Construction of Higher Order Discretely Divergence Free Finite Elements for Incompressible Flow

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Abstract

We consider an hp-like finite element method on a 2D quadrilateral mesh for solving the Stokes problem with continuous Q_r-elements for the velocity and discontinuous P_{r-1}-elements for the pressure where the order r can vary from element to element. We construct a local and practically useful basis for the subspace of the discretely divergence free velocity functions. Thus, the computation of the velocity can be reduced to the solution of a symmetric positive definite problem in this subspace. Our constructed basis is suitable to create a multigrid method for solving the subspace problem efficiently. The pressure can be obtained by a simple post-processing procedure which requires only the solution of local problems. The ideas for the construction of a discretely divergence free basis can be extended easily to the general case of an adaptive mixed mesh with triangular and quadrilateral elements and hanging nodes. The approach can be applied also to the incompressible Navier-Stokes equations.

Keywords: higher order finite elements, discretely divergence free basis, incompressible flow, Stokes equations, Navier-Stokes equations

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1 Introduction

In this paper we consider the finite element approximation of an incompressible flow problem. For simplicity, we take as model problem the stationary two-dimensional Stokes equations with homogeneous Dirichlet boundary conditions, i.e. we want to find a velocity $u : \Omega \to \mathbb{R}^2$

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and a pressure $p : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\triangle u + \nabla p &= f \quad \text{in } \Omega \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma\,, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a connected bounded domain with a polygonal boundary $\Gamma := \partial\Omega$ and f a given body force.

We use a conforming *hp*-like finite element method on a quadrilateral mesh where the polynomial degree can vary from element to element but the grid is supposed to be regular excluding the case of hanging nodes. However, the presented ideas can be generalized in a straightforward way to the case of an adaptive mixed mesh consisting of triangles and quadrilaterals where hanging nodes are allowed. As finite element approximation we use continuous Q_r-elements for the velocity and discontinuous P_{r-1}-elements for the pressure where the order $r \geq 2$ can vary from element to element.

Our aim is to construct a local and practically useful basis for the subspace of the discretely divergence free functions in the velocity space. The advantage of such a basis is that it leads to a symmetric positive definite problem for the velocity in the discretely divergence free subspace. This problem is completely decoupled from the pressure. Thus, the difficulty to find a good solver for the indefinite coupled problem is circumvented. Using our constructed discretely divergence free basis functions, we can create an efficient multigrid-solver for the subspace problem of the velocity which is similar to that in [2], [12], [13]. The most crucial point for this purpose, namely the construction of the intergrid transfer operators, can be solved by means of techniques similar to those developed in [9]. Once the velocity has been determined the pressure can be calculated easily by a post-processing procedure which requires only the solution of a local problem on each element and a marching strategy over all elements to compute local pressure constants.

The idea to use a discretely divergence free basis for the velocity space is not new. We refer to [15], [16] for the case of continuous pressure elements and to [6], [7], [11], [8], [3], [2], [12], [13], [14] for the case of a discontinuous pressure approximation. The new contribution of this paper is the general construction process and the analysis of a local and practically useful basis of the subspace of the discretely divergence free velocity functions in the case of a quite general *hp*-like finite element approximation for the Stokes problem with discontinuous pressure elements. An advantage of a discontinuous pressure space is that the divergence free constraint can be characterized element-wise which simplifies the construction of a discretely divergence free basis. Let us emphasize that a discontinuous pressure approximation is not a disadvantage. By means of a simple post-processing procedure one can compute a continuous piecewise polynomial pressure function which maintains the optimal approximation order of the discontinuous pressure solution [9].

The ideas of this paper can be applied also in the case of the stationary or nonstationary incompressible Navier–Stokes equations since the choice of the basis functions does not change if we go from the Stokes to the Navier–Stokes model.

The paper is organized as follows. We introduce some notation and define the finite element spaces in Section 2. The construction process and the analysis of the discretely divergence free basis as well as some implementation aspects are presented in Section 3. In Section 4 we describe how a given function defined on the boundary can be extended into the domain as a discretely divergence free function having a small support. This is important for the treatment of more general boundary conditions and the management of the continuity requirements for adaptive mesh refinement by means of hanging nodes. In Section 5 we show how the pressure can be computed from a known velocity function by means of a post-processing procedure.

2 Preliminaries and Notation

The weak formulation of problem (1) reads:

$$\begin{array}{rcll} a(u,v) \, + \, b(v,p) & = & (f,v)_\Omega & \forall\, v \in X_0 \\ b(u,q) & = & 0 & \forall\, q \in M_0 \end{array} \tag{2}$$

where $(\cdot,\cdot)_G$ denotes the inner product in $L^2(G)$ or $(L^2(G))^2$, respectively, $X_0 := (H^1_0(\Omega))^2$, $M_0 := L^2_0(\Omega)$ with $L^2_0(G) := \{q \in L^2(G) : (q,1)_G = 0\}$,

$$a(u,v) := (\nabla u, \nabla v)_\Omega \qquad \text{and} \qquad b(v,p) := -(\nabla \cdot v, p)_\Omega \,. \tag{3}$$

Let the domain Ω be decomposed into convex quadrilateral elements $K \in \mathcal{T}^h$. The diameter of an element K is denoted by h_K and the global mesh-size of the grid $\mathcal{T}^h$ is defined as $h := \max_{K \in \mathcal{T}^h} h_K$. By $\mathcal{E}^h$ we denote the set of all edges of the grid $\mathcal{T}^h$, by $\mathcal{E}^h(\Gamma)$ the set of all boundary edges and by $\mathcal{E}^h_0 := \mathcal{E}^h \setminus \mathcal{E}^h(\Gamma)$ the set of all inner edges. For some curve $\gamma \subset \overline{\Omega}$, the inner product in $L^2(\gamma)$ will be denoted by $\langle \cdot,\cdot \rangle_\gamma$.

For an element K, let $F_K : \widehat{K} \to K$ be the one-to-one mapping between the reference element $\widehat{K} := [-1,+1]^2$ and K. We denote by $\mathbb{P}_m(G)$ the space of all polynomials on the domain $G \subset \mathbb{R}^2$ with total degree less or equal to m and by $\mathbb{Q}_m(G)$ the space of those polynomials where the maximum power in each coordinate is less or equal to m. Let $\mathbf{r}$ denote the degree vector

$$\mathbf{r} := \{r_K : K \in \mathcal{T}^h\} \qquad \text{with} \qquad r_K \geq 2 \quad \forall\, K \in \mathcal{T}^h$$

which contains the polynomial degree r_K of the velocity approximation on each element K. Then, the finite element space of the velocity $X^N \subset (H^1(\Omega))^2$ associated with grid $\mathcal{T}^h$ and the degree vector $\mathbf{r}$ is defined as $X^N := S^N \times S^N$ with

$$S^N := S(\Omega; \mathbf{r}, \mathcal{T}^h) := \{\phi \in H^1(\Omega) : \phi\big|_K \circ F_K \in \mathbb{Q}_{r_K}(\widehat{K}) \quad \forall K \in \mathcal{T}^h\} \tag{4}$$

and the finite element space $M^N \subset L^2(\Omega)$ for the approximation of the pressure as

$$M^N := M(\Omega; \mathbf{r}, \mathcal{T}^h) := \{q \in L^2(\Omega) : q\big|_K \in \mathbb{P}_{r_K - 1}(K) \quad \forall K \in \mathcal{T}^h\}. \tag{5}$$

The finite dimensional subspaces for approximating problem (2) are

$$X_0^N := X^N \cap X_0 \qquad \text{and} \qquad M_0^N := M^N \cap M_0 .$$

Now the discrete Stokes problem reads: *Find* $(u^N, p^N) \in (X_0^N, M_0^N)$ *such that*

$$\begin{array}{lcll} a(u^N, v^N) + b(v^N, p^N) & = & (f, v^N)_\Omega & \forall\, v^N \in X_0^N \\ b(u^N, q^N) & = & 0 & \forall\, q^N \in M_0^N . \end{array} \tag{6}$$

From the literature it is known that in case of the h-version, where $r_K = r$ for all $K \in \mathcal{T}^h$, the pair (X_0^N, M_0^N) fulfills the inf-sup-condition uniformly with respect to the mesh-size h (see [4, Sect. II.3.2]) and in the case of the p-version with only one element $\Omega = K$ and $\mathbf{r} = \{r\}$ the pair (X_0^N, M_0^N) is inf-sup-stable uniformly with respect to the polynomial degree r (see [1]). The proof of the inf-sup-condition in [4, Sect. II.3.2] can be generalized in a straightforward way to our case of an hp-version. Thus, we have the existence of a unique and stable solution (u^N, p^N) of problem (6).

Finally, let us introduce some general notation. We denote by $\mathrm{card}(J)$ the number of elements of a finite set J. For a set $G \subset \mathbb{R}^2$, we denote by $\mathrm{int}(G)$ and $\overline{G}$ the interior and closure of G, respectively.

3 Construction of a Discretely Divergence Free Basis

Let $V^N \subset X^N$ and $V_0^N \subset X_0^N$ be the linear subspaces of *discretely divergence free velocity functions*, i.e.

$$V^N := \{v^N \in X^N : b(v^N, q^N) = 0 \quad \forall\, q^N \in M^N\} \qquad \text{and} \qquad V_0^N := V^N \cap X_0^N . \tag{7}$$

In the literature, the space V_0^N often is defined by the constraint $b(v^N, q^N) = 0$ for all $q^N \in M_0^N$. However, due to the homogeneous boundary values of the functions $v^N \in X_0^N$, this is equivalent to the constraint for all $q^N \in M^N$. In cases of mixed boundary conditions, where for X_0^N homogeneous boundary conditions are imposed only on a part of the boundary, one has to take the pressure space $M_0^N := M^N$ for the discretization (6).

It is known that the solution $u^N \in X_0^N$ of the discrete Stokes problem (6) is uniquely determined as the solution of the subspace problem

$$a(u^N, v^N) = (f, v^N)_\Omega \qquad \forall\, v^N \in V_0^N\,, \tag{8}$$

which is symmetric positive definite and completely decoupled from the pressure. If we could construct a locally defined basis of V_0^N it should be possible to create an efficient multigrid solver for the elliptic problem (8) similar to that in [2], [12], [13]. Once the velocity is solved, the pressure $p^N \in M_0^N$ due to the inf-sup-condition is uniquely determined by

$$b(v^N, p^N) = (f, v^N)_\Omega - a(u^N, v^N) \qquad \forall\, v^N \in X_0^N \setminus V_0^N\,. \tag{9}$$

As it will be shown in Section 5, this problem can be solved by a simple post-processing procedure, where the fact is exploited that the pressure is defined element-wise without any continuity constraints at the element interfaces.
Another advantage of the discontinuous pressure space M_0^N is that the divergence free constraint can be characterized element-wise, i.e. for each velocity function $v^N \in X^N$ it holds

$$v^N \in V^N \quad \text{if and only if} \quad (\nabla \cdot v^N, q^N)_K = 0 \quad \forall\, q^N \in \mathbb{P}_{r_K - 1}(K),\ \forall\, K \in \mathcal{T}^h\,. \tag{10}$$

The same is true if we replace the spaces X^N and V^N by X_0^N and V_0^N, respectively.

Now we want to construct the basis functions of V_0^N. Let us denote the usual basis functions of X_0^N (see [10]) by φ_j where the indices $j \in J$ are called the "nodes" of X_0^N. Here and in the following we will omit for all basis functions and sets of nodes the index N indicating the corresponding finite element space X_0^N or V_0^N. Our aim is to find the basis functions $\widetilde{\varphi}_j \subset V_0^N$, $j \in \widetilde{J}$, where $\widetilde{J}$ is the set of nodes of V_0^N with

$$\mathrm{card}(\widetilde{J}) = \dim(V_0^N) = \dim(X_0^N) - \dim(M_0^N) = \mathrm{card}(J) - \dim(M_0^N)\,. \tag{11}$$

At first, for a large subset of nodes $j \in J$, we want to modify the basis function $\varphi_j \in X_0^N$ into a discretely divergence free basis function $\widetilde{\varphi}_j \in V_0^N$. Suppose that each node $j \in J$ of the vector-valued basis $\{\varphi_j\}$ of X_0^N is associated with the scalar basis function $\phi_j \in S^N \cap H_0^1(\Omega)$ and the vector $y^{(j)} \in \mathbb{R}^2$ such that

$$\varphi_j(x) = \phi_j(x)\, y^{(j)}\,, \tag{12}$$

where $y^{(j)} = (1,0)^T$ if the index j is an odd integer and $y^{(j)} = (0,1)^T$ if j is even. That means that the index set J consists of pairs of nodes having the same scalar basis function. Let us subdivide the set of nodes J into disjoint subsets

$$J = J_v \cup J_e \cup J_b \qquad \text{with} \qquad J_b = J_c \cup J_r\,,$$

where J_v is the set of nodes associated with a vertex of $\mathcal{T}^h$, J_e the set of nodes associated with the interior of an edge $E \in \mathcal{E}_0^h$ and J_b the set of *bubble nodes* associated with the interior of an element. J_b is decomposed into a set J_c of *corrector bubble nodes* and a set J_r of *rest bubble nodes*. The *space of corrector bubble functions* $X_c^N \subset X_0^N$ defined as

$$X_c^N := \operatorname{span}\{\varphi_j : j \in J_c\} \tag{13}$$

plays an essential role for the construction of the basis functions $\widetilde{\varphi}_j \in V_0^N$. For a given element $K \in \mathcal{T}^h$, we denote by $J(K)$ the set of nodes related to K, i.e.

$$J(K) := \{j \in J : \operatorname{supp}(\varphi_j) \cap \operatorname{int}(K) \neq \emptyset\}\,. \tag{14}$$

and by $J_s(K) := J(K) \cap J_s$ the corresponding subset for each index $s \in \{v, e, b, c, r\}$. Furthermore, we introduce the local pressure space orthogonal to the constant functions and its basis functions χ_i^K :

$$M_0^N(K) := \mathbb{P}_{r_K-1}(K) \cap L_0^2(K) = \operatorname{span}\{\chi_i^K : i = 1, \dots, n_0^K\}\,. \tag{15}$$

For the choice of $J_c = \bigcup_{K\in\mathcal{T}^h} J_c(K)$, we assume that

$$J_c(K) = \{j_m(K) : 1 \leq m \leq n_0^K = \dim(M_0^N(K))\} \qquad \forall\, K \in \mathcal{T}^h \tag{16}$$

and that the following matrices $D^K \in \mathbb{R}^{n_0^K \times n_0^K}$ are regular, i.e.

$$\det(D^K) \neq 0 \quad \forall\, K \in \mathcal{T}^h \quad \text{where} \quad D^K = (D_{im}^K) := \big((\nabla\cdot\varphi_{j_m(K)}, \chi_i^K)_K\big)\,. \tag{17}$$

In the following we show that there exists a space X_c^N satisfying (16) and (17). However, in practice one has some freedom to choose X_c^N since, in general, there is more than one possible space X_c^N.

Lemma 1 *Let the mesh $\mathcal{T}^h$ be regular in the usual sense (see e.g. [4]). From the subset of basis functions $\varphi_j \in X_0^N$, $j \in J_b$, which are bubble functions in the sense that there exists an element $K \in \mathcal{T}^h$ such that* $\operatorname{supp}(\varphi_j) \subset \overline{K}$, *there can be constructed a subspace of corrector bubble functions $X_c^N \subset X_0^N$ generated by (13) with a subset $J_c \subset J_b$ such that*

$$X_c^N = \{v \in X_0^N : v\big|_K \in X_c^N(K) \quad \forall K \in \mathcal{T}^h\}\,, \tag{18}$$

where $X_c^N(K)$ is the local space of corrector bubble functions

$$X_c^N(K) := \operatorname{span}\{\varphi_j\big|_K : j \in J_c(K)\} \tag{19}$$

and the assumptions (16), (17) are satisfied.

Proof. Let $K \in \mathcal{T}^h$ be an arbitrary fixed element, $J_b(K) = \{\alpha_1, \ldots, \alpha_s\}$ and B the matrix $B = (B_{ik}) \in \mathbb{R}^{n \times s}$ with $n := n_0^K = \dim(M_0^N(K))$ and $B_{ik} := (\nabla \cdot \varphi_{\alpha_k}, \chi_i^K)_K$. We consider the local space of bubble functions on K

$$X_b^N(K) := \mathrm{span}\{\varphi_j : j \in J_b(K)\} = \{v \in (H_0^1(K))^2 : v \circ F_K \in (\mathbb{Q}_{r_K}(\widehat{K}))^2\}.$$

In [4, Th. II.3.2] it has been proven that the pair of local spaces $(X_b^N(K), M_0^N(K))$ is inf-sup-stable. This implies that $\ker(B^T) = \{0\}$ such that $\mathrm{rank}(B) = n$. Therefore, among the column vectors $b^k \in \mathbb{R}^n$, $k = 1, \ldots, s$, of the matrix B there exist n vectors $b^{k_1}, \ldots, b^{k_n}$ that are linearly independent. Let us define the indices $j_m(K) := \alpha_{k_m}$ for $m = 1, \ldots, n$ and the set $J_c(K) := \{j_1(K), \ldots, j_n(K)\} \subset J_b(K)$. Then, for the matrix D^K defined by (17), it holds

$$D_{im}^K = (\nabla \cdot \varphi_{j_m(K)}, \chi_i^K)_K = b_i^{k_m},$$

i.e. the column vectors of D^K are $b^{k_1}, \ldots, b^{k_n}$ which implies that $\det(D^K) \neq 0$. The global set J_c of corrector bubble nodes is defined as $J_c := \bigcup_{K \in \mathcal{T}^h} J_c(K)$. ■

Now, let us define a projection operator $P_c : (H^1(\Omega))^2 \to X_c^N$ that assigns to a given function v a corrector bubble function $P_c v$ which is, on each element $K \in \mathcal{T}^h$, determined by

$$(\nabla \cdot P_c v, q^N)_K = (\nabla \cdot v, q^N)_K \quad \forall\, q^N \in M_0^N(K)\,. \tag{20}$$

Here, the term *projection operator* denotes an operator which satisfies $P_c^2 = P_c$. For the practical computation of the function $P_c v\big|_K \in X_c^N(K)$ satisfying (20), one solves the following equivalent linear system of equations for the coefficients $\underline{v}_m^K \in \mathbb{R}$, $m = 1, \ldots, n_0^K$, of the ansatz $P_c v\big|_K = \sum_{m=1}^{n_0^K} \underline{v}_m^K \varphi_{j_m(K)}$:

$$\sum_{m=1}^{n_0^K} D_{im}^K \underline{v}_m^K = (\nabla \cdot v, \chi_i^K)_K \qquad \forall\, i = 1, \ldots, n_0^K\,, \tag{21}$$

which is a system of small dimension with the system matrix D^K from (17). Because of the assumptions (16), (17), made for X_c^N, there exists for each element K a unique solution.

The next lemma is essential for the construction of our discretely divergence free basis functions.

Lemma 2 *Let $\overline{X}^N$ and $\overline{X}_0^N$ be the following subspaces of X^N*

$$\overline{X}^N := \{v \in X^N : \langle v \cdot n^K, 1 \rangle_{\partial K} = 0 \quad \forall K \in \mathcal{T}^h\}, \qquad \overline{X}_0^N := \overline{X}^N \cap X_0^N\,, \tag{22}$$

where n^K denotes the unit normal vector on ∂K directed outward with respect to K. Assume that the space X_c^N satisfies the conditions (16) and (17). Let $I : (H^1(\Omega))^2 \to (H^1(\Omega))^2$ be *the identity operator. Then, by means of the element-wise computable operator $I - P_c$, each discrete function $v^N \in \overline{X}^N$ can be projected into the subspace of discretely divergence free functions. More precisely, it holds*

$$(I - P_c)\overline{X}^N \subset V^N\,, \qquad V^N \subset \overline{X}^N\,, \tag{23}$$

$$(I - P_c)\overline{X}_0^N \subset V_0^N\,, \qquad V_0^N \subset \overline{X}_0^N\,. \tag{24}$$

Proof. We show (23); the proof of (24) can be done in a completely analogous way. Let $v^N \in \overline{X}^N$ be given. Due to the definition of P_c it holds by (20)

$$(\nabla\cdot(v - P_c v), q^N)_K = 0 \quad \forall\, q^N \in \mathbb{P}_{r_K-1}(K) \cap L_0^2(K)\,. \tag{25}$$

Since $(P_c v)\big|_K \in (H_0^1(K))^2$ and $v \in \overline{X}^N$ we get that (25) is satisfied also for constant functions q^N. Using the element-wise characterization (10) for discretely divergence free functions we get that $v^N - P_c v^N \in V^N$ which shows the first part of (23). For an arbitrary function $v^N \in V^N$ and an element $K \in \mathcal{T}^h$, it holds

$$(\nabla\cdot v^N, 1)_K = \big\langle\, v^N \cdot n^K, 1\,\big\rangle_{\partial K} = 0$$

implying $v^N \in \overline{X}^N$ which shows the second part of (23). ■

Now, we want to characterize V_0^N by the action of the operator $I - P_c$ applied to a subspace $\overline{Z}_0^N$ of $\overline{X}_0^N$.

Lemma 3 *Let $Z_0^N \subset X_0^N$ and $\overline{Z}_0^N \subset \overline{X}_0^N$ be the subspaces*

$$Z_0^N := \operatorname{span}\{\varphi_j \;:\; j \in J_v \cup J_e \cup J_r\}\,, \qquad \overline{Z}_0^N := Z_0^N \cap \overline{X}_0^N\,. \tag{26}$$

Then, $\overline{X}_0^N$ is the direct sum

$$\overline{X}_0^N = X_c^N \oplus \overline{Z}_0^N \tag{27}$$

and the operator $I - P_c : \overline{Z}_0^N \to V_0^N$ is an isomorphism, i.e. in particular

$$V_0^N = (I - P_c)\overline{Z}_0^N\,. \tag{28}$$

Proof. By the definition of the spaces Z_0^N and X_c^N, we have that X_0^N is the direct sum

$$X_0^N = X_c^N \oplus Z_0^N\,.$$

Therefore, a function $v^N \in \overline{X}_0^N$ can be written as $v^N = v_c^N + w^N$ with $v_c^N \in X_c^N$ and $w^N \in Z_0^N$. Since $v_c^N\big|_{\partial K} = 0$ for all $K \in \mathcal{T}^h$, we have $v_c^N \in \overline{X}_0^N$ which implies $w^N \in \overline{Z}_0^N$. Thus, we have shown $\overline{X}_0^N \subset X_c^N \oplus \overline{Z}_0^N$. Since both spaces X_c^N and $\overline{Z}_0^N$ are subspaces of $\overline{X}_0^N$, it holds $X_c^N \oplus \overline{Z}_0^N \subset \overline{X}_0^N$, which proves (27).

Now, we show that the restriction of the operator $I - P_c$ to $\overline{Z}_0^N$ is an isomorphism onto V_0^N. First, since $\overline{Z}_0^N \subset \overline{X}_0^N$, we get by (24) that $(I - P_c)\overline{Z}_0^N \subset V_0^N$. Let v^N be an arbitrary function of V_0^N. Then, by means of $V_0^N \subset \overline{X}_0^N$ (see (24)) and (27), there exist functions $v_c^N \in X_c^N$ and $w^N \in \overline{Z}_0^N$ such that $v^N = v_c^N + w^N$. Since $P_c v^N = 0$ and $P_c v_c^N = v_c^N$, we get

$$(I - P_c)v^N = v^N = (I - P_c)(v_c^N + w^N) = (I - P_c)w^N\,.$$

This shows (28), i.e. the operator $I - P_c : \overline{Z}_0^N \to V_0^N$ is surjective. Now, let $w^N \in \overline{Z}_0^N$ satisfy $(I - P_c)w^N = 0$. That means that $w^N = P_c w^N \in X_c^N$. Since $X_c^N \cap \overline{Z}_0^N = \{0\}$, it follows that $w^N = 0$. This shows that the operator $I - P_c$ restricted to $\overline{Z}_0^N$ is injective. ∎

Lemma 3 says that finding a basis for V_0^N is equivalent to finding a basis for the space

$$\overline{Z}_0^N = \operatorname{span}\{\varphi_j : j \in J_v \cup J_e \cup J_r\} \;\cap\; \{v \in X_0^N : \langle\, v \cdot n^K, 1\,\rangle_{\partial K} = 0 \;\; \forall K \in \mathcal{T}^h\}$$

which consists of finite element functions in X_0^N that satisfy only one constraint per element.

In the following, we want to construct, for a large subset of local basis functions $\varphi_j \in Z_0^N$, associated local basis functions $\overline{\varphi}_j \in \overline{Z}_0^N$. Then, these basis functions $\overline{\varphi}_j$ will produce local discretely divergence free basis functions $\widetilde{\varphi}_j = (I - P_c)\overline{\varphi}_j \in V_0^N$. The vector-valued function $\overline{\varphi}_j$ will be defined as

$$\overline{\varphi}_j(x) := \overline{\phi}_j(x)\,\overline{y}^{(j)} \tag{29}$$

with a modified scalar basis function $\overline{\phi}_j \in S^N$ and some unit vector $\overline{y}^{(j)} \in \mathbb{R}^2$. These functions $\overline{\phi}_j$, except two per edge, are chosen such that $\langle\, \overline{\phi}_j, 1\,\rangle_E = 0$ for all edges $E \in \mathcal{E}^h$ which implies that the vector-valued function $\overline{\varphi}_j$ is contained in $\overline{Z}_0^N$.

Now, for the different types of nodes $j \in J_v \cup J_e \cup J_r$, we will define the associated functions $\overline{\phi}_j$ and vectors $\overline{y}^{(j)}$.

Type 1: $\mathbf{j \in J_e}$. To each edge $E \in \mathcal{E}_0^h$ we uniquely assign a unit vector $n_E = (n_{E,1}, n_{E,2})^T$ normal to E which is directed outward with respect to one of the two adjacent elements $K, K' \in \mathcal{T}^h$ with $E = \partial K \cap \partial K'$. As an associated tangential vector we define $t_E := (n_{E,2}, -n_{E,1})^T$. We introduce, for each edge $E \in \mathcal{E}^h$, the scalar quadratic edge bubble function $b_E \in S^N$ defined by $b_E\big|_K \circ F_K \in \mathbb{Q}_2(\widehat{K})$ for all $K \in \mathcal{T}^h$ and the conditions

$$b_E\big|_{E'} = 0 \quad \forall\, E' \in \mathcal{E}^h \setminus \{E\} \qquad \text{and} \qquad \langle\, b_E, 1\,\rangle_E = |E|\,, \tag{30}$$

where $|E|$ denotes the length of edge E. The support of b_E consists of the two elements adjacent to E. Furthermore, let

$$J_e(E) := \{j_1(E), j_2(E), \ldots j_{2\rho_E - 2}(E)\}$$

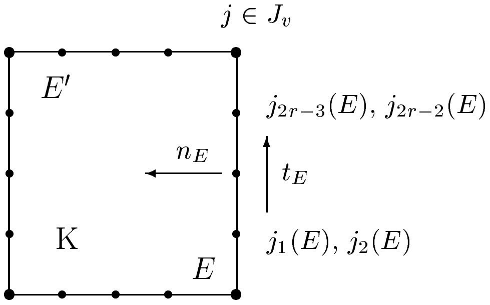

Figure 1: *vertex node $j \in J_v$ and inner edge nodes $j_m(E)$, $m = 1, \dots, 2r-2$, $r := \rho_E$*

be the set of all nodes associated with the interior of the edge E and enumerated in the direction of the tangential vector t_E (see Figure 1). Here the integer ρ_E denotes to the so-called "edge degree" [10] defined as

$$\rho_E := \min\{r_K, r_{K'}\} \quad \text{where} \quad E = \partial K \cap \partial K'\,,\ K, K' \in \mathcal{T}^h\,. \tag{31}$$

This means that the restriction $w\big|_E$ of a function $w \in S^N$ is a polynomial of degree ρ_E and that, in the case when $r_K > r_{K'}$, some nodes of K associated with the interior of edge E are so-called "hanging nodes" whose nodal values are prescribed by the continuity constraint.

We assume that the first two nodes of $J_e(E)$ are associated with the quadratic edge bubble function b_E, i.e. $\phi_j(x) := b_E(x)$ for $j \in \{j_1(E), j_2(E)\}$. Now we define for all $j \in J_e(E)$

$$\overline{\phi}_j(x) := \begin{cases} b_E(x) & \text{if } j \in \{j_1(E), j_2(E)\} \\ \phi_j(x) \,-\, |E|^{-1} \langle\, \phi_j, 1 \,\rangle_E \; b_E(x) & \text{otherwise} \end{cases} \tag{32}$$

and $\overline{y}^{(j)} := n_E$ for $j = j_1(E)$, $\overline{y}^{(j)} := t_E$ for $j = j_2(E)$ and $\overline{y}^{(j)} := y^{(j)}$ otherwise where $y^{(j)}$ denotes the unit vector corresponding to (12).

Type 2: $\mathbf{j \in J_v}$. Let $K \in \mathcal{T}^h$ be an arbitrary element such that $j \in J(K)$ and E, E' are the two edges of K associated with the node j (see Figure 1). Then we define the function $\overline{\phi}_j(x)$ for all $x \in \overline{K}$ by

$$\overline{\phi}_j(x) := \phi_j(x) \,-\, |E|^{-1} \langle\, \phi_j, 1 \,\rangle_E \; b_E(x) \,-\, |E'|^{-1} \langle\, \phi_j, 1 \,\rangle_{E'} \; b_{E'}(x) \tag{33}$$

and $\overline{y}^{(j)} := y^{(j)}$. On elements $K \in \mathcal{T}^h$ with $j \notin J(K)$ we define $\overline{\phi}_j := \phi_j = 0$. Note that $\overline{\phi}_j : \Omega \to \mathbb{R}$ is a continuous function that is contained in the space S^N.

Type 3: $\mathbf{j \in J_r}$. Here we have $\phi_j(x) = 0$ for all $x \in \partial K$, $K \in \mathcal{T}^h$. That means it is sufficient to choose $\overline{\phi}_j(x) := \phi_j(x)$ and $\overline{y}^{(j)} := y^{(j)}$ in order to ensure $\overline{\varphi}_j \in \overline{Z}_0^N$.

Now, let us define the index sets

$$\bar{J}_e := \bigcup_{E \in \mathcal{E}_0^h} \big(J_e(E) \setminus \{j_1(E)\}\big)\,, \qquad \bar{J} := J_v \cup \bar{J}_e \cup J_r\,. \tag{34}$$

Then, for all nodes $j \in \bar{J}$, we have by construction that $\overline{\varphi}_j \in \overline{Z}_0^N$. Therefore, using Lemma 2, we get the discretely divergence free modification of φ_j as

$$\widetilde{\varphi}_j := \overline{\varphi}_j - P_c \overline{\varphi}_j \in V_0^N \qquad \forall\, j \in \bar{J}\,. \tag{35}$$

Lemma 4 *The above constructed functions $\overline{\varphi}_j \in \overline{Z}_0^N$, $j \in \bar{J}$, are linearly independent. Let W_0^N and $\overline{W}_0^N$ be the spaces constructed by the quadratic edge bubble functions $b_E \in S^N$, i.e.*

$$W_0^N := \operatorname{span}\{b_E n_E : E \in \mathcal{E}_0^h\}\,, \qquad \overline{W}_0^N := W_0^N \cap \overline{X}_0^N\,, \tag{36}$$

where n_E denotes the uniquely assigned normal unit vector to edge E. Then, it holds

$$\overline{Z}_0^N = \operatorname{span}\{\overline{\varphi}_j : j \in \bar{J}\} \oplus \overline{W}_0^N\,. \tag{37}$$

Let NEL *denote the number of elements,* NE_0 *the number of inner edges and* NV_0 *the number of the inner vertices of $\mathcal{T}^h$. Furthermore, let the boundary of Ω consist of the disjoint closed components Γ_k, $k = 0, \ldots, \mathrm{NH}$, where Γ_0 denotes the exterior boundary and Γ_k, $k = 1, \ldots, \mathrm{NH}$, the boundary of the k-th hole of Ω. Then, it holds*

$$\dim(\overline{W}_0^N) = \mathrm{NE}_0 - \mathrm{NEL} + 1 = NV_0 + \mathrm{NH}\,. \tag{38}$$

Proof. Let us assume that there is a linear combination with coefficients $c_j \in \mathbb{R}$ such that $\sum_{j \in \bar{J}} c_j \overline{\varphi}_j = 0$. Using the notation of (32) we define the index sets $J_m := \bigcup_{E \in \mathcal{E}_0^h} \{j_m(E)\}$ for $m \in \{1, 2\}$. Then, it holds $\bar{J} = J \setminus J_1$ and $\overline{y}^{(j)} = y^{(j)}$ for all $j \in \bar{J} \setminus J_2$. Therefore, by construction of the $\overline{\varphi}_j$, there exist some constants $d_j \in \mathbb{R}$ such that

$$0 = \sum_{j \in \bar{J}} c_j \overline{\varphi}_j = \sum_{j \in J \setminus (J_1 \cup J_2)} c_j \varphi_j + \sum_{j \in J_1 \cup J_2} d_j \varphi_j\,.$$

Since the basis functions φ_j, $j \in J$, are linearly independent, we conclude that $c_j = 0$ for all $j \in \bar{J} \setminus J_2$. For a node $j \in J_2$, let E_j be the edge such that $j = j_2(E_j)$. Then, using (32), we have for all $x \in \Omega$

$$0 = \sum_{j \in J_2} c_j \overline{\varphi}_j(x) = \sum_{j \in J_2} c_j b_{E_j}(x) t_{E_j}\,.$$

If we choose x as the midpoint of the edge E_j and use the property (30) of the quadratic edge bubble functions b_E we get $0 = c_j (3/2) t_{E_j}$, i.e. $c_j = 0$. Thus, we have proved that the functions $\overline{\varphi}_j$, $j \in \bar{J}$, are linearly independent.

Now we show that Z_0^N is the direct sum of the spaces $U^N := \operatorname{span}\{\overline{\varphi}_j : j \in \bar{J}\}$ and W_0^N. Let $v^N \in U^N \cap W_0^N$. Then, there exist coefficients $\alpha_j, c_E \in \mathbb{R}$ such that

$$v^N(x) = \sum_{j \in \bar{J}} \alpha_j \overline{\varphi}_j(x) = \sum_{E \in \mathcal{E}_0^h} c_E b_E(x) n_E\,.$$

For a fixed edge $E \in \mathcal{E}_0^h$, by construction of the $\overline{\varphi}_j$ it holds $\left\langle \overline{\varphi}_j \cdot n_E, 1 \right\rangle_E = 0$ for all $j \in \bar{J}$. Thus, using (30) it follows

$$\left\langle v^N \cdot n_E, 1 \right\rangle_E = 0 = c_E |E| \,.$$

This implies $c_E = 0$ for all $E \in \mathcal{E}_0^h$, i.e. $v^N = 0$. Hence, the sum of the spaces U^N and W_0^N is a direct sum. Since both spaces are subspaces of Z_0^N we have $U^N \oplus W_0^N \subset Z_0^N$. By definition (34) of $\bar{J}$ we have $\mathrm{card}(\bar{J}) = \mathrm{card}(J_v \cup J_e \cup J_r) - \mathrm{card}(\mathcal{E}_0^h)$. Using (26) we get $\dim(Z_0^N) = \mathrm{card}(J_v \cup J_e \cup J_r)$ and

$$\dim(U^N \oplus W_0^N) = \dim(U^N) + \dim(W_0^N) = \mathrm{card}(\bar{J}) + \mathrm{card}(\mathcal{E}_0^h) = \dim(Z_0^N) \,.$$

Thus, we have shown that $Z_0^N = U^N \oplus W_0^N$.

The spaces U^N and $\overline{W}_0^N$, by construction of the $\overline{\varphi}_j$ and by definition (36), are subspaces of $\overline{Z}_0^N$. Therefore, we have $U^N \oplus \overline{W}_0^N \subset \overline{Z}_0^N$. Due to $\overline{Z}_0^N \subset Z_0^N$ an arbitrary function $v^N \in \overline{Z}_0^N$ can be written as $v^N = u^N + w^N$ with $u^N \in U^N$ and $w^N \in W_0^N$. Since $U^N \subset \overline{Z}_0^N$, we get $w^N = v^N - u^N \in \overline{Z}_0^N$, i.e. $w^N \in \overline{W}_0^N$. Hence $v^N \in U^N \oplus \overline{W}_0^N$ which proves (37).

Finally, we want to prove (38). By construction of the space X_c^N we have $\dim(X_c^N) = \dim(M_0^N) + 1 - \mathrm{NEL}$. Using the fact that $X_0^N = X_c^N \oplus Z_0^N$ together with (11) we get

$$\dim(Z_0^N) = \dim(X_0^N) - \dim(X_c^N) = \dim(V_0^N) + \mathrm{NEL} - 1 \,.$$

On the other hand, by $Z_0^N = U^N \oplus W_0^N$ and $\dim(W_0^N) = \mathrm{NE}_0$ we have

$$\dim(Z_0^N) = \mathrm{card}(\bar{J}) + \mathrm{NE}_0 \,.$$

Due to Lemma 3 the spaces $\overline{Z}_0^N$ and V_0^N have the same dimension. Using (37) we can conclude

$$\dim(\overline{Z}_0^N) = \mathrm{card}(\bar{J}) + \dim(\overline{W}_0^N) = \dim(V_0^N) = \mathrm{card}(\bar{J}) + \mathrm{NE}_0 - \mathrm{NEL} + 1 \,,$$

which proves the first equation of (38). The second part of (38) is a consequence of a well-known theorem on the relation between the numbers of the d-dimensional objects, $d = 0, 1, 2$, of the mesh $\mathcal{T}^h$. ■

This shows that we have to construct $\mathrm{NV}_0 + \mathrm{NH}$ further basis functions of $\overline{Z}_0^N$ in addition to the functions $\overline{\varphi}_j$ with $j \in \bar{J}$. These new basis functions will be constructed as basis functions of the space $\overline{W}_0^N$ which consists of vector-valued edge bubble functions satisfying one constraint per element.

For the construction of the first NV_0 functions, we can use the same idea as in [6], [2] where to each inner vertex of the mesh a vector-valued basis function is assigned for the approximation of the **curl** of a scalar stream function. In the following, we want to describe

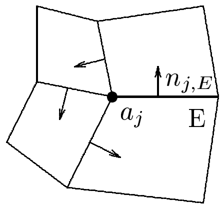

Figure 2: *vertex node* a_j *and associated normal vectors* $n_{j,E}$ *for* $E \in \mathcal{E}^h(a_j)$

the construction of these *vortex basis functions* which we will denote by ψ_j with $j \in J_\psi$. Each node $j \in J_\psi$ is uniquely associated with an inner vertex a_j of the mesh $\mathcal{T}^h$. Let us denote by $\mathcal{E}^h(a_j)$ the set of all edges $E \in \mathcal{E}^h$ corresponding to the vertex a_j in the sense that $a_j \in \overline{E}$. Furthermore, to a given vertex a_j and an edge $E \in \mathcal{E}^h(a_j)$, we assign the unit vector $n_{j,E}$ which is normal to edge E and directed counterclockwise around a_j (see Figure 2). Then we define the function $\psi_j \in W_0^N$ as

$$\psi_j(x) := \sum_{E \in \mathcal{E}^h(a_j)} |E|^{-1} b_E(x)\, n_{j,E}\,. \tag{39}$$

One can easily verify that $\psi_j \in \overline{W}_0^N \subset \overline{Z}_0^N$. Thus, for each node $j \in J_\psi$, we define the discretely divergence free function $\widetilde{\varphi}_j \in V_0^N$ as

$$\widetilde{\varphi}_j := s_j(\psi_j - P_c\psi_j) \qquad \text{with} \qquad s_j := \{\mathrm{card}(\mathcal{E}^h(a_j))\}^{-1} \sum_{E \in \mathcal{E}^h(a_j)} |E|\,. \tag{40}$$

In contrast to [6], [2], we have introduced the scaling factors $s_j = O(h)$ in order to ensure that the L^2-norm of these functions is of the same order (with respect to h) as for the former constructed functions $\widetilde{\varphi}_j$ with $j \in \bar{J}$.

To complete the basis for $\overline{W}_0^N$, we have to construct additionally NH many functions, say $\theta_j \in \overline{W}_0^N$ with $j \in J_\Gamma$. Each node $j \in J_\Gamma$ is uniquely assigned to a boundary component $\Gamma_{k(j)}$ with $1 \leq k(j) \leq \mathrm{NH}$. Let us denote by $\mathcal{E}_0^h(\Gamma_{k(j)})$ the set of all inner edges $E \in \mathcal{E}_0^h$ touching the boundary component $\Gamma_{k(j)}$ in the sense that $\overline{E} \cap \Gamma_{k(j)} \neq \emptyset$. Furthermore, to each edge $E \in \mathcal{E}_0^h(\Gamma_{k(j)})$, we assign the unit vector $n_{j,E}$ which is normal to the edge E and directed counterclockwise around the uniquely determined vertex a_ℓ with $a_\ell = \overline{E} \cap \Gamma_{k(j)}$. Then we define the function $\theta_j \in W_0^N$ as

$$\theta_j(x) := \sum_{E \in \mathcal{E}_0^h(\Gamma_{k(j)})} |E|^{-1} b_E(x)\, n_{j,E}\,, \qquad j \in J_\Gamma\,. \tag{41}$$

For this function, it is easy to verify that $\theta_j \in \overline{W}_0^N$. Thus, for a node $j \in J_\Gamma$, we define the discretely divergence free function $\widetilde{\varphi}_j \in V_0^N$ by

$$\widetilde{\varphi}_j := s_j(\theta_j - P_c\theta_j) \qquad \text{with} \qquad s_j := \{\operatorname{card}(\mathcal{E}_0^h(\Gamma_{k(j)}))\}^{-1} \sum_{E\in\mathcal{E}_0^h(\Gamma_{k(j)})} |E| \,. \tag{42}$$

Lemma 5 *The functions ψ_j, $j \in J_\psi$, and θ_j, $j \in J_\Gamma$, defined by (39) and (41), form a basis of $\overline{W}_0^N$, i.e.*

$$\overline{W}_0^N = \operatorname{span}\{\psi_j : j \in J_\psi\} \oplus \operatorname{span}\{\theta_j : j \in J_\Gamma\} \,. \tag{43}$$

Proof. By (38) it holds

$$\dim(\overline{W}_0^N) = \mathrm{NV}_0 + \mathrm{NH} = \operatorname{card}(J_\psi) + \operatorname{card}(J_\Gamma) \,.$$

Thus, the lemma is proved if we show that each function of $\overline{W}_0^N$ can be represented as a linear combination of the functions ψ_j and θ_j. Let an arbitrary $w^N \in \overline{W}_0^N$ be given having the representation

$$w^N(x) = \sum_{E\in\mathcal{E}_0^h} c_E b_E(x) n_E$$

with some coefficients $c_E \in \mathbb{R}$. Since $\overline{W}_0^N \subset \overline{X}^N$ it holds by the definition (22) of $\overline{X}^N$ that $\left\langle w^N \cdot n^K, 1 \right\rangle_{\partial K} = 0$ for each $K \in \mathcal{T}^h$. Applying Lemma I.2.2 of [4] to the case that the domain is the element K, we get that there exists a function $v_K \in (H^1(K))^2$ such that

$$\nabla \cdot v_K = 0 \quad \text{in } K \,, \qquad v_K = w^N \quad \text{on } \partial K \,. \tag{44}$$

Since $w^N \in (H_0^1(\Omega))^2$ we can compose all the local functions v_K to one global function $v \in (H_0^1(\Omega))^2$ such that $v|_K := v_K$ for all $K \in \mathcal{T}^h$. Due to (44) and the homogeneous boundary conditions of w^N we get that v satisfies the conditions

$$\nabla \cdot v = 0 \quad \text{in } \Omega \,, \qquad \langle v \cdot n, 1 \rangle_{\Gamma_k} = 0 \quad \forall\, k = 0, 1, \ldots, \mathrm{NH} \,, \tag{45}$$

where the Γ_k denote the closed boundary components of Γ and n the normal unit vector with respect to Γ. Applying Theorem I.3.1 of [4] and the consequence after its proof, we know that there exists a stream function $\Phi \in H^2(\Omega)$ such that $v = \mathbf{curl}\,\Phi$ where $\mathbf{curl}\,\Phi := (\partial\Phi/\partial x_2, -\partial\Phi/\partial x_1)^T$. By the embedding theorem this implies that Φ is continuous on $\overline{\Omega}$. Furthermore, since $v|_\Gamma = 0$, we have

$$\Phi(x) = \Phi_k \qquad \forall\, x \in \Gamma_k \,, \quad k = 0, 1, \ldots, \mathrm{NH} \,, \tag{46}$$

where $\Phi_k \in \mathbb{R}$ are some constants with the normalization $\Phi_0 = 0$. We conclude that in the sense of traces it holds

$$w^N\big|_E = \mathbf{curl}\,\Phi\big|_E \qquad \forall\, E \in \mathcal{E}_0^h \,,$$

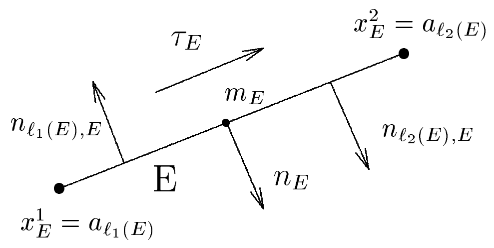

Figure 3: *edge E with associated midpoint, vertices, tangential and normal vectors*

which implies

$$\langle w^N \cdot n_E, 1 \rangle_E = c_E|E| = \langle \mathbf{curl}\,\Phi \cdot n_E, 1 \rangle_E = \Phi(x_E^2) - \Phi(x_E^1)\,, \tag{47}$$

where x_E^1 and x_E^2 denote the vertices at the end points of edge E defined as follows (see Figure 3). For the uniquely assigned normal unit vector $n_E = (n_{E,1}, n_{E,2})^T$ to edge E, we define the tangential vector $\tau_E := (-n_{E,2}, n_{E,1})^T$. Let m_E denote the midpoint of edge E. Then, we define $x_E^1 := m_E - (|E|/2)\tau_E$ and $x_E^2 := m_E + (|E|/2)\tau_E$. Let us denote the two nodes of J_ψ, which are associated with the vertices x_E^1 and x_E^2, by $\ell_1(E)$ and $\ell_2(E)$, i.e. let $a_{\ell_i(E)} = x_E^i$ with $\ell_i(E) \in J_\psi$ for $i = 1, 2$ and $E \in \mathcal{E}_0^h$. Then, according to the definition of the normal vector $n_{j,E}$ for equation (39), we have $n_{\ell_1(E),E} = -n_E$ and $n_{\ell_2(E),E} = n_E$. Thus, using (47) it follows

$$\begin{aligned} w^N(x) &= \sum_{E \in \mathcal{E}_0^h} |E|^{-1}\{\Phi(a_{\ell_1(E)})\, n_{\ell_1(E),E} + \Phi(a_{\ell_2(E)})\, n_{\ell_2(E),E}\}\, b_E(x) \\ &= \sum_{j \in J_\psi \cup J_V(\Gamma)} \Phi(a_j) \Big\{ \sum_{E \in \mathcal{E}_0^h \cap \mathcal{E}^h(a_j)} |E|^{-1} b_E(x) n_{j,E} \Big\}\,, \end{aligned}$$

where, here and in the following, for a subset $\gamma \subseteq \Gamma$, we denote by $J_V(\gamma)$ the set of indices j of vertices $a_j \in \gamma$. For a node $j \in J_\psi$, by definition a_j is an interior vertex such that $\mathcal{E}_0^h \cap \mathcal{E}^h(a_j) = \mathcal{E}^h(a_j)$. Therefore, with (39), (46) and $\Phi_0 = 0$, it follows

$$w^N(x) = \sum_{j \in J_\psi} \Phi(a_j)\, \psi_j(x) + \sum_{k=1}^{\mathrm{NH}} \Phi_k \sum_{\ell \in J_V(\Gamma_k)} \Big\{ \sum_{E \in \mathcal{E}_0^h \cap \mathcal{E}^h(a_\ell)} |E|^{-1} b_E(x) n_{\ell,E} \Big\}\,. \tag{48}$$

Now, let $j \in J_\Gamma$ be a node associated with the function θ_j in (41) and the boundary component $\Gamma_{k(j)}$. Then, each edge $E \in \mathcal{E}_0^h(\Gamma_{k(j)})$ is associated with exactly one addend in the sum $\sum_{E \in \mathcal{E}_0^h \cap \mathcal{E}^h(a_\ell)}$ in (48), where $\ell \in J_V(\Gamma_{k(j)})$ is the uniquely determined index such that $a_\ell = \overline{E} \cap \Gamma_{k(j)}$. Furthermore, the normal unit vector $n_{j,E}$ defined in front of (41) is the same as the vector $n_{\ell,E}$ in (48). Thus, we have

$$\sum_{\ell \in J_V(\Gamma_{k(j)})} \Big\{ \sum_{E \in \mathcal{E}_0^h \cap \mathcal{E}^h(a_\ell)} |E|^{-1} b_E(x) n_{\ell,E} \Big\} = \sum_{E \in \mathcal{E}_0^h(\Gamma_{k(j)})} |E|^{-1} b_E(x)\, n_{j,E} = \theta_j(x)\,,$$

which shows that the function w^N is a linear combination of the functions ψ_j, $j \in J_\psi$, and θ_j, $j \in J_\Gamma$. ■

Now, we can easily prove our main result.

Theorem 6 *The functions $\widetilde{\varphi}_j \in X_0^N$ defined by (35), (40) and (42) form a basis of the subspace of discretely divergence free functions $V_0^N \subset X_0^N$, i.e.*

$$V_0^N = \operatorname{span}\{\, \widetilde{\varphi}_j \;:\; j \in \widetilde{J}\} \qquad \textit{with} \qquad \widetilde{J} := \bar{J} \cup J_\psi \cup J_\Gamma \,. \tag{49}$$

Proof. From Lemma 4 and Lemma 5 it follows that

$$\overline{Z}_0^N = \operatorname{span}\{\overline{\varphi}_j : j \in \bar{J}\} \oplus \operatorname{span}\{\psi_j : j \in J_\psi\} \oplus \operatorname{span}\{\theta_j : j \in J_\Gamma\}$$

and that the functions $\overline{\varphi}_j$, ψ_j and θ_j are linearly independent. Due to Lemma 3 the operator $I - P_c : \overline{Z}_0^N \to V_0^N$ is an isomorphism. This implies that the functions $\widetilde{\varphi}_j = (I - P_c)\overline{\varphi}_j$ for $j \in \bar{J}$, $\widetilde{\varphi}_j = s_j(I - P_c)\psi_j$ for $j \in J_\psi$ and $\widetilde{\varphi}_j = s_j(I - P_c)\theta_j$ for $j \in J_\Gamma$ are linearly independent functions in V_0^N. Since

$$\operatorname{card}(\widetilde{J}) = \operatorname{card}(\bar{J}) + \operatorname{card}(J_\psi) + \operatorname{card}(J_\Gamma) = \dim(\overline{Z}_0^N) = \dim(V_0^N)\,,$$

it follows that the functions $\widetilde{\varphi}_j$ with $j \in \widetilde{J}$ form a basis of V_0^N. ■

The functions $\widetilde{\varphi}_j$, $j \in J_\Gamma$, in contrast to the other functions $\widetilde{\varphi}_j$, $j \in \widetilde{J} \setminus J_\Gamma$, have a non-local support which is a strip of elements located adjacent to the corresponding boundary component $\Gamma_{k(j)}$. This would be a disadvantage concerning the sparsity pattern of the stiffness matrix associated with the basis functions $\widetilde{\varphi}_j$. However, this disadvantage can be avoided in the same way as in [12]. There it has been proposed to use, instead of the functions $\widetilde{\varphi}_j$, $j \in J_\Gamma$, the local functions $\widetilde{\varphi}_j$ of the same type as defined in (40) and (39) where each node j is associated with a vertex of $\mathcal{T}^h$ located at one of the boundary components $\Gamma_1, \ldots, \Gamma_{\mathrm{NH}}$. Then, all these local basis functions generate a larger space, say $\widetilde{V}^N$, that does not satisfy the boundary conditions of X_0^N. However, a function $v = \sum_j \underline{v}_j \widetilde{\varphi}_j \in \widetilde{V}^N$ is contained in the right space V_0^N if the coefficients $\underline{v}_j \in \mathbb{R}$ of the new local "vortex basis functions" are identical for all vertices a_j that are located at the same boundary component Γ_k, $1 \le k \le \mathrm{NH}$. This can be managed by a projection technique [12].

The presented technique for the construction of a discretely divergence free basis can be applied in a straightforward way to the case of triangular finite element pairs which are inf-sup-stable and use a discontinuous pressure approximation (see, e.g., Section II.2.2 in [4]). Moreover, adaptive mesh refinement based on the use of hanging nodes as well as more general boundary conditions can be treated.

The implementation of the discretely divergence free basis functions within an existing finite element code can be done as follows. We assume that this code is able to compute the entries of the element stiffness matrices $A^K = (a_{ij}^K)$ and $B^K = (b_{ij}^K)$ associated with the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ in (3), i.e.

$$a_{ij}^K := (\nabla\varphi_j, \nabla\varphi_i)_K \qquad \text{and} \qquad b_{ij}^K := -(\nabla\cdot\varphi_j, \chi_i^K)_K\,,$$

where the φ_j are the original basis functions of the velocity space X_0^N and the χ_i^K are the local basis functions of the pressure space M_0^N corresponding to element K. Now, the restriction to element K of each discretely divergence free basis function $\widetilde{\varphi}_i$, $i \in \widetilde{J}(K)$, associated with K, can be represented as a linear combination of the original basis functions φ_j, $j \in J(K)$, i.e. there exist constants $c_{i,j}^K \in \mathbb{R}$ such that

$$\widetilde{\varphi}_i(x) = \sum_{j\in J(K)} c_{i,j}^K \varphi_j(x) \qquad \forall\, x \in K\,. \tag{50}$$

Note that in this formula, for $i \in J_\psi \cup J_\Gamma$, we use at first the scaling factors $s_i = 1$ instead of the true values defined in (40) and (42). From the construction process for the $\widetilde{\varphi}_i$ it follows that many of the constants $c_{i,j}^K$ are a priori zero. The non-zero constants can be computed easily by means of the inverse of the matrix D^K defined in (17). D^K can be generated by selecting some of the columns of the element stiffness matrix B^K. Now, in order to assemble the global stiffness matrix $\tilde{A}$ of the subspace problem (8), we have to compute the entries of the element stiffness matrix $\tilde{A}^K = (\tilde{a}_{ij}^K)$ defined by the basis functions $\widetilde{\varphi}_j$, $j \in \widetilde{J}(K)$, as $\tilde{a}_{ij}^K := (\nabla\widetilde{\varphi}_j, \nabla\widetilde{\varphi}_i)_K$. Applying (50), the values $\tilde{a}_{ij}^K$ can be determined by simple algebraic calculations from the entries of the stiffness matrix A^K and the coefficients $c_{i,j}^K$. At the end, once the global stiffness matrix $\tilde{A}$ has been assembled, we modify $\tilde{A}$ corresponding to the correct scaling factors s_j, $j \in J_\psi \cup J_\Gamma$. These scaling factors are computed beforehand within one global loop over all elements of the mesh.

4 Nonhomogeneous Boundary Values

In this section, we will treat the following problem. Let a boundary function $g^N : \Gamma \to \mathbb{R}^2$ be given which is assumed to be the trace of some finite element function in X^N, i.e. we know that there exists an extension $\widetilde{g}^N : \Omega \to \mathbb{R}^2$ of g^N with $\widetilde{g}^N \in X^N$. Now, the problem is to construct a discretely divergence free extension of g^N, i.e. to find, in a practicable and rather inexpensive way, a function $\widetilde{g}^N$ such that

$$\widetilde{g}^N \in V^N \qquad \text{with} \qquad \widetilde{g}^N(x) = g^N(x) \quad \forall\, x \in \Gamma\,. \tag{51}$$

One important application for this problem is the treatment of nonhomogeneous Dirichlet boundary conditions for the Stokes problem (1). Let the boundary condition of (1) be changed into

$$u = g \qquad \text{on } \Gamma\,, \tag{52}$$

where $g : \Gamma \to \mathbb{R}^2$ is some given function satisfying $\langle\, g \cdot n, 1\,\rangle_\Gamma = 0$ with the normal unit vector n on Γ directed outward with respect to Ω.

Now, the discrete velocity u^N is no longer in V_0^N but has to satisfy $u^N \in V^N$ and $u^N|_\Gamma = g^N$ where g^N is a suitable approximation of g such that g^N is the trace of some finite element function in X^N and $\langle\, g^N \cdot n, 1\,\rangle_\Gamma = \langle\, g \cdot n, 1\,\rangle_\Gamma$. If we have constructed an extension $\widetilde{g}^N \in V^N$ of g^N we can split u^N as

$$u^N = u_0^N + \widetilde{g}^N \qquad \text{with} \qquad u_0^N \in V_0^N$$

and our discrete Stokes problem is equivalent to the modified subspace problem: *Find* $u_0^N \in V_0^N$ *such that*

$$a(u_0^N, v^N) = (f, v^N)_\Omega - a(\widetilde{g}^N, v^N) \qquad \forall\, v^N \in V_0^N\,. \tag{53}$$

Another important application for problem (51) is the management of the continuity requirements for adaptive mesh refinement based on the use of hanging nodes. Here the considered domain, say G, is only one coarse grid element, which has been subdivided into several fine grid elements. The given boundary function $g^N : \partial G \to \mathbb{R}^2$ is the trace on ∂G of the finite element function defined outside of G. Then, the discretely divergence free extension of the boundary values g^N into G fulfills the continuity requirements within the space V^N.

A necessary condition to find a $\widetilde{g}^N$ satisfying (51) is obviously

$$\sum_{k=0}^{\mathrm{NH}} \langle\, g^N \cdot n, 1\,\rangle_{\Gamma_k} = 0\,, \tag{54}$$

where the Γ_k denote the closed boundary components of Γ.

At first, we present the construction of $\widetilde{g}^N$ in the special case where we assume

$$\langle\, g^N \cdot n, 1\,\rangle_{\Gamma_k} = 0 \qquad \forall\, k = 0, 1, \ldots, \mathrm{NH}\,. \tag{55}$$

For situations like flow around obstacles, where the interior closed boundary components Γ_k, $k \geq 1$, represent boundaries of obstacles at which no-slip boundary conditions are imposed, the assumption (55) is satisfied independent of the boundary conditions imposed at the exterior boundary Γ_0. In order to describe the construction of $\widetilde{g}^N$ we have to introduce some notation. Recalling that the functions φ_j, $j \in J$, denote the basis functions of the space X_0^N, we denote by $J(\Gamma)$ the set of nodes associated with the boundary Γ and assume that the functions φ_j,

$j \in J \cup J(\Gamma)$, form a basis of the space X^N. Since g^N is the trace of a finite element function in X^N there exist coefficients $g_j \in \mathbb{R}$ such that

$$g^N(x) = \sum_{j \in J(\Gamma)} g_j \varphi_j(x) \qquad \forall\, x \in \Gamma\,. \tag{56}$$

In the same way as in Section 3 we can define for each basis function φ_j, $j \in J(\Gamma)$, an associated function $\overline{\varphi}_j \in \overline{X}^N$ and the index sets

$$\bar{J}(\Gamma) := J_v(\Gamma) \cup \bar{J}_e(\Gamma) \qquad \text{with} \qquad \bar{J}_e(\Gamma) := \bigcup_{E \in \mathcal{E}^h(\Gamma)} \big(J_e(E) \setminus \{j_1(E)\}\big)\,, \tag{57}$$

where $J_v(\Gamma)$ denotes the set of nodes in $J(\Gamma)$ associated with a boundary vertex of $\mathcal{T}^h$, $J_e(E)$ the set of nodes associated with the interior of the boundary edge $E \in \mathcal{E}^h(\Gamma)$ and $j_1(E)$ that node $j \in J_e(E)$ for which the associated function $\overline{\varphi}_j$ is defined via the quadratic edge bubble function b_E, i.e. $\overline{\varphi}_j(x) := b_E(x) n_E$. Here and in the following we suppose that, for a boundary edge $E \in \mathcal{E}^h(\Gamma)$, the associated normal unit vector is $n_E = n$ where n denotes the normal unit vector on Γ in outward direction with respect to Ω. Due to the construction of the functions $\overline{\varphi}_j$, we get from (56) that there exist coefficients $c_E \in \mathbb{R}$ such that

$$g^N(x) = \sum_{j \in \bar{J}(\Gamma)} g_j \overline{\varphi}_j(x) \;+ \sum_{E \in \mathcal{E}^h(\Gamma)} c_E b_E(x) n_E \qquad \forall\, x \in \Gamma\,. \tag{58}$$

Since, for an arbitrary edge E, it holds $\big\langle\, \overline{\varphi}_j \cdot n_E, 1 \,\big\rangle_E = 0$ for all $j \in \bar{J}(\Gamma)$, we conclude that the constants c_E are determined by

$$\big\langle\, g^N \cdot n, 1 \,\big\rangle_E = c_E |E| \qquad \forall\, E \in \mathcal{E}^h(\Gamma)\,. \tag{59}$$

Now, our aim is to write the sum of the b_E-terms in (58) as a sum of functions $\psi_j \in \overline{X}^N$. Let $J_V(\Gamma_k)$ denote the set of indices j of vertices $a_j \in \Gamma_k$. Then, to each boundary vertex $a_j \in \Gamma_k$, $0 \le k \le \mathrm{NH}$, we can assign a function $\psi_j \in \overline{X}^N$ in the same way as in Section 3 by equation (39). For each closed boundary component Γ_k, we denote by $a_{j_{k,m}} \in \Gamma_k$, $m = 1, \ldots, \nu_k$ with $\nu_k := \mathrm{card}(J_V(\Gamma_k))$, the chain of all vertices on Γ_k defined as follows. We start at an arbitrary vertex $a_{j_{k,1}} \in \Gamma_k$. Then, we move along Γ_k, in counterclockwise direction if $k = 0$ and in clockwise direction if $k > 0$, and the next vertex that we pass is $a_{j_{k,2}} \in \Gamma_k$. We continue this process to get the other vertices $a_{j_{k,3}}$ until the last vertex on Γ_k. We define formally $j_{k,0} := j_{k,\nu_k}$. Then, for each $m = 1, \ldots, \nu_k$, the vertices $a_{j_{k,m-1}}$ and $a_{j_{k,m}}$ are the end points of a boundary edge on Γ_k which we denote by $E_{k,m}$. Now, let us consider the function $w^N \in \overline{X}^N$ defined by

$$w^N(x) := \sum_{k=0}^{\mathrm{NH}} \sum_{m=1}^{\nu_k} \Phi_{k,m} \psi_{j_{k,m}}(x)\,, \tag{60}$$

where $\Phi_{k,m} \in \mathbb{R}$ are some suitable constants which will be determined such that w^N is the extension of the sum of the b_E-terms in (58)

Using the structure of the functions ψ_j defined in (39), we get for an arbitrary edge $E_{k,m} \subset \Gamma_k$, $m = 1, \ldots, \nu_k$,

$$w^N(x) = |E_{k,m}|^{-1}\big\{\Phi_{k,m} - \Phi_{k,m-1}\big\} b_{E_{k,m}}(x) n_{E_{k,m}} \qquad \forall\, x \in E_{k,m}\,, \tag{61}$$

where $\Phi_{k,0} := \Phi_{k,\nu_k}$. Comparing with (58), this leads to the following choice for the constants $\Phi_{k,m} \in \mathbb{R}$

$$\Phi_{k,0} := 0\,, \qquad \Phi_{k,m} := \Phi_{k,m-1} + |E_{k,m}| c_{E_{k,m}} \qquad \forall\, m = 1, \ldots, \nu_k\,, \tag{62}$$

where the constants $c_{E_{k,m}}$ are determined by (59). Due to the assumption (55) this choice guarantees that $\Phi_{k,\nu_k} = 0 = \Phi_{k,0}$. Therefore, by (61) and (58) we get

$$g^N(x) = \sum_{j \in \bar{J}(\Gamma)} g_j \overline{\varphi}_j(x) \;+\; \sum_{k=0}^{\mathrm{NH}} \sum_{m=1}^{\nu_k} \Phi_{k,m} \psi_{j_{k,m}}(x) \qquad \forall\, x \in \Gamma\,. \tag{63}$$

In the same way as in (35) and (40) we can define the discretely divergence free functions $\widetilde{\varphi}_j \in V^N$ by

$$\widetilde{\varphi}_j := \begin{cases} (I - P_c)\overline{\varphi}_j & \text{for} \quad j \in \bar{J}(\Gamma) \\ s_j (I - P_c)\psi_j & \text{for} \quad j = j_{k,m} \;\text{ with } 0 \le k \le \mathrm{NH},\, 1 \le m \le \nu_k\,. \end{cases} \tag{64}$$

By construction the function $P_c v^N$ is zero at the boundary Γ for each $v^N \in X^N$. Thus, it follows from (63) that the function

$$\widetilde{g}^N(x) := \sum_{j \in \bar{J}(\Gamma)} g_j \widetilde{\varphi}_j(x) \;+\; \sum_{k=0}^{\mathrm{NH}} \sum_{m=1}^{\nu_k} s_{j_{k,m}}^{-1} \Phi_{k,m} \widetilde{\varphi}_{j_{k,m}}(x) \tag{65}$$

is a discretely divergence free extension of the boundary function g^N, i.e. $\widetilde{g}^N$ is a solution of (51). Moreover, the support of $\widetilde{g}^N$ consists only of one layer of elements located at the boundary Γ.

Now, we consider the general case that, instead of the assumption (55), it holds

$$\big\langle\, g^N \cdot n, 1 \,\big\rangle_{\Gamma_k} = \delta_k \qquad \forall\, k = 0, 1, \ldots, \mathrm{NH}\,, \tag{66}$$

where the δ_k are some computable constants satisfying $\sum_{k=0}^{\mathrm{NH}} \delta_k = 0$ due to the necessary condition (54). The idea to solve problem (51) is to construct functions $\sigma_k \in \overline{X}^N$ such that the function $g_0^N : \Gamma \to \mathbb{R}^2$, defined as

$$g_0^N(x) := g^N(x) + \sum_{k=1}^{\mathrm{NH}} \alpha_k \sigma_k(x) \qquad \forall\, x \in \Gamma \tag{67}$$

with suitable constants $\alpha_k \in \mathbb{R}$, satisfies the assumption (55). To construct the function σ_k we define a chain of edge-neighboured elements going from Γ_{k-1} to Γ_k,

$$K_{k,1} \to \cdots \to K_{k,m} \to K_{k,m+1} \to \cdots \to K_{k,\mu_k} ,$$

such that each element $K_{k,m}$, $m = 1, \ldots, \mu_k$, has exactly one "inflow edge" $e_{k,m-1} \in \mathcal{E}^h$ and one "outflow edge" $e_{k,m} \in \mathcal{E}^h$ where, for $0 < m < \mu_k$, the edge $e_{k,m}$ is the common edge of the elements $K_{k,m}$ and $K_{k,m+1}$, $e_{k,0} \subset \Gamma_{k-1}$ and $e_{k,\mu_k} \subset \Gamma_k$. To each edge $e_{k,m}$ we assign a normal unit vector $n_{e_{k,m}}$ which is directed outward with respect to element $K_{k,m}$ for $m \geq 1$ and inward with respect to element $K_{k,1}$ if $m = 0$. Then, by means of the quadratic edge bubble functions b_E (see (30)), we define the function $\sigma_k \in X^N$ as

$$\sigma_k(x) := \sum_{m=0}^{\mu_k} |e_{k,m}|^{-1} b_{e_{k,m}}(x) n_{e_{k,m}} , \qquad \forall\, k = 1, \ldots, \mathrm{NH}. \tag{68}$$

It is easy to verify that $\sigma_k \in \overline{X}^N$ and

$$\langle \sigma_k \cdot n, 1 \rangle_{\Gamma_i} = \begin{cases} -1 & \text{for} \quad i = k-1 \\ 1 & \text{for} \quad i = k \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, i = 0, \ldots, \mathrm{NH}\,. \tag{69}$$

Thus, choosing the constants

$$\alpha_k := \sum_{m=0}^{k-1} \delta_m , \qquad \forall\, k = 1, \ldots, \mathrm{NH}, \tag{70}$$

and using the fact that $\sum_{k=0}^{\mathrm{NH}} \delta_k = 0$, we can show that the function g_0^N defined by (67) satisfies the assumption (55). Hence, a discretely divergence free extension $\widetilde{g}_0^N \in V^N$ of g_0^N can be constructed. Furthermore, the functions $(I - P_c)\sigma_k$ are discretely divergence free and have the same boundary values on Γ as the functions σ_k. Therefore, the function

$$\widetilde{g}^N(x) := \widetilde{g}_0^N(x) - \sum_{k=1}^{\mathrm{NH}} \alpha_k (I - P_c)\sigma_k(x) \tag{71}$$

is a solution of problem (51).

For the practical implementation, the chain of elements associated with the function σ_k can be constructed by a marching process starting from an element located at Γ_{k-1} and going towards Γ_k where each next element that is added to the chain has to be adjacent to the previously chosen element.

5 Computation of the Pressure

Once the velocity $u^N \in V_0^N$ has been computed (e.g. by a multigrid algorithm), the pressure $p^N \in M_0^N$ can be calculated by the following post-processing procedure without solving any global system of equations. For all elements $K \in \mathcal{T}^h$, the local pressure $p^N\big|_K$ is decomposed uniquely into a constant $p_{0,K} \in \mathbb{R}$ and its orthogonal complement $\widetilde{p}_K \in L_0^2(K) \cap \mathbb{P}_{r_K-1}(K)$:

$$p^N(x) := p_{0,K} + \widetilde{p}_K(x) \qquad \forall\, x \in K\,,\ K \in \mathcal{T}^h\,. \tag{72}$$

In a first step, for each element K, the local pressure $\widetilde{p}_K$ can be determined from (9) by solving the local problem: *Find* $\widetilde{p}_K \in M_0^N(K) = \mathbb{P}_{r_K-1}(K) \cap L_0^2(K)$ *such that*

$$b(\varphi_i, \widetilde{p}_K) = b(\varphi_i, p^N) = (f, \varphi_i)_\Omega - a(u^N, \varphi_i) \qquad \forall\, i \in J_c(K)\,. \tag{73}$$

The stiffness matrix of this local problem is the $n_0^K \times n_0^K$ matrix $(-D^K)^T$ (see (16)) which is regular due to assumption (17).

Then, in a second step, all pressure constants $p_{0,K} \in \mathbb{R}$ for $K \in \mathcal{T}^h$ are determined. We introduce the splitting $p_{0,K} = c + \overline{p}_K$ where $c \in \mathbb{R}$ is an unknown global pressure constant which will be determined at the end. Now we compute the constants $\overline{p}_K$ for all elements $K \in \mathcal{T}^h$. We start by setting $\overline{p}_{K_0} = 0$ for an arbitrary element K_0 which could be located, for instance, at the exterior boundary of Ω. This setting means that the value of the unknown constant is $c = p_{0,K_0}$. Now, starting from K_0, we pass through all other elements by a marching strategy where the next element K is chosen adjacent to an element K' that has been passed already. Let $E \in \mathcal{E}_0^h$ be the common edge of K and K'. We choose in equation (9) as a test function $v_E^N(x) := b_E(x) n_E'$ where n_E' is the unit vector normal to E and directed outward with respect to K', This gives

$$\{\overline{p}_K - \overline{p}_{K'}\}|E| = (f, v_E^N)_\Omega - a(u^N, v_E^N) + (\nabla \cdot v_E^N, \widetilde{p}_K)_K + (\nabla \cdot v_E^N, \widetilde{p}_{K'})_{K'}\,. \tag{74}$$

From this equation the constant $\overline{p}_K$ can be computed since $\overline{p}_{K'}$ is known and all terms on the right hand side are computable. At the end, the global constant c is determined from the condition that the discrete pressure solution of (6) satisfies $p^N \in L_0^2(\Omega)$. This leads to

$$0 = (p^N, 1)_\Omega = \sum_{K \in \mathcal{T}^h} |K| \{c + \overline{p}_K\}\,,$$

where $|K|$ denotes the area of element K. Thus, once the constant c has been computed from this equation, the real pressure constants $p_{0,K}$ can be calculated from the previously computed constants $\overline{p}_K$.

6 Concluding Remarks

The details of the implementation of the discretely divergence free basis functions as well as the description and numerical study of a corresponding multigrid algorithm will be presented in a forthcoming paper.

Many of the presented ideas for the construction of a discretely divergence free basis can be applied also in the 3D case. In particular, as we will show in a forthcoming paper, we can construct a generating system for the space V_0^N, i.e. we can find local functions $\widetilde{\varphi}_j \in V_0^N$ such that $V_0^N = \operatorname{span}\{\widetilde{\varphi}_j : j \in \widetilde{J}\}$, but some of these functions $\widetilde{\varphi}_j$ are linearly dependent. However, it should be possible to create a multigrid method also for the case of a generating system. Such methods can be found for example in [5].

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