Equal-order finite elements with local projection stabilization for the Darcy-Brinkman equations *

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Abstract

For the Darcy-Brinkman equations, which model porous media flow, we present an equal-order H^1-conforming finite element method for approximating velocity and pressure based on a local projection stabilization technique. The method is stable and accurate uniformly with respect to the coefficients of the viscosity and the zeroth order term in the momentum equation. We prove a priori error estimates in a mesh-dependent norm as well as in the L^2-norm for velocity and pressure. In particular, we obtain optimal order of convergence in L^2 for the pressure in the Darcy case with vanishing viscosity and for the velocity in the general case with a positive viscosity coefficient. Numerical results for different values of the coefficients in the Darcy-Brinkman model are presented which confirm the theoretical results and indicate nearly optimal order also in cases which are not covered by the theory.

Keywords: Porous media flow, Darcy-Brinkman equations, Stokes, equal-order finite elements, local projection stabilization

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1 Introduction

The Darcy-Brinkman equations are widely used for modeling porous media flow. In contrast to Darcy's equation, an extra viscous term in Brinkman's equation (introduced by Brinkman [8] in 1947) allows to prescribe all components of the velocity at the boundary, see also Liu et al. [16]. Due to this additional term, the Darcy-Brinkman equations can also be interpreted as a generalization of the Stokes equations where a zeroth order term of the velocity appears in the momentum equation. Hence, they can be viewed as a relatively general model which

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covers as limit cases the Darcy system (zero viscosity coefficient) and the Stokes system (zero coefficient in front of the zeroth order term).

An important issue is to find a discretization which is accurate and stable uniformly with respect to the values of the coefficients of viscosity and the zeroth order term. In Mardal et al. [17], it has been shown by numerical examples that traditional $H(\text{div})$-conforming mixed finite elements for the Darcy system (see for instance the book of Brezzi and Fortin [7]) will not automatically work well for the Stokes system and vice versa that well-known H^1-conforming inf-sup stable Stokes elements like the MINI element can have convergence problems when they are applied to the Darcy system. To overcome this problem modified $H(\text{div})$-conforming mixed finite elements have been introduced in [17, 20]. Another way to achieve robustness of the discretization with respect to the values of the coefficients is to add some stabilization terms to the variational formulation of the discrete problem. Burman and Hansbo [9] considered the lowest order $\mathbb{P}_1/\mathbb{P}_0$-element pair (continuous piecewise linear elements for the velocity and piecewise constant approximation for the pressure) and added stabilization terms involving jumps of the pressure over the element edges. In Xie et al. [20], a stabilizing Galerkin Least-Squares (GLS) term involving the divergence of the velocity has been proposed for usual H^1-conforming inf-sup stable Stokes elements. From the practical point of view, equal-order elements for velocity and pressure are more attractive due to simpler implementation. Their missing inf-sup stability can be compensated by suitable GLS terms in the discrete variational formulation. Recently, Badia and Codina [1] have proposed and analyzed two GLS-type techniques for the Darcy-Brinkman system which on the one hand allow equal-order element pairs and on the other hand guarantee a uniformly stable and accurate discrete solution. They call these approaches the *algebraic subgrid scale* and the *orthogonal subscale stabilization* method.

In this paper, we follow the approach of the so-called *local projection stabilization* (LPS) techniques which move more into the focus of actual research in Computational Fluid Dynamics [2, 5, 18, 3, 11, 14, 15]. There are several reasons, for example, the avoidance of second order derivatives in the variational formulation, the avoidance of artificial couplings between pressure and velocity, the symmetry of all stabilizing terms, which is advantageous for optimization problems [4] and solver aspects, or the possible extension to anisotropic meshes [3]. Here, we present and analyze an equal-order H^1-conforming finite element scheme of local projection type for the Darcy-Brinkman equations which is stable and accurate uniformly with respect to the coefficients of the viscosity and the zeroth order term. In particular, we obtain stability for the limit cases of the Darcy and Stokes system. For the Darcy case (vanishing viscosity), we obtain a new stable discretization for porous media flow. By means of a duality argument we will prove an optimal error estimate for the pressure in the L^2-norm. In the Stokes case (vanishing zeroth order term in the momentum equation), the original LPS method [2] is recovered. For the general Darcy-Brinkman equations with a positive viscosity coefficient, we prove an optimal error estimate for the velocity in the H^1- and L^2-norm. By a numerical experiment with equal-order continuous $\mathbb{Q}_1/\mathbb{Q}_1$-elements we confirm the optimal velocity approximation and show that the error of the pressure in the L^2-norm is at least of the order 7/4 in the general case and of optimal second order in the Darcy case.

The paper is organized as follows. In Section 2, we describe the Darcy-Brinkman model with its variational formulation in appropriate function spaces and present the stabilized discrete finite element scheme. Section 3 contains the analysis of the method in a suitable mesh-dependent norm defined on the product space of velocity and pressure. We prove an a priori error estimate in this norm. This estimate gives us information about the appropriate choice of the stabilization parameters. In Section 4, we use duality arguments in order to show optimal a priori error estimates for velocity and pressure in the L^2-norm. Finally, in Section 5, we illustrate the accuracy of the method for different values of the coefficients in the Darcy-Brinkman model by means of a numerical test problem with known exact solution.

2 The Darcy-Brinkman problem

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with polyhedral boundary. We consider the following *Darcy-Brinkman model*: Find a velocity field $v : \Omega \to \mathbb{R}^d$ and a pressure field $p : \Omega \to \mathbb{R}$ such that

$$-\nu\Delta v + \sigma v + \nabla p = f \quad \text{in } \Omega\,, \tag{1}$$

$$\operatorname{div} v = g \quad \text{in } \Omega\,, \tag{2}$$

$$v \cdot n = 0 \quad \text{on } \partial\Omega\,, \tag{3}$$

$$\nu v \cdot t = 0 \quad \text{on } \partial\Omega\,, \tag{4}$$

where ν and σ denote non-negative constants such that $\nu + \sigma > 0$. The data $f : \Omega \to \mathbb{R}^d$ and $g : \Omega \to \mathbb{R}$ are given functions satisfying the solvability condition $\int_\Omega g\,dx = 0$. By n we denote the outer normal unit vector at the boundary $\partial\Omega$ and by t the corresponding tangential vector (or vectors for $d = 3$). In the case $\nu = 0$, the last condition (4) disappears since it is automatically satisfied. For ease of presentation, we restrict ourselves to homogeneous boundary conditions. However, all results carry over directly to the non-homogeneous case.

This model covers the following important problems:

- *stationary Stokes system:* $\nu > 0$, $\sigma = 0$, $g = 0$
- *Darcy system:* $\nu = 0$, $\sigma > 0$, $f = 0$
- *generalized (or time-dependent) Stokes system:* $\nu > 0$, $\sigma = \frac{c}{\tau} > 0$, $g = 0$, where τ is the time step size in a time discretization of Rothe type.

The stationary Stokes problem is the simplest model of an incompressible viscous flow with viscosity $\nu > 0$. It involves one of the basic difficulties of finite element methods for such flows – the stability of the coupling between velocity and pressure approximation. The non-stationary Stokes problem includes in addition the parameter $\sigma > 0$ for the zeroth order term which arises from the time discretization. Here the task is to get stability with respect to σ. The Darcy problem represents the limit case of vanishing viscosity $\nu = 0$. Whereas we have Dirichlet boundary conditions for all velocity components in the case $\nu > 0$, we have to prescribe only the normal component of the velocity in the Darcy case since the last condition (4) disappears.

2.1 Weak formulation

For the data, we assume $f \in L^2(\Omega)^d$ and $g \in L^2(\Omega)$ where we use standard notations for Lebesgue and Sobolev spaces (see e.g. Girault and Raviart [12]). We will use a context-sensitive notation for the L^2-inner product applied to scalar, vector-valued and tensor-valued functions. Let $\omega \subset \Omega$ a measurable subset of the domain Ω. Then, for scalar functions $p, q \in L^2(\Omega)$, we denote by $(p,q)_\omega$ the usual inner product in $L^2(\omega)$. If $v, w : \Omega \to \mathbb{R}^d$ are vector-valued functions with the components $v_i, w_i \in L^2(\Omega)$ and $A, B : \Omega \to \mathbb{R}^{d\times d}$ are matrix-valued functions with components $A_{i,j}, B_{i,j} \in L^2(\Omega)$ we define

$$(v,w)_\omega := \sum_{i=1}^d (v_i, w_i)_\omega \qquad \text{and} \qquad (A,B)_\omega := \sum_{i,j=1}^d (A_{i,j}, B_{i,j})_\omega .$$

This context-sensitive L^2-inner product induces the corresponding context-sensitive L^2-norm notation $\|\cdot\|_{0,\omega}$ and H^1-semi-norm or norm notation $|v|_{1,\omega}^2 := (\nabla v, \nabla v)_\omega$ or $\|v\|_{1,\omega}^2 := |v|_{1,\omega}^2 + \|v\|_{0,\omega}^2$. In the case $\omega = \Omega$, we will omit the index ω and write simply $(\cdot,\cdot)$ and $\|\cdot\|_r$ or $|\cdot|_r$ for functions in $H^r(\Omega)^n$ where $n \in \{1, d\}$ and H^0 is formally identified with L^2. The natural weak solution space for the velocity v is

$$V := \begin{cases} H_0^1(\Omega)^d = \{v \in H^1(\Omega)^d : \ v = 0 \text{ on } \partial\Omega\}, & \text{if} \quad \nu > 0, \\ H_0(\operatorname{div}, \Omega) = \{v \in L^2(\Omega)^d : \ \operatorname{div} v \in L^2(\Omega), \ v\cdot n = 0 \text{ on } \partial\Omega\}, & \text{if} \quad \nu = 0. \end{cases} \tag{5}$$

The pressure space $Q := L_0^2(\Omega)$ is the subspace of the $L^2(\Omega)$ functions with zero integral mean value.

For a compact notation, we sample the velocity v and pressure variable p together in one variable $y = (v,p) \in Y := V \times Q$. For test functions $z = (z_v, z_p)$ we denote the velocity part by $z_v \in V$ and the pressure part by $z_p \in Q$. Then, the standard variational formulation of the Darcy-Brinkman problem (1)-(4) can be written as:

$$\text{Find } y \in Y \text{ such that} \quad A(y,z) = \langle \ell, z\rangle \quad \forall z \in Y\,, \tag{6}$$

where the bilinear form $A : Y \times Y \to \mathbb{R}$ and the linear functional $\ell \in Y'$ are defined by

$$\begin{aligned} A(y,z) &:= (\sigma v, z_v) + (\nu \nabla v, \nabla z_v) - (p, \operatorname{div} z_v) + (\operatorname{div} v, z_p) \qquad \text{for} \quad y = (v,p)\,, \\ \langle \ell, z\rangle &:= (f, z_v) + (g, z_p)\,. \end{aligned} \tag{7}$$

For our analysis, we will use the following variant of the well-known continuous inf-sup condition:

$$\forall\, p \in L_0^2(\Omega) \ \exists\, v^p \in H_0^1(\Omega)^d \ \text{ with } \ \|v^p\|_1 \le \gamma_c^{-1}\|p\|_0 \ \text{ and } \ (p, \operatorname{div} v^p) = \|p\|_0^2\,, \tag{8}$$

where the constant $\gamma_c > 0$ is independent of p.

2.2 Finite Element spaces

Let $\mathcal{T}_h$ be a shape-regular, admissible decomposition of Ω into quadrilaterals for $d = 2$ or hexahedra for $d = 3$. The diameter of a cell $K \in \mathcal{T}_h$ will be denoted by h_K and the global mesh size is defined as $h := \max\{h_K : K \in \mathcal{T}_h\}$. Let $\hat{K} := (-1, 1)^d$ denote the reference element and $\mathbb{Q}_r(\hat{K})$ the space of all polynomials on $\hat{K}$ with maximal degree $r \geq 0$ in each coordinate direction. We will use the discontinuous finite element space

$$Q_r^{dc}(\mathcal{T}_h) := \{v_h \in L^2(\Omega) : \; v_h|_K \circ F_K \in \mathbb{Q}_r(\hat{K}) \quad \forall K \in \mathcal{T}_h\}\,,$$

as well as the corresponding continuous H^1-conforming finite element spaces

$$Q_r(\mathcal{T}_h) := Q_r^{dc}(\mathcal{T}_h) \cap H^1(\Omega)\,.$$

For the discrete solution $y_h = (v_h, p_h) \in Y_h$, we use the continuous equal-order finite element spaces of degree $r \geq 1$

$$V_h := Q_r(\mathcal{T}_h)^d \cap V, \quad Q_h := Q_r(\mathcal{T}_h) \cap Q \quad \text{and} \quad Y_h := V_h \times Q_h\,,$$

where the homogeneous boundary conditions for the discrete velocity and the zero integral mean of the discrete pressure are already installed in the spaces V_h and Q_h.

2.3 Stabilized Galerkin discretization by local projection

It is known that the standard Galerkin discretization of (6) is not stable for equal-order elements. This instability stems from the violation of the discrete inf-sup condition. One possibility to circumvent this condition is to work with a modified bilinear form $A_h(\cdot, \cdot)$ by adding a stabilization term $S_h(\cdot, \cdot)$, i.e.,

$$A_h(y, z) \;\; := \;\; A(y, z) + S_h(y, z)\,,$$

such that the stabilized discrete problem reads:

$$\text{Find } y_h \in Y_h \text{ such that} \quad A_h(y_h, z_h) = \langle \ell, z_h \rangle \quad \forall z_h \in Y_h\,. \tag{9}$$

Several possibilities of stabilization terms are investigated and analyzed in literature. Such a method is called *strongly consistent* if the term $S_h(y, \cdot)$ vanishes for the exact solution $y = (v, p)$. This is usually achieved by defining S_h in terms of the strong residual multiplied with some suitable test functions. In this work, we will use a local projection of pressure gradients and of the divergence of the velocity. This leads to a *weaker consistent* method, but the consistency error decreases for mesh sizes $h \to 0$ with the correct power of h according to the optimal order of approximation. In this paper, we will consider the following form of a symmetric stabilization:

$$S_h(y, z) = \big(\delta_v \kappa_v^h(\operatorname{div} v), \kappa_v^h(\operatorname{div} z_v)\big) + \big(\delta_p \kappa_p^h(\nabla p), \kappa_p^h(\nabla z_p)\big) \tag{10}$$

for $y = (v, p), z = (z_v, z_p) \in H^1(\Omega)^d \times H^1(\Omega)$ with so-called locally acting *fluctuation operators*

$$\kappa_v^h : L^2(\Omega) \to L^2(\Omega) \quad \text{and} \quad \kappa_p^h : L^2(\Omega)^d \to L^2(\Omega)^d$$

which determine the fine scale part of the function they are applied to. So the meaning of $S_h(\cdot,\cdot)$ in (10) is that we add somehow artificial viscosity but only for the fine scale part of the function. This is the fundamental difference to the classical method of artificial viscosity which is only first order accurate. The parameter functions $\delta_v, \delta_p : \Omega \to \mathbb{R}^+$ are patch-wise (on each macro-element, see below) constant. These parameters will be chosen such that on the one hand the consistency error will be small with the right power of h and on the other hand the method gets enough stability.

To define the operators κ_v^h and κ_p^h, let $\mathcal{M}_h$ be a non-overlapping decomposition of the domain Ω into quadrilateral or hexahedral macro-elements $M \in \mathcal{M}_h$ such that each macro-element M is the union of elements K of the original mesh $\mathcal{T}_h$. For the so-called *two-level method*, each $M \in \mathcal{M}_h$ with diameter h_M consists of 2^d many "son-elements" $K \in \mathcal{T}_h$, i.e., the grid $\mathcal{M}_h$ corresponds to a grid level $L-1$ in a grid-hierarchy if $\mathcal{T}_h$ corresponds to grid level L. For the *one-level method*, it holds $\mathcal{M}_h = \mathcal{T}_h$ but in this case the discrete solution space has to be enriched by some bubble functions [18]. In each case, there exists a constant $C > 0$ such that for all $M \in \mathcal{M}_h$ it holds $h_M \le C h_K$ for all those $K \in \mathcal{T}_h$ with $K \subset M$.

The definition of the fluctuation operators κ_v^h and κ_p^h will be explained simultaneously by means of a scalar version $\kappa_r^h : L^2(\Omega) \to L^2(\Omega)$ where $r \in \mathbb{N}$ is the polynomial degree of the equal-order finite element spaces. Based on the choice of κ_r^h, we use $\kappa_v^h := \kappa_r^h$ and $\kappa_p^h q := (\kappa_r^h q_1, \ldots, \kappa_r^h q_d)$ for vector-valued functions $q \in L^2(\Omega)^d$. In order to define κ_r^h, we introduce for $r \ge 1$ the scalar projection space D_r^h (representing the large scales) as

$$D_r^h := Q_{r-1}^{dc}(\mathcal{M}_h)\,.$$

The operator κ_r^h will be chosen of the form

$$\kappa_r^h(\phi) \;=\; \phi - \pi_r^h \phi \qquad \forall \phi \in L^2(\Omega) \tag{11}$$

with a linear projection operator $\pi_r^h : L^2(\Omega) \to D_r^h$ which locally computes a "large scale part" of a given function. It is not necessary that π_r^h is an orthogonal projector; its choice will still be flexible. However, we need the following properties:

(P1) κ_r^h is locally L^2-stable, i.e., there exists an h independent constant c_κ such that

$$\|\kappa_r^h(\phi)\|_{0,M} \le c_\kappa \|\phi\|_{0,M} \qquad \forall M \in \mathcal{M}_h, \quad \forall \phi \in L^2(\Omega), \tag{12}$$

(P2) $\kappa_r^h(\phi)$ is small for smooth functions ϕ, i.e.,

$$\|\kappa_r^h(\phi)\|_{0,M} \lesssim h_M^k |\phi|_{k,M} \qquad \forall M \in \mathcal{M}_h, \quad \forall \phi \in H^k(\Omega),\; k \le r.$$

Here and in the error analysis below, we use the notation $a \lesssim b$ for the fact that there exists a suitable constant $C > 0$ such that $a \leq Cb$, where C is independent of a and b, the local and global mesh size h_K and h and the model parameters ν and σ.

The existence of such a fluctuation operator κ_r^h is guaranteed by the following obvious lemma.

Lemma 1 *Let the linear projection operator $\pi_r^h : L^2(\Omega) \to D_r^h = Q_{r-1}^{dc}(\mathcal{M}_h)$ be defined patch-wise by local L^2-projection, i.e., $\pi_r^h\phi|_M := \pi_r^M\phi$ for all $M \in \mathcal{M}_h$, where $\pi_r^M\phi \in \mathbb{Q}_{r-1}(M)$ with*

$$(\pi_r^M\phi, \psi)_M \quad = \quad (\phi, \psi)_M \qquad \forall \psi \in \mathbb{Q}_{r-1}(M) := D_r^h\big|_M .$$

Then, the fluctuation operator κ_r^h defined as $\kappa_r^h\phi := \phi - \pi_r^h\phi$ fulfills the properties (P1) and (P2).

Remark 2 *An alternative choice of the stabilizing bilinear form $S_h(\cdot,\cdot)$ would be*

$$S_h(y, z) \quad = \quad \big(\delta_v\kappa_v^h(\nabla v), \kappa_v^h(\nabla z_v)\big) + \big(\delta_p\kappa_p^h(\nabla p), \kappa_p^h(\nabla z_p)\big).$$

Here, κ_v^h acts on the tensor space $L^2(\Omega)^{d\times d}$. For this variant, we could prove the same error estimates as for $S_h(\cdot,\cdot)$ defined by (10). *The matrix structure for the global system of this variant is more efficient than* (10) *in the sense that there are no couplings between the different velocity components $v_1, \dots, v_d$.*

3 Analysis of the finite element method

In this section, we first show stability of the discrete bilinear form $A_h(\cdot,\cdot)$ on the product space Y_h by proving an inf-sup condition with respect to a mesh dependent norm. Then, using this stability result, we will prove an a priori error estimate for the discrete solution in this norm. For our analysis, we also need the existence of a special scalar interpolation operator ($r \in \mathbb{N}$):

$$j_r^h : H^1(\Omega) \to Q_r(\mathcal{T}_h)\,,$$

which preserves homogeneous boundary values and satisfies the usual approximation and stability properties

$$\|\phi - j_r^h\phi\|_m \quad \leq \quad c_i h^{r+1-m}|\phi|_{r+1} \qquad \forall\, \phi \in H^{r+1}(\Omega),\ m \in \{0,1\}, \tag{13}$$

$$\|j_r^h\phi\|_1 \quad \leq \quad c_s\|\phi\|_1 \qquad \forall\, \phi \in H^1(\Omega)\,, \tag{14}$$

as well as the orthogonality condition

$$(\phi - j_r^h\phi, q_h) \quad = \quad 0 \qquad \forall\, q_h \in D_r^h,\ \phi \in H^1(\Omega)\,. \tag{15}$$

The existence of such operators has been shown in [18]. Based on j_r^h, we define interpolation operators for the velocity and pressure space

$$j_v^h : V \cap H^1(\Omega)^d \to V_h \qquad \text{and} \qquad j_p^h : Q \cap H^1(\Omega) \to Q_h\,,$$

as $j_p^h := j_r^h$ and $j_v^h w := (j_r^h w_1, \dots, j_r^h w_d)$ for vector-valued functions $w \in H^1(\Omega)^d$. Furthermore, we use the operator notation $j_h := (j_v^h, j_p^h)$ for the interpolation in the product space

$$j_h : Y \cap \Big(H^1(\Omega)^d \times H^1(\Omega) \Big) \to Y_h \,.$$

Note that the property $j_p^h p \in Q_h$ for $p \in Q \cap H^1(\Omega)$ is a consequence of the orthogonality condition (15) and the fact that the constant function $q_h = 1$ is an element of D_r^h.

Important tools for our analysis are the following orthogonality properties.

Lemma 3 *There exist fluctuation operators κ_v^h, κ_p^h of type (11) with the properties (P1), (P2), and interpolation operators j_v^h, j_p^h, such that for all $p \in H^1(\Omega)$ and all $v \in H^1(\Omega)^d$ it holds*

$$(\kappa_p^h(\nabla p), v - j_v^h v) = (\nabla p, v - j_v^h v)\,, \tag{16}$$

$$(\kappa_v^h(\operatorname{div} v), p - j_p^h p) = (\operatorname{div} v, p - j_p^h p)\,. \tag{17}$$

Proof. These assertions are simple conclusions of (15) and the fact that

$$\phi - \kappa_r^h(\phi) \in D_r^h \qquad \forall \phi \in L^2(\Omega)$$

due the construction (11). $\square$

3.1 Stability

A technical tool to show the inf-sup stability of the bilinear form A_h is the following abstract lemma.

Lemma 4 *Let $(H, \|\cdot\|_H)$ be a Hilbert space where the norm of $y \in H$ is defined by means of two semi-norms $|\cdot|_a$ and $|\cdot|_b$ as $\|y\|_H^2 := |y|_a^2 + |y|_b^2$. For a given bilinear form $B : H \times H \to \mathbb{R}$, we assume*

$$\begin{aligned} \forall y \in H : \qquad & B(y,y) \ \geq\ c_0 |y|_a^2 \\ \forall y \in H \ \exists x \in H : \qquad & B(y,x) \ \geq\ c_2 |y|_b^2 - c_1 |y|_a^2 \quad \text{and} \quad \|x\|_H \leq \|y\|_H \end{aligned}$$

with constants $c_0, c_2 > 0$ and $c_1 \geq 0$. Then, the bilinear form fulfills the inf-sup condition

$$\forall y \in H \ \exists z \in H \setminus \{0\} : \qquad B(y,z) \ \geq\ \gamma \|y\|_H \|z\|_H$$

with the constant

$$\gamma = \min\left\{ \frac{c_2}{1+\varrho}, \frac{c_2}{1 + (c_1 + c_2)/c_0} \right\} > 0,$$

where $\varrho > 1$ is arbitrary.

Proof. In the case $y = 0$, the inf-sup condition is trivial. For any $y \in H \setminus \{0\}$, we choose $z := y + \varepsilon x$ with a suitable $\varepsilon > 0$ specified later. In order to guarantee $z \neq 0$, we use the estimate

$$\|z\|_H \geq \|y\|_H - \varepsilon\|x\|_H \geq (1-\varepsilon)\|y\|_H$$

and require that $\varepsilon \in (0,1)$. Using the assumptions on $B(\cdot,\cdot)$ and the abbreviation $\delta(\varepsilon) := \min\{c_0 - \varepsilon c_1, \varepsilon c_2\}$, we obtain

$$B(y,z) \;\geq\; (c_0 - \varepsilon c_1)|y|_a^2 + \varepsilon c_2 |y|_b^2 \;\geq\; \delta(\varepsilon)\|y\|_H^2 \;\geq\; \delta(\varepsilon)/(1+\varepsilon)\|y\|_H\|z\|_H\,.$$

Thus, the inf-sup condition follows with $\gamma = \gamma(\varepsilon) := \delta(\varepsilon)/(1+\varepsilon)$. It is an elementary task to show that, for $\varepsilon > 0$, the largest value of $\gamma(\varepsilon)$ is $\gamma^* = c_2\varepsilon^*/(1+\varepsilon^*)$ with $\varepsilon^* = c_0/(c_1+c_2)$. Taking into account the requirement $\varepsilon \in (0,1)$, we can express the best value for ε as $\varepsilon = \min\{1/\varrho, \varepsilon^*\}$ where $\varrho > 1$ is an arbitrary constant close to 1. With this choice of ε and $\gamma := \gamma(\varepsilon)$, we get $\varepsilon \in (0,1)$ and

$$\gamma \;=\; \frac{c_2\varepsilon}{1+\varepsilon} = \frac{c_2}{1+\varepsilon^{-1}} = \min\left\{\frac{c_2}{1+\varrho}, \frac{c_2}{1+(c_1+c_2)/c_0}\right\}.$$

□

Now we will apply Lemma 4 to the bilinear form $B = A_h$, the Hilbert space $H = Y_h$ and the norm $\|\cdot\|_H = |||\cdot|||$ which is defined as follows. For an arbitrary function $y = (v,p) \in H^1(\Omega)^d \times H^1(\Omega)$, its mesh-dependent norm is given by

$$|||y|||^2 \;:=\; \nu|v|_1^2 + \sigma\|v\|_0^2 + S_h(y,y) + \theta^2\|p\|_0^2\,.$$

The semi-norm $|\cdot|_b$ will be $|y|_b := \theta\|p\|_0$. The constant $\theta > 0$ has been introduced to contain all the dependencies on σ, ν and h such that the corresponding inf-sup constant with respect to the norm $|||\cdot|||$ is completely parameter independent.

Theorem 5 *Let the parameter functions $\delta_p, \delta_v : \Omega \to \mathbb{R}$ be patch-wise constant and $\delta_{p,K} > 0$ and $\delta_{v,K} \geq 0$ denote the constant values on element $K \in \mathcal{T}_{2h}$. Furthermore, let β and $\overline{\delta}_v$ be any upper bounds satisfying*

$$\begin{aligned} h_K\delta_{p,K}^{-1/2} &\leq \beta \qquad \forall\, K \in \mathcal{T}_{2h}\,, \\ \delta_{v,K} &\leq \overline{\delta}_v \qquad \forall\, K \in \mathcal{T}_{2h}\,. \end{aligned} \tag{18}$$

Then, with the norm parameter θ defined as

$$\theta := \frac{4\gamma_c}{5c_i\beta + 9c_s(\sigma + \nu + c_\kappa^2\overline{\delta}_v)^{1/2}} > 0\,, \tag{19}$$

where c_i, c_s and c_κ denote the constants in (13), (14) *and* (12), *the stabilized bilinear form A_h satisfies the inf-sup condition*

$$\forall\, y_h \in Y_h\ \exists\, z_h \in Y_h \setminus \{0\} : \qquad A_h(y_h, z_h) \geq \frac{1}{2}|||y_h|||\,|||z_h|||$$

and, for the the discrete problem (9), *there exists a unique solution.*

Proof. We use Lemma 4 where the semi-norms of an element $y=(v,p)$ are defined as

$$|y|_a \;:=\; \big(\sigma\|v\|_0^2+\nu|v|_1^2+S_h(y,y)\big)^{1/2} \qquad\text{and}\qquad |y|_b \;:=\; \theta\|p\|_0\,.$$

Let $y_h=(v_h,p_h)\in Y_h$ be given. Then, because of

$$A_h(y_h,y_h) \;=\; |y_h|_a^2\,,$$

the first assumption on the bilinear form is satisfied with $c_0=1$. To show the second assumption we consider the function $v^p\in H_0^1(\Omega)^d$ which has, due to the continuous inf-sup condition (8), the properties

$$(p_h,\operatorname{div} v^p)=\|p_h\|_0^2 \qquad\text{and}\qquad \|v^p\|_1\le\gamma_c^{-1}\|p_h\|_0\,.$$

Based on this v^p we choose the function $x_h\in Y_h$ defined as

$$x_h:=(w_h,0):=(-\mu\theta j_v^h v^p,0)\,,$$

where the constant $\mu>0$ is adjusted such that $|x_h|_a\le|y_h|_b$. To this end, we estimate by means of (14) and (12)

$$\begin{aligned}
|x_h|_a &\le \mu\theta\left\{\nu|j_v^h v^p|_1^2+\sigma\|j_v^h v^p\|_0^2+\overline{\delta}_v\|\kappa_v^h(\operatorname{div} j_v^h v^p)\|_0^2\right\}^{1/2}\\
&\le \mu\theta(\nu+\sigma+c_\kappa^2\overline{\delta}_v)^{1/2}c_s\gamma_c^{-1}\|p_h\|_0\\
&\le \mu(\nu+\sigma+c_\kappa^2\overline{\delta}_v)^{1/2}c_s\gamma_c^{-1}|y_h|_b\,.
\end{aligned}$$

Therefore, it holds $|\!|\!|x_h|\!|\!|=|x_h|_a\le|y_h|_b\le|\!|\!|y_h|\!|\!|$ for

$$\mu=(\nu+\sigma+c_\kappa^2\overline{\delta}_v)^{-1/2}c_s^{-1}\gamma_c.$$

For the component w_h of x_h, we get with partial integration, (16), (13) and Young's inequality the estimate

$$\begin{aligned}
-(p_h,\operatorname{div} w_h) &= \mu\theta\left\{(p_h,\operatorname{div} v^p)-\big(p_h,\operatorname{div}(v^p-j_v^h v^p)\big)\right\}\\
&= \mu\theta\left\{\|p_h\|_0^2+\big(\kappa_p^h(\nabla p_h),v^p-j_v^h v^p\big)\right\}\\
&\ge \mu\theta\left\{\|p_h\|_0^2-|y_h|_a\beta c_i\gamma_c^{-1}\|p_h\|_0\right\}\\
&= \mu\left\{\theta^{-1}|y_h|_b^2-\beta c_i\gamma_c^{-1}|y_h|_a|y_h|_b\right\}\\
&\ge \mu\Big(\theta^{-1}-\frac14\beta c_i\gamma_c^{-1}\Big)|y_h|_b^2-\mu\beta c_i\gamma_c^{-1}|y_h|_a^2\,.
\end{aligned}$$

With the abbreviation $\tilde\beta:=\beta c_i\gamma_c^{-1}$ and the estimate $|x_h|_a\le|y_h|_b$ this implies

$$\begin{aligned}
A_h(y_h,x_h) &= (\nu\nabla v_h,\nabla w_h)+(\sigma v_h,w_h)-(p_h,\operatorname{div}\widehat{w}_h)+S_h(y_h,x_h)\\
&\ge -|y_h|_a|x_h|_a+\mu\Big(\theta^{-1}-\frac14\tilde\beta\Big)|y_h|_b^2-\mu\tilde\beta|y_h|_a^2\,.\\
&\ge c_2|y_h|_b^2-c_1|y_h|_a^2\,,
\end{aligned}$$

where

$$c_2 := \mu\Big(\theta^{-1} - \frac{1}{4}\tilde{\beta}\Big) - \frac{1}{2} \qquad \text{and} \qquad c_1 := \mu\tilde{\beta} + \frac{1}{2}\,.$$

If we apply Lemma 4 with $c_0 = 1$ and $\varrho = \frac{3}{2}$, we get an inf-sup constant $\gamma \geq \frac{1}{2}$ under the condition

$$c_2 \geq \max\left\{\frac{5}{4}, 1 + c_1\right\}\,.$$

We satisfy this condition by the choice $c_2 = \frac{5}{4} + c_1$ which is just the equation for the definition of the constant θ in (19).
Since the discrete problem (9) is equivalent to an algebraic linear system of equations, the existence of a discrete solution follows from its uniqueness. The uniqueness of the discrete solution is a simple consequence of the inf-sup condition proved before. □

3.2 A priori error estimates

In the following, we will use the "perturbed" Galerkin orthogonality

$$A(y - y_h, z_h) - S_h(y_h, z_h) \;=\; 0 \qquad \forall z_h \in Y_h \tag{20}$$

to estimate the discretization error $y - y_h$. For ease of presentation, we will not write the subsequent estimates in a "localized version" where a power of the local mesh size h_K is multiplied with the local Sobolev norm of the exact solution on element K. This would be more relevant for the justification of locally adapted grids. But let us note here that this localized version can be obtained in a straightforward manner.

Theorem 6 *Assume that the parameter functions δ_v and δ_p are element-wise constant and let $\delta_{p,K}$ and $\delta_{v,K}$ denote the constant values on element $K \in \mathcal{T}_h$. Furthermore, assume that there exist constants $\delta_{v,i}$, $\delta_{p,i}$, $i = 0, 1$, such that for all elements $K \in \mathcal{T}_h$ it holds*

$$0 \leq \delta_{v,0} \leq \delta_{v,K} \leq \delta_{v,1} \qquad \textit{and} \qquad 0 < \delta_{p,0} \leq \delta_{p,K} \leq \delta_{p,1}.$$

Let the parameter θ in the mesh-dependent norm $\|\!|\cdot|\!\|$ be defined by (19) with $\overline{\delta}_v = \delta_{v,1}$ and some bound β from (18). Then, if the solution $y = (v, p)$ of problem (6) satisfies the smoothness assumptions $v \in H^{r+1}(\Omega)^d$ and $p \in H^{r+1}(\Omega)$, it holds the a priori error estimate

$$\|\!|y - y_h|\!\| \lesssim c_{1,h} h^r |v|_{r+1} + c_{2,h} h^r |p|_{r+1} \tag{21}$$

with the parameter dependent constants

$$\begin{aligned} c_{1,h} &= \sigma^{1/2} h + \nu^{1/2} + \delta_{v,1}^{1/2} + \delta_{p,0}^{-1/2} h\,, \\ c_{2,h} &= \delta_{p,1}^{1/2} + \big(\delta_{v,0} + \nu\big)^{-1/2} h + \theta h\,. \end{aligned}$$

Proof. We split the error $y - y_h$ into an interpolation error η and a projection error ξ by

$$\eta \;:=\; y - j_h y \quad \text{and} \quad \xi \;:=\; j_h y - y_h \,.$$

Due to the smoothness assumptions and the continuous discrete spaces V_h and Q_h, we have that $y - y_h$, η and ξ belong to the space $H^1(\Omega)^d \times H^1(\Omega)$. Therefore, the mesh-dependent triple norm is defined for these elements even in the case $\nu = 0$, i.e. for $V = H_0(\operatorname{div}, \Omega)$. For the interpolation error $\eta = (\eta_v, \eta_p) = (v - j_v^h v, p - j_p^h p)$ we obtain by (P1) and (13)

$$\begin{aligned} \|\!|\eta\|\!|^2 &\leq \sigma \|\eta_v\|_0^2 + \nu |\eta_v|_1^2 + \delta_{v,1} \|\kappa_v^h(\nabla \eta_v)\|_0^2 + \delta_{p,1}\|\kappa_p^h \nabla \eta_p\|_0^2 + \theta^2 \|\eta_p\|_0^2 \\ &\lesssim (\sigma h^2 + \nu + \delta_{v,1}) h^{2r} |v|_{r+1}^2 + \left(\delta_{p,1} + \theta^2 h^2\right) h^{2r} |p|_{r+1}^2 \,, \end{aligned}$$

which gives

$$\|\!|\eta\|\!| \;\lesssim\; \left(\sigma^{1/2} h + \nu^{1/2} + \delta_{v,1}^{1/2}\right) h^r |v|_{r+1} + \left(\delta_{p,1}^{1/2} + \theta h\right) h^r |p|_{r+1} \,.$$

Now we want to estimate $\|\!|\xi\|\!|$. Due to Theorem 5 there exists a non-zero function $z = (z_v, z_p) \in Y_h$ such that by means of the perturbed Galerkin orthogonality (20) we get:

$$\begin{aligned} \frac{1}{2} \|\!|\xi\|\!| \|\!|z\|\!| &\leq A_h(\xi, z) \;=\; -A(\eta, z) + S_h(j_h y, z) \\ &\leq \|\!|\eta\|\!| \, \|\!|z\|\!| + |(\eta_p, \operatorname{div} z_v)| + |(\eta_v, \nabla z_p)| + |S_h(j_h y, z)| \,. \end{aligned}$$

We will bound the terms on the right hand side separately. Using the Cauchy-Schwarz inequality for the symmetric non-negative bilinear form $S_h(\cdot,\cdot)$, we obtain

$$\begin{aligned} |S_h(j_h y, z)| &\leq S_h(j_h y, j_h y)^{1/2} S_h(z,z)^{1/2} \\ &\leq \left\{ \delta_{v,1}^{1/2} \|\kappa_v^h(\operatorname{div} j_v^h v)\|_0 + \delta_{p,1}^{1/2} \|\kappa_p^h(\nabla\, j_p^h p)\|_0 \right\} \|\!|z\|\!|. \end{aligned}$$

For the scalar version of the fluctuation operator $\kappa_r^h : L^2(\Omega) \to L^2(\Omega)$ and the corresponding interpolation operator $j_r^h : H^{r+1}(\Omega) \to Q_r(\mathcal{M}_h)$, it holds for a smooth function $\phi \in H^{r+1}(\Omega)$ and a partial derivative $\partial_i := \frac{\partial}{\partial x_i}$, $i = 1, \ldots, d$, due to the properties (P1), (13) and (P2)

$$\begin{aligned} \|\kappa_r^h(\partial_i j_r^h \phi)\|_0 &\leq \|\kappa_r^h(\partial_i(\phi - j_r^h \phi))\|_0 + \|\kappa_r^h(\partial_i \phi)\|_0 \\ &\lesssim |\phi - j_r^h \phi|_1 + h^r |\partial_i \phi|_r \\ &\lesssim h^r |\phi|_{r+1}. \end{aligned}$$

Thus, we get

$$|S_h(j_h y, z)| \lesssim \left(\delta_{v,1}^{1/2} h^r |v|_{r+1} + \delta_{p,1}^{1/2} h^r |p|_{r+1}\right) \|\!|z\|\!|.$$

Using the orthogonality properties (16)-(17) of the interpolation operator j_h, we obtain

$$\begin{aligned} |(\eta_v, \nabla z_p)| &= |\left(\eta_v, \kappa_p^h(\nabla z_p)\right)| \leq \delta_{p,0}^{-1/2} \|\eta_v\|_0 \|\!|z\|\!| \lesssim \delta_{p,0}^{-1/2} h^{r+1} |v|_{r+1} \|\!|z\|\!| \,, \\ |(\eta_p, \operatorname{div} z_v)| &= |(\eta_p, \kappa_v^h(\operatorname{div} z_v))| \lesssim \delta_{v,0}^{-1/2} h^{r+1} |p|_{r+1} \, \|\!|z\|\!| \,. \end{aligned}$$

The last estimate is only valid for $\delta_{v,0} > 0$. However, for $\nu > 0$, the last term can be bounded alternatively by the Cauchy-Schwarz inequality:

$$|(\eta_p, \operatorname{div} z_v)| \;\lesssim\; \|\eta_p\|_0 \, \|\nabla z_v\|_0 \;\lesssim \nu^{-1/2} h^{r+1} |p|_{r+1} |\!|\!| z |\!|\!| \,.$$

Summarizing all estimates and dividing by $|\!|\!| z |\!|\!|$ leads to

$$|\!|\!| \xi |\!|\!| \;\lesssim\; |\!|\!| \eta |\!|\!| + \left\{ \delta_{v,1}^{1/2} + \delta_{p,0}^{-1/2} h \right\} h^r |v|_{r+1} + \left\{ \delta_{p,1}^{1/2} + (\delta_{v,0} + \nu)^{-1/2} h \right\} h^r |p|_{r+1} \,.$$

Finally, we obtain by the triangle inequality

$$|\!|\!| y - y_h |\!|\!| \;\lesssim\; \left\{ \sigma^{1/2} h + \nu^{1/2} + \delta_{v,1}^{1/2} + \delta_{p,0}^{-1/2} h \right\} h^r |v|_{r+1} + \left\{ \delta_{p,1}^{1/2} + (\delta_{v,0} + \nu)^{-1/2} h + \theta h \right\} h^r |p|_{r+1} \,.$$

□

3.3 Optimal stabilization constants

We use now the a priori estimate of Theorem 6 to calibrate the stabilization parameters δ_v and δ_p in dependency of the mesh size h and the model parameters ν and σ. For simplicity, we restrict the presentation to the case of quasi-uniform grids $\mathcal{T}_h$ where the quotient h/h_K for all elements $K \in \mathcal{T}_h$ is bounded by an h-independent constant. In the case of general shape-regular meshes, the parameter h occurring in the below derived choices of δ_v, δ_p has to be replaced by the local mesh size h_K.
Our aim is to obtain an optimal estimate in the following sense. On the one hand, we want to choose δ_v and δ_p as small as possible since then the stabilization term $S_h(\cdot,\cdot)$, which causes a consistency error, will be as small as possible. On the other hand, there appear negative powers of δ_p and δ_v in the right hand side of the a priori estimate (21). These terms have to be bounded to a size which is comparable to those terms in the estimate that do not contain δ_v or δ_p. To be more precisely, let us recall the parameter dependent error constants

$$c_{1,h} = \sigma^{1/2} h + \nu^{1/2} + \delta_{v,1}^{1/2} + \delta_{p,0}^{-1/2} h \qquad \text{and} \qquad c_{2,h} = \delta_{p,1}^{1/2} + \left(\delta_{v,0} + \nu\right)^{-1/2} h + \theta h \,.$$

Now, in order to bound $c_{1,h}$, it should hold

$$\delta_{p,0}^{-1/2} h \;\lesssim\; \sigma^{1/2} h + \nu^{1/2}$$

which leads to the lower bound

$$\delta_{p,0} \;\gtrsim\; \left(\sigma^{1/2} h + \nu^{1/2}\right)^{-2} h^2 \gtrsim \left(\sigma h^2 + \nu\right)^{-1} h^2 \,.$$

This motivates the choice $\delta_p = (\sigma h^2 + \nu)^{-1} h^2$. In order to bound $c_{2,h}$ with this choice of δ_p, it should hold

$$\left(\delta_{v,0} + \nu\right)^{-1/2} h \lesssim \delta_p^{1/2} = \left(\sigma h^2 + \nu\right)^{-1/2} h \,,$$

which leads to the lower bound

$$\delta_{v,0} \gtrsim \sigma h^2$$

and motivates the choice $\delta_v = \sigma h^2$.

Theorem 7 *Let the stabilization parameters be chosen as*

$$\delta_p = (\sigma h^2+\nu)^{-1}h^2 \qquad \textit{and} \qquad \delta_v = \sigma h^2\,, \tag{22}$$

and assume that the solution $y=(v,p)$ of problem (6) satisfies the smoothness assumption $(v,p)\in H^{r+1}(\Omega)^{d+1}$. Then, it holds the a priori error estimate

$$|||y-y_h||| \lesssim \varrho_h h^r|v|_{r+1} + \varrho_h^{-1}h^{r+1}|p|_{r+1}\,,$$

where $\varrho_h := \nu^{1/2}+\sigma^{1/2}h$.

Proof. At first we want to check the parameter θ from (19). For our choices of δ_v and δ_p, we can use the bounds

$$\beta = \delta_p^{-1/2}h = (\sigma h^2+\nu)^{1/2} \qquad \text{and} \qquad \overline{\delta}_v = \sigma h^2.$$

which implies

$$\theta = \frac{4\gamma_c}{5c_i\beta + 9c_s(\sigma+\nu+c_\kappa^2\overline{\delta}_v)^{1/2}} \le \frac{4\gamma_c}{5c_i(\sigma h^2+\nu)^{1/2}} \lesssim \frac{1}{\varrho_h}.$$

Now, the constants $c_{1,h}$ and $c_{2,h}$ in Theorem 6 can be estimated as

$$\begin{aligned}
c_{1,h} &\lesssim \sigma^{1/2}h+\nu^{1/2} = \varrho_h\,,\\
c_{2,h} &= \delta_p^{1/2} + (\sigma h^2+\nu)^{-1/2}h + \theta h\\
&\le \delta_p^{1/2} + 2(\sigma^{1/2}h+\nu^{1/2})^{-1}h + \theta h \lesssim \varrho_h^{-1}h\,.
\end{aligned}$$

□

For the Darcy case, we obtain the following result.

Corollary 8 *In the extreme case of vanishing viscosity $\nu=0$ with the choice $\delta_p=\sigma^{-1}$ and $\delta_v=\sigma h^2$ and under the smoothness assumption $(v,p)\in H^{r+1}(\Omega)^{d+1}$, it holds the a priori error estimate*

$$\sigma^{1/2}\|v-v_h\|_0 + \sigma^{-1/2}\|p-p_h\|_0 + S_h(y-y_h,y-y_h)^{1/2} \lesssim \sigma^{1/2}h^{r+1}|v|_{r+1} + \sigma^{-1/2}h^r|p|_{r+1}\,.$$

Proof. We apply Theorem 7 with $\nu=0$ and get $\varrho_h=\sigma^{1/2}h$, $\beta=\sigma^{1/2}h$ and $\sigma^{1/2}\lesssim\theta$ which together imply the assertion. □

This estimate is still suboptimal, because the error bound is only of the size $\mathcal{O}(h^r)$. A further factor of h will be obtained by a duality argument in the next section.

The next corollary shows the result that Theorem 7 generates for the Stokes system.

Corollary 9 *In the extreme case of the Stokes system, i.e., $\sigma=0$, $\nu>0$, with the choice $\delta_p=h^2\nu^{-1}$ and $\delta_v=0$ and under the assumption $(v,p)\in H^{r+1}(\Omega)^{d+1}$, it holds the a priori error estimate*

$$\nu^{1/2}|v-v_h|_1 + \nu^{-1/2}\|p-p_h\|_0 + S_h(y-y_h,y-y_h)^{1/2} \lesssim \nu^{1/2}h^r|v|_{r+1} + \nu^{-1/2}h^{r+1}|p|_{r+1}\,.$$

Proof. For $\nu > 0$ and $\sigma = 0$ we can choose in (18) the bound $\beta = \nu^{1/2}$. Then, we get with $\overline{\delta}_v = 0$ the estimate $\nu^{-1/2} \lesssim \theta$. Together with $\varrho_h = \nu^{1/2}$ the assertion follows from Theorem 7. □

Note, that the choice of the parameters in these two Corollaries can be written in general form as in (22). A distinction of cases is not necessary.

4 Duality arguments

In order to get an improved L^2-error estimate for the pressure or the velocity, we consider the following continuous dual problem. For a given $\varphi = (\varphi_v, \varphi_p)$ with $\varphi_v \in L^2(\Omega)^d$ and $\varphi_p \in L^2_0(\Omega)$, find a solution z of the problem

$$z \in Y: \qquad A(w,z) \;=\; \langle \varphi, w\rangle := (\varphi_v, w_v) + (\varphi_p, w_p) \qquad \forall w = (w_v, w_p) \in Y\,. \tag{23}$$

Using the definition of $A(\cdot,\cdot)$ in (7) and partial integration we see that this problem is equivalent to the formulation: Find $(z_v, z_p) \in V \times Q$ such that

$$-\nu\Delta z_v + \sigma z_v - \nabla z_p \;=\; \varphi_v \text{ and } \operatorname{div} z_v \;=\; \varphi_p \qquad \text{in} \quad \Omega\,, \tag{24}$$

where the boundary conditions for z_v are contained in the definition (5) of the space V and the differential operators are defined in the distributional sense.

4.1 Duality argument for the L^2-error in the pressure for $\nu = 0$

In this subsection we consider the Darcy case with $\nu = 0$ and $\sigma > 0$. We choose $\varphi = (\varphi_v, \varphi_p) = (0, p - p_h)$ in the dual problem (23) in order to estimate $\|p - p_h\|_0^2$ by taking $w_p = \varphi_p$. For the solution of (23) or (24), respectively, we need the following regularity property in the case $\nu = 0$, $\sigma > 0$:

(R1) For $\varphi_v = 0$ and $\varphi_p \in L^2_0(\Omega)$, the solution of (23) satisfies $(z_v, z_p) \in H^1(\Omega)^d \times H^2(\Omega)$ and it holds the estimate

$$\|z_v\|_1 + \sigma^{-1}\|z_p\|_2 \;\lesssim\; \|\varphi_p\|_0\,.$$

This property is guaranteed under some assumptions on the domain Ω which are given in the next lemma.

Lemma 10 *Let Ω be an open bounded domain with a C^2-boundary or a convex polyhedron in $\mathbb{R}^d$, $d \in \{2,3\}$. Then the solution of the dual problem* (23) *with $\nu = 0$ and $\sigma > 0$ has the regularity property (R1).*

Proof. From the theory of mixed methods [7] it is clear that for $\varphi_v = 0$ the problem (23) has a unique solution $(z_v, z_p) \in H_0(\operatorname{div}, \Omega) \times L^2_0(\Omega)$ satisfying the stability estimate

$$\|z_v\|_0 + \|\operatorname{div} z_v\|_0 \lesssim \|\varphi_p\|_0.$$

We consider the homogeneous Neumann problem: Find $\phi \in H^1(\Omega) \cap L_0^2(\Omega)$ such that

$$\begin{aligned} -\Delta\phi &= -\sigma \operatorname{div} z_v && \text{in} \quad \Omega, \\ \frac{\partial \phi}{\partial n} &= 0 && \text{on} \quad \partial\Omega. \end{aligned}$$

Since (z_v, z_p) is also a solution of (24) we have $\operatorname{div} z_v = \varphi_p$ which implies by the regularity result for the Neumann problem [12, Theorem I.1.10] the estimate

$$\|\phi\|_2 \;\lesssim\; \|\sigma \operatorname{div} z_v\|_0 \;=\; \sigma\|\varphi_p\|_0. \tag{25}$$

It is easy to verify that $z = (z_v, \phi)$ is a solution of (24) with $\nu = 0$ and $\varphi_v = 0$. Therefore, (25) yields $\|z_v\|_1 = \sigma^{-1}\|\nabla z_p\|_1 \lesssim \|\varphi_p\|_0$ which concludes the proof. $\square$

Theorem 11 *Let $\nu = 0$, $\sigma > 0$ and assume that the regularity property (R1) is satisfied. Then, it holds the following a priori error estimate for the pressure:*

$$\|p - p_h\|_0 \;\lesssim\; \left(h(\sigma^{1/2} + \delta_p^{-1/2} + \delta_p^{1/2}\sigma) + \delta_v^{1/2}\right) |\!|\!| y - y_h |\!|\!| + \delta_p \sigma h^{r+1}|p|_{r+1} + \delta_v h^r |v|_{r+1}\,.$$

Proof. In (23) we choose $\varphi_v = 0$, $\varphi_p = p - p_h$ and $w = y - y_h$, use the perturbed Galerkin orthogonality (20) and obtain:

$$\|p - p_h\|_0^2 \;=\; A(y - y_h, z) \;=\; A(y - y_h, z - j_h z) + S_h(y_h, j_h z)\,.$$

We estimate the two terms separately:

$$\begin{aligned} A(y - y_h, z - j_h z) \;\leq\;& \sigma\|v - v_h\|_0\|z_v - j_v^h z_v\|_0 + (p - p_h, \operatorname{div}(z_v - j_v^h z_v)) \\ &+(\operatorname{div}(v - v_h), z_p - j_p^h z_p)\,. \end{aligned}$$

For the dual solution z, it holds due to (R1):

$$\begin{aligned} \|z_v - j_v^h z_v\|_0 &\lesssim h\|\nabla z_v\|_0 \;\leq\; h\|\varphi_p\|_0 \;=\; h\|p - p_h\|_0\,, \\ (\nabla(p - p_h), z_v - j_v^h z_v) &= (\kappa_p^h \nabla(p - p_h), z_v - j_v^h z_v) \;\leq\; \delta_p^{-1/2} |\!|\!| y - y_h |\!|\!| \|z_v - j_v^h z_v\|_0 \\ &\lesssim \delta_p^{-1/2} |\!|\!| y - y_h |\!|\!| h\|p - p_h\|_0\,, \\ (\operatorname{div}(v - v_h), z_p - j_p^h z_p) &\lesssim \|v - v_h\|_0 h|z_p|_2 \;\lesssim\; \sigma^{1/2} h |\!|\!| y - y_h |\!|\!| \|p - p_h\|_0\,. \end{aligned}$$

By summation we obtain

$$A(y - y_h, z - j_h z) \;\lesssim\; h(\sigma^{1/2} + \delta_p^{-1/2}) |\!|\!| y - y_h |\!|\!| \|p - p_h\|_0\,.$$

We split the stabilization terms into several parts:

$$S_h(y_h, j_h z) \;=\; S_h(y_h - y, j_h z - z) + S_h(y, j_h z - z) + S_h(y_h - y, z) + S_h(y, z)\,.$$

We bound these terms individually by means of (R1):

$$\begin{aligned}
S_h(y_h - y, j_h z - z) &\le S_h(y_h - y, y_h - y)^{1/2} S_h(j_h z - z, j_h z - z)^{1/2} \\
&\le \|y - y_h\| \big(\delta_p^{1/2} \|\kappa_p^h(\nabla(z_p - j_p^h z_p))\|_0 + \delta_v^{1/2} \|\kappa_v^h(\operatorname{div}(z_v - j_v^h z_v))\|_0\big) \\
&\lesssim \|y - y_h\| (\delta_p^{1/2} h |z_p|_2 + \delta_v^{1/2} |z_v|_1) \\
&\lesssim \|y - y_h\| (\delta_p^{1/2} h\sigma + \delta_v^{1/2}) \|p - p_h\|_0 \,, \\
S_h(y, j_h z - z) &\le \delta_p \|\kappa_p^h(\nabla p)\|_0 \|\kappa_p^h(\nabla(j_p^h z_p - z_p))\|_0 + \delta_v \|\kappa_v^h(\operatorname{div} v)\|_0 \|\kappa_v^h(\operatorname{div}(z_v - j_v^h z_v))\|_0 \\
&\lesssim \delta_p h^r |p|_{r+1} h |z_p|_2 + \delta_v h^r |v|_{r+1} |z_v|_1 \\
&\lesssim (\delta_p h^{r+1} \sigma |p|_{r+1} + \delta_v h^r |v|_{r+1}) \|p - p_h\|_0 \,, \\
S_h(y_h - y, z) &\le \|y - y_h\| S_h(z, z)^{1/2} \\
&\lesssim \|y - y_h\| \big(\delta_p^{1/2} \|\kappa_p^h(\nabla z_p)\|_0 + \delta_v^{1/2} \|\kappa_v^h(\operatorname{div} z_v)\|_0\big) \\
&\lesssim \|y - y_h\| \big(\delta_p^{1/2} h |z_p|_2 + \delta_v^{1/2} |z_v|_1\big) \\
&\lesssim \|y - y_h\| \big(\delta_p^{1/2} h\sigma + \delta_v^{1/2}\big) \|p - p_h\|_0 \,, \\
S_h(y, z) &\le \delta_p \|\kappa_p^h(\nabla p)\|_0 \|\kappa_p^h(\nabla z_p)\|_0 + \delta_v \|\kappa_v^h(\operatorname{div} v)\|_0 \|\kappa_v^h(\operatorname{div} z_v)\|_0 \\
&\lesssim \delta_p h^r |p|_{r+1} h |z_p|_2 + \delta_v . h^r |v|_{r+1} |z_v|_1 \\
&\lesssim (\delta_p \sigma h^{r+1} |p|_{r+1} + \delta_v h^r |v|_{r+1}) \|p - p_h\|_0 \,.
\end{aligned}$$

Now, the assertion follows by summation of the upper bounds of all terms. □

If we apply Theorem 7 for the estimation of $\|y - y_h\|$, we obtain the following optimal result for the pressure error.

Corollary 12 *Let $\nu = 0$, $\sigma > 0$ and assume that the regularity property (R1) is satisfied. Then, for the parameter choice (22), i.e., $\delta_p = \sigma^{-1}$ and $\delta_v = \sigma h^2$, it holds the following optimal a priori error estimate for the pressure:*

$$\|p - p_h\|_0 \lesssim \sigma h^{r+2} |v|_{r+1} + h^{r+1} |p|_{r+1} \,.$$

Proof. The assertion is a direct consequence of Theorems 11 and 7 :

$$\begin{aligned}
\|p - p_h\|_0 &\lesssim \sigma^{1/2} h \left(\sigma^{1/2} h^{r+1} |v|_{r+1} + \sigma^{-1/2} h^r |p|_{r+1}\right) + h^{r+1} |p|_{r+1} + \sigma h^{r+2} |v|_{r+1} \\
&\lesssim \sigma h^{r+2} |v|_{r+1} + h^{r+1} |p|_{r+1} \,.
\end{aligned}$$

□

4.2 Duality argument for the L^2-error in velocity for $\nu > 0$

Now, we treat the case where $\nu > 0$ and $\sigma \ge 0$. In order to estimate $\|v - v_h\|_0^2$ we consider the dual problem (23) with $\varphi = (\varphi_v, \varphi_p) = (v - v_h, 0)$ and choose the test function $w = y - y_h = (v - v_h, p - p_h)$. For the solution, we need the following regularity property:

(R2) For $\varphi_v \in L^2(\Omega)^d$ and $\varphi_p = 0$, the solution of (23) satisfies $(z_v, z_p) \in H^2(\Omega)^d \times H^1(\Omega)$ and it holds the estimate

$$\nu \|z_v\|_2 + \|z_p\|_1 \lesssim \|\varphi_v\|_0 \,.$$

In the literature, this property has been proven for the classical Stokes problem, i.e., for $\nu = 1$ and $\sigma = 0$, under some assumptions on the domain Ω. We can find a proof for a bounded domain $\Omega \subset \mathbb{R}^d$ with a C^2-boundary in [19] and for a bounded convex polygonal or polyhedral domain in [13, 10]. In the following lemma, we derive from these results a generalization to our parameter dependent Stokes problem (24) and show that the constant in the right hand side of the estimate in (R2) is independent of ν and σ.

Lemma 13 *Let Ω be an open bounded domain with a C^2-boundary or a convex polyhedron in $\mathbb{R}^d$, $d \in \{2, 3\}$. Then, the solution of the dual problem* (23) *in the case $\nu > 0$, $\sigma \geq 0$ has the regularity property (R2).*

Proof. From the theory of mixed methods [7] it is clear that problem (23) has a unique solution $(z_v, z_p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ for $\varphi_p = 0$ and arbitrary $\varphi_v \in L^2(\Omega)^d$. In order to derive a stability estimate for z_v we use in (23) the test function $(w_v, w_p) = (z_v, 0)$, exploit the fact that $\operatorname{div} z_v = 0$ and obtain

$$\nu |z_v|_1^2 + \sigma \|z_v\|_0^2 \leq \|\varphi_v\|_0 \|z_v\|_0,$$

which implies

$$\sigma \|z_v\|_0 \leq \|\varphi_v\|_0. \tag{26}$$

Now, we consider the classical Stokes problem: Find $(\xi_v, \xi_p) \in V \times Q$ such that

$$\begin{aligned} -\Delta \xi_v + \nabla \xi_p &= \psi_v \quad \text{in} \quad \Omega \,, \\ \operatorname{div} \xi_v &= 0 \qquad \text{in} \quad \Omega \,, \end{aligned} \tag{27}$$

where $\psi_v \in L^2(\Omega)^d$ is a given arbitrary function. Under the assumptions on the domain Ω it has been proven in [19, 13, 10] that for the solution it holds $(\xi_v, \xi_p) \in H^2(\Omega)^d \times H^1(\Omega)$ and

$$\|\xi_v\|_2 + \|\xi_p\|_1 \lesssim \|\psi_v\|_0. \tag{28}$$

If we choose in (27) the right hand side $\psi_v := \varphi_v - \sigma z_v \in L^2(\Omega)^d$, where z_v is the solution of (23), we can easily verify that the solution is $(\xi_v, \xi_p) = (\nu z_v, -z_p)$. Thus, from (28) and (26) we obtain

$$\nu \|z_v\|_2 + \|z_p\|_1 \lesssim \|\psi_v\|_0 \leq 2 \|\varphi_v\|_0,$$

which concludes the proof. $\square$

Under the assumptions of Lemma 13 on the domain Ω we get the following optimal error estimate in the L^2-norm of the velocity.

Theorem 14 *Let $\nu > 0$, $\sigma \geq 0$ and assume that the regularity property (R2) is satisfied. Then, for the parameter choice of Theorem 7, i.e., $\delta_p = (\sigma h^2 + \nu)^{-1} h^2$ and $\delta_v = \sigma h^2$, it holds the a priori error estimate for the velocity:*

$$\|v - v_h\|_0 \;\lesssim\; h^{r+1}\Big(1 + \frac{\sigma}{\nu} h^2\Big)|v|_{r+1} + \frac{h^{r+2}}{\nu}|p|_{r+1}\,.$$

Proof. In the dual problem (23), we choose $\varphi = (\varphi_v, \varphi_p) = (v - v_h, 0)$ and the test function $w = y - y_h = (v - v_h, p - p_h)$. Due to the perturbed Galerkin orthogonality (20) we get

$$\|v - v_h\|_0^2 \;=\; A(y - y_h, z) \;=\; A(y - y_h, z - j_h z) + S_h(y_h, j_h z)\,, \tag{29}$$

where the interpolate $j_h z = (j_v^h z_v, j_p^h z_p) \in Y_h$ is chosen such that it satisfies the orthogonality equations (16) and (17). In the following, we will use the abbreviations $\eta_v := z_v - j_v^h z_v$ and $\eta_p := z_p - j_v^h z_p$. Then, the definition (7) of $A(\cdot,\cdot)$ and partial integration due to $\eta_v \in H_0^1(\Omega)$ yield

$$A(y - y_h, z - j_h z) \;\leq\; \nu|v - v_h|_1|\eta_v|_1 + \sigma\|v - v_h\|_0\|\eta_v\|_0 + (\nabla(p - p_h), \eta_v) + (\operatorname{div}(v - v_h), \eta_p)\,.$$

With the orthogonality equation (16) and the regularity property (R2) it follows:

$$\begin{aligned}
\|\eta_v\|_0 &\lesssim h^2\|z_v\|_2 \;\leq\; \nu^{-1}h^2\|\varphi_v\|_0\,,\\
(\nabla(p - p_h), \eta_v) &= (\kappa_p^h \nabla(p - p_h), \eta_v) \;\lesssim\; \delta_p^{-1/2}|\!|\!|y - y_h|\!|\!|\nu^{-1}h^2\|\varphi_v\|_0\,,\\
(\operatorname{div}(v - v_h), \eta_p) &\lesssim |v - v_h|_1 h\|z_p\|_1 \lesssim \nu^{-1/2}|\!|\!|y - y_h|\!|\!| h\|\varphi_v\|_0.
\end{aligned}$$

By summation we obtain

$$\begin{aligned}
A(y - y_h, z - j_h z) &\lesssim \frac{h}{\nu}\Big[\nu^{1/2} + \sigma^{1/2}h + \delta_p^{-1/2}h\Big]|\!|\!|y - y_h|\!|\!|\|\varphi_v\|_0\\
&\lesssim \frac{h}{\nu}\Big[\nu^{1/2} + \sigma^{1/2}h\Big]|\!|\!|y - y_h|\!|\!|\|\varphi_v\|_0\,.
\end{aligned}$$

For the stabilization term in (29), we get

$$S_h(y_h, j_h z) \;\leq\; S_h(y_h, y_h)^{1/2} S_h(j_h z, j_h z)^{1/2}.$$

The fact that $S_h(\cdot,\cdot)^{1/2}$ is a semi-norm implies

$$\begin{aligned}
S_h(y_h, y_h)^{1/2} &\leq S_h(y_h - y, y_h - y)^{1/2} + S_h(y, y)^{1/2}\\
&\leq |\!|\!|y - y_h|\!|\!| + \delta_p^{1/2}\|\kappa_p^h(\nabla p)\|_0 + \delta_v^{1/2}\|\kappa_v^h(\operatorname{div} v)\|_0\\
&\lesssim |\!|\!|y - y_h|\!|\!| + \delta_p^{1/2}h^r|p|_{r+1} + \delta_v^{1/2}h^r|v|_{r+1}\,,
\end{aligned}$$

and the regularity property (R2) yields

$$\begin{aligned}
S_h(j_h z, j_h z)^{1/2} &\leq \delta_p^{1/2}\Big\{\|\kappa_p^h(\nabla(j_p^h z_p - z_p))\|_0 + \|\kappa_p^h(\nabla z_p)\|_0\Big\} +\\
&\qquad \delta_v^{1/2}\Big\{\|\kappa_v^h(\operatorname{div}(j_v^h z_v - z_v))\|_0 + \|\kappa_v^h(\operatorname{div} z_v)\|_0\Big\}\\
&\lesssim \delta_p^{1/2}|z_p|_1 + \delta_v^{1/2}h|z_v|_2\\
&\lesssim \Big(\delta_p^{1/2} + \delta_v^{1/2}h\nu^{-1}\Big)\|\varphi_v\|_0.
\end{aligned}$$

For the choice of δ_p and δ_v, it holds

$$\delta_p^{1/2} + \delta_v^{1/2} h\nu^{-1} \le h\nu^{-1/2} + \sigma^{1/2}h^2\nu^{-1} = \frac{h}{\nu}\left[\nu^{1/2} + \sigma^{1/2}h\right]$$

and

$$\delta_p^{1/2} h^r |p|_{r+1} + \delta_v^{1/2} h^r |v|_{r+1} \lesssim h^{r+1}\left[\nu^{1/2} + \sigma^{1/2}h\right]^{-1} |p|_{r+1} + \sigma^{1/2}h^{r+1}|v|_{r+1}.$$

If we summarize the estimates for the terms on the right hand side of (29) and divide by $\|\varphi_v\|_0$, we obtain

$$\|v - v_h\|_0 \lesssim \frac{h}{\nu}\left[\nu^{1/2} + \sigma^{1/2}h\right] |||y - y_h||| + \frac{h^{r+2}}{\nu}|p|_{r+1} + \frac{\sigma^{1/2}}{\nu}\left[\nu^{1/2} + \sigma^{1/2}h\right] h^{r+2}|v|_{r+1}.$$

The choice of δ_p and δ_v implies by Theorem 7 the following estimation of $y - y_h$:

$$\begin{aligned} \frac{h}{\nu}\left[\nu^{1/2} + \sigma^{1/2}h\right] |||y - y_h||| &\lesssim \frac{h}{\nu}\left[\nu^{1/2} + \sigma^{1/2}h\right]^2 h^r |v|_{r+1} + \frac{h^{r+2}}{\nu}|p|_{r+1} \\ &= h^{r+1}\left[1 + \left(\frac{\sigma}{\nu}\right)^{1/2} h\right]^2 |v|_{r+1} + \frac{h^{r+2}}{\nu}|p|_{r+1}. \end{aligned}$$

Now, the assertion follows immediately. □

5 Numerical results

For numerical tests, we consider problem (1)-(2) on the unit square $\Omega = (0,1)^2$ where the data f and g are chosen such that we have the following σ- and ν-dependent exact solution:

$$\begin{aligned} v_1(x,y) &= (1-\sigma)\sin(x)\sin(y) + (1-\nu)\cos(x)\cos(y)\,, \\ v_2(x,y) &= \cos(x)\cos(y)\,, \\ p(x,y) &= 2\cos(x)\sin(y) - p_0\,. \end{aligned}$$

The boundary conditions for the velocity are of Dirichlet-type as in (3) and (4) but non-homogeneous according to the exact solution. The constant pressure part p_0 is defined in such a way that p has zero integral mean value. For the Stokes case, $\nu = 1$ and $\sigma = 0$, we recover the example of Braess and Sarazin [6] which is also considered as a test problem in Ganesan et al. [11]. In Figure 1, we show the pressure and velocity field for the extreme cases $\nu = 0$, $\sigma = 1$ (left) and $\sigma = 0$, $\nu = 1$ (right).

Several tests on a hierarchy of uniformly refined equidistant axis-parallel meshes are carried out with different choices of σ and ν. In all calculations, the polynomial degree for velocity and pressure is $r = 1$ (bilinear finite elements).

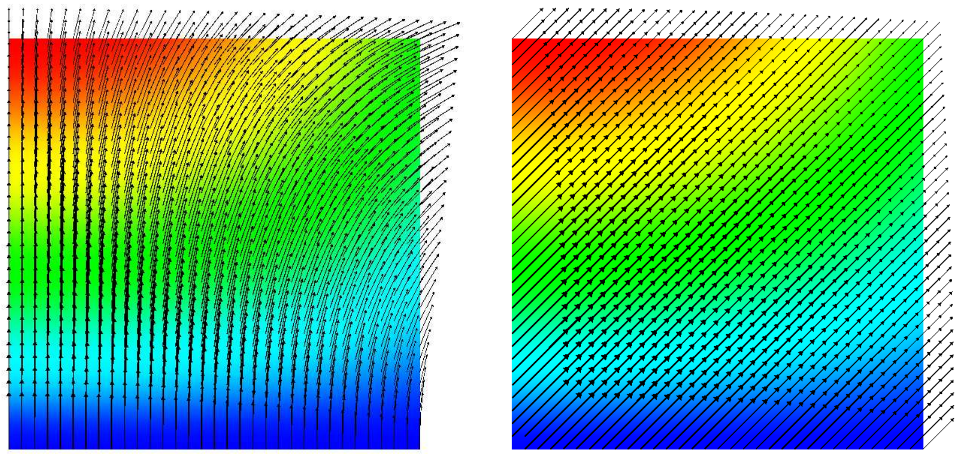

Figure 1: Pressure (color) and velocity field (arrows) of the test problem with $\nu = 0$ and $\sigma = 1$ (left) and $\nu = 1$ and $\sigma = 0$ (right).

5.1 Error in the pressure

Figures 2 and 3 show the L^2- and L^∞-error in the pressure, respectively. For the case $\nu = 0$ (Darcy), these errors behave like h^2 in both norms. This corresponds exactly to the optimal L^2-norm estimate proven above by a duality argument for $\nu = 0$. In the case $\nu = 10^{-6}$, the optimal order of 2 can still be seen. However, for the case $\nu = 1$ and $\sigma = 0$ (Stokes), the convergence order decreases to $7/4$ in the L^2-norm and to 1 in L^∞-norm. On the other hand, this is still better than the result that the error estimate of Theorem 7 provides for the pressure in the L^2-norm.

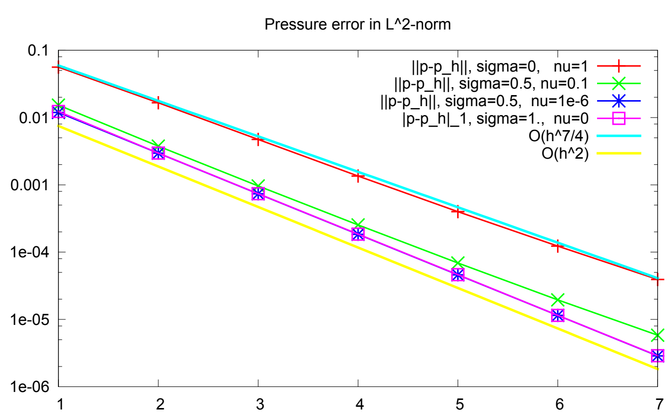

Figure 2: L^2-error of pressure.

In Figure 4 the H^1-semi-norm of the pressure error is shown. We observe order 1 for $\sigma > 0$ and order $3/4$ for $\sigma = 0$.

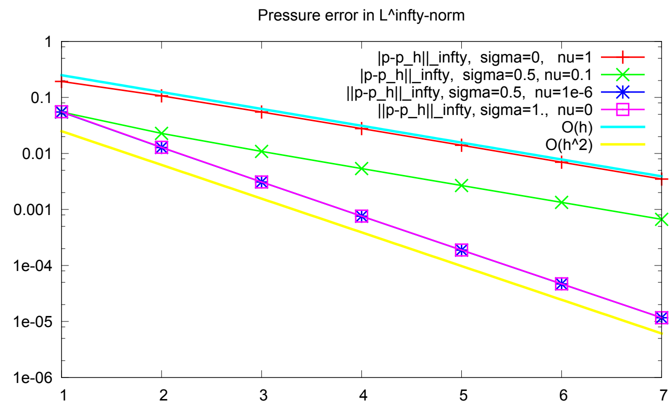

Figure 3: L^∞-error of pressure.

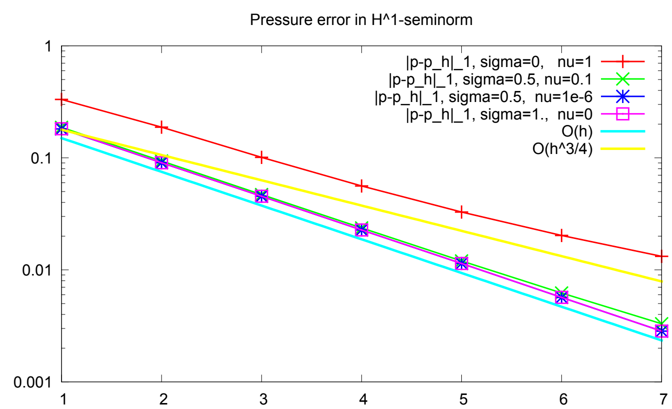

Figure 4: H^1-errors of pressure.

5.2 Error in the velocity

Figures 5 and 6 show the error of the velocity in the L^2- and L^∞-norm, respectively. We see second order of convergence completely independent of the parameters σ and ν. In particular, this confirms the optimal order of convergence that we have proved in Theorem 14 by means of a duality argument for the L^2-norm error of the velocity in the case $\nu > 0$.

Figure 7 shows the velocity error in the H^1-semi-norm. We observe first order of convergence independently of the parameters σ and ν.

6 Conclusion

We have presented and analyzed a symmetric stabilized finite element method for the Darcy-Brinkman system, which is a generalization of both the Stokes and the Darcy system. The method is robust with respect to the model parameters, even in the extremal cases where one of the parameters vanishes. In particular, we recover a well-known local projection scheme

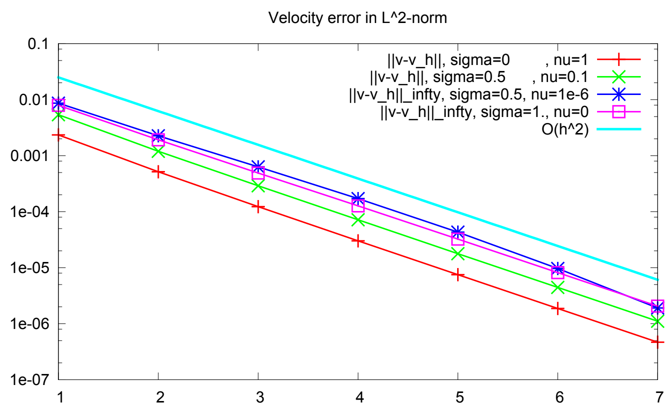

Figure 5: L^2-error of velocity.

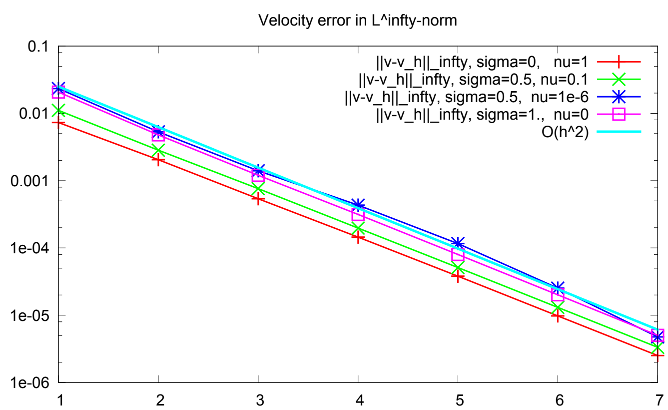

Figure 6: L^∞-error of velocity.

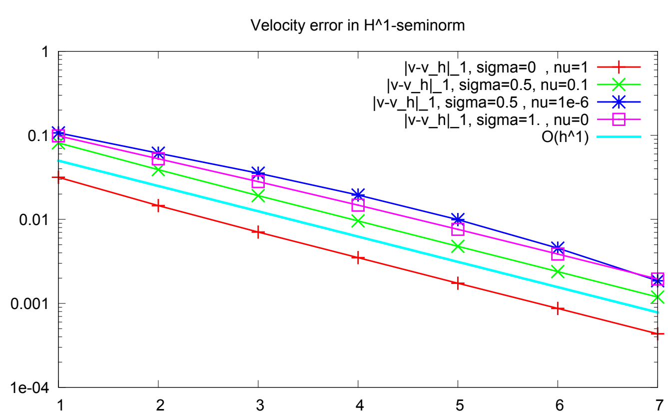

Figure 7: H^1-errors of velocity.

for the Stokes system and obtain a new stable scheme with an optimal order of convergence for the Darcy case. A priori estimates in the L^2-norm are obtained by duality arguments for velocity and pressure. Numerical experiments confirm our theoretical results for a wide regime of the model parameters. These results are summarized in Table 1.

Table 1: Numerical behavior of the errors for different choices of σ and ν obtained for a test problem with smooth solution.

σ	ν	$\|p-p_h\|_{L^2}$	$\|p-p_h\|_{H^1}$	$\|v-v_h\|_{L^2}$	$\|v-v_h\|_{H^1}$
0	1	$h^{7/4}$	$h^{3/4}$	h^2	h
0.5	0.1	$h^{7/4}$	h	h^2	h
0.5	10^{-6}	h^2	h	h^2	h
1	0	h^2	h	h^2	h

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