A note on the approximation properties of a nonconforming quadrilateral finite element

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Abstract

Recently, Cai, Douglas and Ye have proposed a new nonconforming point-value oriented quadrilateral finite element with the property that the integral mean value of the jump of a finite element function vanishes over each edge of the grid. For a corresponding nonconforming finite element discretisation of the Laplacian operator, this property guarantees an optimal estimate of the consistency error which does not depend on the variation of the shape of the quadrilateral mesh cells from the shape of a parallelogram. This is an advantage in comparison to the so-called "parametric" version of the "point-value oriented rotated bilinear" element introduced by Rannacher and Turek. However, we prove that for the new proposed quadrilateral element, the interpolation error is not of optimal order unless the mesh is "nearly" of parallelogram type.

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1 Introduction

As an analogue of the well-known nonconforming piecewise linear simplicial finite element of Crouzeix and Raviart (see [2]) in the case of quadrilateral or hexahedral elements, a so-called element of "nodal type" has been proposed and analysed for scalar equations by Hennart, Jaffre and Roberts in [3] and a

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so-called "rotated bilinear" element for the Stokes problem by Rannacher and Turek in [5]. Four different types of finite elements for the approximation of each component of the velocity have been discussed in [5]. These four types arise if one combines the two types "parametric" and "non-parametric" for the used transformation to the reference element with the two types "point-value oriented" and "mean-value oriented" for the choice of the degrees of freedom. The consistency error and the interpolation error for these four types of elements have been thoroughly investigated in a unified framework in [6].

The analysis for the new nonconforming quadrilateral element in [1] and [4] is mainly concentrated on the consistency error. By modification of the Rannacher-Turek element, a consistency error of optimal order can be guaranteed. For the analysis of the interpolation error, only the case of an affine reference transformation is treated and it is claimed that an extension to arbitrary shape regular quadrilateral elements is immediate. However, this statement is wrong! In this note, we prove by an explicit example that both the local and global interpolation error in general can be of non-optimal order unless the mesh is "nearly" of parallelogram type.

The plan of the paper is as follows. In Section 2, we introduce some notations and summarise the results that are known from the literature for the different types of the nonconforming quadrilateral element studied in [5] and [6]. Then, in Section 3 we recall from [1] the definition of the modified nonconforming finite element for which a consistency error of optimal order can be guaranteed. Finally, in Section 4 an explicit example is designed to show that both the local and global interpolation error can be of non-optimal order.

2 Known results about the Rannacher-Turek elements

For simplicity we restrict the presentation to the two-dimensional case. In order to explain the different types of elements proposed by Rannacher and Turek in [5], we have to introduce some notation. Let us consider the simple elliptic model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} &= g_N \quad \text{on } \Gamma_N, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ denotes a bounded polygonal domain with the boundary $\Gamma = \Gamma_D \cup \Gamma_N$. We assume that $\Gamma_D \cap \Gamma_N = \emptyset$ with $\text{meas}(\Gamma_D) > 0$ and $f \in L^2(\Omega)$, $g \in H^2(\Omega)$, $g_N \in H^{3/2}(\Gamma_N)$. We are looking for weak solutions $u \in V(g)$, where

$$V(g) := \{v \in H^1(\Omega) \ : \ v = g \text{ on } \Gamma_D\}.$$

Note that regularity results for elliptic problems on convex domains ensure that the weak solution u belongs to $V(g) \cap H^2(\Omega)$. Let $\mathcal{T}_h$ be a regular decomposition of the domain Ω into quadrilateral elements $K \in \mathcal{T}_h$ which are assumed to be shape-regular. The diameter of element K is denoted by h_K and the global mesh-size is defined as $h := \max_{K \in \mathcal{T}_h} h_K$. Let $\mathcal{E}$ denote the set of all edges of the elements $K \in \mathcal{T}_h$, $\mathcal{E}_0$ the set of all inner edges, and $\mathcal{E}_D$ the set of all boundary edges belonging to Γ_D (we assume that a boundary edge belongs either to Γ_D or to Γ_N). For $K \in \mathcal{T}_h$, let $F_K : \widehat{K} \to K$ be a bilinear bijective mapping between the reference element $\widehat{K} := [-1, +1]^2$ and K. Then, in both papers [3] and [5], the polynomial space of scalar shape functions on the reference element is defined by

$$\widehat{\mathcal{P}} := \text{span}\{1, \widehat{x}_1, \widehat{x}_2, \widehat{x}_1^2 - \widehat{x}_2^2\}.$$

Let us denote by $\mathcal{P}_K$ the local space of finite element functions restricted to the element $K \in \mathcal{T}_h$. Then, the finite element corresponding to the mesh cell K is called "parametric" if

$$\mathcal{P}_K = \left\{ p = \widehat{p} \circ F_K^{-1} \ : \ \widehat{p} \in \widehat{\mathcal{P}} \right\} . \tag{2}$$

In order to define the "non-parametric" version of this element, Rannacher and Turek have introduced in [5] a local coordinate system on the quadrilateral cell $K \in \mathcal{T}_h$ by connecting the midpoints of the opposite edges of K. Let us denote the corresponding new coordinates by $\xi = \xi(x_1, x_2)$ and $\eta = \eta(x_1, x_2)$. Then, the local space of shape functions for the "non-parametric" version of the nonconforming quadrilateral element is defined by

$$\mathcal{P}_K = \text{span}\{1, \xi, \eta, \xi^2 - \eta^2\} . \tag{3}$$

Now, we want to explain the two types "point-value oriented" and "mean-value oriented" for the choice of the degrees of freedom. For both types, each degree of freedom is uniquely associated with an edge $E \in \mathcal{E}$. Let $\mathcal{E}(K)$ denote the set of all edges $E \in \mathcal{E}$ with $E \subset \partial K$. Then, the local degrees of freedom with respect to element K are given by the local nodal functionals $N_{K,E} \in (C^0(\overline{K}))'$

for $E \in \mathcal{E}(K)$, where X' denotes the dual space of the space X. In case of point-value oriented finite elements, these local nodal functionals are defined as

$$N_{K,E}(v) := v\big|_K(m_E) \tag{4}$$

and in case of mean-value oriented finite elements by

$$N_{K,E}(v) := |E|^{-1} \int_E v\big|_K \, d\gamma\,, \tag{5}$$

where m_E denotes the midpoint of the edge E and $|E|$ its one-dimensional measure. The scalar finite element space $V_h(g)$ which approximates the solution space $V(g)$ is defined by

$$\begin{aligned} V_h(g) \quad := \{ \quad & v \in L^2(\Omega) \,:\, v\big|_K \in \mathcal{P}_K \quad \forall\, K \in \mathcal{T}_h, \quad N_{K,E}(v) = N_{K',E}(v) \\ & \forall\, E \in \mathcal{E}_0 \text{ with } E = \partial K \cap \partial K'\,,\ K, K' \in \mathcal{T}_h \\ & \text{and} \quad N_{K,E}(v) = N_{K,E}(g) \quad \forall\, E \in \mathcal{E}(K) \cap \mathcal{E}_D \ \} \end{aligned} \tag{6}$$

for all four types of the nonconforming quadrilateral element. The corresponding finite element interpolation operator $i_h : H^2(\Omega) \to V_h(g)$ is uniquely defined by the conditions

$$N_{K,E}(i_h u) = N_{K,E}(u) \qquad \forall\, E \in \mathcal{E}(K),\ K \in \mathcal{T}_h\,. \tag{7}$$

Now, we can summarise the results that are known from the literature ([5],[6]) for the four different types of the nonconforming quadrilateral element. For a given cell $K \in \mathcal{T}_h$, let us denote by σ_K the largest angle between opposite edges $E, E' \in \mathcal{E}(K)$. The first known statement concerns the consistency error of the Laplacian operator. For a given $u \in H^2(\Omega)$, let $\theta(u;\cdot) \in (V_h)'$ be the consistency error functional defined by

$$\begin{aligned} \theta(u; v_h) \quad &:= \quad (-\triangle u, v_h)_\Omega - \sum_{K \in \mathcal{T}_h} (\nabla u, \nabla v_h)_K \\ &= \quad \sum_{K \in \mathcal{T}_h} \sum_{E \in \mathcal{E}(K)} \int_E (\nabla u \cdot n_{K,E})\, v_h \, d\gamma, \qquad \forall\, v_h \in V_h\,, \end{aligned} \tag{8}$$

where $(\cdot,\cdot)_G$ denotes the inner product in $L^2(G)$ and $n_{K,E}$ the unit normal vector on the edge $E \in \mathcal{E}(K)$ directed outward with respect to element K. Furthermore, let $V_{h,0}$ denote the test space $V_{h,0} := V_h(0)$. Then, it is known that for all $u \in H^2(\Omega)$ the estimate for the consistency error

$$\overline{\theta}_h(u) := \sup_{v_h \in V_{h,0} \setminus \{0\}} \frac{|\theta(u; v_h)|}{|v_h|_{1,h,\Omega}} \leq C \Big(\sum_{K \in \mathcal{T}_h} (h_K^{(or)})^2 \|u\|_{2,K}^2 \Big)^{1/2} \tag{9}$$

holds where $h_K^{(or)}$ is defined by

$$h_K^{(or)} := \begin{cases} h_K + \sigma_K & , \quad \text{for the point-value oriented case,} \\ h_K & , \quad \text{for the mean-value oriented case,} \end{cases} \tag{10}$$

and $|\cdot|_{1,h,\Omega}$ is the "broken" H^1-seminorm given as

$$|v|_{1,h,\Omega} := \Big(\sum_{K \in \mathcal{T}_h} |v|_{1,K}^2 \Big)^{1/2} . \tag{11}$$

Here and in the following, we use the standard notation $|.|_{m,K}$ and $\|.\|_{m,K}$ for the semi-norm and norm in the Sobolev space $H^m(K)$, respectively. From standard techniques of the finite element analysis in the nonconforming case, it is known that the consistency error $\overline{\theta}_h(u)$ enters the discretisation error as an additive quantity. Thus, the above result in (9) says that one should use mean-value oriented finite elements on general quadrilateral meshes. On the other hand, for a grid $\mathcal{T}_h$ with

$$\sigma_K \leq C\, h_K \qquad \forall\, K \in \mathcal{T}_h\, , \tag{12}$$

the estimate (9) of the consistency error remains optimal also for the point-value oriented elements of Rannacher and Turek. In [6], it was proved that (12) is fulfilled for grids $\mathcal{T}_h = \mathcal{T}_{h_\ell}$ which are generated from a shape-regular coarse grid $\mathcal{T}_{h_0}$ by a sequence of , say ℓ, uniform refinement steps where in step $k \in \{1, \dots, \ell\}$ each quadrilateral cell $K \in \mathcal{T}_{h_{k-1}}$ is divided into four cells of the next finer grid $\mathcal{T}_{h_k}$ by connecting the midpoints of opposite edges of the cell K.

The second known statement concerns the interpolation error. For an arbitrary element $K \in \mathcal{T}_h$, it holds the estimate

$$|u - i_h u|_{m,K} \leq C\, h_K^{1-m} h_K^{(par)}\, \|u\|_{2,K}, \quad \forall u \in H^2(K), \quad m \in \{0, 1\} \tag{13}$$

with

$$h_K^{(par)} := \begin{cases} h_K + \sigma_K & , \quad \text{for the parametric case,} \\ h_K & , \quad \text{for the non-parametric case.} \end{cases} \tag{14}$$

Rannacher and Turek have proved the results (13) and (9) in [5] for the parametric case and claimed the corresponding estimates for the non-parametric version. In [6], the results for all four types of the nonconforming quadrilateral element (generated from the reference space $\widehat{\mathcal{P}}$) have been proved in a unified framework and have been applied for the analysis of an upwind discretisation of the incompressible Navier-Stokes equations.

3 The modified nonconforming quadrilateral element

Cai, Douglas and Ye have constructed in [1] and [4] a modified nonconforming point-value oriented quadrilateral element. This element satisfies the same property ((16) below) as the nonconforming piecewise linear element of Crouzeix and Raviart [2] and the mean-value oriented element of Rannacher and Turek. More precisely, for each edge $E \in \mathcal{E}$ let $\mathbf{n}_E$ be a unit vector normal to E which coincides with the outer normal with respect to Ω if E is a boundary edge. Then, for a point x in the interior of E the jump of a function v on E is defined by

$$[v]_E(x) := \lim_{t\to+0} v(x+t\mathbf{n}_E) - \lim_{t\to+0} v(x-t\mathbf{n}_E). \tag{15}$$

Now, in contrast to the point-value oriented element of Rannacher and Turek, the modified finite element functions $v_h \in V_h^{mod}$ in [1], [4] have the property that

$$\int_E [v_h]_E \, d\gamma = 0 \quad \forall E \in \mathcal{E}. \tag{16}$$

Moreover, for $v_h \in V_h^{mod}$ it holds

$$v_h\big|_K(m_E) = |E|^{-1} \int_E v_h\big|_K \, d\gamma \quad \forall E \in \mathcal{E}(K). \tag{17}$$

On the other hand, the mean-value oriented element of Rannacher and Turek also satisfies (16) due to the definition of V_h by (5) and (6). However, it does not satisfy (17). As a consequence, for the modified element, the point-value oriented and the mean-value oriented version coincide.

Now let us shortly recall from [1], [4] the definition of the modified nonconforming quadrilateral element. On $\hat{K} = [-1,+1]^2$ we introduce the space

$$\widehat{\mathcal{P}}^{mod} := \operatorname{span}\left\{1, \hat{x}_1, \hat{x}_2, \left(\hat{x}_1^2 - \hat{x}_2^2\right)\left(1 - \frac{5}{3}\hat{x}_1^2 - \frac{5}{3}\hat{x}_2^2\right)\right\}.$$

Note that the modified element differs from the rotated bilinear element by replacing the polynomial $\hat{x}_1^2 - \hat{x}_2^2$ by $\left(\hat{x}_1^2 - \hat{x}_2^2\right)\left(1 - \frac{5}{3}\hat{x}_1^2 - \frac{5}{3}\hat{x}_2^2\right)$ The local space of shape functions in the "parametric" case is given by

$$\mathcal{P}_K := \{v \ : \ v = \hat{v} \circ F_K^{-1}, \ \hat{v} \in \widehat{\mathcal{P}}^{mod}\}. \tag{18}$$

Then, the modified nonconforming finite element space $V_h^{mod}(g)$ is defined in the same way as $V_h(g)$ in (6) with $\mathcal{P}_K$ from (18) and the nodal functionals defined by (4). An advantage of the modified element is that, due to the property (16), one can prove in a straightforward way (see [5], [1]) the following estimate for the consistency error

$$\overline{\theta}_h(u) := \sup_{v_h \in V_{h,0}^{mod} \setminus \{0\}} \frac{|\theta(u; v_h)|}{|v_h|_{1,h,\Omega}} \leq C \, \Big(\sum_{K \in \mathcal{T}_h} h_K^2 \|u\|_{2,K}^2 \Big)^{1/2} . \tag{19}$$

This estimate shows that, for the modified space V_h^{mod}, the consistency error is independent of the quantities σ_K, i.e., the cells $K \in \mathcal{T}_h$ need not to be "nearly" of parallelogram type in order to ensure that the consistency error is of optimal order.

4 Interpolation error on general quadrilateral meshes

If we apply the techniques of [5] and [6] we can derive in a straightforward way the following interpolation properties for the modified nonconforming finite element space $V_h^{mod}(g)$.

Lemma 1 *Assume that the mesh $\mathcal{T}_h$ consists of shape regular quadrilateral elements $K \in \mathcal{T}_h$ and that the interpolation operator $i_h : H^2(\Omega) \to V_h^{mod}(g)$ is defined by (7). Then, for $m \in \{0, 1\}$ and arbitrary $u \in H^2(\Omega) \cap V(g)$, the estimates*

$$|u - i_h u|_{m,K} \leq C h_K^{1-m} (h_K + \sigma_K) \|u\|_{2,K}, \tag{20}$$

and

$$|u - i_h u|_{m,h,\Omega} \leq C \left(\sum_{K \in \mathcal{T}_h} h_K^{2-2m} (h_K + \sigma_K)^2 \|u\|_{2,K}^2 \right)^{1/2} , \tag{21}$$

are satisfied with constants C independent of u, $K \in \mathcal{T}_h$, h_K, and σ_K.

Proof. The proof is completely analogous to [5], [6] and will be omitted. ∎

In case of an affine reference transformation $F_K : \widehat{K} \to K$, the mesh cell K has the shape of parallelogram, i.e., it holds $\sigma_K = 0$ and the estimates (20) and (21) are of optimal order. However, if the analysis is sharp, a non-optimal order of the interpolation error has to be expected for the case of a general shape-regular quadrilateral element where the quantity σ_K does not satisfy (12). In the following, we prove by an explicit example that both the local and global interpolation error in general can be of non-optimal order unless the mesh $\mathcal{T}_h$ is "nearly" of parallelogram type in the sense that (12) holds.

Let us consider the following special case of problem (1)

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega = (0,1)^2, \\ u(0,y) = g(0,y) = 4-2y, \qquad & \frac{\partial u}{\partial x}(1,y) = g_N(1,y) = y-2,\ y \in (0,1), \\ u(x,0) = g(x,0) = 4-2x, \qquad & \frac{\partial u}{\partial y}(x,1) = g_N(x,1) = x-2,\ x \in (0,1). \end{aligned} \tag{22}$$

The solution of this problem is $u(x,y) = (2-x)(2-y)$. We consider the following special family of meshes $\mathcal{T}_h^*$ for $h \to 0$. At first, the domain $\Omega = (0,1)^2$ is decomposed into rectangles

$$\bar{\Omega} = \bigcup_{i=1}^{M} \bigcup_{j=1}^{2M} R_{i,j}$$

with

$$R_{i,j} := [x_i - 2H, x_i + 2H] \times [y_j - H, y_j + H]$$

$$x_i := 2H(2i-1), \quad y_j = H(2j-1), \quad H := 1/(4M).$$

Then, each rectangle $R_{i,j}$ is decomposed into two quadrilateral elements $K_{i,j}$ and $K'_{i,j}$ by the straight line through the point (x_i, y_j) with the slope $1/s$ if j is odd and the slope $-1/s$ if j is even (see Figure 1).

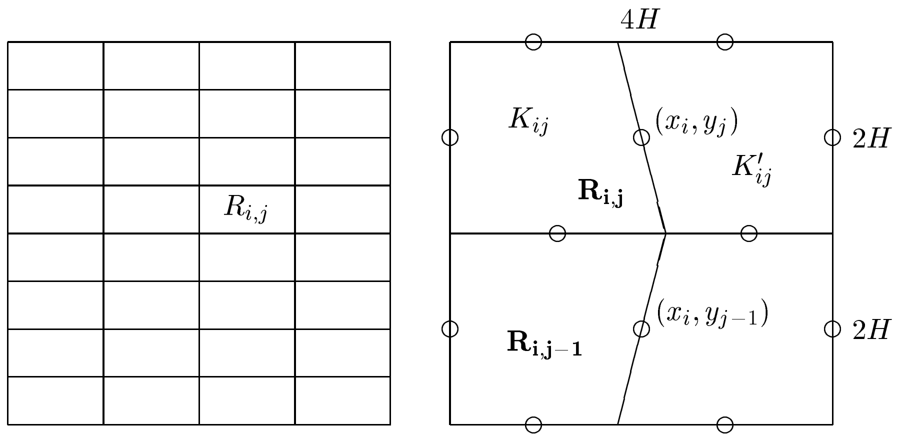

Figure 1: Partition of Ω into rectangles and their decomposition into elements

Thus, our special mesh is

$$\mathcal{T}_h^* = \bigcup_{i=1}^{M} \bigcup_{j=1}^{2M} \{K_{i,j}, K'_{i,j}\}.$$

We assume that $s \in (0,1)$. Then, the mesh $\mathcal{T}_h^*$ has the property that

$$\sigma_K = \sigma := \arctan s \in (0, \pi/4) \quad \forall K \in \mathcal{T}_h^* .$$

Now the local and global interpolation error for our special situation can be characterised as follows.

Theorem 2 *Let $u \in H^2(\Omega) \cap V(g)$ be the solution of the special problem (22) and $\mathcal{T}_h^*$ the special quadrilateral mesh of $\Omega = (0,1)^2$ as defined above. Then, for the interpolation operator $i_h : H^2(\Omega) \to V_h^{mod}(g)$ defined by (7), the following local and global interpolation error estimates are satisfied for for $m \in \{0,1\}$ and $h \leq h_0$*

$$\begin{aligned} |u - i_h u|_{m,K} &\geq C_1 \sigma_K h_K^{1-m} \, \|u\|_{2,K} \quad \forall K \in \mathcal{T}_h^*, & (23)\\ |u - i_h u|_{m,h,\Omega} &\geq C_2 \left(\sum_{K \in \mathcal{T}_h^*} \sigma_K^2 h_K^{2-2m} \, \|u\|_{2,K}^2 \right)^{1/2}, & (24) \end{aligned}$$

where $h_0 > 0$ is sufficiently small, $|.|_{0,h,\Omega}$ denotes the $L^2(\Omega)$-norm, $|.|_{1,h,\Omega}$ the broken H^1-seminorm defined by (11).

Proof. The bilinear mapping

$$F_{K_{i,j}}(\hat{x}_1, \hat{x}_2) = \begin{pmatrix} x_i - H \\ y_j \end{pmatrix} + \begin{pmatrix} H \\ 0 \end{pmatrix} \hat{x}_1 + \begin{pmatrix} \pm sH/2 \\ H \end{pmatrix} \hat{x}_2 + \begin{pmatrix} \pm sH/2 \\ 0 \end{pmatrix} \hat{x}_1 \hat{x}_2$$

transforms the reference element $\hat{K}$ onto the convex quadrilateral $K_{i,j}$, where in the notation "$\pm$" the upper sign corresponds to the case that j is odd and the lower sign to the case that j is even. Similarly the bilinear mapping $F_{K'_{i,j}} : \hat{K} \to K'_{i,j}$ is given as

$$F_{K'_{i,j}}(\hat{x}_1, \hat{x}_2) = \begin{pmatrix} x_i + H \\ y_j \end{pmatrix} + \begin{pmatrix} -H \\ 0 \end{pmatrix} \hat{x}_1 + \begin{pmatrix} \mp sH/2 \\ -H \end{pmatrix} \hat{x}_2 + \begin{pmatrix} \mp sH/2 \\ 0 \end{pmatrix} \hat{x}_1 \hat{x}_2.$$

Now a direct calculation shows that for $K \in \{K_{i,j}, K'_{i,j}\}$

$$(u - i_h u)^2 (F_K(\hat{x}_1, \hat{x}_2)) = \frac{s^2 H^2}{4} \hat{x}_1^2 \hat{x}_2^2 (2 - y_j)^2 + O(H^3).$$

Thus, using

$$|\det(DF_K)| = H^2 (1 \pm \frac{s}{2} \hat{x}_2) \quad \text{for all } K \in \mathcal{T}_h^*,$$

we have for $i = 1, \ldots, M$, $j = 1, \ldots, 2M$,

$$\begin{aligned} \|u - i_h u\|_{0,K_{i,j}}^2 &= (2 - y_j)^2 \int_{\hat{K}} H^2 (1 \pm \frac{s}{2} \hat{x}_2) \left\{ \frac{s^2 H^2}{4} \hat{x}_1^2 \hat{x}_2^2 + O(H^3) \right\} \, d\hat{x}_1 d\hat{x}_2 \\ &= \frac{s^2 H^4}{9} (2 - y_j)^2 + O(H^5). & (25) \end{aligned}$$

In the same way, we get for $i = 1, \ldots, M$, $j = 1, \ldots, 2M$,

$$\|u - i_h u\|_{0,K'_{i,j}}^2 = \frac{s^2 H^4}{9}(2 - y_j)^2 + O(H^5). \tag{26}$$

Consequently, there is a positive H_0 such that for all $H \le H_0$ the estimate

$$\|u - i_h u\|_{0,K} \ge \frac{sH^2}{4} \quad \forall K \in \mathcal{T}_h^* \tag{27}$$

holds. Since u and its derivatives are bounded on Ω we have

$$\|u\|_{2,K} \le C_3 H \quad \text{for all } K \in \mathcal{T}_h^*.$$

Note that

$$s = \tan(\sigma) = \tan(\sigma_K) \ge \sigma,$$

and that for $h_K = \mathrm{diam}(K)$ it holds

$$2\sqrt{2}H \le h_K = \sqrt{4H^2 + H^2(2+s)^2} \le 2\sqrt{5}H.$$

Consequently, we have

$$\|u - i_h u\|_{0,K} \ge \frac{\sigma_K h_K}{8\sqrt{5}C_3}\|u\|_{2,K} \quad \forall h \le h_0 = 2\sqrt{2}H_0,$$

which proves (23) for $m = 0$ and $C_1 = 1/(8\sqrt{5}C_3)$.

Summation of the estimates (27) over all elements yields

$$\|u - i_h u\|_{0,\Omega}^2 = \sum_{K \in \mathcal{T}_h} \|u - i_h u\|_{0,K}^2 \ge \frac{s^2 H^4}{16} \sum_{K \in \mathcal{T}_h} 1 = \frac{s^2 H^2}{64}, \tag{28}$$

which proves (24) for $m = 0$ and a suitable constant C_2.

To study the approximation properties in the H^1-seminorm we start with the error representation

$$\frac{\partial}{\partial x}(u - i_h u)(F_{K_{i,j}}(\hat{x}_1, \hat{x}_2)) = \frac{s(2 - y_j)}{2} \, \frac{\hat{x}_2}{1 \pm \dfrac{s}{2}\hat{x}_2} + O(H),$$

$$\frac{\partial}{\partial y}(u - i_h u)(F_{K_{i,j}}(\hat{x}_1, \hat{x}_2)) = \frac{s(2 - y_j)}{2} \, \frac{\hat{x}_1 \mp \dfrac{s}{2}\hat{x}_2}{1 \pm \dfrac{s}{2}\hat{x}_2} + O(H),$$

and find that there is a positive constant H_1 such that for all $H \le H_1$ it holds

$$\begin{aligned} |u - i_h u|_{1,K_{i,j}}^2 &= \frac{s^2 H^2 (2 - y_j)^2}{4} \int_{\hat{K}} \frac{\hat{x}_1^2 + (1 + \frac{s^2}{4})\hat{x}_2^2}{1 \pm \frac{s}{2}\hat{x}_2} \, d\hat{x} + O(H^3) \\ &\ge \frac{s^2 H^2}{8(1 + \frac{s}{2})} \int_{\hat{K}} (\hat{x}_1^2 + \hat{x}_2^2) \, d\hat{x} \ge \frac{2s^2 H^2}{9}, \end{aligned}$$

where we used the assumption $s \in (0,1)$. Note that the same estimate holds for $|u - i_h u|^2_{1,K'_{i,j}}$ and therefore

$$|u - i_h u|_{1,K} \geq \frac{sH}{3} \quad H \leq H_1, \, \forall K \in \mathcal{T}_h^*. \tag{29}$$

Using the same arguments as for the L^2-norm estimates, we obtain (23) and (24) for $m = 1$. ■

Remark 3 *Our theorem shows that for the family of meshes an inequality of type $\sigma_K \leq C\, h_K$ is in general necessary for getting the optimal order of the local and global interpolation error. We call a mesh to be "nearly" of parallelogram type, if such an inequality holds. On the other hand, we see from Lemma1 that meshes, which satisfy this condition, guarantee an optimal order of the interpolation error. Thus, for the modified nonconforming quadrilateral element recently proposed by Cai, Douglas and Ye, the interpolation error in general is of optimal order only for meshes which are "nearly" of parallelogram type.*

Remark 4 *In order to avoid the dependence of the interpolation error on the angles σ_K, $K \in \mathcal{T}_h$, one can modify the space $\hat{\mathcal{P}}$ on an appropriate "reference element". More precisely, for each convex quadrilateral K there is a unique (up to notations) affine transformation $F_K^* : \hat{K} \to K$ which maps the points $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$, onto the midpoints of edges of K. Under this affine transformation F_K^* the vertices of K become the images of the vertices of the reference quadrilateral $\hat{K}$, i.e., the images of the points $(1+\varepsilon_1, 1+\varepsilon_2)$, $(-1-\varepsilon_1, 1-\varepsilon_2)$, $(-1+\varepsilon_1, -1+\varepsilon_2)$, $(1-\varepsilon_1, -1-\varepsilon_2)$. Note that now $\hat{K} = \hat{K}(\varepsilon)$ is no longer a fixed reference quadrilateral since ε depends on $K \in \mathcal{T}_h$. The vector $\varepsilon = (\varepsilon_1, \varepsilon_2)$ can be considered as deviation of the quadrilateral K from a parallelogram, in particular if K is a parallelogram we get $\varepsilon = (0,0)$. Moreover, K is convex if and only if $-1 < \varepsilon_1 + \varepsilon_2 < +1$ and $-1 < \varepsilon_1 - \varepsilon_2 < +1$. Now, we explain the basic idea for constructing a new nonconforming finite element space, which allows σ_K-independent estimates of the interpolation error. We introduce a space $\widehat{\mathcal{P}}^{mod}(\varepsilon)$ of functions over $\hat{K}(\varepsilon)$ by*

$$\widehat{\mathcal{P}}^{mod}(\varepsilon) := \operatorname{span}\left\{1, \hat{x}_1, \hat{x}_2, \Phi(\varepsilon; \hat{x}_1, \hat{x}_2)\right\} \tag{30}$$

where the (non-linear) function $\Phi(\varepsilon; .,.)$ is chosen in such a way that the property

$$\hat{v}_h\big|_{\hat{K}}(\hat{m}_E) = |\hat{E}|^{-1} \int_{\hat{E}} \hat{v}_h\big|_{\hat{K}}\, d\hat{\gamma} \quad \forall \hat{E} \in \mathcal{E}(\hat{K}) \tag{31}$$

is satisfied for all $\hat{v}_h \in \widehat{\mathcal{P}}^{mod}(\varepsilon)$. *Since the functions* $\hat{v}_h = 1$, $\hat{v}_h = \hat{x}_1$, *and* $\hat{v}_h = \hat{x}_2$, *always satisfy (31), the property needs only to hold for the unknown function* $\Phi(\varepsilon;.,.)$. *Without going into the details we only mention that in a neighbourhood of* $\varepsilon = (0,0)$ *such a function* $\Phi(\varepsilon;.,.)$ *really exist. Indeed, taking*

$$\Phi(\varepsilon;\hat{x}_1,\hat{x}_2) = \left(\hat{x}_1^2 - \hat{x}_2^2\right)\left(1 + A(\varepsilon)\hat{x}_1 + B(\varepsilon)\hat{x}_2 + C(\varepsilon)\hat{x}_1^2 + D(\varepsilon)\hat{x}_2^2\right),$$

we can show that for sufficiently small $|\varepsilon_1|$, $|\varepsilon_2|$, *the coefficients* $A(\varepsilon)$, $B(\varepsilon)$, $C(\varepsilon)$, *and* $D(\varepsilon)$ *can uniquely be determined such that (31) holds. For example, in the case* $\varepsilon = (\varepsilon_1, 0)$, *in which* K *and* $\hat{K}$ *are trapezoids, we obtain for* ε_1 *small enough*

$$A(\varepsilon_1,0) = 0, \quad B(\varepsilon_1,0) = \frac{2\varepsilon_1(7-2\varepsilon_1^2)}{(7-25\varepsilon_1^2-2\varepsilon_1^4)},$$

$$C(\varepsilon_1,0) = -\frac{5}{3}\frac{7-2\varepsilon_1^2}{(7-25\varepsilon_1^2-2\varepsilon_1^4)}, \quad D(\varepsilon_1,0) = -\frac{5}{3}\frac{7-20\varepsilon_1^2}{(7-25\varepsilon_1^2-2\varepsilon_1^4)}.$$

Moreover, we see that for $\varepsilon_1 \to 0$ *the case can be recovered, in which* K *is a parallelogram. Indeed, it holds* $\widehat{\mathcal{P}}^{mod}(0,0) = \widehat{\mathcal{P}}^{mod}$, *where* $\widehat{\mathcal{P}}^{mod}$ *is the space introduced by Cai, Douglas and Ye [1]. Now, a new nonconforming finite element space* $V_h^*(g)$ *can be defined in the same way as* $V_h(g)$ *in (6) with* $\mathcal{P}_K$ *replaced by*

$$\mathcal{P}_K^* := \{v \ : \ v = \hat{v} \circ (F_K^*)^{-1}, \ \hat{v} \in \widehat{\mathcal{P}}^{mod}(\varepsilon)\}$$

and the nodal functionals defined by (4). The corresponding interpolation operator $i_h^* : H^2(\Omega) \to V_h^*(g)$ *is uniquely defined by the conditions (7). Then, in an analogous way as in [6], we can prove the optimal estimate for the interpolation error*

$$|u - i_h^* u|_{m,K} \leq C h_K^{2-m} \|u\|_{2,K} \quad \forall K \in \mathcal{T}_h,$$

where $m \in \{0,1\}$, $u \in H^2(\Omega)$ *and* C *is a constant independent of* $K \in \mathcal{T}_h$, h_K *and* u. *In this way we could obtain a nonconforming, "point-value oriented" finite element space* $V_h^*(g)$ *on meshes* $\mathcal{T}_h$ *of arbitrary quadrilateral cells* K *with optimal order of the consistency error functional and the interpolation error, respectively. However, this advantage has to be paid by calculating the local basis functions on* $\hat{K} = \hat{K}(\varepsilon(K))$ *for each* $K \in \mathcal{T}_h$ *separately.*

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