Higher order variational time discretizations for nonlinear systems of ordinary differential equations *

G. Matthies†, F. Schieweck‡

Abstract

Starting from the known continuous Galerkin-Petrov (cGP) and discontinuous Galerkin (dG) time discretization method of some polynomial order k, we propose two new variational time discretizations for the system of ordinary differential equations (ODE) associated with the semi-discrete finite element solution of a parabolic partial differential equation. Both methods are based on a time polynomial ansatz of higher order $k+1$ where the global smoothness of the discrete solution is also one level higher than that of the original method cGP or dG, respectively. Therefore, the total number of unknowns and the computational costs are not increasing but the accuracy is improved by one order. For the new methods, we prove optimal order a priori error estimates in the maximum norm for a general nonlinear ODE-system with a Lipschitz-continuous right-hand side. We show that the C^1-continuous Galerkin-Petrov methods are A-stable and that the C^0-continuous dG-methods are strongly A-stable (L-stable). Moreover, we prove that, in each case, the new higher order method and the original method coincide at the endpoints of the time intervals and that their difference can be computed by a simple post-processing step with low computational costs. With this relationship we have proven in the nonlinear case that, in particular, the cGP(2)- and dG(1)-method are superconvergent of order 4 and 3, respectively, at the endpoints of the time intervals. Finally, we present numerical results for the Burgers equation which confirm the theoretical results.

Keywords: discontinuous Galerkin method, continuous Galerkin–Petrov method, A-stability, L-stability, error estimates

2000 Mathematics Subject Classification (MSC): 65M12, 65M60

1 Introduction

For the numerical solution of parabolic partial differential equations by means of finite element methods, the first step is often a semi-discretization in space leading to a large system of ordinary differential equations (ODE) for the time-dependent nodal vector of the finite element solution. The second step is then to solve this ODE-system numerically by a suitable time discretization. If the mesh size in space becomes small then often the ODE-system becomes

*appears as Preprint No. 23/2011, Fakultät für Mathematik, Otto-von-Guericke Universität Magdeburg

†Institut für Mathematik, Universität Kassel, Heinrich-Plett-Str. 40, D-34132 Kassel, Germany

‡Institut für Analysis und Numerik, Otto-von-Guericke-Universität Magdeburg, Postfach 4120, D-39016 Magdeburg, Germany

more and more stiff. Therefore, explicit time discretizations are not a good choice and implicit, at least A-stable methods are needed. If we look, for instance, to A-stable BDF-methods, then their order is restricted to two. If we want to have an A-stable higher order time discretization, we arrive at implicit Runge-Kutta methods or at discontinuous Galerkin (dG) methods with higher polynomial order.

In this paper, we will consider only time discretizations of variational type like the dG-method. We call them *variational type methods* since the technique of their derivation is the same as for finite element methods. This means that we multiply the ODE with a test function and integrate over the whole time interval. Then, we subdivide the whole time interval into small subintervals and assume that the time-discrete solution is contained in an ansatz space which is time-polynomial on each subinterval. For the global smoothness of the time-discrete ansatz space, we consider three different cases – the discontinuous, the continuous and the C^1-continuous one. We choose this variational approach because of the following advantages:

- we have a uniform variational approach in space and time which may be advantageous in future for the analysis of the fully discrete problem and the construction of simultaneous space-time adaptive methods;
- it is very natural to construct methods of higher order;
- we can use the well-known finite element stability concepts of the Galerkin-Petrov or discontinuous Galerkin methods to obtain at least A-stable methods;
- for future developments, we can apply the well-known adaptive finite element techniques for changing the polynomial degree as well as the length of the time intervals.

In the literature, the construction and the a priori as well as the a posteriori error analysis of dG-methods for time discretization is well-known, see for instance the monographs of Thomée [8] and Eriksson et al. [3]. In order to obtain a time-marching process for a variational type method, it is sufficient that the test space is discontinuous in time. In the discontinuous Galerkin method where solution and test space are the same, also the discrete solution space consists of discontinuous piecewise polynomials in time. Therefore, some jump terms appear in this discretization. However, they can be avoided if a continuous discrete solution space is combined with a discontinuous test space. In [3], this method has been called *continuous Galerkin method* (cG-method) and in [7] *discontinuous Galerkin-Petrov method* since the test space is discontinuous and different from the ansatz space. As a compromise, it has been called *continuous Galerkin-Petrov method* (cGP-method) in [5, 6] and we will also use this name in this paper.

In the following, let us denote by cGP(k) the method where the discrete solution space consists of continuous piecewise polynomials in time of degree $k \geq 1$ and the discrete test space of discontinuous polynomials of degree $k-1$ and by dG(k) the method where both the solution and test space are constructed by means of discontinuous polynomials of degree k. Concerning the cGP(k)-method, some analysis (mostly a posteriori estimates) can be found for $k = 1$ and linear ODEs in [3, Sect. 9.3]. For the full discretization of the heat equation, Aziz and Monk proved in [2] optimal error estimates as well as superconvergence results in the endpoints of the time intervals. In [7], the cGP(k)-method was analyzed for the linear case in an abstract Hilbert-space setting as well as for a nonlinear ODE-system in the d-dimensional Euclidean space. It was shown by some energy arguments that the method is A-stable for any $k \geq 1$ and that it has an energy decreasing property for the gradient flow equation of an energy functional. Moreover, optimal error estimates have been proven.

A numerical comparison between the dG(1)-, cGP(1)- and cGP(2)-method has been done in [5] for the heat equation and in [6] for the non-stationary Stokes equations. Moreover, efficient multigrid solvers for the block-systems on each time interval have been proposed. With respect to the computational costs, the cGP(2)-method is comparable to the dG(1)-method. However, concerning the discretization error in time, the accuracy of the cGP(2)-method is one order higher than that of the dG(1)-method, i.e., at the endpoints of the time intervals the cGP(2)-method is of order 4 whereas the dG(1)-method has order 3. On the other hand, it is known that the dG(k)-methods are even "strongly A-stable" (or *L-stable* according to [4]), i.e., the dG-methods have better damping properties with respect to high frequency error components.

The main contribution of this paper is the development and analysis of two new time discretizations of variational type combined with some collocation conditions. We prove that the new methods can be interpreted also in two other ways: as pure collocation methods and as pure variational methods with a special numerical integration.

The first new method is derived from the cGP(k)-method by increasing the level of the global smoothness of the discrete solution from C^0- to C^1-continuity. Due to the continuity constraints of the time derivative of discrete solution at the endpoints of the time intervals we can increase the polynomial order to $k+1$ without increasing the total number of free unknowns in the system. Since the new method is now an approximation by polynomials with a one higher degree it is natural that this method has one order of accuracy more than the original cGP(k)-method. So we get a method with nearly the same computational costs as for the cGP(k)-method but with a one order higher accuracy. Moreover, we will prove that the new method, which we call cGP-C1($k+1$)-method, coincides with the cGP(k)-method at the endpoints of the time intervals. Therefore, we obtain the A-stability of cGP-C1($k+1$) from the known A-stability of cGP(k) and the superconvergence of cGP(k) at the endpoints of the time intervals from the optimal convergence of cGP-C1($k+1$) which we will prove in this paper.

The second new method is derived from the dG(k)-method by increasing the level of the global smoothness from C^{-1}- to C^0-continuity. Again, due to the continuity constraints of the discrete solution at the endpoints of the time intervals, we can increase the polynomial order to $k+1$ without increasing the total number of free unknowns in the system. Thus, we obtain a new method called dG-C0($k+1$) with nearly the same computational costs as for the dG(k)-method but with a one order higher accuracy. Again, we can prove that the new method generates the same values at the endpoints of the time intervals as the dG(k)-method. Therefore, we obtain the strong A-stability of dG-C0($k+1$) from the strong A-stability of dG(k) which we will show by means of energy type arguments. Moreover, the superconvergence of dG(k) at the endpoints of the time intervals follows from the optimal convergence of dG-C0($k+1$) which we also prove in this paper.

Let us note that our theoretical superconvergence results for cGP(k) and dG(k) are not optimal for higher values of k. However, we get – also for the nonlinear case – the optimal result that the cGP(2)- and the dG(1)-method are superconvergent of order 4 and 3, respectively. Our numerical results for the Burgers equation confirm these optimal orders and indicate also, for higher values of k, the higher order of superconvergence as it has been proven for the linear case in [2] and [8].

Finally, we would like to remark that the discrete solutions of the new, one order higher methods cGP-C1(k+1) or dG-C0(k+1) can be computed by means of a simple post-processing step with low computational costs from the original methods cGP(k) or dG(k), respectively. This means that the difference of the two solutions with different accuracy is easy to compute and its norm can be used as a cheap error indicator.

The paper is organized as follows. Section 2 describes the considered problem and introduces the basic notation. The continuous Galerkin–Petrov methods cGP(k) and the discontinuous Galerkin methods dG(k) together with their generalizations cGP-C1($k+1$) and dG-C0($k+1$) are considered in Sections 3 and 4. The relationship between cGP(k) and cGP-C1($k+1$) is discussed in Section 5 while the connection between dG(k) and dG-C0($k+1$) is shown in Section 6. Error estimates for the methods cGP-C1 and dG-C0 are proven in Section 7. Finally, Section 8 presents numerical results for the Burgers equation which confirm the theoretical results.

2 Problem and notation

As a model problem we consider the ODE system: *Find* $u : [0,T] \to V$ *such that*

$$\begin{aligned} M d_t u(t) &= F(t, u(t)) \qquad \forall\, t \in (0,T), \\ u(0) &= u_0, \end{aligned} \tag{1}$$

where $u(t) = \big(u_j(t)\big) \in V$ with $V := \mathbb{R}^{n_h}$ denotes the nodal vector of the semi-discrete finite element solution $u_h(t) \in V_h$ of a parabolic partial differential equation with

$$u_h(t) = \sum_{j=1}^{n_h} u_j(t) b_j$$

and $V_h = \operatorname{span}\{b_j : j = 1, \dots, n_h\}$. By $d_t u$ we denote the derivative of $u(t)$ with respect to time, M is the time-constant mass matrix $M = (M_{i,j}) := (b_j, b_i)$ with the L^2 inner product $(\cdot,\cdot)$ and $u_0 \in V$ the initial value at time $t = 0$. The function $F : [0,T] \times V \to V$ can be nonlinear and is assumed to be sufficiently smooth.

To describe the time discretization of problem (1) let us introduce the following notation. We denote by $I = [0,T]$ the time interval with some positive final time T. In order to characterize the smoothness of functions $t \mapsto u(t)$ we introduce, for a subinterval $J \subset I$, the space $C^r(J,V)$ as the space of r times continuously differentiable, V-valued functions on J equipped with the following norm and semi-norm

$$\|u\|_{C^r(J,V)} := \max_{0 \le k \le r} \sup_{t \in J} \|d_t^k u(t)\|_V, \qquad |u|_{C^r(J,V)} := \sup_{t \in J} \|d_t^r u(t)\|_V.$$

In the case $r = 0$, we will simply write $C(J, V)$ instead of $C^0(J, V)$. Moreover, we will use the space $L^2(I, V)$ defined as

$$L^2(I,V) := \{u : I \to V : \ \|u\|_{L^2(I,V)} < \infty\} \qquad \text{with} \qquad \|u\|_{L^2(I,V)} := \left(\int_I \|u(t)\|_V^2 dt\right)^{1/2}.$$

We decompose the time interval I into N subintervals $I_n := (t_{n-1}, t_n]$, where $n = 1, \ldots, N$ and $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$. The symbol τ will denote the *time discretization parameter* and will also be used as the maximum time step size $\tau := \max_{1\le n\le N} \tau_n$, where $\tau_n := t_n - t_{n-1}$. In the following, the set of the time intervals $\mathcal{M}_\tau := \{I_1, \ldots, I_N\}$ will be called the *time-mesh*. We approximate the solution $u : I \to V$ by means of a function $u_\tau : I \to V$ which is a piecewise polynomial of some order k with respect to time, i.e., we are looking for u_τ in either the discontinuous time-discrete space

$$\mathbb{P}_k^{dc}(\mathcal{M}_\tau) := \{v \in L^2(I,V) : \ v\big|_{I_n} \in \mathbb{P}_k(I_n, V) \quad \forall\, I_n \in \mathcal{M}_\tau\}, \tag{2}$$

the continuous time-discrete space

$$\mathbb{P}_k^{c}(\mathcal{M}_\tau) := \{u \in C(I,V) : \ u\big|_{I_n} \in \mathbb{P}_k(I_n, V) \quad \forall\, I_n \in \mathcal{M}_\tau\} \tag{3}$$

or the C^1-continuous time-discrete space

$$\mathbb{P}_k^{C1}(\mathcal{M}_\tau) := \{u \in C^1(I,V) : \ u\big|_{I_n} \in \mathbb{P}_k(I_n, V) \quad \forall\, I_n \in \mathcal{M}_\tau\}, \tag{4}$$

where

$$\mathbb{P}_k(I_n, V) := \Big\{u : I_n \to V \ : \ u(t) = \sum_{j=0}^{k} U^j t^j \ , \ \forall\, t \in I_n, \ U^j \in V, \ \forall\, j\Big\}$$

denotes the space of vector-valued polynomials in time of order k. For the case of discrete functions $u_\tau : I \to V$, which are discontinuous at the end points of the time intervals, we define the left-sided and right-sided values u_n^- and u_n^+ and the jump $[u_\tau]_n$ as

$$u_n^- := \lim_{t\to t_n - 0} u_\tau(t), \qquad u_n^+ := \lim_{t\to t_n + 0} u_\tau(t), \qquad [u_\tau]_n := u_n^+ - u_n^-.$$

Furthermore, we define the function value $u_\tau(t_n)$ with $n \ge 1$ as the value coming from interval $I_n = (t_{n-1}, t_n]$, i.e. as $u_\tau(t_n) := u_n^-$, and the value $u_\tau(0)$ as the initial value u_0, i.e. $u_\tau(0) := u_0$.

3 Continuous Galerkin–Petrov methods

3.1 The cGP(k)-method

Here we use the space $\mathbb{P}_k^c(\mathcal{M}_\tau)$ as the time-discrete solution space and $\mathbb{P}_{k-1}^{dc}(\mathcal{M}_\tau)$ as the discrete test space. Due to the initial condition in (1), the discrete solution u_τ has $N \cdot k$ many unknown degrees of freedom and the dimension of the test space is also $N \cdot k$.

In order to determine the unknown coefficients of u_τ we multiply the first equation in (1) with a test function $v_\tau \in \mathbb{P}_{k-1}^{dc}(\mathcal{M}_\tau)$, integrate over I and obtain the following **time-discrete global problem**:

Find $u_\tau \in \mathbb{P}_k^c(\mathcal{M}_\tau)$ *such that* $u_\tau(0) = u_0$ *and*

$$\int_0^T \big\langle M u_\tau'(t), v_\tau(t)\big\rangle \, dt = \int_0^T \langle F(t, u_\tau(t)), v_\tau(t)\rangle \, dt \qquad \forall\, v_\tau \in \mathbb{P}_{k-1}^{dc}(\mathcal{M}_\tau) \tag{5}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $V = \mathbb{R}^{n_h}$.

We will call this discretization the *exact continuous Galerkin–Petrov method* of *order* k or briefly the "**exact cGP(k)-method**". The name Galerkin–Petrov is due to the fact that the test space $\mathbb{P}_{k-1}^{dc}(\mathcal{M}_\tau)$ is different from the ansatz space $\mathbb{P}_k^c(\mathcal{M}_\tau)$. With "exact" we indicate that the time integral at the right-hand side is evaluated exactly.

Since the discrete test space $\mathbb{P}_{k-1}^{dc}(\mathcal{M}_\tau)$ is discontinuous, problem (5) can be solved by a time-marching process where successively local problems on the time intervals are solved. Therefore, we choose test functions $v_\tau(t) = v\psi(t)$ with an arbitrary time independent $v \in V$ and a scalar function $\psi : I \to \mathbb{R}$ which is zero on $I \setminus I_n$ and a polynomial of order less than or equal to $k-1$ on I_n. Then, we obtain from (5) the "**I_n-problem of the exact cGP(k)-method**":

Find $u_\tau|_{I_n} \in \mathbb{P}_k(I_n, V)$ *such that* $u_\tau|_{I_n}(t_{n-1}) = u_{n-1}^-$ *and*

$$\int_{I_n} \big\langle M u_\tau'(t), v\big\rangle \, \psi(t)\, dt = \int_{I_n} \langle F(t, u_\tau(t)), v\rangle \, \psi(t)\, dt \qquad \forall\, v \in V \quad \forall\, \psi \in \mathbb{P}_{k-1}(I_n) \tag{6}$$

with $u_{n-1}^- := u_\tau|_{I_{n-1}}(t_{n-1})$ *for* $n \geq 2$ *and* $u_{n-1}^- := u_0$ *for* $n = 1$.

In the general case of a nonlinear function $F(\cdot,\cdot)$, we have to compute the integrals on the right-hand side of (6) numerically. The $(k+1)$-point Gauß–Lobatto formula is exact if the function to be integrated is a polynomial of degree less than or equal $2k-1$. So, this formula applied to the integral on the left-hand side of (6) will give the exact value. Let $\hat{w}_j$ and $\hat{t}_j$, $j = 0,\ldots,k$, denote the corresponding weights and nodes on the reference interval $[-1,1]$. Then, the "**I_n-problem of the numerically integrated cGP(k)-method** " reads:

Find $u_\tau|_{I_n} \in \mathbb{P}_k(I_n, V)$ *such that*

$$\begin{aligned} u_\tau(t_{n-1}) &= u_{n-1}, \\ \sum_{j=0}^{k} \hat{w}_j M d_t u_\tau(t_{n,j}) \psi(t_{n,j}) &= \sum_{j=0}^{k} \hat{w}_j F\big(t_{n,j}, u_\tau(t_{n,j})\big)\psi(t_{n,j}) \qquad \forall \psi \in \mathbb{P}_{k-1}(I_n). \end{aligned} \tag{7}$$

To determine $u_\tau|_{I_n}$ we represent it by a polynomial ansatz

$$u_\tau(t) := \sum_{j=0}^{k} U_n^j \phi_{n,j}(t) \qquad \forall\, t \in I_n, \tag{8}$$

where the "coefficients" U_n^j are elements of $V = \mathbb{R}^{n_h}$ and the real-valued functions $\phi_{n,j} \in \mathbb{P}_k(I_n)$ are Lagrange basis functions with respect to $k+1$ suitable nodal points $t_{n,j} \in \overline{I_n}$

satisfying the conditions

$$\phi_{n,j}(t_{n,i}) = \delta_{i,j}, \qquad i,j = 0,\dots,k, \tag{9}$$

with the Kronecker symbol $\delta_{i,j}$. Like in [7], we choose the nodes $t_{n,j}$ as the quadrature points of the $(k+1)$-point Gauß–Lobatto formula on I_n. In particular, this means $t_{n,0} = t_{n-1}$ and $t_{n,k} = t_n$. Then, the initial condition for (6) is equivalent to the condition

$$U_n^0 = u_\tau|_{I_{n-1}}(t_{n-1}) \quad \text{if} \quad n \geq 2 \qquad \text{or} \qquad U_n^0 = u_0 \quad \text{if} \quad n = 1. \tag{10}$$

We define the basis functions $\phi_{n,j} \in \mathbb{P}_k(I_n)$ via the affine reference transformation $T_n : \hat{I} \to I_n$ where $\hat{I} := (-1,1]$ and

$$t = T_n(\hat{t}) := \frac{t_{n-1}+t_n}{2} + \frac{\tau_n}{2}\hat{t} \in I_n \qquad \forall\, \hat{t} \in \hat{I},\ n = 1,\dots,N. \tag{11}$$

Let $\hat{\phi}_j \in \mathbb{P}_k(\hat{I})$, $j = 0,\dots,k$, denote the basis functions satisfying the conditions

$$\hat{\phi}_j(\hat{t}_i) = \delta_{i,j}, \qquad i,j = 0,\dots,k,$$

where $\hat{t}_i$, $i = 0,\dots,k$, are the quadrature points for the reference interval $\hat{I}$. Then, we define the basis functions on the original time interval I_n by the standard mapping

$$\phi_{n,j}(t) := \hat{\phi}_j(\hat{t}) \qquad \text{with} \qquad \hat{t} := T_n^{-1}(t) = \frac{2}{\tau_n}\left(t - \frac{t_n + t_{n-1}}{2}\right) \in \hat{I}.$$

Similarly, we define the test basis functions $\psi_{n,i}$ by suitable reference basis functions $\hat{\psi}_i \in \mathbb{P}_{k-1}(\hat{I})$, i.e.,

$$\psi_{n,i}(t) := \hat{\psi}_i\big(T_n^{-1}(t)\big) \qquad \forall\, t \in I_n,\ i = 1,\dots,k.$$

From the representation (8) of u_τ on I_n we get for $d_t u_\tau$

$$d_t u_\tau(t) = \sum_{j=0}^{k} U_n^j \phi'_{n,j}(t) \qquad \forall\, t \in I_n, \tag{12}$$

which leads, for each test basis function $\psi \in \mathbb{P}_{k-1}(I_n)$ and all $v \in V$, to the formula

$$\int_{I_n} \langle M d_t u_\tau(t), v\rangle\, \psi(t) dt = \sum_{\mu=0}^{k} \hat{w}_\mu \sum_{j=0}^{k} \left\langle M U_n^j, v\right\rangle \hat{\phi}'_j(\hat{t}_\mu)\hat{\psi}(\hat{t}_\mu). \tag{13}$$

If we choose the test functions $\psi_{n,i} \in \mathbb{P}_{k-1}(I_n)$ such that

$$\hat{\psi}_i(\hat{t}_\mu) = (\hat{w}_\mu)^{-1}\delta_{i,\mu} \qquad i,\mu = 1,\dots,k,$$

we get the "**special form of the numerically integrated $\boldsymbol{I_n}$-problem of cGP($\boldsymbol{k}$)**":

Find the coefficients $U_n^j \in V$, $j = 1, \ldots, k$, *as the solution of the following nonlinear* $(k \times k)$*-block-system:*

$$\sum_{j=0}^{k} \alpha_{i,j} M U_n^j = \frac{\tau_n}{2} \Big\{ F\big(t_{n,i}, U_n^i\big) + \beta_i F\big(t_{n,0}, U_n^0\big) \Big\} \qquad \forall\, i = 1, \ldots, k, \tag{14}$$

where $U_n^0 := u_{n-1}^-$ *is the "initial value" and* $\alpha_{i,j}$ *and* β_i *are defined by*

$$\alpha_{i,j} = \hat{\phi}_j'(\hat{t}_i) + \beta_i \hat{\phi}_j'(\hat{t}_0)\,, \qquad \beta_i := \hat{w}_0 \hat{\psi}_i(\hat{t}_0)\,. \tag{15}$$

The advantage of using Lagrange basis functions $\phi_{n,j}$ with respect to some points $t_{n,j} \in \overline{I_n}$ is that the coefficients $U_n^j \in V$ have the meaning $U_n^j = u_\tau(t_{n,j})$.

In the following, we will refer to the numerically integrated cGP(k)-method simply as the "**cGP(k)-method**". We present this method for the cases $k = 1$ and $k = 2$ in the next two subsections.

3.1.1 cGP(1)-method

We use the 2-point Gauß–Lobatto formula with the points $t_{n,0} = t_{n-1}$, $t_{n,1} = t_n$ and weights $\hat{w}_0 = \hat{w}_1 = 1$ which yields the well-known *Trapezoidal rule*. Then, we get $\alpha_{1,0} = -1$, $\alpha_{1,1} = 1$, $\beta_1 = 1$ and the I_n-problem leads to the following block-equation for the one coefficient $U_n^1 = u_\tau(t_n) \in V$

$$\boxed{M U_n^1 - M U_n^0 = \frac{\tau_n}{2} \Big\{ F\big(t_n, U_n^1\big) + F\big(t_{n-1}, U_n^0\big) \Big\}.} \tag{16}$$

3.1.2 cGP(2)-method

Here we apply the 3-point Gauß–Lobatto formula with the points $t_{n,0} = t_{n-1}$, $t_{n,1} = (t_{n-1} + t_n)/2$, $t_{n,2} = t_n$ and the reference weights $\hat{w}_0 = \hat{w}_2 = 1/3$, $\hat{w}_1 = 4/3$, which is known as *Simpson's rule*. Then, we obtain the coefficients

$$(\alpha_{i,j}) = \begin{pmatrix} -\frac{5}{4} & 1 & \frac{1}{4} \\ 2 & -4 & 2 \end{pmatrix}, \qquad (\beta_i) = \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix}, \qquad i = 1, 2,\ j = 0, 1, 2.$$

On the time interval $I_n = (t_{n-1}, t_n]$ we have to solve for the two unknowns $U_n^j = u_\tau(t_{n,j})$ with $t_{n,j} := T_n(\hat{t}_j)$ for $j = 1, 2$. The coupled (2×2)-block-system for $U_n^1, U_n^2 \in V$ reads:

$$\boxed{\begin{aligned} M U_n^1 + \tfrac{1}{4} M U_n^2 &= \tfrac{5}{4} M U_n^0 + \tfrac{\tau_n}{2} \Big\{ F\big(t_{n,1}, U_n^1\big) + \tfrac{1}{2} F\big(t_{n,0}, U_n^0\big) \Big\}, \\ -4 M U_n^1 + 2 M U_n^2 &= -2 M U_n^0 + \tfrac{\tau_n}{2} \Big\{ F\big(t_{n,2}, U_n^2\big) - F\big(t_{n,0}, U_n^0\big) \Big\}. \end{aligned}} \tag{17}$$

3.2 The cGP-C1(k)-method

We assume that $k \geq 3$. The idea to use a C^1-continuous time-discrete solution u_τ can be easily demonstrated on the first interval $I_1 = (0, t_1]$. To determine a V-valued polynomial of order k on I_1 we need $k+1$ many values. One value is the initial value $u_\tau(0) = u_0$ but we also know the time derivative of u_τ at $t_0 = 0$ by means of the differential equation:

$$M d_t u_\tau(t_0) = F(t_0, u_0).$$

If we can quickly solve a system with the mass matrix we get the derivative $d_t u_\tau(t_0)$ with low computational costs. Moreover, if we use as unknowns the value $u_1 := u_\tau(t_1)$ and the derivative $d_t u_\tau(t_1)$ which are coupled in the quasi-explicit way by

$$M d_t u_\tau(t_1) = F(t_1, u_1).$$

Thus, the number of the really implicitly coupled unknowns is $k-2$ and the numerical costs for the solution of the block system of the cGP-C1(k)-method is comparable to the cGP($k-1$)-method.

The "**global cGP-C1(k)-method**", for $k \geq 3$, reads:

Find $u_\tau \in \mathbb{P}_k^{C1}(\mathcal{M}_\tau)$ *such that* $u_\tau(0) = u_0$ *and*

$$\begin{aligned} \int_0^T \langle M d_t u_\tau(t), v_\tau(t)\rangle \, dt &= \int_0^T \big\langle F\big(t, u_\tau(t)\big), v_\tau(t)\big\rangle \, dt \qquad \forall v_\tau \in \mathbb{P}_{k-3}^{dG}(\mathcal{M}_\tau) \\ M d_t u_\tau(t_n) &= F\big(t_n, u_\tau(t_n)\big), \qquad n = 0, \ldots, N. \end{aligned} \tag{18}$$

The dimension of the space $\mathbb{P}_k^{C1}(\mathcal{M}_\tau)$ is given by $(k+1)N - 2(N-1) = (k-1)N + 2$. We have $k-2$ Galerkin conditions on each subinterval, $N+1$ points where the differential equation has to be fulfilled, and one initial conditions. This gives in total $(k-1)N + 2$ conditions. Hence, the number of degrees of freedom of $\mathbb{P}_k^{C1}(\mathcal{M}_\tau)$ and the number of conditions coincide.

Due to the discontinuity of the test functions v_τ, the global cGP-C1(k) problem can be solved by means of a time-marching process. The "***I_n*-problem of the exact cGP-C1(k)-method**" reads:

Find $u_\tau|_{I_n} \in \mathbb{P}_k(I_n, V)$ *such that*

$$\begin{aligned} u_\tau(t_{n-1}) &= u_{n-1}, \\ M d_t u_\tau(t_{n-1}) &= F\big(t_{n-1}, u_\tau(t_{n-1})\big), \\ M d_t u_\tau(t_n) &= F\big(t_n, u_\tau(t_n)\big), \\ \int_{I_n} M d_t u_\tau(t) \psi(t) \, dt &= \int_{I_n} F\big(t, u_\tau(t)\big) \psi(t) \, dt \qquad \forall \psi \in \mathbb{P}_{k-3}(I_n) \end{aligned} \tag{19}$$

For a general nonlinear function $F(\cdot,\cdot)$, we have to use numerical integration in equation (19). We have found that a Gauß–Lobatto formula with k nodes, which is exact for polynomials of degree less than or equal to $2k-3$, is accurate enough. Thus, the "***I_n*-problem of the numerically integrated cGP-C1(k)-method** " reads:

Find $u_\tau|_{I_n} \in \mathbb{P}_k(I_n, V)$ *such that*

$$\begin{aligned} u_\tau(t_{n-1}) &= u_{n-1}, \\ M d_t u_\tau(t_{n-1}) &= F\big(t_{n-1}, u_\tau(t_{n-1})\big), \\ M d_t u_\tau(t_n) &= F\big(t_n, u_\tau(t_n)\big), \\ \sum_{j=0}^{k-1} \hat{w}_j M d_t u_\tau(t_{n,j}) \psi(t_{n,j}) &= \sum_{j=0}^{k-1} \hat{w}_j F\big(t_{n,j}, u_\tau(t_{n,j})\big) \psi(t_{n,j}) \qquad \forall \psi \in \mathbb{P}_{k-3}(I_n). \end{aligned} \tag{20}$$

In Section 5, we prove that there is a strong relation between the solutions of the cGP($k-1$)-method and the cGP-C1(k)-method. Therefore, we do not present a practical scheme for the cGP-C1(k)-method.

Remark 1 *In [7, Th. 5.1], it has been shown by some energy arguments that the cGP(k)-method is A-stable for all* $k \in \mathbb{N}$. *The proof, which was done for the usual complex model problem, can be generalized in a straightforward way also to a linear ODE system with a negative definite matrix which need not to be normal (see Sect. 4.1.3). We will prove in Section 5 that, for* $k \geq 3$, *the numerically integrated versions of the cGP(k − 1)-method and the cGP-C1(k)-method generate the same values at the endpoints of the time intervals. This implies that also the cGP-C1(k)-method is A-stable for all* $k \geq 3$.

In the following lemma, we will prove that the interval-wise computed local discrete solutions $u_\tau|_{I_n}$ really form a global C^1-solution and that the method can be interpreted also as a collocation method as well as a cGP(k)-method with a reduced Gauß–Lobatto integration rule.

Lemma 2 *Let* $u_\tau \in L^2(I, V)$ *denote the formally discontinuous function such that its restriction to time interval* $I_n = (t_{n-1}, t_n]$ *is the solution of the numerically integrated* I_n*-problem* (20) *of the cGP-C1(k)-method. Then, the function* u_τ *is a global* C^1*-function with respect to time, i.e.* $u_\tau \in C^1(I, V)$. *Moreover, the solution* $u_\tau|_{I_n} \in \mathbb{P}_k(I_n, V)$ *of* (20) *can also be characterized as the solution of the collocation problem according to the k Gauß–Lobatto points* $t_{n,0} = t_{n-1}, \ldots, t_{n,k-1} = t_n$

$$\begin{aligned} u_\tau(t_{n-1}) &= u_{n-1}, \\ M d_t u_\tau(t_{n,j}) &= F\big(t_{n,j}, u_\tau(t_{n,j})\big) \qquad \forall\, j = 0, \ldots, k-1 \end{aligned} \tag{21}$$

or the numerically integrated cGP(k)-method according to the k-point Gauß–Lobatto quadrature formula

$$\begin{aligned} u_\tau(t_{n-1}) &= u_{n-1}, \\ \sum_{j=0}^{k-1} \hat{w}_j M d_t u_\tau(t_{n,j}) \psi(t_{n,j}) &= \sum_{j=0}^{k-1} \hat{w}_j F\big(t, u_\tau(t_{n,j})\big) \psi(t_{n,j}) \qquad \forall \psi \in \mathbb{P}_{k-1}(I_n). \end{aligned} \tag{22}$$

Proof. First, we show the C^1-continuity of $u_\tau(t)$. In the interior of the time intervals, the polynomial function $u_\tau|_{I_n}$ is arbitrarily smooth. It remains to show that the left-sided derivative $d_t^- u_\tau(t_n)$ and the right-sided one $d_t^+ u_\tau(t_n)$ coincide for all $n = 1, \ldots, N-1$. Due to the condition $u_\tau|_{I_{n+1}}(t_n) = u_\tau|_{I_n}(t_n) =: u_\tau(t_n)$ we get from (20)

$$d_t^- u_\tau(t_n) = d_t u_\tau|_{I_n}(t_n) = M^{-1} F(t_n, u_\tau(t_n)) = d_t u_\tau|_{I_{n+1}}(t_n) = d_t^+ u_\tau(t_n).$$

Thus, $u_\tau(\cdot)$ is differentiable at t_n and $d_t u_\tau(\cdot)$ is continuous at t_n which proves $u_\tau \in C^1(I, V)$.

Now, we will prove the collocation conditions (21). The second and third equation line in (20) show this condition for $j \in \{0, k-1\}$. This implies

$$\sum_{j=1}^{k-2} \hat{w}_j M d_t u_\tau(t_{n,j}) \psi(t_{n,j}) = \sum_{j=1}^{k-2} \hat{w}_j F\big(t_{n,j}, u_\tau(t_{n,j})\big) \psi(t_{n,j}) \qquad \forall\, \psi \in \mathbb{P}_{k-3}(I_n).$$

In order to prove (21) for $j = i \in \{1, \ldots, k-2\}$, we choose the test function $\psi_i \in \mathbb{P}_{k-3}(I_n)$ defined by the $k-2$ conditions $\psi_i(t_{n,i}) = 1$ and $\psi_i(t_{n,j}) = 0$ for all $j \in \{1, \ldots, k-2\} \setminus \{i\}$ and obtain

$$\hat{w}_i M d_t u_\tau(t_{n,i}) = \hat{w}_i F\big(t_{n,i}, u_\tau(t_{n,i})\big).$$

Finally, we show (22). For each $i \in \{0, \ldots, k-1\}$, let the function $\psi_i \in \mathbb{P}_{k-1}(I_n)$ be defined by the k conditions $\psi_i(t_{n,i}) = 1$ and $\psi_i(t_{n,j}) = 0$ for all $j \in \{0, \ldots, k-1\} \setminus \{i\}$. These functions form a basis of the polynomial space $\mathbb{P}_{k-1}(I_n)$ and the variational condition in (22) is equivalent to the k equations for the basis functions ψ_i. The equation (22) for $\psi = \psi_i$ is equivalent to the collocation equation (21) for $j = i$. Thus, the problems (22) and (21) are equivalent to problem (20). □

4 Discontinuous Galerkin methods

4.1 The dG(k)-method

In this section we describe the details of the discontinuous Galerkin method dG(k). Here, the discrete solution space is the same as the test space, namely $\mathbb{P}_k^{dc}(\mathcal{M}_\tau)$, and the "**global discontinuous Galerkin method dG(k)**" reads:

Find $u_\tau \in \mathbb{P}_k^{dc}(\mathcal{M}_\tau)$ such that $u_\tau(0) = u_0^- = u_0$ and

$$\sum_{n=1}^{N} \int_{I_n} \langle M d_t u_\tau(t), v_\tau(t) \rangle \, dt + \sum_{n=1}^{N} \big\langle M[u_\tau]_{n-1}, v_{n-1}^+ \big\rangle = \int_0^T \langle F(t, u_\tau(t)), v_\tau(t) \rangle \, dt \qquad \forall v_\tau \in \mathbb{P}_k^{dc}(\mathcal{M}_\tau) \quad (23)$$

To decouple this formulation we choose test functions $v_\tau(t) = v\psi(t)$ with an arbitrary time-independent $v \in V$ and scalar basis functions $\psi(t)$ of the polynomial space $\mathbb{P}_k(I_n)$ which are extended by zero outside of $\overline{I_n}$. Then, the solution of the dG(k)-method can be determined by successively solving a local problem on each time interval I_n. Using the known value $U_n^0 := u_{n-1}^-$, the "***I_n*-problem of the exact dG(*k*)-method** " reads:

Find $u_\tau|_{I_n} \in \mathbb{P}_k(I_n, V)$ *such that*

$$\int_{I_n} \langle M d_t u_\tau(t), v\rangle\, \psi(t)\, dt + \left\langle M[u_\tau]_{n-1}, v_{n-1}^+\right\rangle \psi(t_{n-1}) = \int_{I_n} \langle F(t, u_\tau(t)), v\rangle\, \psi(t)\, dt \qquad \forall v \in V,\ \psi \in \mathbb{P}_k(I_n). \quad (24)$$

The discrete solution u_τ, on each time interval I_n, is a V-valued polynomial of degree k with the representation

$$u_\tau(t) := \sum_{j=1}^{k+1} U_n^j \phi_{n,j}(t) \qquad \forall\, t \in I_n, \quad (25)$$

where the $k+1$ "coefficients" U_n^j are elements of the space V and the real-valued functions $\phi_{n,j} \in \mathbb{P}_k(I_n)$ are the Lagrange basis functions with respect to $k+1$ suitable nodal points $t_{n,j} \in \overline{I_n}$ satisfying the conditions $\phi_{n,j}(t_{n,i}) = \delta_{i,j}$ for all $i, j = 1, \ldots, k+1$. In this paper, we will choose the $t_{n,j}$ as the integration points of the $(k+1)$-point right-sided Gauß–Radau quadrature formula on the interval I_n which is exact if the function to be integrated is a polynomial of order less than or equal to $2k$. If we apply this formula to the integrals in (24) we get the "***I_n*-problem of the numerically integrated dG(*k*)-method** ":

Find $u_\tau|_{I_n} \in \mathbb{P}_k(I_n, V)$ *such that*

$$\frac{\tau_n}{2} \sum_{j=1}^{k+1} \hat{w}_j M d_t u_\tau(t_{n,j}) \psi(t_{n,j}) + M\big(u_\tau(t_{n-1}) - u_{n-1}^-\big)\psi(t_{n-1}) = \frac{\tau_n}{2} \sum_{j=1}^{k+1} \hat{w}_j F\big(t_{n,j}, u_\tau(t_{n,j})\big)\psi(t_{n,j}) \qquad \forall \psi \in \mathbb{P}_k(I_n). \quad (26)$$

We define the test functions $\psi_{n,i}$ by means of the reference functions $\hat{\psi}_i := (\hat{w}_i)^{-1}\hat{\phi}_i$ where the $\hat{w}_i$ are the weights of the Gauß–Radau formula on $\hat{I}$. If we insert the representation (25) of u_τ into the I_n-problem (26), transform the integrals to the reference interval $\hat{I}$ and apply the Gauß–Radau-quadrature formula, we obtain the following "**special form of the numerically integrated *I_n*-problem of dG(*k*)**":

Find the coefficients $U_n^j \in V$, $j = 1, \ldots, k+1$, *such that*

$$\sum_{j=1}^{k+1} \alpha_{i,j} \left\langle MU_n^j, v \right\rangle = d_i \left\langle MU_n^0, v \right\rangle + \frac{\tau_n}{2} \left\langle F(t_{n,i}, U_n^i), v \right\rangle, \qquad i = 1, \ldots, k+1, \tag{27}$$

where

$$\alpha_{i,j} := \hat{\phi}_j'(\hat{t}_i) + c_j d_i, \qquad c_j := \hat{\phi}_j(-1), \qquad d_i := (\hat{w}_i)^{-1} \hat{\phi}_i(-1), \qquad U_n^0 := u_{n-1}^-.$$

Once we have solved this system, we can compute by means of the ansatz (25) the left side value u_n^- of u_τ at time t_n. Then, we enter the next time interval $I_{n+1} = (t_n, t_{n+1}]$ and set the initial value to $U_{n+1}^0 := u_n^-$. In the following, we present the schemes of the numerically integrated methods dG(0) and dG(1).

4.1.1 dG(0)-method

We use the 1-point Gauß–Radau formula with the point $t_{n,1} = t_n$, and weight $\hat{w}_1 = 2$ which yields the well-known *implicit Euler method*, i.e., the I_n-problem is the following 1-block-equation for the one coefficient $U_n^1 = u_\tau(t_n) \in V$:

$$\boxed{MU_n^1 - MU_n^0 = \tau_n F\big(t_n, U_n^1\big).} \tag{28}$$

4.1.2 dG(1)-method

Here we apply the 2-point Gauß–Radau formula with the points $t_{n,1} = t_{n-1} + \tau_n/3$, $t_{n,1} = t_n$ and the reference weights $\hat{w}_1 = 3/2$, $\hat{w}_2 = 1/2$. On the time interval $I_n = (t_{n-1}, t_n]$ we have to solve for the two unknowns $U_n^j = u_\tau(t_{n,j})$ for $j = 1, 2$. The coupled (2×2)-block-system for $U_n^1, U_n^2 \in V$ reads:

$$\boxed{\begin{array}{rcl} \frac{3}{4}MU_n^1 + \frac{1}{4}MU_n^2 &=& MU_n^0 + \frac{\tau_n}{2}F\big(t_{n,1}, U_n^1\big), \\ -\frac{9}{4}MU_n^1 + \frac{5}{4}MU_n^2 &=& -MU_n^0 + \frac{\tau_n}{2}F\big(t_{n,2}, U_n^2\big). \end{array}} \tag{29}$$

4.1.3 Strong A-stability of the dG(k)-method

We analyze the stability of the dG(k)-method applied to the linear model problem:

$$\begin{array}{rcll} \multicolumn{4}{l}{\textit{Find } u : [0,T] \to \mathbb{R}^d \textit{ such that}} \\ d_t u(t) &=& Au(t) & \forall\, t \in (0,T) \\ u(0) &=& u_0, & \end{array} \tag{30}$$

where $A \in \mathbb{R}^{d \times d}$ is a negative definite matrix, i.e.,

$$\exists\, \alpha > 0 : \qquad \langle Av, v \rangle \le -\alpha \|v\|^2 \qquad \forall\, v \in \mathbb{R}^d. \tag{31}$$

For this model problem, the mass matrix M is the identity matrix and the function F is linear, i.e., $F(t,u) := Au$. Here, the numerically integrated version of the dG(k)-method is the same as the exact version due to the properties of the $(k+1)$-point Gauß–Radau quadrature formula. In the following lemma, we prove the A-stability of the dG(k)-method by means of a stability estimate. Note, that our analysis does not assume that the matrix A can be diagonalized by means of eigenvectors.

Lemma 3 (A-stability) *Let $u_0 \in \mathbb{R}^d$ denote the initial value of model problem* (30) *and $u_1 := u_\tau|_{I_1}(t_1) = u_1^-$ with $t_1 := \tau$ and $I_1 := (0, t_1]$ the left-sided value at $t = t_1$ of the discrete solution u_τ of the dG(k)-method applied to* (30). *Then, under the assumption* (31) *with the parameter $\alpha > 0$ for the matrix A, it holds the estimate*

$$\|u_1\| \le \frac{1}{\sqrt{1+\hat{w}_{k+1}\alpha\tau}}\|u_0\| < \|u_0\| \qquad \forall\, \tau > 0, \tag{32}$$

where $\hat{w}_{k+1} > 0$ denotes the weight of the $(k+1)$-point Gauß–Radau formula on the unit interval $[-1,1]$ which is related to the point $\hat{t}_{k+1} = 1$. Thus, the dG(k)-method is A-stable.

Proof. We apply the dG(k)-method (23) with $F(t,u) = Au$ and M as the identity matrix to problem (30) and obtain for all $v_\tau \in \mathbb{P}_k^{dc}(\mathcal{M}_\tau)$ the equation

$$\sum_{n=1}^{N}\int_{I_n} \langle d_t u_\tau(t), v_\tau(t)\rangle\, dt + \sum_{n=1}^{N}\left\langle [u_\tau]_{n-1}, v_{n-1}^+\right\rangle = \int_0^T \langle Au_\tau(t)), v_\tau(t)\rangle\, dt.$$

We choose v_τ such that $v_\tau = u_\tau$ on I_1 and $v_\tau = 0$ outside of I_1 and get by means of (31) and the properties of the $(k+1)$-point Gauß–Radau integration formula

$$\int_{I_1} \langle d_t u_\tau(t), u_\tau(t)\rangle\, dt + \left\langle [u_\tau]_0, u_0^+\right\rangle \le -\alpha \int_{I_1} \|u_\tau(t))\|^2\, dt \le -\alpha\frac{\tau_n}{2}\hat{w}_{k+1}\|u_1\|^2,$$

which implies

$$\frac{1}{2}\|u_1\|^2 - \frac{1}{2}\|u_0^+\|^2 + \left\langle u_0^+ - u_0, u_0^+\right\rangle \le -\alpha\frac{\tau_n}{2}\hat{w}_{k+1}\|u_1\|^2.$$

Rearranging the terms yields

$$\left(1 + \alpha\tau_n\hat{w}_{k+1}\right)\|u_1\|^2 \le -\|u_0^+\|^2 + 2\left\langle u_0, u_0^+\right\rangle \le \|u_0\|^2,$$

which proves (32). □

Remark 4 *If we use [1, Sect. 25.4.31] and the relationship between the left-sided and right-sided Gauß–Radau formula, we obtain the formula*

$$\hat{w}_{k+1} = \frac{2}{(k+1)^2}.$$

From Lemma 3 we can easily derive the so-called *strong A-stability* of the dG(k)-method which is also called *L-stability* in [4, Chapter IV.3].

Lemma 5 (L-stability) *Let $u_0 \in \mathbb{C}$ denote the initial value of the complex model problem*

$$\begin{array}{l} \text{Find } u : [0,T] \to \mathbb{C} \text{ such that} \\ \begin{aligned} d_t u(t) &= \lambda u(t) \qquad \forall\, t \in (0,T) \\ u(0) &= u_0 \end{aligned} \end{array} \tag{33}$$

for some given $\lambda \in \mathbb{C}$. Furthermore, let $u_1(\lambda, \tau) := u_\tau|_{I_1}(\tau)$ with $I_1 := (0, \tau]$ be the left-sided value at $t = \tau$ of the discrete solution u_τ of the dG(k)-method applied to (33)*. Then, it holds*

$$\lim_{Re(\lambda)\tau \to -\infty} \frac{|u_1(\lambda, \tau)|}{|u_0|} = 0, \tag{34}$$

i.e., the dG(k)-method is L-stable.

Proof. Using real and imaginary parts we introduce:

$$\lambda = \alpha + i\beta, \qquad A := \begin{pmatrix} -\alpha & -\beta \\ \beta & -\alpha \end{pmatrix}, \qquad \vec{u}_\tau(t) := \begin{pmatrix} \mathrm{Re}(u_\tau(t)) \\ \mathrm{Im}(u_\tau(t)) \end{pmatrix}, \qquad \vec{u}_0 := \begin{pmatrix} \mathrm{Re}(u_0) \\ \mathrm{Im}(u_0) \end{pmatrix}.$$

Then, the complex model problem (33) is equivalent to the system: *Find $\vec{u}_\tau : [0,T] \to \mathbb{R}^2$ such that*

$$d_t \vec{u}_\tau(t) = A\vec{u}_\tau(t) \qquad \forall\, t \in (0,T), \qquad \vec{u}_\tau(0) = \vec{u}_0.$$

It holds $v^T A v = -\alpha \|v\|^2$ for all $v \in \mathbb{R}^2$. Thus, Lemma 3 yields

$$0 \le \frac{|u_1(\lambda,\tau)|}{|u_0|} = \frac{\|\vec{u}_\tau(\tau)\|}{\|\vec{u}_0\|} \le \frac{1}{\sqrt{1 + \hat{w}_{k+1}\alpha\tau}},$$

which proves (34). $\square$

4.2 The dG-C0(k)-method

The idea for this method was similar as for the change from the cGP(k)-method to the cGP-C1(k)-method, namely: take over as much as possible the variational construction of the method but make the discrete solution one level smoother in time such that in the time-marching process more information of the solution from the previous time interval can be reused in the new time interval in order to reduce the computational costs. That means in our case, that we have to change from a discontinuous solution in the dG(k)-method to a continuous solution in the dG-C0(k)-method. The result is a method that combines variational equations with left-sided collocation conditions at the rightmost Gauß–Radau points on the time intervals. The "**global dG-C0(k)-method** " reads:

Find $u_\tau \in \mathbb{P}_k^c(\mathcal{M}_\tau)$ *such that* $u_\tau(0) = u_0$,

$$\int_0^T \langle M d_t u_\tau(t), v_\tau(t)\rangle \, dt = \int_0^T \big\langle F\big(t, u_\tau(t)\big), v_\tau(t)\big\rangle \, dt \qquad \forall v_\tau \in \mathbb{P}_{k-2}^{dG}(\mathcal{M}_\tau) \tag{35}$$

and

$$M d_t u_n^- = F\big(t_n, u_n^-)\big), \qquad n = 1, \ldots, N, \tag{36}$$

where

$$d_t u_n^- := \lim_{t \to t_n - 0} d_t \left(u_\tau|_{I_n}\right)(t)$$

is the left-sided derivative in t_n. The dimension of the space $\mathbb{P}_k^c(\mathcal{M}_\tau)$ is given by $(k+1)N - (N-1) = kN + 1$. There are $k-1$ Galerkin conditions on each interval I_n, N points where the left-sided derivative fulfills the differential equation, and one initial condition. In total, we have $kN + 1$ conditions. Hence, there are as many conditions as degrees of freedom.

The discontinuity of the test space allows to calculate the solution of the dG-C0(k)-method in a time-marching process. We obtain the following "**I_n-problem of the exact dG-C0(k)-method** ":

Find $u_\tau|_{I_n} \in \mathbb{P}_k(I_n, V)$ *such that*

$$\begin{aligned} u_\tau(t_{n-1}) &= u_{n-1}^-, \\ M d_t u_\tau(t_n) &= F\big(t_n, u_\tau(t_n)\big), \\ \int_{I_n} M d_t u_\tau(t)\psi(t)\, dt &= \int_{I_n} F\big(t, u_\tau(t)\big)\psi(t)\, dt \qquad \forall \psi \in \mathbb{P}_{k-2}(I_n). \end{aligned} \tag{37}$$

If we apply the right-sided k-point Gauß–Radau quadrature formula, we get the following "**I_n-problem of the numerically integrated dG-C0(k)-method** ":

Find $u_\tau|_{I_n} \in \mathbb{P}_k(I_n, V)$ *such that*

$$\begin{aligned} u_\tau(t_{n-1}) &= u_{n-1}^-, \\ M d_t u_\tau(t_n) &= F\big(t_n, u_\tau(t_n)\big), \\ \sum_{j=1}^k \hat{w}_j M d_t u_\tau(t_{n,j})\psi(t_{n,j}) &= \sum_{j=1}^k \hat{w}_j F\big(t_{n,j}, u_\tau(t_{n,j})\big)\psi(t_{n,j}) \qquad \forall \psi \in \mathbb{P}_{k-2}(I_n). \end{aligned} \tag{38}$$

Remark 6 *We will prove in Section 6 that, for* $k \geq 2$*, the numerically integrated versions of the dG(*$k-1$*)-method and the dG-C0(k)-method generate the same left-sided values at the endpoints of the time intervals. This implies that the dG-C0(k)-methods are also A-stable and L-stable. Moreover, the numerical solution based on the dG-C0(k)-method can be easily* *computed from the solution generated by the dG(k−1)-method. Therefore, we will not present a practical scheme for the dG-C0(k)-method.*

In the following lemma, we will prove that the I_n-problems of the numerically integrated dG-C0(k)-method are equivalent to a collocation method as well as to a cGP(k)-method with a reduced Gauß–Radau integration rule.

Lemma 7 *Let $u_\tau \in C(I,V)$ denote the globally continuous function such that its restriction to time interval $I_n = (t_{n-1}, t_n]$ is the solution of the numerically integrated I_n-problem (38) of the dG-C0(k)-method. Then, the solution $u_\tau|_{I_n} \in \mathbb{P}_k(I_n, V)$ of (38) can also be characterized as the solution of the collocation problem according to the k right-sided Gauß–Radau-points $t_{n,1}, \ldots, t_{n,k} = t_n$*

$$\begin{aligned} u_\tau(t_{n-1}) &= u_{n-1}, \\ M d_t u_\tau(t_{n,j}) &= F\big(t_{n,j}, u_\tau(t_{n,j})\big) \qquad \forall\, j = 1, \ldots, k \end{aligned} \tag{39}$$

or the numerically integrated cGP(k)-method according to the k-point Gauß–Radau quadrature formula

$$\begin{aligned} u_\tau(t_{n-1}) &= u_{n-1}, \\ \sum_{j=1}^{k} \hat{w}_j M d_t u_\tau(t_{n,j}) \psi(t_{n,j}) &= \sum_{j=1}^{k} \hat{w}_j F\big(t, u_\tau(t_{n,j})\big) \psi(t_{n,j}) \qquad \forall \psi \in \mathbb{P}_{k-1}(I_n). \end{aligned} \tag{40}$$

Proof. First, we will prove the collocation conditions (39). The second equation line in (38) shows this condition for $j = k$. This implies

$$\sum_{j=1}^{k-1} \hat{w}_j M d_t u_\tau(t_{n,j}) \psi(t_{n,j}) = \sum_{j=1}^{k-1} \hat{w}_j F\big(t_{n,j}, u_\tau(t_{n,j})\big) \psi(t_{n,j}) \qquad \forall\, \psi \in \mathbb{P}_{k-2}(I_n).$$

In order to prove (39) for $j = i \in \{1, \ldots, k-1\}$, we choose the test function $\psi_i \in \mathbb{P}_{k-2}(I_n)$ defined by the $k-1$ conditions $\psi_i(t_{n,i}) = 1$ and $\psi_i(t_{n,j}) = 0$ for all $j \in \{1, \ldots, k-1\} \setminus \{i\}$ and obtain the collocation condition (39) for $j = i$. Finally, we show (40). For each $i \in \{1, \ldots, k\}$, let the function $\psi_i \in \mathbb{P}_{k-1}(I_n)$ be defined by the k conditions $\psi_i(t_{n,i}) = 1$ and $\psi_i(t_{n,j}) = 0$ for all $j \in \{1, \ldots, k\} \setminus \{i\}$. These functions form a basis of the polynomial space $\mathbb{P}_{k-1}(I_n)$ and the variational condition in (40) is equivalent to the k equations for the basis functions ψ_i. The equation (40) for $\psi = \psi_i$ is equivalent to the collocation equation (39) for $j = i$. Thus, the problems (40) and (39) are equivalent to problem (38). □

5 Connection between cGP($k-1$) and cGP-C1(k)

We will present an important connection between the solutions of discrete cGP($k-1$)-problem and the discrete cGP-C1(k)-problem.

Theorem 8 *Let u_τ^C and u_τ^{C1} denote the solutions of the numerically integrated variants of the discrete methods cGP($k-1$) and cGP-C1(k), respectively. Assume that the time steps τ_n are sufficiently small such that the I_n-problems of both methods have a unique solution for all $n = 1, \dots, N$. Then, for all time intervals $I_n = (t_{n-1}, t_n]$, it holds*

$$u_\tau^{C1}(t) = u_\tau^C(t) + a_n \zeta_n(t) \qquad \forall t \in I_n, \tag{41}$$

where the vector $a_n \in V$ can be computed as

$$a_n := M^{-1}\Big\{F\big(t_n, u_\tau^C(t_n)\big) - M d_t u_\tau^C(t_n)\Big\} \tag{42}$$

and

$$\zeta_n(t) := \frac{\tau_n}{2}\hat{\zeta}(\hat{t}) \qquad \text{with} \qquad \hat{t} := T_n^{-1}(t),$$

where $T_n : (-1,1] \to I_n$ denotes the affine reference mapping and $\hat{\zeta} \in P_k([-1,1])$ is the polynomial defined by the conditions $\hat{\zeta}'(1) = 1$ and $\hat{\zeta}(\hat{t}_j) = 0$ for all $j = 0, \dots, k-1$ with $\hat{t}_j$ denoting the integration points of the k-point Gauß–Lobatto formula.

Proof. We define $z(t) := u_\tau^C(t) + a_n\zeta_n(t)$ and show that this z fulfills all conditions in (20) on the solution of discrete cGP-C1(k)-problem. Since this solution is unique, the proof is done. Due to $\zeta_n(t_{n,j}) = 0$ for all Gauß–Lobatto points $t_{n,j}$, we have

$$z(t_{n,j}) = u_\tau^C(t_{n,j}), \qquad j = 0, \dots, k-1.$$

From $u_\tau^C(t_{n-1}) = u_{n-1}$ and $t_{n,0} = t_{n-1}$ we obtain $z(t_{n-1}) = u_{n-1}$. Now we consider the numerically integrated variational conditions. Let $\psi \in \mathbb{P}_{k-3}(I_n)$ be arbitrary. Then, we get

$$\begin{aligned}\sum_{j=0}^{k-1} \hat{w}_j M d_t z(t_{n,j})\psi(t_{n,j}) &= \sum_{j=0}^{k-1} \hat{w}_j M d_t u_\tau^C(t_{n,j})\psi(t_{n,j}) + \sum_{j=0}^{k-1} \hat{w}_j M a_n \zeta_n'(t_{n,j})\psi(t_{n,j}) \\ &= \sum_{j=0}^{k-1} \hat{w}_j F\big(t_{n,j}, u_\tau^C(t_{n,j})\big)\psi(t_{n,j}) + \frac{2}{\tau_n} M a_n \int_{I_n} \zeta_n'(t)\psi(t)\,dt\end{aligned}$$

where we have used that u_τ^C solves the numerically integrated I_n-problem (7) of the cGP($k-1$)-method and, for the last sum, that the k-point Gauß–Lobatto formula is exact for polynomials of degree less than or equal to $2k-3$. An integration by parts shows that

$$\begin{aligned}\int_{I_n} \zeta_n'(t)\psi(t)\,dt &= -\int_{I_n} \zeta_n(t)\psi'(t)\,dt + \zeta_n(t_n)\psi(t_n) - \zeta_n(t_{n-1})\psi(t_{n-1}) \\ &= -\frac{\tau_n}{2}\sum_{j=0}^{k-1} \hat{w}_j \zeta_n(t_{n,j})\psi'(t_{n,j}) = 0\end{aligned}$$

where we used that t_{n-1} and t_n are quadrature points. If we summarize the above results we obtain

$$\sum_{j=0}^{k-1} \hat{w}_j M d_t z(t_{n,j})\psi(t_{n,j}) = \sum_{j=0}^{k-1} \hat{w}_j F\big(t_{n,j}, u_\tau^C(t_{n,j})\big)\psi(t_{n,j}) = \sum_{j=0}^{k-1} \hat{w}_j F\big(t_{n,j}, z(t_{n,j})\big)\psi(t_{n,j}),$$

i.e., the function z fulfills the variational conditions in (20).

Now, we show that the function z satisfies the differential equation in t_n. Indeed, using (42) and $\zeta_n'(t_n) = 1$, we get

$$\begin{aligned} M d_t z(t_n) &= M d_t u_\tau^C(t_n) + M a_n \zeta_n'(t_n) = M d_t u_\tau^C(t_n) + \Big\{ F\big(t_n, u_\tau^C(t_n)\big) - M d_t u_\tau^C(t_n) \Big\} \\ &= F\big(t_n, z(t_n)\big). \end{aligned}$$

In order to proof that z fulfills the differential equation in t_{n-1}, we use an additional relation. To this end, we choose in the discrete cGP($k-1$)-problem as test function the polynomial $\varrho_n \in \mathbb{P}_{k-2}$ which vanishes in all inner Gauß–Lobatto points $t_{n,j}$, $j = 1, \ldots, k-2$, and which satisfies $\varrho_n(t_n) = 1$. Since ϱ_n is an admissible test function in the I_n-problem of the cGP($k-1$)-method, we obtain from (7) with $k \mapsto k-1$ and exploiting the fact that the terms with index $j = 1, \ldots, k-2$ vanish the following equation

$$\begin{aligned} &\hat{w}_0 M d_t u_\tau^C(t_{n-1}) \varrho_n(t_{n-1}) + \hat{w}_{k-1} M d_t u_\tau^C(t_n) \varrho_n(t_n) \\ &\qquad = \hat{w}_0 F\big(t_{n-1}, u_\tau^C(t_{n-1})\big) \varrho_n(t_{n-1}) + \hat{w}_{k-1} F\big(t_n, u_\tau^C(t_n)\big) \varrho_n(t_n) \end{aligned}$$

which simplifies by means of $\hat{w}_0 = \hat{w}_{k-1} \neq 0$ and $\varrho_n(t_n) = 1$ to

$$\varrho_n(t_{n-1}) \Big\{ M d_t u_\tau^C(t_{n-1}) - F\big(t_{n-1}, u_\tau^C(t_{n-1})\big) \Big\} = F\big(t_n, u_\tau^C(t_n)\big) - M d_t u_\tau^C(t_n) = M a_n.$$

This implies with the definition of z

$$\begin{aligned} M d_t z(t_{n-1}) &= M d_t u_\tau^C(t_{n-1}) + M a_n \zeta_n'(t_{n-1}) \\ &= M d_t u_\tau^C(t_{n-1}) + \zeta_n'(t_{n-1}) \varrho_n(t_{n-1}) \Big\{ M d_t u_\tau^C(t_{n-1}) - F\big(t_{n-1}, u_\tau^C(t_{n-1})\big) \Big\}. \end{aligned} \tag{43}$$

We show that $\zeta_n'(t_{n-1}) \varrho_n(t_{n-1}) = -1$. Indeed, an integration by parts results in

$$\begin{aligned} 0 &= \int_{I_n} \zeta_n(t) \varrho_n'(t)\, dt = - \int_{I_n} \zeta_n'(t) \varrho_n(t)\, dt + \zeta_n(t_n) \varrho_n(t_n) - \zeta_n(t_{n-1}) \varrho_n(t_{n-1}) \\ &= -\frac{\tau_n}{2} \hat{w}_0 \zeta_n'(t_{n-1}) \varrho_n(t_{n-1}) - \frac{\tau_n}{2} \hat{w}_{k-1} \zeta_n'(t_n) \varrho_n(t_n) \\ &= -\frac{\tau_n}{2} \hat{w}_0 \zeta_n'(t_{n-1}) \varrho_n(t_{n-1}) - \frac{\tau_n}{2} \hat{w}_{k-1} \end{aligned}$$

due to the properties of ζ_n and ϱ_n. With $\hat{w}_0 = \hat{w}_{k-1} \neq 0$, we conclude $\zeta_n'(t_{n-1}) \varrho_n(t_{n-1}) = -1$. From (43) and $z(t_{n-1}) = u_\tau^C(t_{n-1})$ we obtain that the differential equation holds for z also in the point t_{n-1}.

Thus, we have shown that the function z fulfills all conditions in (20) for the solution of the I_n-problem of the cGP-C1(k)-method. By the uniqueness assumption it follows $z = u_\tau^{C1}$ which concludes the proof. □

Remark 9 *The existence and uniqueness of the solution of exact I_n-problem of the cGP(k)-method has been shown in [7, Th. 6.5] for a sufficiently small time step by means of Banach's fixed point theorem. The proof can be generalized in a straightforward way also to the numerically integrated versions of the cGP($k-1$)-method and the cGP-C1(k)-method.*

6 Connection between dG($k-1$) and dG-C0(k)

Similar to Section 5 we will prove now a relation between the solutions of discrete dG($k-1$)- and dG-C0(k)-problem.

Theorem 10 *Let u_τ^{dG} and u_τ^{C0} denote the solutions of the numerically integrated variants of the discrete methods dG(k − 1) and dG-C0(k), respectively. Assume that the time steps τ_n are sufficiently small such that the I_n-problems of both methods have a unique solution for all $n = 1, \dots, N$. Then, for all time intervals $I_n = (t_{n-1}, t_n]$, it holds*

$$u_\tau^{C0}(t) = u_\tau^{dG}(t) + b_n \vartheta_n(t) \qquad \forall t \in I_n, \tag{44}$$

where the vector $b_n \in V$ can be computed as

$$b_n := \frac{1}{\vartheta_n(t_{n-1})} \Big\{ u_{n-1}^- - u_\tau^{dG}(t_{n-1}) \Big\} \tag{45}$$

and

$$\vartheta_n(t) := \frac{\tau_n}{2} \hat{\vartheta}(\hat{t}) \qquad \text{with} \qquad \hat{t} := T_n^{-1}(t),$$

where $T_n : (-1, 1] \to I_n$ denotes the affine reference mapping and $\hat{\vartheta} \in P_k([-1, 1])$ is the polynomial defined by the conditions $\hat{\vartheta}'(1) = 1$ and $\hat{\vartheta}(\hat{t}_j) = 0$ for all $j = 1, \dots, k$ with $\hat{t}_j$ denoting the integration points of the k-point right-sided Gauß–Radau formula.

Proof. We define $z(t) := u_\tau^{dG}(t) + b_n \vartheta_n(t)$ and show that this z fulfills all conditions in (38) on the solution of the I_n-problem of the dG-C0(k)-method. Since this solution is unique, the proof is done. Due to $\vartheta_n(t_{n,j}) = 0$ for all Gauß–Radau points $t_{n,j}$, we have

$$z(t_{n,j}) = u_\tau^{dG}(t_{n,j}) \qquad \forall\, j = 1, \dots, k. \tag{46}$$

First we consider the condition at t_{n-1} and obtain by (45)

$$z(t_{n-1}) = u_\tau^{dG}(t_{n-1}) + b_n \vartheta_n(t_{n-1}) = u_\tau^{dG}(t_{n-1}) + \Big\{ u_{n-1}^- - u_\tau^{dG}(t_{n-1}) \Big\} = u_{n-1}^-.$$

Now we look at the variational conditions in (38). Choosing $\psi \in \mathbb{P}_{k-2}(I_n)$ arbitrarily, we get

$$\begin{aligned} \sum_{j=1}^{k} \hat{w}_j M d_t z(t_{n,j}) \psi(t_{n,j}) &= \sum_{j=1}^{k} \hat{w}_j M d_t u_\tau^{dG}(t_{n,j}) \psi(t_{n,j}) + \sum_{j=1}^{k} \hat{w}_j M b_n \vartheta_n'(t_{n,j}) \psi(t_{n,j}) \\ &= \sum_{j=1}^{k} \hat{w}_j F\big(t_{n,j}, u_\tau^{dG}(t_{n,j})\big) \psi(t_{n,j}) - \frac{2}{\tau_n} M \big(u_\tau^{dG}(t_{n-1}) - u_{n-1}^- \big) \psi(t_{n-1}) \\ &\quad + \frac{2}{\tau_n} M b_n \int_{I_n} \vartheta_n'(t) \psi(t)\, dt, \end{aligned} \tag{47}$$

where we use that u_τ^{dG} solves the numerically integrated I_n-problem (26) of the dG($k-1$)-method and that the k-point Gauß–Radau formula is exact for polynomials of degree less than or equal to $2k-2$. Using an integration by parts and exploiting that $\vartheta_n(t_{n,j}) = 0$ for $j = 1, \dots, k$, we obtain

$$\int_{I_n} \vartheta_n'(t)\psi(t)\,dt = -\int_{I_n} \vartheta_n(t)\psi'(t)\,dt + \vartheta_n(t_n)\psi(t_n) - \vartheta_n(t_{n-1})\psi(t_{n-1})$$
$$= -\frac{\tau_n}{2}\sum_{j=1}^{k} \hat{w}_j \vartheta_n(t_{n,j})\psi'(t_{n,j}) - \vartheta_n(t_{n-1})\psi(t_{n-1}) = -\vartheta_n(t_{n-1})\psi(t_{n-1})$$

since t_n is a quadrature point. This result and the definition of b_n yield

$$-M\big(u_\tau^{dG}(t_{n-1}) - u_{n-1}^-\big)\psi(t_{n-1}) + Mb_n \int_{I_n} \vartheta_n'(t)\psi(t)\,dt$$
$$= -M\big(u_\tau^{dG}(t_{n-1}) - u_{n-1}^-\big)\psi(t_{n-1}) - M\Big\{u_{n-1}^- - u_\tau^{dG}(t_{n-1})\Big\}\psi(t_{n-1}) = 0.$$

Hence, the last two term in (47) cancel and we get with (46)

$$\sum_{j=1}^{k} \hat{w}_j M d_t z(t_{n,j})\psi(t_{n,j}) = \sum_{j=1}^{k} \hat{w}_j F\big(t_{n,j}, u_\tau^{dG}(t_{n,j})\big)\psi(t_{n,j}) = \sum_{j=1}^{k} \hat{w}_j F\big(t_{n,j}, z(t_{n,j})\big)\psi(t_{n,j}),$$

i.e., z fulfills the variational conditions in (38).

It remains to check whether z fulfills the differential equation in t_n with the left-sided derivative. First, we derive an auxiliary equation. Let $\varrho_n \in \mathbb{P}_{k-1}$ be that polynomial defined by $\varrho_n(t_{n,k}) = 1$ and $\varrho(t_{n,j}) = 0$ for $j = 1, \dots, k-1$. Due to $t_{n,k} = t_n$ and the properties of ϑ_n and ϱ_n, we have

$$0 = \frac{\tau_n}{2}\sum_{j=1}^{k} \hat{w}_j \vartheta_n(t_{n,j})\varrho_n'(t_{n,j}) = \int_{I_n} \vartheta_n(t)\varrho_n'(t)\,dt$$
$$= -\int_{I_n} \vartheta_n'(t)\varrho(t)\,dt + \vartheta_n(t_n)\varrho_n(t_n) - \vartheta_n(t_{n-1})\varrho_n(t_{n-1}) = -\frac{\tau_n}{2}\hat{w}_k - \vartheta_n(t_{n-1})\varrho_n(t_{n-1}).$$

This results in

$$\vartheta_n(t_{n-1})\varrho_n(t_{n-1}) = -\frac{\tau_n}{2}\hat{w}_k.$$

Now we choose ϱ_n as a test function in the I_n-problem (26) with $k \mapsto k-1$, i.e., for the dG$(k-1)$-method, and get due to the properties of ϱ_n

$$\frac{\tau_n}{2}\hat{w}_k F\big(t_n, u_\tau^{dG}(t_n)\big) = \frac{\tau_n}{2}\hat{w}_k M d_t u_\tau^{dG}(t_n) + M\Big\{u_\tau^{dG}(t_{n-1}) - u_{n-1}^-\Big\}\varrho_n(t_{n-1})$$
$$= \frac{\tau_n}{2}\hat{w}_k M d_t u_\tau^{dG}(t_n) - Mb_n\vartheta_n(t_{n-1})\varrho_n(t_{n-1})$$
$$= \frac{\tau_n}{2}\hat{w}_k M d_t u_\tau^{dG}(t_n) + \frac{\tau_n}{2}\hat{w}_k M b_n.$$

Dividing this by $\hat{w}_k\tau_n/2$, we obtain with (46) and $\vartheta_n'(t_n) = 1$

$$F\big(t_n, z(t_n)\big) = F\big(t_n, u_\tau^{dG}(t_n)\big) = M d_t u_\tau^{dG}(t_n) + Mb_n\vartheta_n'(t_n) = M d_t z(t_n).$$

Hence, z fulfills the differential equation in t_n.

Thus, we have shown that the function z fulfills all conditions in (38) for the solution of the I_n-problem of the dG-C0(k)-method. By the uniqueness assumption it follows that z coincides with u_τ^{C0} which concludes the proof. □

Let us note that Remark 9 on the proof of existence and uniqueness of the discrete solutions can be stated in the same way also for the dG(k)- and dG-C0(k)-method.

Remark 11 *The results of Theorem 8 and 10 show that we can compute the discrete solutions of the new modified methods by means of a post-process applied to the discrete solutions of the original methods with an order reduced by one. This means*

$$u_\tau^{cGP\text{-}C1(k+1)}(t) = u_\tau^{cGP(k)}(t) + \zeta_n(t) M^{-1}\Big\{F\big(t_n, u_\tau^{cGP(k)}(t_n)\big) - M d_t u_\tau^{cGP(k)}(t_n)\Big\} \tag{48}$$

and

$$u_\tau^{dG\text{-}C0(k+1)}(t) = u_\tau^{dG(k)}(t) + \frac{\vartheta_n(t)}{\vartheta_n(t_{n-1})}\Big\{u_{n-1}^- - u_\tau^{dG(k)}(t_{n-1})\Big\}. \tag{49}$$

Thus, we get the accuracy of the higher order methods cGP-C1($k+1$) or dG-C0($k+1$) with nearly the same computational costs as for the lower order methods cGP-C1(k) or dG-C0(k), respectively. Moreover, we can use the difference between the higher order and lower order solution as an error indicator for the control of the time step size.

7 Error estimates for the cGP-C1- and dG-C0-method

In the following, we want to present a unified error analysis which covers simultaneously the case of the cGP-C1(k)- and dG-C0(k)-method. To this end, we shift the indices of the above used k-point Gauß–Lobatto formula to the notation $t_{n,j} \in [t_{n-1}, t_n]$ and $\hat{w}_j$, $j = 1, \dots, k$, for the integration points and the weights. So the right-sided Gauß–Radau and the Gauß–Lobatto formula with k points use the same notation. Let r_{ex} denote the maximum degree of a polynomial such that the used integration formula is exact. Then, we known that

$$r_{\text{ex}} := \begin{cases} 2k-3, & \text{for the } k\text{-point Gauß–Lobatto formula,} \\ 2k-2, & \text{for the } k\text{-point Gauß–Radau formula.} \end{cases} \tag{50}$$

Moreover, it is well-known that the quadrature error for an arbitrary function $\varphi \in C^{r+1}(I_\ell, V)$ on an interval I_ℓ can be estimated by

$$\left\| \int_{I_\ell} \varphi(t)\,dt - \frac{\tau_\ell}{2}\sum_{j=1}^{k} \hat{w}_j \varphi(t_{\ell,j}) \right\|_V \le C_q \frac{\tau_\ell^{r+2}}{(r+1)!} |\varphi|_{C^{r+1}(I_\ell,V)} \qquad \forall\, r \le r_{\text{ex}}, \tag{51}$$

where the vector-valued integral and quadrature formula are defined in a component-wise sense. This estimate will be an essential tool in the proof of the following theorem.

Theorem 12 *Let u_τ denote either the solution of the cGP-C1(k)-method with $k \ge 3$ or of the dG-C0(k)-method with $k \ge 2$ and let u be the continuous solution of* (1) *which is* *assumed to be sufficiently smooth. Furthermore, assume that the right-hand side function* $\widetilde{F}(t,u) := M^{-1}F(t,u)$ *of the explicit form of problem* (1) *is Lipschitz continuous, i.e., there exists a positive constant* L *such that*

$$\big\|\widetilde{F}(t,u_1) - \widetilde{F}(t,u_2)\big\|_V \le L\,\|u_1 - u_2\|_V \qquad \forall\, u_1, u_2 \in V,\ t \in [0,T], \tag{52}$$

and that the maximum time step size τ *satisfies that* $L\tau$ *is sufficiently small. Then, for the error* $e(t) := u(t) - u_\tau(t)$ *on a local interval* $I_n = (t_{n-1}, t_n]$*, it holds the estimate*

$$\|e\|_{C(I_n,V)} \le C_{\exp}\Big\{\frac{2C_q}{k+1}t_{n-1} + 2\Big\} \max_{\ell=1,\dots,n} \frac{\tau_\ell^{k+1}}{k!}\|u\|_{C^{k+2}(I_\ell,V)}, \qquad n = 1,\dots,N, \tag{53}$$

where the constant $C_{\exp} := \exp(2Lt_{n-1})$ *does not dependent on* τ*,* u *and* n *but it depends in general exponentially on the length* T *of the time interval and the Lipschitz constant* L*.*

Proof. We obtain for $t \in I_n = (t_{n-1}, t_n]$ the relation

$$e(t) = \int_{t_0}^{t} d_t e(s)\,ds = \sum_{\ell=1}^{n-1}\int_{I_\ell} d_t e(s)\,ds + \int_{t_{n-1}}^{t} d_t e(s)\,ds, \tag{54}$$

where the sum is defined as zero if $n = 1$. At first, we consider an arbitrary summand from the sum and get

$$S_\ell := \int_{I_\ell} d_t e(s)\,ds = \int_{I_\ell} d_t u(s)\,ds - \int_{I_\ell} d_t u_\tau(s)\,ds.$$

Since $u_\tau|_{I_n} \in P_k(I_n, V)$, the term $d_t u_\tau$ is integrated exactly by the quadrature formula and by means of (51) we can estimate

$$\|S_\ell\|_V \le \left\|\frac{\tau_\ell}{2}\sum_{j=1}^{k}\hat{w}_j\big(d_t u(t_{\ell,j}) - d_t u_\tau(t_{\ell,j})\big)\right\|_V + C_q\frac{\tau_\ell^{r+2}}{(r+1)!}|d_t u|_{C^{r+1}(I_\ell,V)}.$$

Using the collocation conditions of the method, see (21) or (39), and the Lipschitz continuity for $\widetilde{F}$ we obtain

$$\begin{aligned}
\|S_\ell\|_V &\le \frac{\tau_\ell}{2}\sum_{j=1}^{k}\hat{w}_j\big\|\widetilde{F}\big(t_{\ell,j}, u(t_{\ell,j})\big) - \widetilde{F}\big(t_{\ell,j}, u_\tau(t_{\ell,j})\big)\big\|_V + C_q\frac{\tau_\ell^{r+2}}{(r+1)!}|u|_{C^{r+2}(I_\ell,V)} \\
&\le \frac{L\tau_\ell}{2}\sum_{j=1}^{k}\hat{w}_j\big\|u(t_{\ell,j}) - u_\tau(t_{\ell,j})\big\|_V + C_q\frac{\tau_\ell^{r+2}}{(r+1)!}|u|_{C^{r+2}(I_\ell,V)}.
\end{aligned}$$

With the abbreviation $e_\ell := \|e\|_{C(I_\ell,V)}$ and the fact that $\sum_{j=1}^{k}\hat{w}_j = 2$ this implies the estimate

$$\|S_\ell\|_V \le L\tau_\ell e_\ell + C_q\frac{\tau_\ell^{r+2}}{(r+1)!}|u|_{C^{r+2}(I_\ell,V)} \qquad \forall\, r \le r_{\text{ex}}. \tag{55}$$

In order to bound the remaining term of (54), we apply a different technique. The Gauß–Lobatto or Gauß–Radau formula can't be applied since not the whole subinterval I_n is covered by $(t_{n-1}, t]$ in the case $t < t_n$. We denote by $\Pi_{k-1}v \in P_{k-1}(I_n, V)$ the Lagrange interpolate of $v \in C(I_n, V)$ based on the k Gauß–Lobatto- or right-sided Gauß–Radau-points. Let $\hat{t} := T_n^{-1}(t) \in \hat{I}$ denote the reference value corresponding to $t \in I_n$ and $\hat{\phi}_j \in \mathbb{P}_k(\hat{I})$, $j = 1, \ldots, k$, are the Lagrange basis functions according to the integration points $\hat{t}_j \in [-1, 1]$. Then, the corresponding interpolatory quadrature formula reads

$$\int_{t_{n-1}}^{t} v(s)\, ds \sim Q_{n,t}(v) := \int_{t_{n-1}}^{t} \Pi_{k-1}v(s)\, ds = \frac{\tau_n}{2} \sum_{j=1}^{k} \widetilde{w}_j(\hat{t}) v(t_{n,j}) \tag{56}$$

with the $\hat{t}$-dependent weights

$$\widetilde{w}_j(\hat{t}) := \int_{-1}^{\hat{t}} \hat{\phi}_j(\hat{s})\, d\hat{s}, \qquad j = 1, \ldots, k.$$

This formula is obviously exact for all $v \in \mathbb{P}_{k-1}(I_n, V)$. Furthermore, from the well-known interpolation error estimate, we get for all $t \in I_n$

$$\left\| \int_{t_{n-1}}^{t} v(s)\, ds - Q_{n,t}(v) \right\|_V \leq \frac{\tau_n^{k+1}}{k!} |v|_{C^k(I_n,V)} \qquad \forall\, v \in C^k(I_n, V). \tag{57}$$

We are now able to estimate the remainder term in (54)

$$R_n(t) := \int_{t_{n-1}}^{t} d_t e(s)\, ds = \int_{t_{n-1}}^{t} d_t u(s)\, ds - \int_{t_{n-1}}^{t} d_t u_\tau(s)\, ds.$$

Since the quadrature formula (56) is exact for $d_t u_\tau \in P_{k-1}(I_n, V)$ we get

$$\begin{aligned} \|R_n(t)\|_V &\leq \|Q_{n,t}(d_t u) - Q_{n,t}(d_t u_\tau)\|_V + \frac{\tau_n^{k+1}}{k!} |d_t u|_{C^k(I_n,V)} \\ &\leq \|Q_{n,t}(d_t u - d_t u_\tau)\|_V + \frac{\tau_n^{k+1}}{k!} |u|_{C^{k+1}(I_n,V)}. \end{aligned}$$

The collocation conditions (21) or (39) and the Lipschitz continuity of $\widetilde{F}$ yield

$$\|R_n(t)\|_V \leq \frac{L\tau_n}{2} e_n \sum_{j=1}^{k} \left|\widetilde{w}_j(\hat{t})\right| + \frac{\tau_n^{k+1}}{k!} |u|_{C^{k+1}(I_n,V)}.$$

If we combine this estimate with the estimates (55) for $r = k \leq r_{\text{ex}}$ and introduce the constants

$$C_w := \sum_{j=1}^{k} \max_{\hat{t} \in \hat{I}} \left|\widetilde{w}_j(\hat{t})\right|, \qquad M_n(u) := \max_{\ell=1,\ldots,n} \frac{\tau_\ell^{k+1}}{k!} \|u\|_{C^{k+2}(I_\ell,V)},$$

we obtain

$$\begin{aligned} \|e(t)\|_V &\leq \sum_{\ell=1}^{n-1} \|S_\ell\|_V + \|R_n(t)\|_V \\ &\leq \sum_{\ell=1}^{n-1} \left\{ L\tau_\ell e_\ell + \frac{C_q}{k+1} \tau_\ell M_n(u) \right\} + \frac{L\tau_n}{2} C_w e_n + M_n(u). \end{aligned}$$

Taking the maximum on the left-hand side for $t \in I_n$, we get the inequality

$$e_n \leq L \sum_{\ell=1}^{n-1} \tau_\ell e_\ell + C_w \frac{L\tau_n}{2} e_n + \Big\{ \frac{C_q}{k+1} t_{n-1} + 1 \Big\} M_n(u).$$

If we now assume that the local time step size τ_n is sufficiently small such that

$$L\tau_n \leq \frac{1}{C_w},$$

the e_n-term on the right-hand side can be absorbed by the left-hand side which implies

$$e_n \leq 2L \sum_{\ell=1}^{n-1} \tau_\ell e_\ell + \Big\{ \frac{2C_q}{k+1} t_{n-1} + 2 \Big\} M_n(u).$$

Then, the application of the discrete Gronwall lemma yields the estimate (53). □

Remark 13 *The error estimate* (53) *of Theorem 12 justifies an adaptive choice of the time step size since a large local norm* $\|u\|_{C^{k+2}(I_\ell,V)}$ *of the exact solution can be compensated by a small time step size* τ_ℓ*. Furthermore, Theorem 12 shows that the maximum-norm of the discretization error is of optimal order* $k+1$ *for the methods cGP-C1(k) and dG-C0(k), respectively. Moreover, using the connection to the lower order methods cGP(*$k-1$*) and dG(*$k-1$*), we have proven their superconvergence order of* $k+1$ *at the integration points of the k-point Gauß–Lobatto- or right-sided Gauß–Radau quadrature formula.*

8 Numerical results

As a nonlinear model problem for numerical tests we choose the viscous Burgers equation where the space variable is one-dimensional. In order to be able to compute the true error we construct an academic test situation where we prescribe the exact solution as an analytic expression in the space and time variables and compute the associated right-hand side as well as boundary and initial data from the equations of the model problem. Thus, for the space domain $\Omega = (0,1)$ and the time interval $I = (0,T)$, we consider the problem

$$\frac{\partial u}{\partial t} u(x,t) - \varepsilon \frac{\partial^2 u}{\partial x^2}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = f(x,t) \qquad \text{in } \Omega \times I, \tag{58}$$

$$u(0,t) = g_0(t) \qquad \forall\, t \in I, \tag{59}$$

$$u(1,t) = g_1(t) \qquad \forall\, t \in I, \tag{60}$$

$$u(x,0) = u_0(x) \qquad \forall\, x \in \Omega, \tag{61}$$

where ε is a positive diffusion parameter and f, g_0, g_1 and u_0 are given data.

For the space discretization, let $V_h = \operatorname{span}\{b_0, \dots, b_{n+1}\}$ denote a finite element space where $\{b_0, \dots, b_{n+1}\}$ are the Lagrange basis functions of a suitable polynomial degree and b_0 and b_{n+1} correspond to the boundary points $x = 0$ and $x = 1$, respectively. Then, the semi-discrete solution in space $u_h(t) \in V_h$ can be represented by means of a nodal vector $\underline{u}(t) := (u_1(t), \dots, u_n(t)) \in V := \mathbb{R}^n$ as

$$u_h(t) = U_h(\underline{u}(t)) := \sum_{j=1}^{n} u_j(t) b_j + u_h^b(t), \tag{62}$$

where

$$u_h^b(t) := g_0(t) b_0 + g_1(t) b_{n+1}. \tag{63}$$

If we insert this ansatz in (58), multiply by b_i, integrate over Ω, and apply an integration by parts, we get for $i = 1, \dots, n$:

$$\sum_{j=1}^{n} (b_j, b_i)_\Omega \, d_t u_j(t) + \varepsilon \sum_{j=1}^{n} \big(b'_j, b'_i\big)_\Omega \, u_j(t) + \sum_{j=1}^{n} \big(U_h(\underline{u}(t))\, b'_j, b_i\big)_\Omega \, u_j(t) = r_i(t, \underline{u}(t)),$$

where $(\cdot,\cdot)_\Omega$ denotes the inner product of $L^2(\Omega)$ and

$$r_i(t, \underline{u}(t)) := (f(t), b_i)_\Omega - \Big(d_t u_h^b(t), b_i\Big)_\Omega - \varepsilon \Big((u_h^b(t))', b'_i\Big)_\Omega - \Big(U_h(\underline{u}(t))\,(u_h^b(t))', b_i\Big)_\Omega .$$

This is a nonlinear ODE-system of type (1) for the nodal vector $\underline{u}(t)$, i.e.,

$$\begin{aligned} M d_t \underline{u}(t) &= F\big(t, \underline{u}(t)\big) \qquad \forall\, t \in I, \\ \underline{u}(0) &= \underline{u}_0, \end{aligned} \tag{64}$$

where $\underline{u}_0$ is the nodal vector of the homogeneous part (w.r.t. to boundary values) of the Lagrange interpolation of the initial solution $u_0(x)$.

The i-th component of the vector-valued right-hand side F is:

$$F_i\big(t, \underline{u}(t)\big) = r_i(t, \underline{u}(t)) - \varepsilon \sum_{j=1}^{n} \big(b'_j, b'_i\big)_\Omega \, u_j(t) - \sum_{j=1}^{n} \big(U_h(\underline{u}(t))\, b'_j, b_i\big)_\Omega \, u_j(t).$$

If $\underline{u}(t)$ denotes the exact solution of the ODE-system (64), then the semi-discrete solution is $u_h(t) = U_h(\underline{u}(t)) \in V_h$ for $t \in I$. We will denote the full discrete solution as

$$u_\tau(t) = U_h(\underline{u}_\tau(t)) := \sum_{j=1}^{n} \left(\underline{u}_\tau\right)_j(t)\, b_j + u_h^b(t),$$

where $\underline{u}_\tau(t) \in \mathbb{R}^n$ is the time-discrete solution of the ODE-system (64). In our theory, we prove error estimates for the time error $\underline{e}(t) := \underline{u}(t) - \underline{u}_\tau(t)$ in the Euclidean norm $\|\cdot\|_{\mathbb{R}^n}$. However, in the following tables, we use for the error norm in time the numerically integrated $L^2(\Omega)$-norm, i.e.,

$$\|e(t)\|_{V_h} \approx \|u(t) - u_\tau(t)\|_{L^2(\Omega)} \qquad \text{and} \qquad \|e\|_{L^2(I,V_h)} := \left(\int_I \|e(t)\|_{V_h}^2 \, dt\right)^{1/2}.$$

As a test problem, we consider the Burgers equations (58)-(61) where the source term f, the boundary conditions g_0, g_1, and the initial condition u_0 have been computed from the prescribed exact solution

$$u(x,t) = \sin(2\pi x) + t \sin(10\pi t) \cos(3\pi x),$$

which is non-polynomial but very smooth with respect to x and t. Since we will not treat numerical problems caused by an unsuitable space discretization, we take the viscosity parameter as $\varepsilon = 1$. Then, the above described standard Galerkin discretization in space will work well. In order to check really the convergence orders of the time errors we try to keep the space error as small as possible. To this end, we have chosen a finite element space V_h based on conforming P_4-elements on a relatively fine uniform mesh consisting of 500 cells. Thus, the dimension of the resulting ODE-system was about 2000 such that from $u(t) \approx u_h(t)$ we have

$$\|e(t)\|_{V_h} \approx \|U_h(\underline{u}(t)) - U_h(\underline{u}_\tau(t))\|_{V_h} \sim h^{d/2}\|\underline{u}(t) - \underline{u}_\tau(t)\|_{\mathbb{R}^n}.$$

Therefore, the error norm $\|e(t)\|_{V_h}$ has the same order with respect to τ as the Euclidean norm of the error $\underline{e}(t) := \underline{u}(t) - \underline{u}_\tau(t)$ which was analyzed in our paper.

The final time in all calculations was set to $T = 1$. The nonlinear block systems, which appear in each time step, have been solved by Newton's method. In all computations its convergence was very fast. At most 3 Newton steps were needed to achieve a nonlinear residual less than 10^{-12}.

Table 1: Results for cGP(2), where $\tilde{e} := u - u_\tau^{\text{cGP-C1(3)}}$.

τ	$\|e\|_{L^2(I,V_h)}$		$\|d_t e\|_{L^2(I,V_h)}$		$\|e\|_{\ell_\infty(I,V_h)}$		$\|\tilde{e}\|_{L^2(I,V_h)}$		$\|d_t\tilde{e}\|_{L^2(I,V_h)}$	
	error	ord	error	ord	error	ord	error	ord	error	ord
1/5	2.892-1		9.170+0		4.791-2		1.891-1		6.076+0	
1/10	2.035-2	3.83	1.408+0	2.70	5.175-2	-0.11	2.564-2	2.88	1.318+0	2.20
1/20	6.062-3	1.75	8.182-1	0.78	4.323-3	3.58	1.832-3	3.81	2.020-1	2.71
1/40	7.867-4	2.95	2.094-1	1.97	3.833-4	3.50	1.358-4	3.75	3.156-2	2.68
1/80	1.006-4	2.97	5.257-2	1.99	2.528-5	3.92	9.484-6	3.84	4.549-3	2.79
1/160	1.266-5	2.99	1.315-2	2.00	1.609-6	3.97	7.009-7	3.76	6.892-4	2.72
theoret. order:		3		2		4		4		3

Table 2: Results for cGP(3), where $\tilde{e} := u - u_\tau^{\text{cGP-C1(4)}}$.

τ	$\|e\|_{L^2(I,V_h)}$		$\|d_t e\|_{L^2(I,V_h)}$		$\|e\|_{\ell_\infty(I,V_h)}$		$\|\tilde{e}\|_{L^2(I,V_h)}$		$\|d_t\tilde{e}\|_{L^2(I,V_h)}$	
	error	ord	error	ord	error	ord	error	ord	error	ord
1/5	5.490-2		3.088+0		1.269-1		6.982-2		3.135+0	
1/10	1.174-2	2.22	1.093+0	1.50	1.376-3	6.53	3.479-3	4.33	3.129-1	3.32
1/20	6.003-4	4.29	1.116-1	3.29	2.021-4	2.77	1.212-4	4.84	1.888-2	4.05
1/40	3.754-5	4.00	1.417-2	2.98	6.777-6	4.90	4.013-6	4.92	1.134-3	4.06
1/80	2.345-6	4.00	1.778-3	2.99	1.665-7	5.35	1.316-7	4.93	7.188-5	3.98
1/160	1.466-7	4.00	2.225-4	3.00	3.874-9	5.43	4.203-9	4.97	4.533-6	3.99
theoret. order:		4		3		6		5		4

Table 3: Results for cGP(4), where $\tilde{e} := u - u_\tau^{\text{cGP-C1(5)}}$.

τ	$\Vert e\Vert_{L^2(I,V_h)}$		$\Vert d_t e\Vert_{L^2(I,V_h)}$		$\Vert e\Vert_{\ell_\infty(I,V_h)}$		$\Vert\tilde{e}\Vert_{L^2(I,V_h)}$		$\Vert d_t\tilde{e}\Vert_{L^2(I,V_h)}$	
	error	ord	error	ord	error	ord	error	ord	error	ord
1/5	4.286-2		2.515+0		6.449-3		1.896-2		1.056+0	
1/10	5.198-4	6.37	7.600-2	5.05	8.143-4	2.99	4.989-4	5.25	5.326-2	4.31
1/20	4.429-5	3.55	1.100-2	2.79	1.045-5	6.28	7.450-6	6.07	1.549-3	5.10
1/40	1.414-6	4.97	7.014-4	3.97	1.431-7	6.19	1.301-7	5.84	5.393-5	4.84
1/80	4.438-8	4.99	4.404-5	3.99	1.716-9	6.38	2.281-9	5.83	1.930-6	4.80
1/160	1.388-9	5.00	2.756-6	4.00	2.247-11	6.25	4.312-11	5.73	7.666-8	4.65
theoret. order:		5		4		8		6		5

Tables 1–3 show the results for the error $e = u - u_\tau^{\text{cGP}(k)}$ obtained by the continuous Galerkin–Petrov method cGP(k) with $k \in \{2, 3, 4\}$. The predicted orders for the error and its time derivative in the integral-based norm $\Vert \cdot \Vert_{L^2(I,V_h)}$ (see Theorem 12) are confirmed. In particular, the "post-processed solution" $u_\tau^{\text{cGP-C1}(k+1)}$ shows one additional order of accuracy compared to the solution $u_\tau^{\text{cGP}(k)}$. We also present the time error $e = u - u_\tau^{\text{cGP}(k)}$ in the discrete maximum norm

$$\Vert e\Vert_{\ell_\infty(I,V_h)} := \max_{n=1,\ldots,N} \Vert e(t_n)\Vert_{V_h}.$$

Due to the fact that the solutions of cGP(k) and cGP-C1($k+1$) coincide at the discrete time points t_n we get at least a superconvergence of order $k+2$ for the cGP(k)-method in the discrete maximum norm. Comparing the absolute values of the error norms in Tables 1–3, we see that the higher order methods cGP(3) and cGP(4) achieve with a quite large time step size τ the same accuracy as the "lower order" method cGP(2) with a much smaller time step size.

For the heat equation, Aziz and Monk [2] proved superconvergence of order $2k$ for the cGP(k)-method in the discrete time points t_n which seems to be optimal. In case of our model problem, we can nearly observe the order $2k$ for cGP(2) and cGP(3) whereas for the cGP(4) this high order cannot be seen probably since the error in space is too high.

Tables 4–6 present the norms of time error $e := u - u_\tau^{\text{dG}(k)}$ for the discontinuous Galerkin method dG(k) with $k \in \{1, 2, 3\}$ and the associated "post-processed solution" $u_\tau^{\text{dG-C0(k+1)}}$. The experimental orders of convergence in the integral-based norm $\Vert \cdot \Vert_{L^2(I,V_h)}$ confirm the theoretical results. Due to the fact that the dG(k)-solution and dG-C0($k+1$)-solution coincide at the discrete time points t_n we get at least superconvergence of order $k+2$ for the dG(k)-method in the discrete maximum norm which can be seen in the tables.

In [8], for an abstract symmetric model problem like the heat equation, superconvergence of order $2k+1$ was proved for the dG(k)-method in the discrete time points t_n which seems to be optimal. In case of our model problem, we can nearly observe the order $2k+1$ for the dG(1) and dG(2) whereas for the dG(3)-method this high order cannot be seen probably due

Table 4: Results for dG(1), where $\tilde{e} := u - u_\tau^{\text{dG-C0(2)}}$.

τ	$\|e\|_{L^2(I,V_h)}$		$\|d_t e\|_{L^2(I,V_h)}$		$\|e\|_{\ell_\infty(I,V_h)}$		$\|\tilde{e}\|_{L^2(I,V_h)}$		$\|d_t\tilde{e}\|_{L^2(I,V_h)}$	
	error	ord	error	ord	error	ord	error	ord	error	ord
1/5	3.814-1		8.636+0		2.814-1		4.137-1		8.953+0	
1/10	1.962-1	0.96	1.036+1	-0.26	4.562-2	2.63	1.813-2	4.51	1.380+0	2.70
1/20	4.121-2	2.25	4.524+0	1.20	1.946-2	1.23	1.005-2	0.85	9.309-1	0.57
1/40	1.071-2	1.94	2.354+0	0.94	3.368-3	2.53	1.230-3	3.03	2.558-1	1.86
1/80	2.715-3	1.98	1.191+0	0.98	5.103-4	2.72	1.571-4	2.97	7.009-2	1.87
1/160	6.817-4	1.99	5.977-1	0.99	7.215-5	2.82	2.026-5	2.95	1.881-2	1.90
theoret. order:		2		1		3		3		2

Table 5: Results for dG(2), where $\tilde{e} := u - u_\tau^{\text{dG-C0(3)}}$.

τ	$\|e\|_{L^2(I,V_h)}$		$\|d_t e\|_{L^2(I,V_h)}$		$\|e\|_{\ell_\infty(I,V_h)}$		$\|\tilde{e}\|_{L^2(I,V_h)}$		$\|d_t\tilde{e}\|_{L^2(I,V_h)}$	
	error	ord	error	ord	error	ord	error	ord	error	ord
1/5	2.657-1		1.225+1		1.190-1		5.734-2		2.994+0	
1/10	1.514-2	4.13	1.405+0	3.12	2.330-2	2.35	1.603-2	1.84	1.196+0	1.32
1/20	5.471-3	1.47	1.095+0	0.36	1.275-3	4.19	9.298-4	4.11	1.316-1	3.18
1/40	6.911-4	2.98	2.809-1	1.96	9.997-5	3.67	6.458-5	3.85	1.799-2	2.87
1/80	8.635-5	3.00	7.056-2	1.99	5.104-6	4.29	4.254-6	3.92	2.400-3	2.91
1/160	1.078-5	3.00	1.766-2	2.00	2.356-7	4.44	2.752-7	3.95	3.151-4	2.93
theoret. order:		3		2		5		4		3

Table 6: Results for dG(3), where $\tilde{e} := u - u_\tau^{\text{dG-C0(4)}}$.

τ	$\|e\|_{L^2(I,V_h)}$		$\|d_t e\|_{L^2(I,V_h)}$		$\|e\|_{\ell_\infty(I,V_h)}$		$\|\tilde{e}\|_{L^2(I,V_h)}$		$\|d_t\tilde{e}\|_{L^2(I,V_h)}$	
	error	ord	error	ord	error	ord	error	ord	error	ord
1/5	5.083-2		3.290+0		6.193-2		5.732-2		2.769+0	
1/10	1.083-2	2.23	1.660+0	0.99	4.984-4	6.96	6.771-4	6.40	7.561-2	5.19
1/20	5.289-4	4.36	1.667-1	3.32	7.506-5	2.73	6.668-5	3.34	1.331-2	2.51
1/40	3.371-5	3.97	2.136-2	2.96	2.095-6	5.16	2.215-6	4.91	8.945-4	3.90
1/80	2.117-6	3.99	2.686-3	2.99	4.799-8	5.45	7.280-8	4.93	5.934-5	3.91
1/160	1.324-7	4.00	3.363-4	3.00	1.160-9	5.37	2.350-9	4.95	3.868-6	3.94
theoret. order:		4		3		7		5		4

to the influence of the error in space.

References

[1] M. Abramowitz and I. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Government Printing Office, Washington, D.C., 1964.

[2] A. K. Aziz and P. Monk. Continuous finite elements in space and time for the heat equation. *Math. Comp.*, 52(186):255–274, 1989.

[3] K. Eriksson, D. Estep, P. Hansbo, and C. Johnson. *Computational differential equations*. Cambridge University Press, Cambridge, 1996.

[4] E. Hairer and G. Wanner. *Solving ordinary differential equations. II*, volume 14 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, second edition, 1996. Stiff and differential-algebraic problems.

[5] S. Hussain, F. Schieweck, and S. Turek. Higher order galerkin time discretizations and fast multigrid solvers for the heat equation. Ergebnisberichte des Instituts für Angewandte Mathematik 416, Fakultät für Mathematik, TU Dortmund, 2010.

[6] S. Hussain, F. Schieweck, and S. Turek. A note on accurate and efficient higher order galerkin time stepping schemes for the nonstationary stokes equations. Ergebnisberichte des Instituts für Angewandte Mathematik 424, Fakultät für Mathematik, TU Dortmund, 2011.

[7] F. Schieweck. A-stable discontinuous Galerkin-Petrov time discretization of higher order. *J. Numer. Math.*, 18(1):25 – 57, 2010.

[8] V. Thomée. *Galerkin finite element methods for parabolic problems*, volume 25 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, second edition, 2006.