolongated correction may be added to the already computed fine grid bubble parts u_b^K.

Now, we will define the prolongation operator $\widetilde{P}^{(\ell)} : \widetilde{X}^{(\ell)} \to \widetilde{X}^{(\ell+1)}$, i.e., we will describe how a fine grid QDF approximation $\widetilde{u}^{(\ell+1)} \in \widetilde{X}^{(\ell+1)}$ of a given coarse grid QDF function $\widetilde{u}^{(\ell)} \in \widetilde{X}^{(\ell)}$ is computed. Schematically, this is shown in Figure 6.2. The function $\widetilde{u}^{(\ell)} = \sum_{j \in \widetilde{J}^{(\ell)}} \widetilde{u}_j^{(\ell)} \psi_j^{(\ell)}$ is uniquely associated with the coefficient vec-

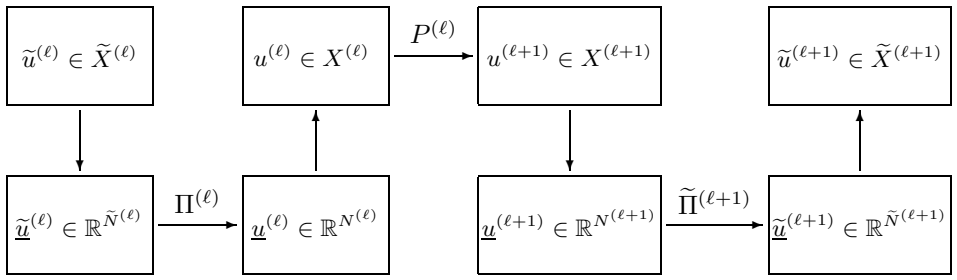

FIG. 6.2. *Computation of the prolongation $\widetilde{u}^{(\ell+1)}$ of a given coarse grid QDF function $\widetilde{u}^{(\ell)}$*

tor $\underline{\widetilde{u}}^{(\ell)} := (\widetilde{u}_j^{(\ell)})$. From this vector we compute the coefficient vector $\underline{u}^{(\ell)} = (u_j^{(\ell)}) := \Pi^{(\ell)} \underline{\widetilde{u}}^{(\ell)}$ that corresponds to the basis representation of $u^{(\ell)} := \sum_{j \in J^{(\ell)}} u_j^{(\ell)} \varphi_j^{(\ell)} = \widetilde{u}^{(\ell)}$ in the original finite element space $X^{(\ell)}$, see (5.11). Now, we use the prolongation operator $P^{(\ell)} : X^{(\ell)} \to X^{(\ell+1)}$ that would be used for a multigrid method with the original spaces $X^{(\ell)}$, $\ell = 0, \ldots, L$. This operator yields a fine grid approximation $u^{(\ell+1)} := P^{(\ell)} u^{(\ell)}$ of $\widetilde{u}^{(\ell)}$ in the original fine grid space $X^{(\ell+1)}$ which again corresponds to a coefficient vector $\underline{u}^{(\ell+1)} = (u_j^{(\ell+1)})$ such that $u^{(\ell+1)} := \sum_{j \in J^{(\ell+1)}} u_j^{(\ell+1)} \varphi_j^{(\ell+1)}$. Finally, we apply our operator $\widetilde{\Pi}^{(\ell+1)}$ in order to compute the coefficient vector $\underline{\widetilde{u}}^{(\ell+1)} = (\widetilde{u}_j^{(\ell+1)}) := \widetilde{\Pi}^{(\ell+1)} \underline{u}^{(\ell+1)}$ of the projection $\widetilde{u}^{(\ell+1)} := \sum_{j \in \widetilde{J}^{(\ell+1)}} \widetilde{u}_j^{(\ell+1)} \psi_j^{(\ell+1)}$ of $u^{(\ell+1)}$ into the subspace $\widetilde{X}^{(\ell+1)}$ of the QDF functions in $X^{(\ell+1)}$. Thus, the definition of the prolongation $\widetilde{u}^{(\ell+1)} = \widetilde{P}^{(\ell)} \widetilde{u}^{(\ell)}$ of the coarse grid QDF function $\widetilde{u}^{(\ell)}$ is completed.

For the defect restriction operator $\widetilde{R}^{(\ell)} : (\widetilde{X}^{(\ell+1)})' \to (\widetilde{X}^{(\ell)})'$, we choose the adjoint of the above defined prolongation operator $\widetilde{P}^{(\ell)} : \widetilde{X}^{(\ell)} \to \widetilde{X}^{(\ell+1)}$. We will omit the technical details here.

7. Numerical Tests. In this section, we will compare by means of numerical tests our new multigrid method based on the usage of QDF basis functions with other multigrid solvers known from the literature. In order to focus on the main properties we restrict ourselves to the two-dimensional case $d = 2$ and second order finite element approximation, i.e. $r = 2$.

As a test problem we consider the Stokes problem (1.1) on the unit square $\Omega = (0,1)^2$ with $\nu = 1$, the prescribed solution

$$u(x,y) := \begin{pmatrix} \sin x & \sin y \\ \cos x & \cos y \end{pmatrix}, \qquad p(x,y) := 2\cos x \ \cos y - 2\big(1 - \cos(1)\big)\sin(1)$$

and the data $f := -\operatorname{div} \mathbb{T}(u,p)$ and $g := u\big|_{\partial\Omega}$. This example was taken from [2].

In the following, we compare four different types of multigrid methods. The first method (further called *usual*) is a fully coupled multigrid solver with a Vanka-type smoother based on the original finite element basis which has been successfully used in the last years [12, 9]. The second method (*MDML*) is a multiple discretization multilevel approach where the higher order finite element discretization is applied only on the finest grid $\mathcal{T}^{(L)}$ whereas a stable low order discretization is used for all coarsenings. Note that in this method we have two different discretizations on the finest grid. For details on this method we refer to [11, 10]. The third method (*QDF*) is our new multigrid method where the QDF basis functions are used on all grid levels. The fourth method (*QDF-MDML*) is an MDML method where the QDF basis functions are used only for the higher order discretization on the finest grid $\mathcal{T}^{(L)}$ and where on all other levels the same low order discretization as in the second method is applied.

In order to get a fair comparison we fix for all methods the multigrid cycle to a W-cycle with 2 presmoothing and 2 postsmoothing steps. Furthermore, we do not use any damping or step length control mechanism in order to accelerate the convergence. In Table 7.1, we will compare the four methods with respect to their total running times (*total time*) measured in seconds on a Linux PC (Pentium IV, 2.8 GHz), the times used for the solver (*solver time*), the number of multigrid cycles (*# cycles*) used to achieve the stopping criterion that the Euclidean norm of the residual vector is less than 10^{-11}. Moreover, we compare the running time in seconds for one multigrid cycle (*time/cycle*), the average multigrid convergence rate (*aver. rate*) and the total memory (*memory*) in Mbyte required for the whole method.

The coarsest grid $\mathcal{T}^{(0)}$ is built by a decomposition of the unit square into 4 congruent squares. Our results are presented for the finest grid $\mathcal{T}_h = \mathcal{T}^{(7)}$ generated by 7 uniform refinement steps. This grid consists of 256×256 quadrilateral elements. For the approximation of the velocity we use Q_2-elements and for the pressure discontinuous P_1-elements which leads on the grid $\mathcal{T}^{(7)}$, in case of using the original finite element basis, to $526,338$ degrees of freedom (d.o.f.) for the velocity and $196,608$ d.o.f. for the pressure while the QDF basis needs only $395,266$ d.o.f. for the velocity and $65,536$ d.o.f. for the piecewise constant pressure within the reduced Stokes problem.

We see from Table 7.1 that the methods involving the QDF basis functions are much faster than the other methods although the convergence rate of the QDF method is not as good as those of the corresponding methods based on the original finite element basis. This shows that the new multigrid method profits a lot from the fact that the QDF basis for the velocity and the piecewise constant pressure basis for the reduced Stokes problem (4.7) lead to much less unknowns and that the local block sizes

TABLE 7.1
Results for different multigrid approaches.

	usual	MDML	QDF	QDF-MDML
total time	158	121	46	68
solver time	150	112	37	59
# cycles	10	9	12	9
time/cycle	15.0	12.8	3.1	6.6
aver. rate	0.0512	0.0332	0.0808	0.0301
memory	330	326	292	296

within the block Gauss-Seidel smoother of the new QDF method are much smaller than those of the usual method. It is to be expected that this effect will become much stronger in the case of a higher polynomial degree or in the 3D case. To show this numerically will be subject of further research.

8. Generalizations. We will briefly discuss the generalization of our multigrid approach based on QDF basis functions to the case of the stationary or non-stationary Navier-Stokes equations. Typical for both is that after a linearization of the nonlinear convection term and an implicit time discretization in the non-stationary case we have to solve *Oseen-type problems* of the following from:

Find $(u_h, p_h) \in (\widetilde{g}_h + X_{h,0}) \times M_{h,0}$ *such that*

$$\begin{array}{lcll} a(u_h, v_h) + b(v_h, p_h) & = & l(v_h) & \forall\, v_h \in X_{h,0} \\ b(u_h, q_h) & = & 0 & \forall\, q_h \in M_{h,0}\,. \end{array} \tag{8.1}$$

where the bilinear form $a(\cdot,\cdot)$ is now defined as

$$a(u, v) := \kappa(u, v)_\Omega + 2\nu\ (\mathbb{D}(u), \mathbb{D}(v))_\Omega + ((\beta \cdot \nabla)u, v)_\Omega \tag{8.2}$$

and $l(\cdot)$ denotes a linear functional that contains, in particular, the term $(f, v)_\Omega$ as well as further terms in the non-stationary case. The parameter κ in (8.2) is defined by $\kappa = 0$ in the stationary case and $\kappa = 1/\triangle t$ in the non-stationary case where $\triangle t$ denotes the size of the time step. The function $\beta \in \widetilde{g}_h + X_{h,0}$ is the previous velocity approximation during the nonlinear iteration process. For solving problem (8.1), the QDF basis functions can be constructed in an analogous way as for the Stokes problem (2.9). The only difference is that we have to work with the bilinear form $a(\cdot,\cdot)$ defined by (8.2) which means, in particular, that this form is no longer symmetric. As a consequence the decomposition of the Oseen-type problem (8.1) contains a difference compared to the Stokes case. For the element bubble part $u_b \in X_b$ of the solution $u_h = u_b + \widetilde{u}_s$ with $\widetilde{u}_s \in \widetilde{X}_s$ we obtain a problem which is completely analogous to (4.5) with the only difference that the right-hand side in (4.5) has to be replaced by the term $l(v_b)$. That means that, like in the Stokes case, the part u_b of the solution can be computed independently from the remaining part $\widetilde{u}_s$. Moreover, this computation can be done in an elementwise fashion during the assembling process of the stiffness matrices. However, for the part $\widetilde{u}_s \in \widetilde{X}_s$ of u_h we get the following *reduced Oseen-type problem* depending on the previously computed part $u_b \in X_b$:

Find $(\widetilde{u}_s, \overline{p}_h) \in (\widetilde{g}_h + \widetilde{X}_{s,0}) \times \overline{M}_{h,0}$ *such that*

$$\begin{array}{lcll} a(\widetilde{u}_s, \widetilde{v}_s) + b(\widetilde{v}_s, \overline{p}_h) & = & l(\widetilde{v}_s) - a(u_b, \widetilde{v}_s) & \forall\, \widetilde{v}_s \in \widetilde{X}_{s,0} \\ b(\widetilde{u}_s, \overline{q}_h) & = & 0 & \forall\, \overline{q}_h \in \overline{M}_{h,0}\,. \end{array} \tag{8.3}$$

In an analogous way as presented for the QDF formulation of the Stokes problem, we can create a multigrid method to compute efficiently the solution $(u_h, \overline{p}_h)$ of the corresponding QDF formulation of the Oseen problem. The pressure part $\widetilde{p}_h$ of $p_h = \overline{p}_h + \widetilde{p}_h$ can be computed in the same way as described in Section 5.

REFERENCES

[1] D. N. Arnold, D. Boffi, and R. S. Falk, *Approximation by quadrilateral finite elements*, Math. Comp., 71 (2002), pp. 909–922.

[2] D. Braess and R. Sarazin, *An efficient smoother for the Stokes problem*, Applied Numer. Math., 23 (1997), pp. 3 – 19.

[3] S. C. Brenner, *A nonconforming multigrid method for the stationary Stokes equations.*, Math. Comput., 55 (1990), pp. 411–437.

[4] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, vol. 15 of Springer Series in Computational Mathematics, Springer-Verlag, 1991.

[5] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier–Stokes equations*, Springer-Verlag, Berlin-Heidelberg-New York, 1986.

[6] D. F. Griffiths, *Finite elements for incompressible flow.*, Math. Methods Appl. Sci., 1 (1979), pp. 16–31.

[7] ———, *An approximately divergence-free 9-node velocity element (with variations) for incompressible flows.*, Int. J. Numer. Methods Fluids, 1 (1981), pp. 323–346.

[8] F. Hecht, *Construction of a basis at free divergence in finite element and application to the Navier-Stokes equations*, in Numerical solutions of nonlinear problems (Rocquencourt, 1983), INRIA, Rocquencourt, 1984, pp. 284–297.

[9] V. John, *Higher order finite element methods and multigrid solvers in a benchmark problem for the 3D Navier-Stokes equations.*, Int. J. Numer. Methods Fluids, 40 (2002), pp. 775–798.

[10] ———, *Large eddy simulation of turbulent incompressible flows. Analytical and numerical results for a class of LES models*, vol. 34 of Lecture Notes in Computational Science and Engineering, Springer-Verlag, Berlin, 2004.

[11] V. John, P. Knobloch, G. Matthies, and L. Tobiska, *Non-nested Multi-level Solvers for Finite Element Discretisations of Mixed Problems*, Computing, 68 (2002), pp. 313–341.

[12] V. John and G. Matthies, *Higher-order finite element discretizations in a benchmark problem for incompressible flows.*, Int. J. Numer. Methods Fluids, 37 (2001), pp. 885–903.

[13] G. Matthies, *Mapped finite elements on hexahedra. Necessary and sufficient conditions for optimal interpolation errors.*, Numer. Algorithms, 27 (2001), pp. 317–327.

[14] G. Matthies and L. Tobiska, *The inf-sup condition for the mapped Q_k-P^{disc}_{k-1} element in arbitrary space dimensions*, Computing, 69 (2002), pp. 119–139.

[15] F. Schieweck, *A general transfer operator for arbitrary finite element spaces*, Preprint 25/00, Otto-von-Guericke Universität Magdeburg, Fakultät für Mathematik, 2000.

[16] F. Schieweck, *Construction of higher order discretely divergence free finite elements for incompressible flow*, Preprint 34/02, Otto-von-Guericke Universität Magdeburg, Fakultät für Mathematik, 2002.

[17] F. Thomasset, *Implementation of finite element methods for Navier-Stokes equations.*, Springer Series in Computational Physics. New York - Heidelberg -Berlin: Springer-Verlag., 1981.

[18] S. Turek, *Multigrid techniques for a divergence-free finite element discretization.*, East-West J. Numer. Math., 2 (1994), pp. 229–255.

[19] ———, *Tools for simulating non-stationary incompressible flow via discretely divergence-free finite element models.*, Int. J. Numer. Methods Fluids, 18 (1994), pp. 71–105.

[20] X. Ye and G. Anderson, *The derivation of minimal support basis functions for the discrete divergence operator.*, J. Comput. Appl. Math., 61 (1995), pp. 105–116.

[21] X. Ye and C. A. Hall, *The construction of an optimal weakly divergence-free macroelement.*, Int. J. Numer. Methods Eng., 36 (1993), pp. 2245–2262.

[22] ———, *A discrete divergence-free basis for finite element methods.*, Numer. Algorithms, 16 (1997), pp. 365–380.