# A General Transfer Operator for Arbitrary Finite Element Spaces

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**Abstract**

We construct and analyze a transfer operator from any given (e.g. nonconforming) to an arbitrary desired (e.g. higher order conforming) finite element space. This transfer operator also defines a stable interpolation operator for element-wise smooth functions satisfying Dirichlet boundary conditions. It can be generalized in a natural way for rough $L^1$-functions. The practical computation of the transfered function can be implemented efficiently. For the transfer operator, we prove local and global stability estimates in the $L^2$-norm and the $H^1$-semi-norm. Furthermore, we prove local and global error estimates between the exact solution of some problem and the "post-processed" numerical solution computed by the action of the transfer operator applied to the discrete solution of the problem in the primary given finite element space. As applications we discuss a posteriori error estimation, graphical output and multigrid prolongation.

**Keywords:** nonconforming and conforming finite elements, interpolation operator, stability and error estimates, a posteriori error estimation, post-processing, multigrid prolongation

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## 1 Introduction

Transfer operators between finite element spaces are needed in several areas for the numerical solution of partial differential equations. One main application is their use within

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multigrid methods where a finite element function from a coarse space has to be transfered into a fine one. Whereas for nested spaces the transfered function is the original function in the new basis representation, some special care has to be taken in the case of non-nested spaces which occurs, for example, for nonconforming finite elements. In this paper, we propose a general transfer operator that is suited for any occurring case within the multigrid method. Despite of its generality this transfer operator can be implemented efficiently.

Other applications of transfer operators are associated with nonconforming finite elements. For mathematical reasons, in certain applications it is useful to weaken the continuity requirement of finite element approximations and to work with nonconforming finite elements. For example, in fourth order problems, one can avoid the necessity to work with $C^1$-smooth elements by using a nonconforming approach. Another example is the Stokes or Navier-Stokes problem where standard low-order conforming element pairs do not satisfy the Babuška-Brezzi stability condition (see [5]). Here the nonconforming $P_1/P_0$ triangular element pair of Crouzeix and Raviart (see [8]) satisfies that condition and provides an optimal order of convergence. The same is true for the analogue of this element pair in the case of quadrilateral elements which has been proposed by Rannacher and Turek (see [14]) and which has proved to be attractive for the numerical solution of the Navier-Stokes equations (see [15], [16], [19]). Moreover, an advantage of these nonconforming finite elements is that each degree of freedom belongs to at most two elements which simplifies the local communication for a parallelization of the method, particularly in the 3D case (see [15], [16]).

However, for practical reasons (e.g. graphical post-processing), one wants to have a finite element approximation of the solution which is "nice" in some sense , i.e. which is at least continuous. The obvious idea used in practice is to compute in a post-processing step from a solution $u_h^N$ in a given nonconforming finite element space $V_h^N$ an approximation $u_h^C$ in some desired conforming finite element space $V_h^C$.

In this paper, we define and analyze a general transfer operator $R_h : V_h^{(1)} \to V_h^{(2)}$ for an arbitrary pair of finite element spaces $V_h^{(1)}$ and $V_h^{(2)}$. Concerning the above mentioned application we have in mind, for example, that $V_h^{(1)}$ is the given nonconforming space $V_h^N$ and $V_h^{(2)}$ the desired conforming space $V_h^C$ which could be even a space of globally $C^1$-smooth functions. Our aim is to show that this operator provides a practically useful connection between the nonconforming and the conforming finite element world which opens up the possibility to work in that world that is most advantageous for the problem under consideration. The operator $R_h$ is based on averaging of local nodal functionals. The computation of the "smoothed function" $u_h^{(2)} = R_h u_h^{(1)}$ can be implemented easily and efficiently by one loop over the elements followed by one loop over the nodes corresponding to the space $V_h^{(2)}$.

The low computational costs associated with our operator $R_h$ are an advantage in comparison to the well-known interpolation operator proposed by Clément [7]. To compute the action of the Clément operator applied to a given function $v$, one has to consider for each node the patch of all elements that contain this node. Then, the $L^2$-projection of $v$ with respect to this patch has to be calculated which would be relatively expensive for practical use. In contrast to that, for our transfer operator, we do not need calculations on patches of elements. We only need the evaluation of the nodal functionals (associated with the target space $V_h^{(2)}$) applied to the given function $v$ on each single element combined with an averaging process. Moreover, we can generalize our transfer operator $R_h : V_h^{(1)} \to V_h^{(2)}$ to an interpolation operator $R_h^{(g)} : L^1(\Omega) \to V_h^{(2)}$ which has the same projection property as $R_h$, i.e. $R_h^{(g)} v = v$ for all $v \in V_h^{(2)}$. Thus, the $V_h^{(2)}$-function $R_h^{(g)} v$ is defined for the same class of "rough" functions $v$ as for the operator of Clément [7]. However, in contrast to the Clément operator, we need for the computation of $R_h^{(g)} v$ only an $L^2$-projection of $v$ on the reference element and not on patches of elements. Furthermore, $R_h^{(g)} v$ satisfies Dirichlet boundary conditions which is not the case for the Clément-operator.
An advantage of our operator $R_h$ compared to the Scott-Zhang operator [18] is that it is defined for a more general class of finite elements. The operator of Scott and Zhang is only defined for Lagrange-type finite elements since its construction exploits the fact that the global basis functions of $V_h^{(2)}$ restricted to a fixed $(d-1)$-dimensional element face $E$ (where $\Omega \subset \mathcal{R}^d$) are identically zero for all nodes $j$ which do not belong to $E$. Moreover, the Scott-Zhang operator is defined for functions $v$ with a weak but global smoothness, say for $v \in W^{\ell,p}(\Omega)$, such that by the trace theorem for each $(d-1)$-dimensional element face $E$, the function $v$ has a trace in $L^1(E)$. In contrast to that, our operator is defined for element-wise smooth functions with no global smoothness assumption.

In contrast to the analysis in [7] and [18], where the interpolation error between a given "rough" function $v$ and its interpolate $R_h v$ is estimated, we present here another point of view. Since our given rough function $v = u_h^{(1)}$ in general is not in $H_0^1(\Omega)$ and only element-wise smooth, we can not expect a high order of the interpolation error $u_h^{(1)} - R_h u_h^{(1)}$. Therefore, we prove local and global estimates for the error between the exact solution $u$

and the "smoothed" approximation $R_h u_h^{(1)}$ in the $L^2$-norm and the $H^1$-semi-norm. These estimates show that this error is controlled by the error of the primary used finite element method $u - u_h^{(1)}$ and the interpolation error of the exact solution $u$ by means of a desired finite element function $v_h^{(2)} \in V_h^{(2)}$. This means that the error of the post-processed discrete solution $u - R_h u_h^{(1)}$ is of a certain order if the primary finite element solution $u_h^{(1)}$ is of that order and the desired finite element space $V_h^{(2)}$ is rich enough to guarantee at least the same high order.

One "nice" feature of our analysis is that it is independent of the concrete problem which has to be solved by the finite element method. The only fact that we have to assume is that our computed primary finite element solution $u_h^{(1)} \in V_h^{(1)}$ is close to a sufficiently smooth solution $u$ of the continuous problem.

At the end of this paper we discuss some applications of the transfer operator $R_h$. The first one is the use of $R_h$ for a posteriori error estimation in the case of nonconforming finite elements. The computable quantities $\eta_K^{post}$, being defined as the local norms of the "post-processing error" $u_h^{(1)} - R_h u_h^{(1)}$ on the elements $K$, can be used for a posteriori error control. For a model problem, we present an a posteriori error estimate for the error $u - R_h u_h^{(1)}$ in the $H^1$-semi-norm which contains the usual quantities $\eta_K^{conf}$, that are well-known for the case of a conforming finite element space (see [20], [1]), and the quantities $\eta_K^{post}$. By the triangle inequality this also gives an a posteriori error estimate for the discretization error $u - u_h^{(1)}$ of the nonconforming finite element method. This estimate can be regarded as an alternative to existing a posteriori error estimates for the nonconforming case which have been proposed for instance in [10], [11], [12]. Moreover, the quantities $\eta_K^{post}$ can be used as local error indicators since they measure in some sense the local "smoothness" of the discrete solution $u_h^{(1)}$. The results concerning the a posteriori error control have been proved in a recent paper [17].

The second application is the use of $R_h$ for graphical output and the third one is its use as a prolongation operator within a multigrid method. For both applications, the definition of $R_h : V_h^{(1)} \to V_h^{(2)}$ is generalized to the case that the finite element space $V_h^{(2)}$ corresponds to a grid $\mathcal{T}_h^{(2)}$, which is derived from the original grid $\mathcal{T}_h$ of the space $V_h^{(1)}$ by a regular refinement process. We present a generalization of the original transfer operator $R_h$ that defines a general prolongation operator from any coarse level space to any fine level space in a multigrid method. The advantage of this prolongation operator is that it can be implemented on an abstract level in an object-oriented program only once for all possible cases of a coarse and a fine level space. Although such an implementation is very general it can be done in an efficient way.

# 2 Preliminaries and Notations

Let $\Omega \subset \mathcal{R}^d$, $d \in \{2,3\}$, be a bounded polyhedral domain and $\mathcal{T}_h$ a regular decomposition of $\Omega$ into elements $K \in \mathcal{T}_h$. The diameter of an element $K$ is denoted by $h_K$ and the elements are assumed to be shape regular in the usual sense (see e.g. [4]). Let us note that this assumption includes the case of a locally refined grid where the ratio $h/h_K$ with $h := \max_{K\in\mathcal{T}_h} h_K$ may be very large for elements $K$ in some critical subregions of $\Omega$.
We consider three finite element spaces defined on the same decomposition $\mathcal{T}_h$, a given primary (e.g. nonconforming) space $V_h^{(1)}$, some desired (e.g. conforming) space $V_h^{(2)}$ and an auxiliary space $V_h^A$. For the definition of these spaces, we use the reference transformation $F_K : \widehat{K} \to K$ that maps a fixed reference element $\widehat{K}$ onto the actual element $K \in \mathcal{T}_h$. We assume, for simplicity, that we have only one reference element $\widehat{K}$. However, our analysis can be generalized to the situation that we have a fixed finite set of reference elements which occurs for example if the decomposition of $\Omega$ contains both triangular and quadrilateral elements.
In the following, for a given domain $G \subset \mathcal{R}^d$, we denote by $\mathbb{P}_m(G)$ the set of all polynomials on $G$ with total degree less or equal to $m$.
Let us characterize the finite element space $V_h^{(i)}$, $i = 1,2$, as a subspace of a corresponding discontinuous space $\widetilde{V}_h^{(i)}$, i.e.

$$V_h^{(i)} \subset \widetilde{V}_h^{(i)} := \left\{ v \in L^2(\Omega) : \; v\big|_K \in \mathcal{P}_K^{(i)} \quad \forall\, K \in \mathcal{T}_h \right\} , \tag{1}$$

where

$$\mathcal{P}_K^{(i)} := \left\{ p = \widehat{p} \circ F_K^{-1} : \; \widehat{p} \in \widehat{\mathcal{P}}^{(i)} \right\} \qquad \text{with} \qquad \mathbb{P}_0(\widehat{K}) \subset \widehat{\mathcal{P}}^{(i)} \subset \mathbb{P}_{r_i}(\widehat{K}) \tag{2}$$

is the space of local shape functions of $V_h^{(i)}$ on element $K$ defined by some given reference space $\widehat{\mathcal{P}}^{(i)}$. A function $v_h \in \widetilde{V}_h^{(i)}$ belongs to $V_h^{(i)}$ if it is continuous with respect to the nodal functionals that are associated with $V_h^{(i)}$ and if it satisfies some desired boundary conditions.

Moreover, in this paper, we use as an auxiliary space $V_h^A$ the following discontinuous finite element space

$$V_h^A := \left\{ v \in L^2(\Omega) : \; v\big|_K \in \mathcal{P}_K^A \quad \forall\, K \in \mathcal{T}_h \right\} , \tag{3}$$

where

$$\mathcal{P}_K^A := \left\{ p = \widehat{p} \circ F_K^{-1} : \; \widehat{p} \in \widehat{\mathcal{P}}^A \right\} \qquad \text{with} \qquad \widehat{\mathcal{P}}^{(1)} + \widehat{\mathcal{P}}^{(2)} \subset \widehat{\mathcal{P}}^A . \tag{4}$$

A possible choice for $\widehat{\mathcal{P}}^A$ would be for instance $\widehat{\mathcal{P}}^A = \mathbb{P}_{r_3}(\widehat{K})$ with $r_3 = \max(r_1, r_2)$. By construction this auxiliary space has the property

$$V_h^{(1)} + V_h^{(2)} \subset V_h^A \tag{5}$$

which will be exploited in the subsequent analysis.

Finally, let us introduce some general notations. We denote by $\mathrm{card}(\mathcal{M})$ the number of elements of a finite set $\mathcal{M}$. Let $Y'$ be the dual space of a given space $Y$. For a set $G \subset \mathcal{R}^d$, we denote by $\mathrm{int}(G)$ and $\overline{G}$ the interior and closure of $G$, respectively. Further, let $\|\cdot\|_{m,p,G}$ and $|\cdot|_{m,p,G}$ denote the usual norm and semi-norm in the Sobolev space $W^{m,p}(G)$ where we omit the index $p$ if $p = 2$. We use the notation $C$ for a general constant which may have different values at different occurrences but which is independent of the mesh parameters $h_K$ of the elements $K \in \mathcal{T}_h$.

# 3 A General Transfer Operator

For the definition of the transfer operator we need a more detailed characterization of the desired finite element space $V_h^{(2)}$ by means of its basis functions and nodal functionals. Let $\{\varphi_j^{(2)} : \; j \in J^{(2)}\}$ be the finite element basis of $V_h^{(2)}$. The indices $j \in J^{(2)}$ are called “nodes”. For a given element $K \in \mathcal{T}_h$, we denote by $J^{(2)}(K)$ the set of all nodes $j \in J^{(2)}$ that are associated with $K$, i.e.

$$J^{(2)}(K) := \left\{ j \in J^{(2)} : \; \mathrm{supp}(\varphi_j^{(2)}) \cap \mathrm{int}(K) \neq \emptyset \right\} . \tag{6}$$

The element nodes $j \in J^{(2)}(K)$ are associated with a set of local basis functions $\varphi_j^{(2)}\big|_K$ and a set of local nodal functionals $N_{j,K} \in (C^s(\overline{K}))'$ which are “dual” (see [4]) in the sense that

$$N_{i,K}(\varphi_j^{(2)}\big|_K) = \delta_{ij} \qquad \forall\, i, j \in J^{(2)}(K), \tag{7}$$

where $\delta_{ij}$ denotes the Kronecker delta. Note that we simply write $N_{i,K}$ instead of $N_{i,K}^{(2)}$ since we do not consider nodal functionals for the finite element spaces $V_h^{(1)}$ and $V_h^A$. The parameter $s$ depends on the type of the nodal functionals and corresponds to the highest order of derivatives that have to be evaluated for a function $v$ in order to define $N_{j,K}(v)$. Typical examples for nodal functionals in the case $s = 0$ are point values $N_{j,K}(v) := v_{|K}(a_j)$, where $a_j \in \overline{K}$ is a given point, or integral mean values $N_{j,K}(v) := |\Gamma_j|^{-1} \int_{\Gamma_j} v\big|_K \omega_j \, d\gamma$, where $\Gamma_j$ is a face of element $K$ with the $(d-1)$-dimensional measure $|\Gamma_j|$ and $\omega_j$ a given polynomial weighting function.

Now, by means of the local nodal functionals $N_{j,K}$, we define global functionals $N_j$ on a general space $Y(\mathcal{T}_h, s)$ of piecewise smooth functions given by

$$Y(\mathcal{T}_h, s) := \left\{ v \in L^2(\Omega) : \; v\big|_K \in C^s(\overline{K}) \quad \forall\, K \in \mathcal{T}_h \right\}. \tag{8}$$

For a node $j \in J^{(2)}$, let

$$\mathcal{T}_{h,j} := \left\{ K \in \mathcal{T}_h : \; j \in J^{(2)}(K) \right\} \tag{9}$$

be the set of all elements that are associated with this node. Then, as a natural choice for $N_j \in (Y(\mathcal{T}_h, s))'$, we define the value $N_j(v)$ as a weighted mean value of all local values $N_{j,K}(v_{|K})$ that are associated with the node $j \in J^{(2)}$, i.e.

$$N_j(v) := (\omega_{\Sigma,j})^{-1} \sum_{K \in \mathcal{T}_{h,j}} \omega_{j,K} N_{j,K}(v\big|_K) \qquad \forall\, v \in Y(\mathcal{T}_h, s) \tag{10}$$

with

$$\omega_{\Sigma,j} := \sum_{K \in \mathcal{T}_{h,j}} \omega_{j,K}\,, \tag{11}$$

where the quantities $\omega_{j,K}$ are some user-chosen positive weights. There are at least two practically useful choices. The first one is $\omega_{j,K} = 1$ for all $j$ and $K$, where an isotropic "spread of information" is guaranteed. For the second one, there is exactly one weight equal to one, say $\omega_{j,K_j^*}$ corresponding to an arbitrary chosen element $K_j^* \in \mathcal{T}_{h,j}$, and the other weights $\omega_{j,K}$ for all $K \in \mathcal{T}_{h,j} \backslash \{K_j^*\}$ are set to zero. For this choice, the computational costs for computing the values $N_j(v)$, $j \in J^{(2)}$, are lower than for the first choice. By means of these global nodal functionals we define a general transfer operator $R_h : Y(\mathcal{T}_h, s) \to V_h^{(2)}$ as

$$R_h v := \sum_{j \in J^{(2)}} N_j(v)\, \varphi_j^{(2)}. \tag{12}$$

Note that, by (12), $R_h v$ is defined in particular for all $v \in V_h^A$ since by definition $V_h^A$ is a subspace of $Y(\mathcal{T}_h, s)$. In the following, we only need the restriction of $R_h$ to $V_h^A$ and denote this operator again by $R_h : V_h^A \to V_h^{(2)}$.

The computation of the function $v = R_h u \in V_h^{(2)}$ for a given piecewise smooth function $u \in Y(\mathcal{T}_h, s)$ can be implemented easily and efficiently in the following way. The task is to compute the vector $\underline{v} = (\underline{v}_j)_{j \in J^{(2)}}$ of the nodal values $\underline{v}_j$ in the basis representation $v = R_h u = \sum_{j \in J^{(2)}} \underline{v}_j \varphi_j^{(2)}$. One needs an auxiliary vector $\underline{c} = (\underline{c}_j)_{j \in J^{(2)}}$ and the functionality to compute the values of the local nodal functionals $N_{j,K}(u)$ for given $u \in Y(\mathcal{T}_h, s)$, $j \in J^{(2)}$ and $K \in \mathcal{T}_h$. Then, the algorithm to compute the vector $\underline{v}$ is the following.

<u>COMPUTATION OF $\underline{v} = (\underline{v}_j)$ FOR $v = R_h u = \sum_{j \in J^{(2)}} \underline{v}_j \varphi_j^{(2)}$ :</u>

(a) initialize $\underline{v} := 0\,;\ \ \underline{c} := 0\,;$

(b) **for** $K \in \mathcal{T}_h$ **do**
  **for** $j \in J^{(2)}(K)$ **do**
    $\underline{v}_j := \underline{v}_j \,+\, N_{j,K}(u)\,;$
    $\underline{c}_j := \underline{c}_j \,+\, \omega_{j,K}\,;$
  **enddo**
**enddo**

(c) **for** $j \in J^{(2)}$ **do**
  $\underline{v}_j := \underline{v}_j / \underline{c}_j\,;$
**enddo**

In the case that $u \in V_h^{(1)}$, the computation of the nodal functionals $N_{j,K}(u)$ can be done by means of the basis representation $u = \sum_{i \in J^{(1)}} \underline{u}_i \varphi_i^{(1)}$ where $\varphi_i^{(1)}$, $i \in J^{(1)}$, are the basis functions of $V_h^{(1)}$ and $\underline{u}_i \in \mathcal{R}$ the nodal values of $u$. For $K \in \mathcal{T}_h$, let $J^{(1)}(K)$ denote the index set of all $V_h^{(1)}$-nodes associated with element $K$. Then we have

$$N_{j,K}(u) = \sum_{i \in J^{(1)}(K)} \underline{u}_i N_{j,K}(\varphi_i^{(1)}) \qquad \forall\, j \in J^{(2)}(K)\,. \tag{13}$$

In many cases, the values $N_{j,K}(\varphi_i^{(1)})$ do not change for all elements $K$ of the same class. Thus, they can be computed and stored in a pre-processing step which simplifies and accelerates the above described computation of $R_h u$.

**Remark 1** *An example of our operator $R_h : V_h^{(1)} \to V_h^{(2)}$ can be found in [13]. Here the space $V_h^{(1)}$ is the space of nonconforming piecewise linear finite elements on a triangular grid $\mathcal{T}_h$ (see [8]) and $V_h^{(2)}$ the space of conforming piecewise linear finite elements on the same grid. For an element $K \in \mathcal{T}_h$, the local nodal functionals $N_{j,K}$ corresponding to $V_h^{(2)}$ are the point values at the vertices $a_j$ of the element $K$, i.e. $N_{j,K}(v) := v\big|_K(a_j)$. Then, the formulas (25) and (26) in [13] for the definition of the transfer operator between the spaces $V_h^{(1)}$ and $V_h^{(2)}$ are equivalent to our definition (9) – (12).*

The next lemma shows the projection property of the operator $R_h$, i.e. that $R_h$ is the identity on the subspace $V_h^{(2)}$ of $V_h^A$, which is an essential property for our analysis.

**Lemma 2** *For the transfer operator $R_h$ defined in (12) it holds*

$$R_h v = v \qquad \forall\, v \in V_h^{(2)}. \tag{14}$$

**Proof.** Suppose that $v \in V_h^{(2)}$ has the basis representation $v = \sum_{j \in J^{(2)}} \underline{v}_j\, \varphi_j^{(2)}$ with $\underline{v}_j \in \mathcal{R}$. The property $\varphi_j^{(2)}\big|_K \equiv 0$ for all $j \in J^{(2)} \setminus J^{(2)}(K)$ implies the following local representation of $v$

$$v\big|_K = \sum_{j \in J^{(2)}(K)} \underline{v}_j\, \varphi_j^{(2)}\big|_K \qquad \forall\, K \in \mathcal{T}_h.$$

Applying the functional $N_{j,K}$ to both sides of this equation and using (7) leads to

$$N_{j,K}(v\big|_K) = \underline{v}_j \qquad \forall\, K \in \mathcal{T}_{h,j}.$$

Thus, by the definition (10) of $N_j$ we obtain $N_j(v) = \underline{v}_j$. Using the local representation of $R_h v$, we get for arbitrary $K \in \mathcal{T}_h$

$$R_h v\big|_K = \sum_{j \in J^{(2)}(K)} N_j(v)\, \varphi_j^{(2)}\big|_K = \sum_{j \in J^{(2)}(K)} \underline{v}_j\, \varphi_j^{(2)}\big|_K = v\big|_K$$

which proves (14). ∎

We can generalize our operator $R_h$ to an operator $R_h^{(g)} : L^1(\Omega) \to V_h^{(2)}$ in the following way. The idea is similar to the approach of Clément [7], i.e. a function $v \in L^1(\Omega)$, which is not smooth enough to apply the local nodal functionals $N_{j,K}(\cdot)$ associated with the space $V_h^{(2)}$, at first will be regularized on each element $K \in \mathcal{T}_h$ by a projection to the local space $\mathcal{P}_K^{(2)}$ (see (2)) of polynomial shape functions of $V_h^{(2)}$. Concerning the practical computation, note that our approach with an element-wise projection is much cheaper than that of Clément since for the computation of the Clément operator one has to calculate for each node an $L^2$-projection on a patch of elements related to this node. In order to guarantee an efficient implementation we will use in our approach an $L^2$-projection corresponding to the reference element $\widehat{K}$ instead of the original element $K$. Let us recall that $F_K : \widehat{K} \to K$ denotes the reference transformation which maps the fixed reference element $\widehat{K}$ onto the actual element $K \in \mathcal{T}_h$. For a given function $v \in L^1(K)$, we associate the function $\widehat{v} \in L^1(\widehat{K})$ defined by $\widehat{v} := v \circ F_K$. Let us assume that the basis functions of the reference space $\widehat{\mathcal{P}}^{(2)}$ (see (2)) are $\{\, \widehat{\varphi}_\mu^{(2)} : \mu \in \widehat{J}^{(2)}\}$ and that for a local node $j \in J^{(2)}(K)$, which is associated with the local basis function $\varphi_j^{(2)}\big|_K \in \mathcal{P}_K^{(2)}$, the “reference node” $\widehat{j} \in \widehat{J}^{(2)}$ is defined such that

$$\widehat{\varphi}_{\widehat{j}}^{(2)} = \widehat{\varphi_j^{(2)}\big|_K} \qquad \forall\, j \in J^{(2)}(K)\,. \tag{15}$$

Now we define the projection operator $\widehat{\pi} : L^1(\widehat{K}) \to \widehat{\mathcal{P}}^{(2)}$ by

$$\int_{\widehat{K}} (\widehat{\pi}\widehat{w})\, \widehat{\varphi}\, d\widehat{x} \;=\; \int_{\widehat{K}} \widehat{w}\widehat{\varphi}\, d\widehat{x} \qquad \forall\, \widehat{\varphi} \in \widehat{\mathcal{P}}^{(2)}\,. \tag{16}$$

Since $\widehat{\mathcal{P}}^{(2)} \subset L^\infty(\widehat{K})$, the projection $\widehat{v} := \widehat{\pi}\widehat{w}$ is uniquely defined for each $\widehat{w} \in L^1(\widehat{K})$. Its practical computation can be done as follows. Let $\widehat{G} = (\widehat{G}_{\nu,\mu}) \in \mathcal{R}^{\widehat{n}\times\widehat{n}}$ with $\widehat{n} := \mathrm{card}(\widehat{\mathcal{P}}^{(2)})$ denote the Gram-matrix defined by $\widehat{G}_{\nu,\mu} := \int_{\widehat{K}} \widehat{\varphi}_\mu^{(2)}\, \widehat{\varphi}_\nu^{(2)}\, d\widehat{x}$ and $\widehat{G}^{-1} = \big((\widehat{G}^{-1})_{\nu,\mu}\big)$ its inverse which can be computed for a practical implementation once only in a pre-processing step. Then, a simple calculation yields the formula

$$\widehat{\pi}\widehat{w} = \sum_{i\in\widehat{J}^{(2)}} \underline{\widehat{v}}_i\, \widehat{\varphi}_i^{(2)} \qquad \text{with} \qquad \underline{\widehat{v}}_i := \sum_{\mu\in\widehat{J}^{(2)}} (\widehat{G}^{-1})_{i,\mu} \int_{\widehat{K}} \widehat{w}\, \widehat{\varphi}_\mu^{(2)}\, d\widehat{x}\,. \tag{17}$$

Now, for each element $K \in \mathcal{T}_h$, we define the projection $\pi_K : L^1(K) \to \mathcal{P}_K^{(2)}$ via the operator $\widehat{\pi}$ by

$$(\pi_K w)(x) := (\widehat{\pi}\widehat{w})(\widehat{x}) \qquad \forall\, x \in K\,, \quad \widehat{x} := F_K^{-1}(x)\,, \tag{18}$$

where $\widehat{w} := w \circ F_K$ is the “reference function” to $w$. If we assume the basis representation $\widehat{v} := \widehat{\pi}\widehat{w} = \sum_{\mu\in\widehat{J}^{(2)}} \underline{\widehat{v}}_\mu \widehat{\varphi}_\mu^{(2)}$ with some nodal values $\underline{\widehat{v}}_\mu \in \mathcal{R}$ then it follows that $v = \pi_K w = \sum_{j\in J^{(2)}(K)} \underline{\widehat{v}}_{\widehat{j}}\, \varphi_j^{(2)}$ where $\widehat{j}$ denotes the reference node to $j \in J^{(2)}(K)$ defined by (15). Using the duality relation (7) of the local nodal functionals and formula (17) we obtain

$$N_{j,K}(\pi_K w) = \sum_{\mu \in \widehat{J}^{(2)}} (\widehat{G}^{-1})_{\widehat{j},\mu} \int_{\widehat{K}} \widehat{w}\, \widehat{\varphi}_\mu^{(2)}\, d\widehat{x}\,, \tag{19}$$

where again $\widehat{j}$ denotes the reference node to $j$ and $\widehat{w} := w \circ F_K$.

By means of the operator $\pi_K$ we define for each local node $j \in J^{(2)}(K)$ the generalized local nodal functional $N_{j,K}^{(g)} \in (L^1(K))'$ from the original nodal functional $N_{j,K} \in (C^s(\overline{K}))'$ by

$$N_{j,K}^{(g)}(w) := N_{j,K}(\pi_K w) \qquad \forall\, w \in L^1(K)\,. \tag{20}$$

For a global node $j \in J^{(2)}$, these local functionals form a global nodal functional $N_j^{(g)} \in (L^1(\Omega))'$ by means of

$$N_j^{(g)}(v) := (\omega_{\Sigma,j})^{-1} \sum_{K \in \mathcal{T}_{h,j}} \omega_{j,K} N_{j,K}^{(g)}(v\big|_K) \qquad \forall\, v \in L^1(\Omega)\,, \tag{21}$$

where the $\omega_{j,K}$ are some user-chosen positive weights and $\omega_{\Sigma,j}$ is defined by (11). Finally, we define our generalized operator $R_h^{(g)} : L^1(\Omega) \to V_h^{(2)}$ as

$$R_h^{(g)} v := \sum_{j \in J^{(2)}} N_j^{(g)}(v)\, \varphi_j^{(2)} \qquad \forall\, v \in L^1(\Omega)\,. \tag{22}$$

Note that, if the weights $\omega_{j,K}$ in (10) and (21) are the same, then it holds $N_j^{(g)}(v) = N_j(v)$ for all $v \in V_h^{(2)}$ since $\pi_K w = w$ for all $w = v\big|_K \in \mathcal{P}_K^{(2)}$. Therefore, the generalized operator $R_h^{(g)}$ maintains the projection property of Lemma 2 that $R_h^{(g)} v = v$ for all $v \in V_h^{(2)}$.

# 4 Local $L^2$-Stability

In this section, we present a local $L^2$-stability estimate of the transfer operator $R_h : V_h^A \to V_h^{(2)}$ which plays an essential role in the subsequent analysis. Let us first introduce some notation. We denote by $\{\varphi_j^A : j \in J^A\}$ the finite element basis of $V_h^A$ and by $J^A(K)$ the set of all nodes $j \in J^A$ that are associated with element $K \in \mathcal{T}_h$ in an analogous manner as in (6). For a given element $K$, let

$$\delta(K) := \bigcup_{j \in J^{(2)}(K)} \bigcup_{\widetilde{K} \in \mathcal{T}_{h,j}} \widetilde{K} \tag{23}$$

be the vicinity of $K$ consisting of all elements $\widetilde{K} \in \mathcal{T}_h$ that have a $V_h^{(2)}$-node in common with $K$, i.e. $J^{(2)}(K) \cap J^{(2)}(\widetilde{K}) \neq \emptyset$.

In the following, we list some basic assumptions that are needed for the proof of our stability estimate. For the basis functions of $V_h^{(2)}$, we assume that there is a constant $C_1$ such that

$$\textbf{(A1)} \qquad \|\varphi_j^{(2)}\|_{0,K} \leq C_1 h_K^{d/2} \qquad \forall\, j \in J^{(2)}(K),\; K \in \mathcal{T}_h.$$

Furthermore, we suppose that the values of the local nodal functionals of the space $V_h^{(2)}$ applied to the basis functions of $V_h^A$ are uniformly bounded, i.e. that there exists a constant $C_2$ such that

$$\textbf{(A2)} \qquad \big|N_{j,K}(\varphi_i^A\big|_K)\big| \leq C_2 \qquad \forall\, j \in J^{(2)}(K),\; i \in J^A(K),\; K \in \mathcal{T}_h.$$

Finally, we require that changes of the mesh size within the vicinity $\delta(K)$ of an element $K$ are uniformly bounded by a constant factor $C_3$, i.e.

$$C_3^{-1}\, h_{\widetilde{K}} \leq h_K \leq C_3\, h_{\widetilde{K}} \qquad \forall\, \widetilde{K} \subset \delta(K), \quad \widetilde{K}, K \in \mathcal{T}_h. \tag{24}$$

These assumptions are not restrictive since (A1) and (A2) are fulfilled for the usual finite element spaces and (24) is satisfied if the mesh $\mathcal{T}_h$ consists of shape regular elements.

Now we can prove the following local $L^2$-stability estimate of the operator $R_h$.

**Lemma 3** *Suppose that the spaces $V_h^{(2)}$ and $V_h^A$ fulfill the assumptions (A1) and (A2) and that the mesh $\mathcal{T}_h$ consists of shape regular elements such that (24) is satisfied. Then, for the transfer operator $R_h : V_h^A \to V_h^{(2)}$ defined in (12), there exists a constant $C_S$ such that*

$$\|R_h v\|_{0,K} \leq C_S\, \|v\|_{0,\delta(K)} \qquad \forall\, v \in V_h^A, \quad K \in \mathcal{T}_h\,. \tag{25}$$

**Proof.** Let $v \in V_h^A$ and $K \in \mathcal{T}_h$ be given. From the local representation of $R_h v$ on $K$ and the estimate (A1) we obtain

$$\|R_h v\|_{0,K} = \|\sum_{j \in J^{(2)}(K)} N_j(v)\, \varphi_j^{(2)}\, \|_{0,K} \;\leq\; C_1 h_K^{d/2} \sum_{j \in J^{(2)}(K)} |N_j(v)|. \tag{26}$$

Let $v = \sum_{i \in J^A} \underline{v}_i\, \varphi_i^A$ be the basis representation of $v$ and $\underline{v} = (\underline{v}_i)_{i \in J^A} \in \mathcal{R}^{J^A}$ the corresponding coefficient vector. Then, from the local representation of $v$ we get

$$N_{j,\widetilde{K}}(v\big|_{\widetilde{K}}) = \sum_{i \in J^A(\widetilde{K})} \underline{v}_i\, N_{j,\widetilde{K}}(\varphi_i^A\big|_{\widetilde{K}}) \qquad \forall\, \widetilde{K} \in \mathcal{T}_{h,j}.$$

For each $\widetilde{K} \in \mathcal{T}_h$, let

$$\|\underline{v}\|_{\ell^2(\widetilde{K})} := \Big( \sum_{i \in J^A(\widetilde{K})} \underline{v}_i^2 \Big)^{1/2}$$

be the Euclidean norm of the local coefficient vector associated with the element $\widetilde{K}$. Using (A2) and the fact that $\mathrm{card}(J^A(\widetilde{K})) \leq d_A := \dim(\widehat{\mathcal{P}}^A)$ for all $\widetilde{K} \in \mathcal{T}_h$, we get

$$\big|N_{j,\widetilde{K}}(v\big|_{\widetilde{K}})\big| \leq C_2\, d_A^{1/2}\, \|\underline{v}\|_{\ell^2(\widetilde{K})} \qquad \forall\, \widetilde{K} \in \mathcal{T}_{h,j}. \tag{27}$$

By well-known arguments as the transformation onto reference element and equivalence of norms on finite dimensional spaces (see e.g. [6]) we can show that there exists a constant $C_4$ such that

$$\|\underline{v}\|_{\ell^2(\widetilde{K})} \leq C_4\, h_{\widetilde{K}}^{-d/2} \|v\|_{0,\widetilde{K}} \qquad \forall\, \widetilde{K} \in \mathcal{T}_h. \tag{28}$$

Combining (28) with (27) and (24), we obtain for each $j \in J^{(2)}(K)$ the estimate

$$\big|N_{j,\widetilde{K}}(v\big|_{\widetilde{K}})\big| \leq C\, h_K^{-d/2} \|v\|_{0,\widetilde{K}} \qquad \forall\, \widetilde{K} \in \mathcal{T}_{h,j}$$

with the constant $C = C_2\, d_A^{1/2} C_4\, C_3^{d/2}$. The fact that $\widetilde{K} \subset \delta(K)$ for all $\widetilde{K} \in \mathcal{T}_{h,j}$, where $j \in J^{(2)}(K)$, and the definition (10) of $N_j(v)$ imply

$$|N_j(v)| \leq C\, h_K^{-d/2} \|v\|_{0,\delta(K)} \qquad \forall\, j \in J^{(2)}(K).$$

Together with (26) this proves (25). ■

**Remark 4** *We can prove the same results of Lemma 3 if we replace the operator* $R_h : V_h^A \to V_h^{(2)}$ *by the generalized operator* $R_h^{(g)} : V_h^A \to V_h^{(2)}$ *defined as the restriction of* $R_h^{(g)}$ *given in (22) to the subspace* $V_h^A \subset L^1(\Omega)$. *For the proof, we have to replace the local nodal functionals* $N_{j,K}(\cdot)$ *by* $N_{j,K}^{(g)}(\cdot)$ *defined in (20). From (19) we see that the assumption (A2) for the functionals* $N_{j,K}^{(g)}(\cdot)$ *will be satisfied for a very general class of spaces* $V_h^A$.

# 5 Error Estimates

## 5.1 $L^2$-Estimates

Using the local $L^2$-stability of $R_h$ and the fact that this operator is the identity on the subspace $V_h^{(2)}$ of $V_h^A$, we can prove the following local error estimate.

**Theorem 5** *Assume that the elements $K \in \mathcal{T}_h$ are shape regular such that (24) is satisfied. Furthermore, suppose that the spaces $V_h^{(2)}$ and $V_h^A$ fulfill the assumptions (A1) and (A2). Let $u \in L^2(\Omega)$ be the exact solution of some problem and $u_h^{(1)} \in V_h^{(1)}$ the finite element solution of a corresponding discrete problem. Then, for the approximation $R_h u_h^{(1)} \in V_h^{(2)}$ of $u_h^{(1)}$, it holds the error estimate*

$$\|u - R_h u_h^{(1)}\|_{0,K} \leq C \left\{ \|u - u_h^{(1)}\|_{0,\delta(K)} + \inf_{v_h^{(2)} \in V_h^{(2)}} \|u - v_h^{(2)}\|_{0,\delta(K)} \right\} \tag{29}$$

*for each element $K \in \mathcal{T}_h$ with a constant $C$ independent of $K$.*

**Proof.** Let $v_h^{(2)}$ be an arbitrary element of $V_h^{(2)}$ and $K \in \mathcal{T}_h$. Using (5) we have $u_h^{(1)} - v_h^{(2)} \in V_h^A$ and the stability estimate (25) yields

$$\|R_h(u_h^{(1)} - v_h^{(2)})\|_{0,K} \leq C_S \|u_h^{(1)} - v_h^{(2)}\|_{0,\delta(K)} \,.$$

With $R_h v_h^{(2)} = v_h^{(2)}$ we obtain

$$\begin{aligned} \|u - R_h u_h^{(1)}\|_{0,K} &= \|(u - v_h^{(2)}) - R_h(u_h^{(1)} - v_h^{(2)})\|_{0,K} \\ &\leq \|u - v_h^{(2)}\|_{0,K} + C_S \|u_h^{(1)} - v_h^{(2)}\|_{0,\delta(K)} \\ &\leq (1 + C_S)\|u - v_h^{(2)}\|_{0,\delta(K)} + C_S \|u - u_h^{(1)}\|_{0,\delta(K)} \end{aligned}$$

which implies the estimate (29). ∎

From this local $L^2$-estimate we immediately get the following global one.

**Theorem 6** *Under the assumptions of Theorem 5, it holds the error estimate*

$$\|u - R_h u_h^{(1)}\|_{0,\Omega} \leq C \left\{ \|u - u_h^{(1)}\|_{0,\Omega} + \inf_{v_h^{(2)} \in V_h^{(2)}} \|u - v_h^{(2)}\|_{0,\Omega} \right\} . \tag{30}$$

**Proof.** Let $v_h^{(2)}$ be a fixed element of $V_h^{(2)}$. Then, from (29) we obtain for each $K \in \mathcal{T}_h$

$$\|u - R_h u_h^{(1)}\|_{0,K}^2 \leq C \sum_{\widetilde{K} \subset \delta(K)} \left\{ \|u - u_h^{(1)}\|_{0,\widetilde{K}}^2 + \|u - v_h^{(2)}\|_{0,\widetilde{K}}^2 \right\} \tag{31}$$

Using the shape regularity of the elements, it follows by geometrical arguments that there is a constant $C_5$ such that

$$\max_{\widetilde{K} \in \mathcal{T}_h} \operatorname{card} \Big( \{ K \in \mathcal{T}_h : \widetilde{K} \subset \delta(K) \} \Big) \leq C_5 \,. \tag{32}$$

Then, summation of (31) for all $K \in \mathcal{T}_h$ yields

$$\|u - R_h u_h^{(1)}\|_{0,\Omega}^2 \leq C C_5 \left\{ \|u - u_h^{(1)}\|_{0,\Omega}^2 + \|u - v_h^{(2)}\|_{0,\Omega}^2 \right\} .$$

Since $v_h^{(2)}$ can be any element of $V_h^{(2)}$, we obtain (30). ■

## 5.2 $H^1$-Estimates

In this section, we want to present error estimates in a discrete $H^1$-semi-norm which is a natural generalization of the usual $H^1$-semi-norm to the case of element-wise smooth functions. For a subdomain $G \subset \Omega$, let

$$\mathcal{T}_h(G) := \{K \in \mathcal{T}_h : \ K \subset G\} \tag{33}$$

be the set of all elements belonging to $G$. Then, for $G \subset \Omega$ with $\overline{G} = \bigcup_{K \in \mathcal{T}_h(G)} \overline{K}$ and for functions $v \in V_h^A + H^1(\Omega)$, we define the discrete $H^1$-semi-norm by

$$|v|_{1,h,G} := \Big( \sum_{K \in \mathcal{T}_h(G)} |v|_{1,K}^2 \Big)^{1/2}. \tag{34}$$

In contrast to $L^2$-estimates, here we have to assume at least the following weak continuity condition for functions of the finite element spaces $V_h^{(1)}$ and $V_h^{(2)}$

**(A3)** $\qquad \int_{\partial K \cap \partial K'} \left( v_h\big|_K - v_h\big|_{K'} \right) ds = 0 \qquad \forall\, v_h \in V_h^{(1)} + V_h^{(2)}, \quad K, K' \in \mathcal{T}_h \,,$

where the integral in (A3) has to be understood as a $(d-1)$-dimensional integral. On the other hand, this condition is a minimal requirement to ensure consistency for a finite element discretization, i.e. it is a very natural assumption. Furthermore, for our proof of $H^1$-estimates, we have to take into account the boundary conditions that are imposed on the discrete spaces $V_h^{(1)}$ and $V_h^{(2)}$. For simplicity, we consider here the case of homogeneous Dirichlet boundary conditions, i.e. we suppose $V_h^{(i)} \approx H_0^1(\Omega)$, $i = 1, 2$, in the following weak sense

**(A4)** $\qquad \int_{\partial K \cap \partial \Omega} v_h \, ds = 0 \qquad \forall\, v_h \in V_h^{(1)} + V_h^{(2)}, \quad K \in \mathcal{T}_h.$

The basic tool for deriving $H^1$-error estimates for the approximation $R_h u_h^{(1)} \in V_h^{(2)}$ of a finite element solution $u_h^{(1)} \in V_h^{(1)}$ is again a local stability estimate of the transfer operator $R_h$ which is presented in the next lemma.

At first, we have to define a new local vicinity $\delta'(K)$ of a given element $K$ which differs from the vicinity $\delta(K)$ defined in (23) only if $K$ is located near the boundary $\partial\Omega$. Two different elements $K, K' \in \mathcal{T}_h$ are called "face-neighbors" if $\partial K \cap \partial K'$ is a common $(d-1)$-dimensional face of $K$ and $K'$. For $K \in \mathcal{T}_h$, let $\gamma^* := \partial\Omega \cap \overline{\delta(K)}$ be a non-empty set of points with a dimension less than $d-1$, i.e. a vertex or, in the 3D case, an edge of the element $K$. Furthermore, in this case, let $K^* \in \mathcal{T}_h(\delta(K))$ be an element in $\delta(K)$ with $\gamma^* \subset \overline{K^*}$. Then, there exists a chain of elements $\kappa_i \in \mathcal{T}_h$, $i = 1, \ldots, m_K$, such that $\kappa_1 := K^*$, $\gamma^* \subset \overline{\kappa_i}$ for all $1 \le i \le m_K$, $\kappa_i$ and $\kappa_{i+1}$ are face-neighbors for all $1 \le i < m_K$ if $m_K > 1$ and the "final chain element" $K^b := \kappa_{m_K}$ has a $(d-1)$-dimensional boundary face $\Gamma^b \subset \partial K^b \cap \partial\Omega$ with $\gamma^* \subset \Gamma^b$. Now, we define

$$\delta'(K) := \delta(K) \cup \bigcup_{i=1}^{m_K} \kappa_i \tag{35}$$

as the new vicinity of $K$ in the above defined case where $K$ is in some sense near to the boundary $\partial\Omega$. In all other cases concerning the location of $K$, we define $\delta'(K) := \delta(K)$. Due to the shape-regularity of the grid $\mathcal{T}_h$, all the numbers $m_K$ are uniformly bounded, i.e. $m_K \le C$ for all $K \in \mathcal{T}_h$, and furthermore we have, similarly to (24), that there exists a constant $C_3'$ such that

$$(C_3')^{-1}\, h_{\widetilde{K}} \le h_K \le C_3'\, h_{\widetilde{K}} \qquad \forall\, \widetilde{K} \subset \delta'(K), \quad \widetilde{K}, K \in \mathcal{T}_h. \tag{36}$$

**Lemma 7** *Let $V_h^{(i)}$, $i = 1, 2$, be subspaces of $\widetilde{V}_h^{(i)}$ defined by (1) such that the assumptions (A1), (A2), (A3) and (A4) are fulfilled. The mesh $\mathcal{T}_h$ is supposed to consist of shape regular elements such that (36) is satisfied. Then, for the transfer operator $R_h$ defined in (12), there exists a constant $C_S'$ such that*

$$|R_h v|_{1,K} \le C_S'\, |v|_{1,h,\delta'(K)} \qquad \forall\, v \in V_h^{(1)} + V_h^{(2)}, \quad K \in \mathcal{T}_h\,. \tag{37}$$

**Proof.** Let $v \in V_h^{(1)} + V_h^{(2)}$ and $K \in \mathcal{T}_h$ be fixed. The proof of (37) consists of 4 steps.
**Step 1.** We construct a function $c_K \in V_h^{(2)}$ which is a constant approximation of $v$ in the vicinity $\delta'(K)$ of the element $K$ (defined as above). More precisely, let

$$c_K(x) = v\big|_{K^*}(z_{K^*}) \qquad \forall\, x \in \overline{\delta'(K)}, \tag{38}$$

where $K^* \in \mathcal{T}_h(\delta'(K))$ is a properly chosen element and $z_{K^*}\ in\overline{K^*}$ a suitable point defined as follows.

In the case, when $\delta'(K)$ has a positive distance to the boundary $\partial\Omega$, i.e. when $\overline{\delta'(K)} \subset \mathrm{int}(\Omega)$, we choose $K^* = K$ and $z_{K^*}$ as the barycentre of $K$. It is clear that in this case there exists an extension (by zero nodal values) of the constant function $c_K\big|_{\delta'(K)}$ to the whole domain $\Omega$ such that $c_K \in V_h^{(2)}$, that means in particular that the homogeneous Dirichlet boundary conditions due to $V_h^{(2)} \approx H_0^1(\Omega)$ are satisfied.

In the remaining case, i.e. when $\overline{\delta'(K)} \cap \partial\Omega \neq \emptyset$, we choose $c_K(x) := 0$ for all $x \in \overline{\Omega}$ in order to guarantee that $c_K \in V_h^{(2)} \approx H_0^1(\Omega)$. Therefore, in order to satisfy (38), we choose $K^* \in \mathcal{T}_h(\delta'(K))$ as a boundary element, that has a $(d-1)$-dimensional boundary face $\Gamma^* \subset \partial K^* \cap \partial\Omega$, and we choose $z_{K^*} \in \Gamma^*$ as a point with $v\big|_{K^*}(z_{K^*}) = 0$. The existence of such an element $K^*$ follows from the definition of the vicinity $\delta'(K)$ and the existence of such point $z_{K^*}$ is a consequence of $v \in V_h^{(1)} + V_h^{(2)}$ and the assumption (A4).

**Step 2.** We will prove the following estimate

$$\|v - c_K\|_{0.\infty,\widetilde{K}} \leq C_6\, h_K \sum_{T \in \mathcal{T}_h(\delta'(K))} |v|_{1,\infty,T} \qquad \forall\, \widetilde{K} \in \mathcal{T}_h(\delta'(K))\,. \tag{39}$$

In the case $\widetilde{K} = K^*$, the standard interpolation estimate

$$\|v - c_K\|_{0.\infty,K^*} = \|v - v(z_{K^*})\|_{0.\infty,K^*} \leq h_{K^*}|v|_{1,\infty,K^*} \tag{40}$$

and relation (36) imply the estimate (39) with $C_6 = C_3'$. In the case $\widetilde{K} \in \mathcal{T}_h(\delta'(K))$ with $\widetilde{K} \neq K^*$, there exists a chain of face-neighbored elements

$$K^* =: K_0 \to \cdots \to K_n := \widetilde{K}$$

such that for all $\mu = 1, \ldots, n$, the elements $K_{\mu-1}$ and $K_\mu$ have a common $(d-1)$-dimensional face $\Gamma_\mu := \partial K_{\mu-1} \cap \partial K_\mu$. From $v \in V_h^{(1)} + V_h^{(2)}$ and the consistency condition (A3), it follows that there exists a point $x_\mu \in \Gamma_\mu$ such that

$$v\big|_{K_{\mu-1}}(x_\mu) = v\big|_{K_\mu}(x_\mu) \qquad \forall\, \mu = 1, \ldots, n\,. \tag{41}$$

Using a standard interpolation estimate and the relation (36) we get for the point $x_n \in \overline{K_n}$

$$\|v - v(x_n)\|_{0.\infty,K_n} \leq h_{K_n}|v|_{1,\infty,K_n} \leq C_3'\, h_K|v|_{1,\infty,K_n}\,.$$

Applying this estimate and using $K_n = \widetilde{K}$ and (38) with $v^* := v\big|_{K^*}(z_{K^*})$ we obtain

$$\|v - c_K\|_{0.\infty,\widetilde{K}} \leq \|v - v(x_n)\|_{0.\infty,\widetilde{K}} + |\, v(x_n) - v^*| \;\leq\; C_3'\, h_K|v|_{1,\infty,\widetilde{K}} + |\, v(x_n) - v^*|\,. \tag{42}$$

Furthermore, we have $x_1 \in \overline{K_0} = \overline{K^*}$, i.e. $x_1$ and $z_{K^*}$ belong to the same element $K^*$, which implies

$$| v(x_1) - v^*| \le C_3' \, h_K |v|_{1,\infty,K^*} . \tag{43}$$

In the case $n = 1$, from (42) and (43) we obtain the estimate (39) with $C_6 = C_3'$. In the case $n \ge 2$, using (43), (36) and the fact that $x_{\mu+1}, x_\mu \in \overline{K_\mu}$ for all $\mu = 1, \dots, n-1$, we can estimate

$$\begin{aligned} | v(x_n) - v^*| &\le \sum_{\mu=1}^{n-1} | v(x_{\mu+1}) - v(x_\mu)| \; + \; | v(x_1) - v^*| \\ &\le \sum_{\mu=1}^{n-1} C_3' \, h_K \, |v|_{1,\infty,K_\mu} \; + \; C_3' \, h_K \, |v|_{1,\infty,K^*} . \end{aligned}$$

Together with (42) this implies the estimate (39) with $C_6 = C_3'$.

**Step 3.** From (39) we will now derive an estimate of $v - c_K$ in the $L^2$-norm. Let $\widetilde{K} \in \mathcal{T}_h(\delta'(K))$ be a given element in the vicinity of $K$. Then, using (39), inverse inequalities (see e.g. [6]) and the local mesh-size relation (36) we get

$$\begin{aligned} \|v - c_K\|^2_{0,\widetilde{K}} &\le C \, h^d_{\widetilde{K}} \|v - c_K\|^2_{0,\infty,\widetilde{K}} \\ &\le C \, h^d_{\widetilde{K}} h^2_K \left( \sum_{T \in \mathcal{T}_h(\delta'(K))} |v|_{1,\infty,T} \right)^2 \\ &\le C \, h^2_K \, \mathrm{card}(\mathcal{T}_h(\delta'(K))) \sum_{T \in \mathcal{T}_h(\delta'(K))} (h_{\widetilde{K}}/h_T)^d \, |v|^2_{1,T} . \end{aligned}$$

Since $h_{\widetilde{K}}/h_T \le (C_3')^2$ for all $\widetilde{K}, T \in \mathcal{T}_h(\delta'(K))$ and $\mathrm{card}(\mathcal{T}_h(\delta'(K))) \le C$ due to the shape-regularity of $\mathcal{T}_h$, it follows the $L^2$-norm estimate

$$\|v - c_K\|_{0,\widetilde{K}} \le C \, h_K \, |v|_{1,h,\delta'(K)} \qquad \forall \, \widetilde{K} \in \mathcal{T}_h(\delta'(K)) . \tag{44}$$

**Step 4.** Now we are able to prove (37). Since $c_K \in V_h^{(2)}$, it follows from Lemma 2 that $c_K = R_h c_K$. Using the fact that $c_K$ is constant on $K$ and applying an inverse inequality for functions of the space $V_h^{(2)}$ we get

$$|R_h v|_{1,K} = |R_h v - c_K|_{1,K} = |R_h(v - c_K)|_{1,K} \le C \, h_K^{-1} \|R_h(v - c_K)\|_{0,K} .$$

The local $L^2$-stability (25) of the operator $R_h$ and the estimate (44) for all elements $\widetilde{K} \subset \delta(K) \subset \delta'(K)$ yield

$$\|R_h(v - c_K)\|_{0,K} \le C_S \, \|v - c_K\|_{0,\delta(K)} \le C \, C_S \, h_K \, |v|_{1,h,\delta'(K)} ,$$

which finally proves the estimate (37). ■

Using the local $H^1$-stability of $R_h$ proved in Lemma 7 and the fact that $R_h$ is the identity operator on the subspace $V_h^{(2)}$ of $V_h^A$, we can prove the following local and global error estimates.

**Theorem 8** *Suppose that the assumptions of Lemma 7 are fulfilled. Let $u \in H_0^1(\Omega)$ be the exact solution of some problem and $u_h^{(1)} \in V_h^{(1)}$ the finite element solution of a corresponding discrete problem. Then, for the approximation $R_h u_h^{(1)} \in V_h^{(2)}$ of $u_h^{(1)}$, it holds for each element $K \in \mathcal{T}_h$ the local error estimate*

$$|u - R_h u_h^{(1)}|_{1,K} \le C \left\{ |u - u_h^{(1)}|_{1,h,\delta'(K)} + \inf_{v_h^{(2)} \in V_h^{(2)}} |u - v_h^{(2)}|_{1,h,\delta'(K)} \right\} \tag{45}$$

*with a constant $C$ independent of $K$. Furthermore, it holds the global estimate*

$$|u - R_h u_h^{(1)}|_{1,h,\Omega} \le C \left\{ |u - u_h^{(1)}|_{1,h,\Omega} + \inf_{v_h^{(2)} \in V_h^{(2)}} |u - v_h^{(2)}|_{1,h,\Omega} \right\} . \tag{46}$$

**Proof.** The proof of the error estimates (45) and (46) is completely analogous to the proof of the corresponding $L^2$-norm estimates (29) and (30) in Theorem 5 and Theorem 6. The only modification is to replace the $L^2$-norm by the discrete $H^1$-semi-norm. ■

# 6 Applications

## 6.1 A Posteriori Error Estimation

Let us consider the following model problem: *Find a function $u$ such that*

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \Omega\,, \\ u &= 0 \quad \text{on} \quad \partial\Omega\,, \end{aligned} \tag{47}$$

where $\Omega$ is a bounded polygonal domain and $f \in L^2(\Omega)$ a given function. If $u_h \in V_h \subset H_0^1(\Omega)$ is a conforming finite element approximation for problem (47), then it is well-known (see e.g. [20], [1]) that

$$\eta^{conf} := \left( \sum_{K \in \mathcal{T}_h} (\eta_K^{conf})^2 \right)^{1/2} \tag{48}$$

with

$$(\eta_K^{conf})^2 := h_K^2 \|f + \triangle u_h\|_{0,K}^2 + \frac{1}{2} \sum_{E \in \mathcal{E}(K)} h_E \, \| \, [n_E \cdot \nabla u_h]_E \, \|_{L^2(E)}^2 \tag{49}$$

is a suitable a posteriori error estimator, where $\mathcal{E}(K)$ denotes the set of the inner $(d-1)$-dimensional faces of the element $K$, $h_E$ denotes the diameter of the face $E \in \mathcal{E}(K)$, $n_E$ is a unit vector normal to $E$ and $[v]_E$ denotes the jump of the function $v(\cdot)$ along the element face $E$. For this estimator, it holds

$$|u - u_h|_{1,\Omega} \leq C_7 \, \eta^{conf} \, . \tag{50}$$

However, if $u_h^N \in V_h^{(1)} \not\subset H_0^1(\Omega)$ is a nonconforming finite element approximation of the weak solution $u \in H_0^1(\Omega)$ of problem (47), then it is also well-known from the literature (see [10], [9], [11]) that the estimators $\eta_K^{conf}$ have to be modified into $\widetilde{\eta}_K$ by adding some extra terms containing the jumps of the derivatives of $u_h^N$ in the tangential directions $t_E$ of the element faces $E \in \mathcal{E}(K)$. For the corresponding modified estimator $\widetilde{\eta} := \big(\sum_{K \in \mathcal{T}_h} \widetilde{\eta}_K^2\big)^{1/2}$ the a posteriori error estimate

$$|u - u_h^N|_{1,h,\Omega} \leq C \, \widetilde{\eta} \tag{51}$$

has been proved for the two-dimensional case. However, the analysis in [10] can not be generalized in a straightforward way to the three-dimensional case.

Using our post-processing operator $R_h : V_h^{(1)} \to V_h^{(2)} \subset H_0^1(\Omega)$ to compute a conforming approximation $R_h u_h^N$ of the solution, we can define the following computable "post-processing error"

$$\eta^{post} := \left( \sum_{K \in \mathcal{T}_h} (\eta_K^{post})^2 \right)^{1/2} \qquad \text{with} \qquad \eta_K^{post} := |u_h^N - R_h u_h^N|_{1,K} \tag{52}$$

Then, we can apply the well-known techniques for the conforming case ([20], [1]) together with some simple additional arguments to prove the following a posteriori error estimate [17]

$$|u - R_h u_h^N|_{1,\Omega} \leq C_7 \, \eta^{conf} + \eta^{post} \, , \tag{53}$$

where $C_7$ and $\eta^{conf}$ are the quantities in (50) and (48), (49). Note that from (53) we directly get the a posteriori error estimate for the nonconforming finite element solution

$u_h^N \in V_h^{(1)}$

$$|u - u_h^N|_{1,h,\Omega} \leq C_7\, \eta^{conf} + 2\eta^{post} \tag{54}$$

and let us emphasize that this error bound can be proved for the general $d$-dimensional case $\Omega \subset \mathcal{R}^d$.

## 6.2 Graphical Output

The basic element of any graphical output consists in drawing a straight line. For example, the visualization of any curved line is realized by drawing a suitable sequence of straight lines. One basic task for the visualization of a scalar finite element function $u_h$ is to plot a set of "contour-lines" $\mathcal{L}_i$, $i = 1, \ldots, M$, where $\mathcal{L}_i$ is the set of those points $x$ in the domain $\Omega$ where the function $u_h$ has a constant value, i.e. $u_h(x) = c_i$ for all $x \in \mathcal{L}_i$.

If for some element $K \in \mathcal{T}_h$, $u_h\big|_K$ is a linear function, then the "$K$-part" of the $i$-th contour-line $\mathcal{L}_i^K := \mathcal{L}_i \cap K$ is a straight line. That means, a piecewise linear finite element function would be desirable for graphical output. This leads to the idea that we take for our desired finite element space $V_h^{(2)}$ the space of piecewise linear continuous functions on a triangular or tetrahedral grid $\mathcal{T}_h^{(2)}$, where $\mathcal{T}_h^{(2)}$ is derived from the original grid $\mathcal{T}_h$ by a regular refinement process. This means that $\mathcal{T}_h^{(2)}$ is a regular decomposition of the domain $\Omega$ such that for each element $K^{(2)} \in \mathcal{T}_h^{(2)}$, there exists exactly one "father-element" $K = \mathcal{F}(K^{(2)}) \in \mathcal{T}_h$ with $K^{(2)} \subset K$. Then, the definition of the transfer operator $R_h : V_h^{(1)} \to V_h^{(2)}$ in Section 3 can be generalized to the situation where $V_h^{(1)}$ is associated with the grid $\mathcal{T}_h$ and $V_h^{(2)}$ with the grid $\mathcal{T}_h^{(2)}$. We simply define, as in (12), $R_h v := \sum_{j \in J^{(2)}} N_j(v)\, \varphi_j^{(2)}$ with

$$N_j(v) := \left(\omega_{\Sigma,j}^{(2)}\right)^{-1} \sum_{K \in \mathcal{T}_{h,j}^{(2)}} \omega_{j,K} N_{j,K}(v\big|_K) \tag{55}$$

where

$$\mathcal{T}_{h,j}^{(2)} := \left\{K \in \mathcal{T}_h^{(2)} \,:\, j \in J^{(2)}(K)\right\} \qquad \text{and} \qquad \omega_{\Sigma,j}^{(2)} := \sum_{K \in \mathcal{T}_{h,j}^{(2)}} \omega_{j,K}\,. \tag{56}$$

Here again $N_{j,K}(\cdot)$ denotes the local (w.r.t. $K \in \mathcal{T}_h^{(2)}$) nodal functional associated with the node $j \in J^{(2)}$ of the space $V_h^{(2)}$, i.e. in our context of a piecewise linear finite element space $V_h^{(2)}$ that $N_{j,K}(\cdot)$ corresponds to a point value $N_{j,K}(v) := v\big|_K(a_j)$ where $a_j$ is the $j$-th vertex-node.

Let us note that the above ideas can be applied in particular to the case where $V_h^{(1)}$ is a conforming finite element space constructed with polynomials of higher degree. Thus, our theory justifies the graphical post-processing for hp-FEM by means of piecewise linear finite elements on a suitably refined grid.

## 6.3 Prolongation for Multigrid Methods

We can generalize the definition of our transfer operator $R_h : V_h^A \to V_h^{(2)}$ such that it can be used as a prolongation operator for a multigrid method. Let us assume that we have at some level $k$ in a multi-level hierarchy a coarse grid $\mathcal{T}_k$ and a fine grid $\mathcal{T}_{k+1}$ where each element $K \in \mathcal{T}_{k+1}$ was created from its uniquely determined "father element" $\mathcal{F}(K) \in \mathcal{T}_k$ by a regular refinement procedure. We denote the finite element spaces corresponding to the grid $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$ by $V_k$ and $V_{k+1}$, respectively. Note that these spaces need not to be nested and that we can treat the case of conforming and nonconforming finite element spaces in the same way. The only thing that is needed is the fact that each element $K \in \mathcal{T}_{k+1}$ is completely contained in its father element $\mathcal{F}(K) \in \mathcal{T}_k$ because this property guarantees that each coarse grid function $v_k \in V_k$ is piecewise smooth with respect to $\mathcal{T}_{k+1}$, i.e.

$$V_k \subset Y(\mathcal{T}_{k+1}, s) := \left\{ v \in L^2(\Omega) : \ v\big|_K \in C^s(\overline{K}) \quad \forall\, K \in \mathcal{T}_{k+1} \right\} , \tag{57}$$

where $Y(\mathcal{T}_{k+1}, s)$ is defined as in (8) for $\mathcal{T}_h = \mathcal{T}_{k+1}$ with some $s$ depending on the type of the nodal functionals associated with the space $V_{k+1}$. Now, we define a transfer operator $R_k : Y(\mathcal{T}_{k+1}, s) \to V_{k+1}$ in the same way as $R_h$ in the previous subsection by (55) and (56) where $\mathcal{T}_h^{(2)}$ corresponds to $\mathcal{T}_{k+1}$ and $V_h^{(2)}$ to $V_{k+1}$. Because of (57) the restriction of the operator $R_k$ to the space $V_k$ defines a prolongation operator $R_k : V_k \to V_{k+1}$.

The computation of $R_k v_k \in V_{k+1}$ for a given coarse grid function $v_k \in V_k$ can be implemented efficiently. Moreover, the advantage of this general prolongation operator is that its implementation on an abstract level within an object-oriented program has to be done once only for all different cases of finite element spaces $V_k$ and $V_{k+1}$. This idea has been realized already in practice and proved to be efficient within multigrid methods [2].

Let us note that the above construction of the prolongation operator $R_k : V_k \to V_{k+1}$ fits into the general framework of [3] for multigrid methods in the case of nonconforming finite element discretizations. This gives a basis for convergence analysis of a multigrid method applied to a model problem where our operator $R_k$ is used for prolongation and its adjoint for defect restriction.

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