A Posteriori Error Estimates with Post-Processing for Nonconforming Finite Elements

F. Schieweck *
Institut für Analysis und Numerik
Otto-von-Guericke-Universität Magdeburg
Postfach 4120, D-39016 Magdeburg, Germany

Abstract

For a nonconforming finite element approximation of an elliptic model problem, we propose an a posteriori error estimate in the energy norm which uses as an additive term the "post-processing error" between the original nonconforming finite element solution and an easy computable conforming approximation of that solution. Thus, for the error analysis, the existing theory for the conforming case can be used together with some simple additional arguments. As an essential point, the property is exploited that the nonconforming finite element space contains as a subspace a conforming finite element space of first order. This property is fulfilled for many known nonconforming spaces. For the a posteriori error bound, we prove that it has the same asymptotic behavior as the energy norm of the real discretization error itself. We show that the "post-processing error" can be used also as an additional error indicator. Besides the error estimates in the global energy norm, we demonstrate that the concept of using a conforming approximation of the nonconforming solution can be applied also to derive an a posteriori error estimate for a linear functional of the solution representing some quantity of physical interest.

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1 Introduction

Nonconforming finite elements are attractive in the field of mixed finite element approximations or saddle point problems like for instance the Navier-Stokes equations as a typical problem in Computational Fluid Dynamics ([12], [9], [5]). An advantage of many nonconforming finite elements compared to conforming ones is that each degree of freedom is associated

*http://david.math.uni-magdeburg.de/home/schieweck

with the interior of an element or the interior of a $(d-1)$-dimensional face of the d-dimensional elements. This implies that each degree of freedom belongs to at most two elements which simplifies the local communication for a parallelization of the method, particularly in the 3D case (see [22], [23]).

In order to get an efficient numerical method for solving a partial differential equation, it is important to use adaptive mesh-refinement. This requires an a posteriori error estimator for evaluating the quality of the numerical solution. Nowadays, there exists a lot of work in the literature concerning the construction and analysis of a posteriori error estimators for the case of conforming finite elements (see [26], [1] and the references cited therein). However, there exist not so much papers treating the nonconforming case. It turned out that in this case some extra terms have to be added to the well-known a posteriori error estimator used for the conforming case. In [11], [10], [7], these extra terms are the jumps across the element faces of the derivatives of the finite element solution in tangential direction with respect to the element faces. In [15], two other approaches for constructing an a posteriori error estimator are considered which are based on the solution of local subproblems or on a two-level splitting of the space of piecewise quadratic conforming finite elements. For deriving an L^2-error estimator, John ([16], [17]) has used as additional term the jumps of the nonconforming finite element solution itself across the element edges.

In this paper, an alternative approach is presented which is based on the usage of a "smoothed" conforming finite element approximation $R_h u_h$ of the nonconforming solution u_h. Then, the computable quantities $\eta_K^{(p)}$, defined as the local norms of the "post-processing error" $u_h - R_h u_h$ on the elements K, are used as additional terms in the a posteriori error estimator. With this idea the existing theory from the conforming case can be used together with some simple additional arguments to prove a posteriori error estimates in the energy norm. This approach, which has been proposed also in [2] in an abstract setting, is based on the property that the nonconforming finite element space contains as a subspace a first order conforming finite element space. This property is fulfilled for many known nonconforming spaces and will be discussed in detail in Section 6.

In Section 4, we prove that our error estimator yields a reliable upper bound for the global discretization error. In order to guarantee that the true error will not be overestimated too much, we will show in Section 5 that our a posteriori error estimator gives an error bound which is "asymptotically correct" when the mesh width h tends to zero. This means that the a posteriori error bound has the same asymptotic order, say $O(h^r)$, as the discretization error.

In Section 7, we will show that the post-processing errors $\eta_K^{(p)}$ can be used as local error indicators since they measure in some sense the local "smoothness" of the discrete solution.

In many applications, one is not interested in all details of the solution but in the value of some physical quantities which can be regarded as functionals of the solution. Therefore, it is important to have an a posteriori error estimate for the error with respect to a functional applied to the solution. In contrast to the case of conforming finite element spaces, where a lot of work on this subject has been done by Rannacher and his coworkers (see [3], [19], [20]), there is not as much known for the nonconforming case. The main problem in the nonconforming case is the loss of Galerkin orthogonality of the discretization error. In Section 8, we show that the idea of using a conforming approximation $R_h u_h$ of the nonconforming solution u_h can be applied to overcome this problem. Together with some simple arguments, that involve the post-processing error $u_h - R_h u_h$, we can use the existing theory for the conforming case. We derive residual based a posteriori error estimates with weight factors obtained from the solution of a dual problem and with additional terms depending on the computable post-processing error $u_h - R_h u_h$. Furthermore, we derive local error indicators for controlling the adaptive procedure to create a suitable locally refined mesh.

Concerning the space dimension and the order of the finite element space, we treat the two- and three-dimensional case and the general case of a finite element space of order $r \geq 1$ in a unified fashion. However, concerning the treated partial differential equation, we restrict the presentation in this paper to the case of a simple Poisson problem in order to avoid technical difficulties and to make the underlying ideas as clear as possible. Generalizations to more general elliptic partial differential equations and also to mixed problems as for instance the Stokes problem can be done in a relatively straightforward way.

2 Preliminaries and Notations

We consider the simple elliptic model problem: *Find a function* $u \in H^2(\Omega)$ *such that*

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \Omega\,, \\ u &= 0 \quad \text{on} \quad \partial\Omega\,, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, is a bounded polygonal domain and $f \in L^2(\Omega)$ a given function.

At first, let us introduce some notation and formulate necessary assumptions. We use the standard notation $|.|_{m,G}$ and $\|.\|_{m,G}$ for the semi-norm and norm in the Sobolev space $H^m(G)$ and we denote by $|\cdot|_{1,h,\Omega}$ the "broken" H^1-seminorm given as

$$|v|_{1,h,\Omega} := \Big(\sum_{K \in \mathcal{T}_h} |v|^2_{1,K} \Big)^{1/2}, \tag{2}$$

where $\mathcal{T}_h$ is the set of mesh cells K forming a regular partition of the domain Ω. The cells are assumed to be shape-regular and supposed to be of simplicial, quadrilateral (for $d = 2$)

or hexahedral (for $d = 3$) shape. The diameter of element K is denoted by h_K and the global mesh-size is defined as $h := \max_{K \in \mathcal{T}_h} h_K$. Let $\mathcal{E}$ denote the set of all $(d-1)$-dimensional faces of the elements $K \in \mathcal{T}_h$, $\mathcal{E}_0$ the set of all inner faces and $\langle \cdot, \cdot \rangle_E$, for $E \in \mathcal{E}$, the inner product in $L^2(E)$. For $K \in \mathcal{T}_h$, let $F_K : \widehat{K} \to K$ be the one-to-one mapping between the reference element $\widehat{K}$ and K, where for instance $\widehat{K} := [-1, +1]^d$ for a quadrilateral $(d = 2)$ or hexahedral $(d = 3)$ element K. By $\widehat{\mathcal{P}}$ we denote the polynomial space of shape functions on the reference element $\widehat{K}$. Then, the local space of finite element functions on the original element K is defined by

$$\mathcal{P}_K := \left\{ p = \widehat{p} \circ F_K^{-1} : \ \widehat{p} \in \widehat{\mathcal{P}} \right\} . \tag{3}$$

These local spaces form the discontinuous discrete space

$$V(\mathcal{T}_h) := \{ v \in L^2(\Omega) : \ v\big|_K \in \mathcal{P}_K \quad \forall \, K \in \mathcal{T}_h \} . \tag{4}$$

The finite element space V_h will be chosen as the subspace of those functions $v_h \in V(\mathcal{T}_h)$ where the jump $[v_h]_E$ on any face $E \in \mathcal{E}$ (defined by (18) in Section 3) satisfies a certain "smallness condition". As an example, it holds $[v_h]_E \equiv 0$ for a conforming space $V_h \subset H_0^1(\Omega)$ or $\int_E [v_h]_E \, d\gamma = 0$ for the nonconforming finite element spaces proposed in [9], [21],[6].

Now, we want to describe in a uniform way the discretization of the model problem (1) for both the conforming and nonconforming case, respectively. We start with the weak formulation of (1) in the function space $V := H_0^1(\Omega)$, which reads

$$u \in V : \qquad a(u, v) = (f, v) \qquad \forall \, v \in V , \tag{5}$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$ and the bilinear form $a : V \times V \to \mathbb{R}$ is defined by

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \forall \, u, v \in V . \tag{6}$$

For the description of the discrete problem, we generalize the bilinear form $a(\cdot, \cdot)$ to a bilinear form $a_h : H^1(\Omega, \mathcal{T}_h) \times H^1(\Omega, \mathcal{T}_h) \to \mathbb{R}$ on the larger "broken H^1-space"

$$H^1(\Omega, \mathcal{T}_h) := \{ v \in L^2(\Omega) : \ v\big|_K \in H^1(K) \quad \forall \, K \in \mathcal{T}_h \} \tag{7}$$

by means of

$$a_h(u, v) := \sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v \, dx \qquad \forall \, u, v \in H^1(\Omega, \mathcal{T}_h) . \tag{8}$$

Obviously, we have that $V_h \subset V(\mathcal{T}_h) \subset H^1(\Omega, \mathcal{T}_h)$,

$$a_h(u, v) := a(u, v) \qquad \forall \, u, v \in V \tag{9}$$

and

$$|v|_{1,h,\Omega} = |v|_{1,\Omega} \qquad \forall\, v \in V. \tag{10}$$

For a given finite element space $V_h \subset V(\mathcal{T}_h)$, the discrete problem reads

$$u_h \in V_h : \qquad a_h(u_h, v_h) = (f, v_h) \qquad \forall\, v_h \in V_h\,. \tag{11}$$

In order to guarantee existence and uniqueness of the solution of problem (5) and (11) and for our error analysis we use the assumption that the bilinear form $a_h(\cdot,\cdot)$ is coercive and continuous, i.e. that there exist h-independent positive constants α and M such that

$$(\mathbf{A1}) \qquad \alpha\,|v|^2_{1,h,\Omega} \;\le\; a_h(v,v) \qquad \forall\, v \in H^1(\Omega, \mathcal{T}_h)$$

and

$$(\mathbf{A2}) \qquad a_h(u,v) \;\le\; M\,|u|_{1,h,\Omega}\,|v|_{1,h,\Omega} \qquad \forall\, u, v \in H^1(\Omega, \mathcal{T}_h).$$

For our particular bilinear form defined in (8), these assumptions are obviously fulfilled with $\alpha = M = 1$.

Finally, let us introduce some general notation. We denote by $\mathrm{card}(\mathcal{M})$ the number of elements of a finite set $\mathcal{M}$. Let X' be the dual space of a given space X. For a set $G \subset \mathbb{R}^d$, we denote by $\mathrm{int}(G)$ and $\overline{G}$ the interior and closure of G, respectively, and by $(\cdot,\cdot)_G$ the inner product in $L^2(G)$. In contrast to indexed fixed constants $C_1, C_2, \ldots$, we use the notation C for a general constant, which may have different values at different occurrences, but all these constants (indexed or not) are independent of the mesh parameters h_K of the grid $\mathcal{T}_h$.

3 Known Results in the Conforming Case

In the conforming case, i.e. $V_h \subset V = H^1_0(\Omega)$, we have the *Galerkin orthogonality* for the discretization error $u - u_h \in V$, i.e.

$$a_h(u - u_h, v_h) = 0 \qquad \forall\, v_h \in V_h\,. \tag{12}$$

For some given $w_h \in V_h$, we define the residual $r_h(w_h) \in V'$ by

$$\langle\, r_h(w_h), v \,\rangle_{V',V} := (f, v) - a_h(w_h, v) \qquad \forall\, v \in V\,. \tag{13}$$

From (5), (9) and the coercivity (A1) of the bilinear form $a_h(\cdot,\cdot)$, we get the error estimate

$$\begin{aligned} |u - u_h|_{1,\Omega} \;&\le\; \alpha^{-1} \sup_{v \in V\setminus\{0\}} \frac{a_h(u - u_h, v)}{|v|_{1,\Omega}} \\ &=\; \alpha^{-1} \sup_{v \in V\setminus\{0\}} \frac{\langle\, r_h(u_h), v \,\rangle_{V',V}}{|v|_{1,\Omega}} \;=\; \alpha^{-1}\, \|r_h(u_h)\|_{V'}\,. \end{aligned} \tag{14}$$

It is well-known (see e.g. [26]) that in the conforming case $V_h \subset V$ the residual $r_h(u_h)$ can be estimated by

$$\langle\, r_h(u_h), v \,\rangle_{V',V} \;\le\; C_1\, \eta^{(c)} |v|_{1,\Omega} \qquad \forall\, v \in V \tag{15}$$

with

$$\eta^{(c)} := \Big(\sum_{K \in \mathcal{T}_h} \eta_K^2 \Big)^{1/2} \tag{16}$$

and

$$\eta_K := \Big\{ h_K^2 \| f + \triangle u_h \|_{0,K}^2 \;+\; \frac{1}{2} \sum_{E \in\, \mathcal{E}(K) \cap \mathcal{E}_0} h_K \,\|\, [\nabla u_h \cdot n_E]_E \,\|_{L^2(E)}^2 \Big\}^{1/2} \tag{17}$$

where $\mathcal{E}(K)$ denotes the set of the $(d-1)$-dimensional faces of element K, n_E a unit vector normal to E and $[v]_E$ the jump of the function v on the face E defined as follows. For an inner face $E \in \mathcal{E}_0$, there exist two uniquely defined elements $K^{out}(E), K^{in}(E) \in \mathcal{T}_h$ with $\bar{E} = \partial K^{out}(E) \cap \partial K^{in}(E)$ and the property that the normal vector n_E points outward with respect to the element $K^{out}(E)$. Then, for an element-wise smooth function v and a point $x \in \mathrm{int}(E)$, we define

$$[v]_E\,(x) := v\big|_{K^{out}(E)}(x) - v\big|_{K^{in}(E)}(x)\,. \tag{18}$$

As a consequence of (14) and (15), we obtain the a posteriori error estimate

$$|u - u_h|_{1,\Omega} \le C_2\, \eta^{(c)} \tag{19}$$

with $C_2 := \alpha^{-1} C_1$. Now we want to recall the main steps for proving (15) because the same arguments can be used also for the nonconforming case in the next section. At first, let $V_h^{C,1} \subset V_h \cap V$ be a conforming finite element space of at least first order, i.e. there exists an interpolation operator $I_h^{C,1} : V \to V_h^{C,1}$ with the approximation properties

$$|v - I_h^{C,1} v|_{m,K} \le C_3\, h_K^{1-m} |v|_{1,\delta(K)} \qquad \forall\, K \in \mathcal{T}_h,\; m \in \{0,1\},\; v \in V \tag{20}$$

and

$$\|v - I_h^{C,1} v\|_{L^2(E)} \le C_4\, h_K^{1/2} |v|_{1,\delta(K)} \qquad \forall\, E \in \mathcal{E}(K),\; K \in \mathcal{T}_h,\; v \in V\,. \tag{21}$$

Here $\delta(K)$ denotes the vicinity of K consisting of all elements $\widetilde{K} \in \mathcal{T}_h$, which have a $V_h^{C,1}$-node in common with K. A possible choice for $I_h^{C,1}$ satisfying (20) and (21) would be the operator of Scott and Zhang proposed in [25] for the usual conforming finite element spaces $V_h^{C,1}$ of Lagrange type.

Now, using (5), (9), (13) the Galerkin orthogonality (12) for $v_h = I_h^{C,1} v \in V_h$ and the fact that $v - I_h^{C,1} v \in V$, we get for an arbitrary $v \in V$

$$\langle\, r_h(u_h), v \,\rangle_{V',V} \;=\; a_h(u - u_h, v) \;=\; a_h(u - u_h, v - I_h^{C,1} v)$$

$$= \sum_{K \in \mathcal{T}_h} \Big\{ (f + \triangle u_h, v - I_h^{C,1} v)_K - \frac{1}{2} \sum_{E \in \mathcal{E}(K) \cap \mathcal{E}_0} \Big\langle\, [\nabla u_h \cdot n_E]_E \,,\, v - I_h^{C,1} v \,\Big\rangle_E \Big\}. \qquad (22)$$

By means of the approximation properties (20) and (21), this implies with η_K from (17)

$$\begin{aligned} \langle\, r_h(u_h), v \,\rangle_{V',V} &\le (C_3^2 + C_5 C_4^2/2)^{1/2} \sum_{K \in \mathcal{T}_h} \eta_K \, |v|_{1,\delta(K)} \\ &\le (C_3^2 + C_5 C_4^2/2)^{1/2} \, \eta^{(c)} \Big\{ \sum_{K \in \mathcal{T}_h} |v|_{1,\delta(K)}^2 \Big\}^{1/2}, \end{aligned} \qquad (23)$$

where $C_5 := \max_{K \in \mathcal{T}_h} (\mathrm{card}\{E : E \in \mathcal{E}(K) \cap \mathcal{E}_0\})$. Since the grid $\mathcal{T}_h$ has been assumed to be shape-regular, the last factor in (23) can be estimated by $C_6^{1/2} |v|_{1,\Omega}$ where $C_6 := \max_{\widetilde{K} \in \mathcal{T}_h} (\mathrm{card}\{K \in \mathcal{T}_h : \widetilde{K} \subset \delta(K)\})$. This proves (15) with $C_1 := (C_6(C_3^2 + C_5 C_4^2/2))^{1/2}$.

For the next section, it is essential to note that in the above arguments we have not exploited the fact that $u_h \in V_h$ is a continuous function. We have only used that $v - I_h^{C,1} v \in H_0^1(\Omega)$ and that $u_h \in V(\mathcal{T}_h)$ is element-wise smooth.

4 General Framework for the Nonconforming Case

In the nonconforming case, i.e. $V_h \not\subset V$, an important difference in contrast to the conforming case is that we have only the "reduced Galerkin orthogonality"

$$a_h(u - u_h, v_h) = 0 \qquad \forall\, v_h \in V_h \cap V\,. \qquad (24)$$

The residual $r_h(u_h) \in V'$ is defined by (13) in the same way as for the conforming case. Inspecting the analysis for the estimation of $\langle\, r_h(u_h), v \,\rangle_{V',V}$ in Section 3, we see that the Galerkin orthogonality is used only for a test function $v_h = I_h^{C,1} v$ in a first order conforming finite element space $V_h^{C,1} \subset V$. Therefore, in order to make the analysis of Section 3 work, we need the assumption that

(**A3**) *there exists a first order finite element space $V_h^{C,1} \subset V_h \cap V$ and an interpolation operator $I_h^{C,1} : V \to V_h^{C,1}$ satisfying (20) and (21).*

In Section 6, we will show that this assumption is satisfied for many known nonconforming finite element spaces.

Another point in the analysis of the conforming case, which has to be modified for a nonconforming method, concerns the first inequality in (14) where the property $u - u_h \in V$

is used in combination with the coercivity of $a(\cdot,\cdot) = a_h(\cdot,\cdot)$ on $V \times V$. The main idea in this paper to circumvent the problem of $u - u_h \notin V$ in the nonconforming case is to analyze at first the error $u - u_h^C \in V$ where $u_h^C = R_h u_h \in V_h^C$ is a "smoothed" approximation of the solution $u_h \in V_h$ in an appropriate conforming finite element space $V_h^C \subset V$. The choice of V_h^C and the post-processing operator $R_h : V_h \to V_h^C$ should guarantee that the order of the discretization error $u - u_h$ is saved, i.e., for the energy norm, that $|u - u_h|_{1,h,\Omega} = O(h^r)$ should imply $|u - R_h u_h|_{1,\Omega} = O(h^r)$. For a general situation of spaces V_h and V_h^C, an operator R_h with this property has been proposed and analyzed in [24]. In the following, we will briefly recall from [24] the definition of the function $R_h v \in V_h^C$ for a given element-wise smooth function $v \in V_h$. At first, let $V_h^C \subset H_0^1(\Omega)$ be a given conforming finite element space with the approximation property that

(**A4**) *there exists an interpolation operator* $I_h^C : H^2(\Omega) \cap V \to V_h^C$ *and an integer* $r \geq 1$ *such that for* $m \in \{0, 1\}$ *it holds*

$$|v - I_h^C v|_{m,K} \leq C\, h_K^{r+1-m} |v|_{r+1,K} \qquad \forall\, K \in \mathcal{T}_h, \quad v \in H^{r+1}(\Omega) \cap V\,.$$

For the definition of the transfer operator R_h, we need a more detailed characterization of V_h^C by means of its basis functions and nodal functionals. Let $\{\varphi_j^C : j \in J^C\}$ be the finite element basis of V_h^C. The indices $j \in J^C$ are called "nodes". For a given element $K \in \mathcal{T}_h$, we denote by $J^C(K)$ the set of all nodes $j \in J^C$ that are associated with K, i.e.

$$J^C(K) := \left\{ j \in J^C : \operatorname{supp}(\varphi_j^C) \cap \operatorname{int}(K) \neq \emptyset \right\}. \tag{25}$$

The element nodes $j \in J^C(K)$ are associated with a set of local basis functions $\varphi_j^C\big|_K$ and a set of local nodal functionals $N_{j,K} \in (C^s(\overline{K}))'$ which are "dual" (see [4]) in the sense that

$$N_{i,K}(\varphi_j^C\big|_K) = \delta_{ij} \qquad \forall\, i, j \in J^C(K), \tag{26}$$

where δ_{ij} denotes the Kronecker delta. Note that we simply write $N_{i,K}$ instead of $N_{i,K}^C$ since we do not consider nodal functionals for the finite element space V_h. The parameter s depends on the type of the nodal functionals and corresponds to the highest order of derivatives that have to be evaluated for a function v in order to define $N_{j,K}(v)$. Typical examples for nodal functionals in the case $s = 0$ are point values $N_{j,K}(v) := v_{|K}(a_j)$, where $a_j \in \overline{K}$ is a given nodal point. Now, by means of the local nodal functionals $N_{j,K}$, we define global functionals N_j for element-wise smooth functions in the following way. For a node $j \in J^C$, let

$$\mathcal{T}_{h,j} := \left\{ K \in \mathcal{T}_h : \; j \in J^C(K) \right\} \tag{27}$$

be the set of all elements that are associated with this node. Then, we define the value $N_j(v)$ as a weighted mean value of all local values $N_{j,K}(v_{|K})$ that are associated with the node

$j \in J^C$, i.e.

$$N_j(v) := \left(\omega_j^\Sigma\right)^{-1} \sum_{K \in \mathcal{T}_{h,j}} \omega_{j,K} N_{j,K}(v\big|_K) \qquad \text{with} \qquad \omega_j^\Sigma := \sum_{K \in \mathcal{T}_{h,j}} \omega_{j,K}\,, \tag{28}$$

where the quantities $\omega_{j,K}$ are some user-chosen non-negative weights. There are at least two practically useful choices. The first one is $\omega_{j,K} = 1$ for all j and K, where an isotropic "spread of information" is guaranteed. For the second one, there is exactly one weight equal to one, say ω_{j,K_j^*} corresponding to an arbitrary chosen element $K_j^* \in \mathcal{T}_{h,j}$, and the other weights $\omega_{j,K}$ for all $K \in \mathcal{T}_{h,j} \setminus \{K_j^*\}$ are set to zero. For this choice, the computational costs for computing the values $N_j(v)$, $j \in J^C$, are lower than for the first choice. By means of these global nodal functionals we define the transfer operator $R_h : V_h \to V_h^C$ as

$$R_h v := \sum_{j \in J^C} N_j(v)\, \varphi_j^C\,. \tag{29}$$

Remark 1 *The computation of the function $R_h v \in V_h^C$ for a given nonconforming finite element function $v \in V_h$ can be implemented easily and efficiently in the following way. One has to compute the nodal values v_j^C, $j \in J^C$, for the basis representation $R_h v = \sum_{j \in J^C} v_j^C\, \varphi_j^C$. At the beginning two vectors $\mathbf{c} = (c_j)_{j \in J^C}$ and $\mathbf{v^C} = (v_j^C)_{j \in J^C}$ are initialized by zero. Then, in a first loop over the elements, for each element $K \in \mathcal{T}_h$, all the contributions $\omega_{j,K} N_{j,K}(v\big|_K)$ are computed and added to v_j^C for $j \in J^C(K)$, i.e. for those nodes $j \in J^C$ that are associated with element K. Furthermore, if $j \in J^C(K)$, the value of c_j is increased by $\omega_{j,K}$. In a final loop over the nodes $j \in J^C$, the right values $v_j^C = N_j(v)$ are computed by $v_j^C := v_j^C / c_j$.*

Using our post-processing operator R_h to compute a "smoothed" conforming approximation $R_h u_h$ of the solution, we can define the following computable "post-processing error"

$$\eta^{(p)} := \Big(\sum_{K \in \mathcal{T}_h} (\eta_K^{(p)})^2 \Big)^{1/2} \qquad \text{with} \qquad \eta_K^{(p)} := |u_h - R_h u_h|_{1,K}\,. \tag{30}$$

Now, we want to prove a posteriori error estimates for the errors $u - R_h u_h$ and $u - u_h$.

Theorem 2 *Assume that the grid $\mathcal{T}_h$ is shape regular. Furthermore, suppose that the bilinear form $a_h(\cdot,\cdot)$ and the nonconforming finite element space V_h fulfill the assumptions (A1), (A2) and (A3). Let $u \in V$ and $u_h \in V_h$ be the solutions of the continuous and discrete problem (5) and (11), respectively. Then, for the smoothed solution $R_h u_h \in V_h^C$ in a suitable conforming finite element space $V_h^C \subset V$, it holds the a posteriori error estimate*

$$|u - R_h u_h|_{1,\Omega} \;\le\; C_2\, \eta^{(c)} + \alpha^{-1} M\, \eta^{(p)}\,, \tag{31}$$

where C_2 and $\eta^{(c)}$ are the quantities in (19), (16), (17) and $\eta^{(p)}$ is defined by (30). For the nonconforming finite element solution $u_h \in V_h$, it holds

$$|u - u_h|_{1,h,\Omega} \;\le\; C_2\, \eta^{(c)} + (\alpha^{-1} M + 1)\, \eta^{(p)}\,. \tag{32}$$

Proof. Since $u - R_h u_h \in V$, we have

$$\begin{aligned} |u - R_h u_h|_{1,\Omega} &\leq \alpha^{-1} \sup_{v \in V \setminus \{0\}} \frac{a_h(u - R_h u_h, v)}{|v|_{1,\Omega}} \\ &= \alpha^{-1} \sup_{v \in V \setminus \{0\}} \frac{\langle\, r_h(u_h), v \,\rangle_{V',V} + a_h(u_h - R_h u_h, v)}{|v|_{1,\Omega}} \,. \end{aligned} \tag{33}$$

The term $\langle\, r_h(u_h), v \,\rangle_{V',V}$ can be estimated by (15) in the same way as in (22), (23). Using the assumption (A2) for $a_h(\cdot,\cdot)$, we obtain the estimate (31) with $C_2 = \alpha^{-1} C_1$. From (31) we immediately get

$$\begin{aligned} |u - u_h|_{1,h,\Omega} &\leq |u - R_h u_h|_{1,\Omega} + |u_h - R_h u_h|_{1,h,\Omega} \\ &\leq C_2\, \eta^{(c)} + \alpha^{-1} M\, \eta^{(p)} + |u_h - R_h u_h|_{1,h,\Omega} \,. \end{aligned} \tag{34}$$

Thus, (32) follows from $|u_h - R_h u_h|_{1,h,\Omega} = \eta^{(p)}$. ■

5 Asymptotic Behavior of the A Posteriori Error Bound

In Section 4, we have proved a posteriori error estimates which give computable upper bounds for the errors $|u - R_h u_h|_{1,\Omega}$ and $|u - u_h|_{1,h,\Omega}$, respectively. For practical reasons, these bounds should not overestimate the true errors too much. Therefore, a minimal requirement is that the bounds (31) and (32) are at least "asymptotically correct" for $h \to 0$. That means that the asymptotic order of the discretization error, say $|u - u_h|_{1,h,\Omega} = O(h^r)$, is also reflected by the a posteriori error bound in the sense that

$$C_2\, \eta^{(c)} + C_7 \eta^{(p)} \;=\; O(h^r) \,, \tag{35}$$

where we have used the abbreviation $C_7 := 1 + \alpha^{-1} M$. In the following, we want to prove (35) under some natural assumptions on the nonconforming finite element space V_h. We assume that V_h satisfies the approximation property that

(A5) *there exists an interpolation operator $I_h : H^2(\Omega) \cap V \to V_h$ and an integer $r \geq 1$ such that for all integers m, s with $0 \leq m \leq s+1$ and $1 \leq s \leq r$ it holds*

$$|v - I_h v|_{m,K} \;\leq\; C\, h_K^{s+1-m} |v|_{s+1,K} \qquad \forall\, K \in \mathcal{T}_h, \;\; v \in H^{s+1}(\Omega) \cap V \,.$$

Furthermore, we need a condition concerning a "weak continuity" of the functions $v_h \in V_h$ along element faces. We assume that

(A6) $$\langle\, [v_h]_E \,, q \,\rangle_E \;=\; 0 \qquad \forall\, q \in \mathbb{P}_{r-1}(E) \qquad \forall\, E \in \mathcal{E}, \;\; v_h \in V_h \,,$$

where $\mathbb{P}_s(E)$ denotes the space of all polynomials in $L^2(E)$ with a degree not larger than s. For an element-wise smooth function v, we denote by $[v]_E$ the jump of v on E which is defined for inner faces $E \in \mathcal{E}_0$ by (18) and for boundary faces in the following way. If $E \subset \partial\Omega$ is a boundary face then there exists exactly one element $K(E) \in \mathcal{T}_h$ with $E \subset \partial K(E)$. We assume that in this case the unit normal vector n_E points outward with respect to $K(E)$. Then, for a point $x \in \mathrm{int}(E)$, we define the jump of v across E by $[v]_E(x) := v\big|_{K(E)}(x)$. For a boundary face $E \in \mathcal{E} \setminus \mathcal{E}_0$, the assumption (A6) represents a weak homogeneous Dirichlet boundary condition for the functions $v_h \in V_h$.

The next theorem shows that the a posteriori error bound $C_2\,\eta^{(c)} + C_7\eta^{(p)}$ of Section 4 for the energy norm of the discretization error will be asymptotically correct for $h \to 0$ in the sense of (35).

Theorem 3 *Assume that the grid $\mathcal{T}_h$ is shape regular and that the bilinear form $a_h(\cdot,\cdot)$ and the nonconforming finite element space V_h fulfill the assumptions (A1), (A2), ..., (A6). Let $u \in V$ and $u_h \in V_h$ be the solutions of the continuous and discrete problem (5) and (11), respectively, and let u satisfy the regularity assumption $u \in H^{r+1}(\Omega)$ with the integer $r \geq 1$ from (A4), (A5), (A6). Then, the energy norm of the discretization error can be estimated by*

$$|u - u_h|_{1,h,\Omega} \;\leq\; C\Big(\sum_{K\in\mathcal{T}_h} h_K^{2r}\,|u|_{r+1,K}^2\Big)^{1/2} \tag{36}$$

and, for the a posteriori error bound in (32) with the estimators $\eta^{(c)}$ and $\eta^{(p)}$ defined by (16), (17) and (30), it holds the estimate

$$C_2\,\eta^{(c)} + C_7\,\eta^{(p)} \;\leq\; C\Big(\sum_{K\in\mathcal{T}_h} h_K^{2r}\,|u|_{r+1,K}^2\Big)^{1/2} \tag{37}$$

where $C_7 = 1 + \alpha^{-1}M$.

Proof. STEP 1. We prove the estimate

$$|I_h u - u_h|_{1,h,\Omega} \;\leq\; C\Big(\sum_{K\in\mathcal{T}_h} h_K^{2r}\,|u|_{r+1,K}^2\Big)^{1/2}. \tag{38}$$

Let us substitute $v_h := I_h u - u_h \in V_h \subset H^1(\Omega,\mathcal{T}_h)$. By the coercivity (A1) and the continuity (A2) of $a_h(\cdot,\cdot)$ we get

$$\begin{aligned}
\alpha\,|v_h|_{1,h,\Omega}^2 &\leq a_h(I_h u - u_h, v_h)\\
&= a_h(I_h u - u, v_h) \,+\, a_h(u - u_h, v_h)\\
&\leq M|u - I_h u|_{1,h,\Omega}|v_h|_{1,h,\Omega} \,+\, a_h(u - u_h, v_h)\,.
\end{aligned} \tag{39}$$

In the following, we prove a "perturbed" Galerkin orthogonality which says that the term $a_h(u-u_h,v_h)$ is small. The regularity assumption $u \in H^{r+1}(\Omega)$ with $r \geq 1$ implies

$$(-\triangle u, v_h) \;=\; (f, v_h) \;=\; a_h(u_h, v_h)\,,$$

which yields

$$a_h(u-u_h,v_h) \;=\; \sum_{K\in\mathcal{T}_h} \sum_{E\in\mathcal{E}(K)} \lambda_E \,\langle\, \nabla u\cdot n_E,\, [v_h]_E \,\rangle_E \tag{40}$$

where

$$\lambda_E := \begin{cases} 1 & , \quad \text{for} \quad E \in \mathcal{E}\setminus\mathcal{E}_0, \\ 1/2 & , \quad \text{for} \quad E \in \mathcal{E}_0\,. \end{cases} \tag{41}$$

For a given element $K \in \mathcal{T}_h$, let the operator $\Pi_K : H^{r+1}(K) \to \mathbb{P}_r(K)$ be the orthogonal projection in $L^2(K)$ which satisfies the well-known error estimate

$$|v-\Pi_K v|_{m,K} \;\leq\; C\,h_K^{r+1-m}|v|_{r+1,K} \qquad \forall\, v \in H^{r+1}(K) \tag{42}$$

for all integers m with $0 \leq m \leq r+1$. Since $(\nabla\Pi_K u)\big|_E \cdot n_E \in \mathbb{P}_{r-1}(E)$ for $E \in \mathcal{E}(K)$, we get from (A6)

$$\langle\, \nabla u\cdot n_E,\, [v_h]_E \,\rangle_E \;=\; \langle\, \nabla(u-\Pi_K u)\cdot n_E,\, [v_h]_E \,\rangle_E\,. \tag{43}$$

By standard techniques as the transformation to the reference element $\widehat{K}$, the trace theorem on $\widehat{K}$ and the transformation back to $K \in \mathcal{T}_h$ (see [8]), we can prove the estimate

$$\|w\|_{L^2(E)} \;\leq\; C\,h_K^{1/2}\,\big(\,h_K^{-1}|w|_{0,K} \;+\; |w|_{1,K}\,\big) \qquad \forall\, w \in H^1(K)\,, \quad E \in \mathcal{E}(K)\,. \tag{44}$$

Applying this estimate for $w = \nabla(u-\Pi_K u)\cdot n_E \in H^1(K)$, we obtain from (43) and (42)

$$\begin{aligned} \langle\, \nabla u\cdot n_E,\, [v_h]_E \,\rangle_E \;&\leq\; C\,h_K^{1/2}\,\big(\,h_K^{-1}|u-\Pi_K u|_{0,K} \;+\; |u-\Pi_K u|_{1,K}\,\big)\,\|\,[v_h]_E\,\|_{L^2(E)} \\ &\leq\; C\,h_K^{r-1/2}\,\|\,[v_h]_E\,\|_{L^2(E)}\,|u|_{r+1,K} \end{aligned} \tag{45}$$

In order to estimate $\|\,[v_h]_E\,\|_{L^2(E)}$ we define for a given element face $E \in \mathcal{E}$

$$\mathcal{T}_h^E := \{\, \widetilde{K} \in \mathcal{T}_h : E \subset \partial\widetilde{K} \,\}$$

as the set of all elements that are associated with E and the constant $v_{h,E} \in \mathbb{R}$ by

$$v_{h,E} := |E|^{-1}\,\langle\, v_h,\, 1\,\rangle_E\,,$$

where $|E|$ denotes the $(d-1)$-dimensional measure of the face E. Then, using (44), we can estimate

$$\| [v_h]_E \|_{L^2(E)} \leq \sum_{\widetilde{K} \in \mathcal{T}_h^E} \| v_h|_{\widetilde{K}} - v_{h,E} \|_{L^2(E)} \leq \sum_{\widetilde{K} \in \mathcal{T}_h^E} C\, h_{\widetilde{K}}^{1/2} |v_h|_{1,\widetilde{K}}\,. \tag{46}$$

For an element $K \in \mathcal{T}_h$, let us define

$$\widetilde{\mathcal{T}}_h(K) := \bigcup_{E \in \mathcal{E}(K)} \mathcal{T}_h^E\,. \tag{47}$$

Then, due to the shape regularity of the grid $\mathcal{T}_h$, we can prove

$$C^{-1}\, h_K \leq h_{\widetilde{K}} \leq C\, h_K \qquad \forall\, \widetilde{K} \in \widetilde{\mathcal{T}}_h(K)\,. \tag{48}$$

Combining (40), (45) and (46), we get

$$a_h(u-u_h, v_h) \leq \sum_{K \in \mathcal{T}_h} \sum_{E \in \mathcal{E}(K)} C\, h_K^{r-1/2} |u|_{r+1,K} \Big(\sum_{\widetilde{K} \in \mathcal{T}_h^E} C\, h_{\widetilde{K}}^{1/2} |v_h|_{1,\widetilde{K}} \Big)\,.$$

Applying (48) and using that $\max_{K \in \mathcal{T}_h} (\mathrm{card}\{E:\, E \in \mathcal{E}(K)\}) \leq C$, we conclude

$$\sum_{E \in \mathcal{E}(K)} \sum_{\widetilde{K} \in \mathcal{T}_h^E} C\, h_{\widetilde{K}}^{1/2} |v_h|_{1,\widetilde{K}} \leq C\, h_K^{1/2} \Big\{ \sum_{\widetilde{K} \in \widetilde{\mathcal{T}}_h(K)} |v_h|_{1,\widetilde{K}}^2 \Big\}^{1/2}$$

which yields

$$\begin{aligned} a_h(u-u_h, v_h) &\leq \sum_{K \in \mathcal{T}_h} C\, h_K^r |u|_{r+1,K} \Big\{ \sum_{\widetilde{K} \in \widetilde{\mathcal{T}}_h(K)} |v_h|_{1,\widetilde{K}}^2 \Big\}^{1/2} \\ &\leq C \Big\{ \sum_{K \in \mathcal{T}_h} h_K^{2r} |u|_{r+1,K}^2 \Big\}^{1/2} \Big\{ \sum_{K \in \mathcal{T}_h} \sum_{\widetilde{K} \in \widetilde{\mathcal{T}}_h(K)} |v_h|_{1,\widetilde{K}}^2 \Big\}^{1/2} \\ &\leq C \Big\{ \sum_{K \in \mathcal{T}_h} h_K^{2r} |u|_{r+1,K}^2 \Big\}^{1/2} |v_h|_{1,h,\Omega}\,. \end{aligned}$$

Together with (39) and the approximation property (A5), this implies the estimate (38). From (38) and (A5) we obtain the estimate (36) for discretization error.

STEP 2. We prove the error bound in (37) for the estimator $\eta^{(c)}$. Using the regularity assumption $u \in H^{r+1}(\Omega)$ with $r \geq 1$, the approximation property (A5), the estimate (38) and an inverse inequality, we get

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} h_K^2 \|f + \triangle u\|_{0,K}^2 &= \sum_{K \in \mathcal{T}_h} h_K^2 \|\triangle(u-u_h)\|_{0,K}^2 \\ &\leq \sum_{K \in \mathcal{T}_h} \Big\{ h_K^2 |u - I_h u|_{2,K}^2 + C\, |I_h u - u_h|_{1,K}^2 \Big\} \\ &\leq C \sum_{K \in \mathcal{T}_h} h_K^{2r} |u|_{r+1,K}^2\,. \end{aligned} \tag{49}$$

For an element $K \in \mathcal{T}_h$ and a face $E \in \mathcal{E}(K) \cap \mathcal{E}_0$, we obtain from $[\nabla u \cdot n_E]_E = 0$, (44) for $w = \nabla(u-u_h)\big|_{\widetilde{K}} \cdot n_E$, (48) and an inverse inequality

$$\begin{aligned}
h_K \,\|\, [\nabla u_h \cdot n_E]_E \,\|^2_{L^2(E)} &= h_K \,\|\, [\nabla(u-u_h)\cdot n_E]_E \,\|^2_{L^2(E)} \\
&\le \sum_{\widetilde{K}\in\mathcal{T}_h^E} 2\,h_K \,\big\|\, \nabla(u-u_h)\big|_{\widetilde{K}} \cdot n_E \,\big\|^2_{L^2(E)} \\
&\le \sum_{\widetilde{K}\in\mathcal{T}_h^E} C\,\Big\{ \|\, \nabla(u-u_h)\cdot n_E \,\|^2_{0,\widetilde{K}} + h^2_{\widetilde{K}}\, |\, \nabla(u-u_h)\cdot n_E \,|^2_{1,\widetilde{K}} \Big\} \\
&\le \sum_{\widetilde{K}\in\mathcal{T}_h^E} C\,\Big\{ |u-u_h|^2_{1,\widetilde{K}} + h^2_{\widetilde{K}}\, |u-u_h|^2_{2,\widetilde{K}} \Big\} \\
&\le \sum_{\widetilde{K}\in\mathcal{T}_h^E} C\,\Big\{ |u-u_h|^2_{1,\widetilde{K}} + h^2_{\widetilde{K}}\, |u-I_h u|^2_{2,\widetilde{K}} + C\,|I_h u - u_h|^2_{1,\widetilde{K}} \Big\}.
\end{aligned}$$

Applying the approximation property (A5) and the estimates (36) and (38), it follows by summation

$$\sum_{K\in\mathcal{T}_h} \sum_{E\in\,\mathcal{E}(K)\cap\mathcal{E}_0} h_K \,\|\, [\nabla u_h \cdot n_E]_E \,\|^2_{L^2(E)} \;\le\; C \sum_{K\in\mathcal{T}_h} h_K^{2r} |u|^2_{r+1,K}\,,$$

which implies with (49) the estimate

$$\eta^{(c)} \;\le\; C \Big\{ \sum_{K\in\mathcal{T}_h} h_K^{2r} |u|^2_{r+1,K} \Big\}^{1/2}. \tag{50}$$

STEP 3. Now we will prove (37). In [24], it has been proved the following estimate for the post-processing operator $R_h : V_h \to V_h^C$

$$|u - R_h u_h|_{1,h,\Omega} \le C|u-u_h|_{1,h,\Omega} + C \inf_{v_h^C \in V_h^C} |u - v_h^C|_{1,\Omega}\,. \tag{51}$$

Using the approximation property (A4) of the conforming finite element space $V_h^C \subset H_0^1(\Omega)$ and the a priori error estimate (36) for the discretization error, we get

$$\begin{aligned}
\eta^{(p)} &= |u_h - R_h u_h|_{1,h,\Omega} \;\le\; |u - R_h u_h|_{1,h,\Omega} + |u-u_h|_{1,h,\Omega} \\
&\le C\,|u-u_h|_{1,h,\Omega} + C\,|u - I_h^C u|_{1,h,\Omega} \\
&\le C \Big\{ \sum_{K\in\mathcal{T}_h} h_K^{2r} |u|^2_{r+1,K} \Big\}^{1/2},
\end{aligned}$$

which proves, together with (50), the estimate (37). ■

6 Discussion of Relevant Nonconforming Spaces

In this section, we present examples of nonconforming finite element spaces V_h which fit into our theoretical framework, i.e. for which the assumptions (A1), ..., (A6) are satisfied. For the bilinear form $a_h(\cdot,\cdot)$ of our model problem defined in (8), the assumptions (A1) and (A2) are obviously satisfied. Since the nonconforming finite element spaces typically consist of element-wise smooth functions, we have $V_h \subset H^1(\Omega, \mathcal{T}_h)$ such that (A1) and (A2) guarantee the existence and uniqueness of a solution $u_h \in V_h$ of the discrete problem (11).

For a discussion of assumption (A3), we start with the definition of the first order conforming finite element space $V_h^{C,1} \subset V$ as

$$V_h^{C,1} := \{v \in C(\overline{\Omega}) \cap V : \quad \widehat{v} = v\big|_K \circ F_K \in \widehat{\mathcal{P}}^{C,1} \quad \forall\, K \in \mathcal{T}_h\} \tag{52}$$

where

$$\widehat{\mathcal{P}}^{C,1} := \begin{cases} \mathbb{P}_1(\widehat{K}) & \text{if } K \text{ is a d-dimensional simplex} \\ \mathbb{Q}_1(\widehat{K}) & \text{if } K \text{ is a quadrilateral (d=2) or a hexahedral (d=3) element} \end{cases} \tag{53}$$

and $F_K : \widehat{K} \to K$ is the mapping between the reference element $\widehat{K}$ and the original element K. In order to explain the polynomial space $\mathbb{Q}_1(\widehat{K})$, let us define for some multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ the quantity $|\alpha|_\infty = \max\{\alpha_i : 1 \le i \le d\}$. Then, for an integer $r \ge 1$, we set

$$\mathbb{Q}_r(\widehat{K}) := \{\widehat{p} \in C(\widehat{K}) : \widehat{p}(\widehat{x}) = \sum_{|\alpha|_\infty \le r} c_\alpha \, \Pi_{i=1}^d \widehat{x}_i^{\alpha_i}\,, \quad \forall\, \widehat{x} \in \widehat{K}\,, \;\; c_\alpha \in \mathbb{R}\}\,. \tag{54}$$

Remark 4 *In the three-dimensional case, the choice (53) means that either all elements $K \in \mathcal{T}_h$ are tetrahedrons or all elements are hexahedrons. The reason is that for a regular grid $\mathcal{T}_h$, the face-neighboring elements of a hexahedral element can not be tetrahedral elements. If we want to have a grid $\mathcal{T}_h$ that has both, tetrahedral and hexahedral elements, respectively, then we have also to allow pyramidal elements with a quadrilateral base.*

For the above defined space $V_h^{C,1}$ of Lagrange type, we can use as the operator $I_h^{C,1} : V \to V_h^{C,1}$ the operator of Scott and Zhang proposed in [25] which satisfies the approximation properties (20) and (21). Therefore, in order to satisfy assumption (A3), it remains to guarantee that the nonconforming finite element space V_h fulfills $V_h^{C,1} \subset V_h$. For this, let us take an arbitrary function $v_h \in V_h^{C,1}$. Then, we have to guarantee that $v_h \in V_h$. Since V_h is a subspace of the discontinuous space $V(\mathcal{T}_h)$ defined in (4), we have to show at first that $v_h\big|_K \in \mathcal{P}_K$ for all elements $K \in \mathcal{T}_h$ where $\mathcal{P}_K$ is defined by (3). This is satisfied if

$$\widehat{\mathcal{P}}^{C,1} \subset \widehat{\mathcal{P}}\,, \tag{55}$$

where that condition has to be understood separately for each occurring type of the reference element, i.e., for instance, separately for the case of a triangular and a quadrilateral reference element. From (53) it follows that (55) is satisfied if

$$\widehat{\mathcal{P}} \supset \begin{cases} \mathbb{P}_1(\widehat{K}) & \text{if } K \text{ is a d-dimensional simplex} \\ \mathbb{Q}_1(\widehat{K}) & \text{if } K \text{ is a quadrilateral (d=2) or a hexahedral (d=3) element.} \end{cases} \tag{56}$$

Thus, condition (56) guarantees that $v_h \in V_h^{C,1}$ implies $v_h \in V(\mathcal{T}_h)$. Now, a discrete function $v_h \in V(\mathcal{T}_h)$ belongs to the nonconforming space V_h if it satisfies a "weak continuity" condition along the element faces $E \in \mathcal{E}$ (for example assumption (A6)). Since a function in $V_h^{C,1}$ is continuous at all points of $\overline{\Omega}$, the weak continuity condition for a given function $v_h \in V_h^{C,1}$ will be satisfied for each nonconforming finite element space V_h. Therefore, assumption (A3) is fulfilled if (56) is satisfied.

Let us now consider assumption (A5) which says that V_h has an approximation property of some order $r \geq 1$. The essential condition to guarantee (A5), at least in the case of affine reference transformations $F_K : \widehat{K} \to K$, is that

$$\mathbb{P}_r(\widehat{K}) \subset \widehat{\mathcal{P}}\,. \tag{57}$$

In the case of quadrilateral or hexahedral elements, where we have a non-affine reference transformation $F_K : \widehat{K} \to K$, the estimate of the local interpolation error on element K contains a term σ_K which measures the variation of the shape of K from the shape of a parallelogram or a parallelepiped (for details see [21], [23]). However, for a grid $\mathcal{T}_h$, which is generated from a shape-regular coarse grid by a number of uniform refinement steps, it has been proved in [23] that $\sigma_K \leq C h_K$ for all $K \in \mathcal{T}_h$ and that the interpolation estimate in (A5) remains valid. Therefore, we assume to have such a grid in the case of quadrilateral or hexahedral elements.

The nonconforming finite element spaces known from the literature (see [9], [5], [14], [13], [21], [6]) satisfy (57). For the interpolation operator $I_h : H^2(\Omega) \cap V \to V_h$ in (A5), the natural finite element interpolation operator can be used.

The assumption (A6) on the "weak continuity" of functions $v_h \in V_h$ along the element faces $E \in \mathcal{E}$ is typically satisfied by the construction of the space V_h (see [9], [14], [13], [21], [6]). Note that for the quadrilateral or hexahedral nonconforming elements proposed in [21], we choose the "mean-value oriented" variant which guarantees (A6).

It remains to discuss assumption (A4) which means that the conforming finite element space V_h^C, used for the post-processing, should have the same order of approximation $r \geq 1$ as the nonconforming space V_h. This is guaranteed by the following choice

$$V_h^C := \{ v \in C(\overline{\Omega}) \cap V : \quad \widehat{v} = v\big|_K \circ F_K \in \widehat{\mathcal{P}}^C \quad \forall\, K \in \mathcal{T}_h \} \tag{58}$$

where

$$\widehat{\mathcal{P}}^C := \begin{cases} \mathbb{P}_r(\widehat{K}) & \text{if } K \text{ is a d-dimensional simplex} \\ \mathbb{Q}_r(\widehat{K}) & \text{if } K \text{ is a quadrilateral (d=2) or a hexahedral (d=3) element.} \end{cases} \tag{59}$$

Note that Remark 4 applies also to (59).

At the end of the above discussion we can summarize that the assumptions of our theoretical framework (A1), ... , (A6) are guaranteed if the nonconforming space V_h satisfies (56),(57) and the weak continuity condition (A6). Now, let us give some examples of such finite element spaces V_h.

At first we consider simplicial elements. The most simple example is the well-known Crouzeix-Raviart-element of order $r = 1$. Here, the reference space of shape functions is $\widehat{\mathcal{P}} = \mathbb{P}_1(\widehat{K})$ and the nodal functionals $N_j(v)$ are the point-values $v(a_j)$ at the barycenters a_j of the faces $E_j \in \mathcal{E}$. In [9, Example 5], a triangular element of order $r = 3$ is given. Here, on each element K, one has 12 degrees of freedom. These are the point-values at three nodes in the interior of K and at three nodes on each edge $E \in \mathcal{E}(K)$. The three nodes on an edge E are chosen as Gaussian integration points with respect to E which guarantees that (A6) is satisfied for $r = 3$.

Now, let us consider quadrilateral elements. The "rotated bilinear" elements proposed in [21] by Rannacher and Turek use the following space of shape functions on $\widehat{K} = [-1, 1]^2$

$$\widehat{\mathcal{P}} = \text{span}\{1,\ \widehat{x}_1,\ \widehat{x}_2,\ \widehat{x}_1^2 - \widehat{x}_2^2\}\,.$$

Here, the condition (56) is violated. Therefore, we modify this finite element space by adding the function $\widehat{x}_1\widehat{x}_2$ to $\widehat{\mathcal{P}}$, i.e. we choose

$$\widehat{\mathcal{P}} = \text{span}\{1,\ \widehat{x}_1,\ \widehat{x}_2,\ \widehat{x}_1\widehat{x}_2,\ \widehat{x}_1^2 - \widehat{x}_2^2\}\,.$$

The nodal functionals $N_j(\cdot)$ of the modified space V_h are the mean-values corresponding to the edges $E_j \in \mathcal{E}$, i.e.

$$N_j(v) := |E_j|^{-1} \int_{E_j} v\, ds\,, \tag{60}$$

and the mean-values corresponding to the elements $K_j \in \mathcal{T}_h$, i.e.

$$N_j(v) := |K_j|^{-1} \int_{K_j} v\, dx\,, \tag{61}$$

where $|K_j|$ denotes the two-dimensional measure of K_j. For this modified space of order $r = 1$, which has been used also in [18], the conditions (56), (57) and (A6) are satisfied. An analogous space of order $r = 2$ was given in [14]. Here, the space $\widehat{\mathcal{P}}$ is defined by

$$\widehat{\mathcal{P}} = \text{span}\{1,\ \widehat{x}_1,\ \widehat{x}_2,\ \widehat{x}_1\widehat{x}_2,\ \widehat{x}_1^2,\ \widehat{x}_2^2,\ \widehat{x}_1^2\widehat{x}_2,\ \widehat{x}_1\widehat{x}_2^2,\ \widehat{x}_1\widehat{x}_2(\widehat{x}_1^2 - \widehat{x}_2^2)\}\,.$$

such that $\mathbb{Q}_1(\widehat{K}) \subset \widehat{\mathcal{P}}$ and $\mathbb{P}_2(\widehat{K}) \subset \widehat{\mathcal{P}}$. The 9 degrees of freedom on each element $K_j \in \mathcal{T}_h$ are one inner node corresponding to the nodal functional in (61) and two nodes on each edge E of K_j corresponding to the nodal functionals

$$N_{E,\mu}(v) := |E|^{-1} \int_E v\, \omega_{E,\mu}\, ds\,, \qquad \mu \in \{1,2\}\,, \tag{62}$$

where $\omega_{E,1}$ and $\omega_{E,2}$ are given basis functions of the polynomial space $\mathbb{P}_1(E)$. This choice of the nodal functionals guarantees that (A6) is satisfied for $r = 2$. In [14], generalizations of this element to arbitrary order r and to the three-dimensional case are given. For these elements, the condition $\mathbb{P}_r(\widehat{K}) \subset \widehat{\mathcal{P}}$ and assumption (A6) are fulfilled by construction. However, concerning the condition $\mathbb{Q}_1(\widehat{K}) \subset \widehat{\mathcal{P}}$, there is a difference between the two- and the three-dimensional case. In the two-dimensional case, it holds $\mathbb{Q}_1(\widehat{K}) \subset \mathbb{P}_2(\widehat{K})$ and therefore the condition $\mathbb{Q}_1(\widehat{K}) \subset \widehat{\mathcal{P}}$ is satisfied for all the constructed spaces of order $r \geq 2$. In the three-dimensional case, we have $\mathbb{Q}_1(\widehat{K}) \subset \mathbb{P}_3(\widehat{K})$ such that the condition $\mathbb{Q}_1(\widehat{K}) \subset \widehat{\mathcal{P}}$ can be guaranteed if the space V_h has an order $r \geq 3$. The nonconforming spaces for the three-dimensional case that have been proposed in [21] and [14] for the order $r = 1$ and $r = 2$ do not satisfy the condition $\mathbb{Q}_1(\widehat{K}) \subset \widehat{\mathcal{P}}$ and have to be modified in order to fit into our framework.

7 Error Indication by Post-Processing

Now we will show that the computable element-wise quantities

$$e_K^{(p)} := \|u_h - R_h u_h\|_K \qquad \forall\, K \in \mathcal{T}_h \tag{63}$$

can be regarded as local error indicators in a sense explained below. Here the term $\|v\|_G$, where $G \subset \Omega$ is a subdomain consisting of a subset of elements $K \in \mathcal{T}_h$, denotes either the L^2-norm $\|v\|_{0,G}$ or the "broken" H^1-seminorm $|v|_{1,h,G}$ defined by

$$|v|_{1,h,G} := \Big(\sum_{K \in \mathcal{T}_h,\, K \subset G} |v|_{1,K}^2 \Big)^{1/2}. \tag{64}$$

At first, for an arbitrary element $K \in \mathcal{T}_h$, we derive an estimate of $e_K^{(p)}$ in terms of the local discretization error and the local interpolation error of u by a function of the conforming finite element space V_h^C. The triangle inequality yields

$$\|u_h - R_h u_h\|_K \;\leq\; \|u - u_h\|_K \;+\; \|u - R_h u_h\|_K\,.$$

In [24], the following estimate was proved

$$\|u - R_h u_h\|_K \;\leq\; C_1 \|u - u_h\|_{\delta(K)} \;+\; C_2\, \|u - v_h^{C,best}\|_{\delta(K)}\,, \tag{65}$$

where $v_h^{C,best} \in V_h^C$ denotes the best approximation of the solution u according to the conforming space V_h^C in a local vicinity $\delta(K)$ of K consisting of all elements which have a V_h^C-node in common with K (and moreover a slight modification near the boundary of Ω in case of the broken H^1-seminorm $\|\cdot\|_{\delta(K)} = |\cdot|_{1,h,\delta(K)}$). Thus, we obtain

$$e_K^{(p)} \;=\; \|u_h - R_h u_h\|_K \;\leq\; (1 + C_1)\,\|u - u_h\|_{\delta(K)} \;+\; C_2\,\|u - v_h^{C,best}\|_{\delta(K)}\,. \tag{66}$$

Now, let us assume that the value $e_K^{(p)}$ is "large", i.e. $e_K^{(p)} \geq \tau$ where τ is a given positive threshold. Then, for arbitrary positive constants α_1 and α_2 with $\alpha_1 + \alpha_2 = 1$, it holds

$$\|u - u_h\|_{\delta(K)} \;\geq\; \frac{\alpha_1}{1 + C_1}\,\tau \qquad \text{or} \qquad \|u - v_h^{C,best}\|_{\delta(K)} \;\geq\; \frac{\alpha_2}{C_2}\,\tau\,. \tag{67}$$

A possible choice of α_1 and α_2 would be to equilibrate the factors $\alpha_1/(1 + C_1)$ and α_2/C_2. The interpretation of (67) is the following. If the value $e_K^{(p)}$ is "large" for a certain element K, then this says for the vicinity $\delta(K)$ of this element that either the discretization error $\|u - u_h\|_{\delta(K)}$ is large or the interpolation error $\|u - v_h^{C,best}\|_{\delta(K)}$ is large. Of course, we have to note that $e_K^{(p)}$ is not a local error estimator since a small value of $e_K^{(p)}$ does not imply that the discretization error $\|u - u_h\|_K$ is small. However, a large value of $e_K^{(p)}$ tells us that there is a problem with the accuracy in the local region $\delta(K)$ of the element K, i.e. $e_K^{(p)}$ can be used as an additional error indicator together with other error indicating terms.

8 A Posteriori Error Estimates for Functionals

In the previous sections, we have constructed and analyzed a posteriori error estimates for the global energy norm of the discretization error. However, in applications one often wants to know with a certain accuracy approximate values for some quantities of physical interest. Therefore, it is important to have also an a posteriori estimate for the error with respect to such a quantity. For the case of conforming finite element spaces, a lot of work on this subject has been done by Rannacher and his coworkers (see [3], [19], [20]). An extension of these techniques to the case of two-dimensional nonconforming first order elements has been done in [18].

In this section, we want to present an alternative approach for nonconforming finite elements which uses again a conforming approximation $R_h u_h \in V_h^C$ of the nonconforming solution $u_h \in V_h$. Let us assume that the quantity of interest q can be computed by applying a linear functional $J \in V'$ to the solution $u \in V$ of the continuous problem, i.e. let $q = J(u)$. Furthermore, we assume that the numerical approximation q_h of the quantity q is computed from the discrete solution u_h by $q_h = J(u_h)$ and that $J \in V_h'$ where $\|J\|_{V_h'} \leq M_J$ with an h-independent constant M_J.

In the following, we show how the smoothed conforming approximation $R_h u_h \in V_h^C$ of the solution $u_h \in V_h$ can also be used for the estimation of the error with respect to our quantity, i.e. for estimating $q - q_h = J(u) - J(u_h)$.

For a given functional $J \in V'$, let $z \in V$ be the solution of the continuous dual problem

$$a(\varphi, z) = J(\varphi) \qquad \forall\, \varphi \in V. \tag{68}$$

Then, we can prove the following error representation.

Lemma 5 *Let the assumptions of Theorem 3 be satisfied. Furthermore, let $R_h u_h \in V_h^C$ be the smoothed approximation of the discrete solution u_h and $I_h^{C,1} z \in V_h^{C,1}$ a first order conforming interpolation of the solution $z \in V$ of the dual problem (68). Then, for the error corresponding to the functional $J(\cdot)$, we have the representation*

$$J(u) - J(u_h) = \sum_{K \in \mathcal{T}_h} \eta_K^{(c)}(u_h, z) \quad + \quad \eta^{(p)}(u_h, z) \tag{69}$$

where

$$\eta_K^{(c)}(u_h, z) := (f + \triangle u_h, z - I_h^{C,1} z)_K - \frac{1}{2} \sum_{E \in \mathcal{E}(K) \cap \mathcal{E}_0} \Big\langle \, [\nabla u_h \cdot n_E]_E \, , \, z - I_h^{C,1} z \, \Big\rangle_E \, , \tag{70}$$

$$\eta^{(p)}(u_h, z) := a_h(e_h^{(p)}, z) \; - \; J(e_h^{(p)}) \qquad \textit{with} \qquad e_h^{(p)} := u_h - R_h u_h \, . \tag{71}$$

Proof. Since $V_h^C \subset V$, we have that $e_h^{(c)} := u - R_h u_h \in V$. Taking $\varphi = e_h^{(c)}$ in (68) and using the Galerkin orthogonality (24) for $v_h = I_h^{C,1} z \in V_h^{C,1} \subset V \cap V_h$, we obtain

$$\begin{aligned} J(e_h^{(c)}) &= a(e_h^{(c)}, z) \; = \; a_h(e_h^{(c)}, z) \\ &= a_h(u - u_h, z - I_h^{C,1} z) \; + \; a_h(e_h^{(p)}, z) \\ &= \sum_{K \in \mathcal{T}_h} \big(\nabla(u - u_h), \nabla(z - I_h^{C,1} z) \big)_K \quad + \quad a_h(e_h^{(p)}, z) \, . \end{aligned} \tag{72}$$

Partial integration and the fact that $z - I_h^{C,1} z$ is zero on the boundary $\partial\Omega$, imply that the sum in (72) is equal to the sum of all $\eta_K^{(c)}(u_h, z)$. Thus, we get

$$J(u) - J(u_h) = J(e_h^{(c)}) \; - \; J(e_h^{(p)}) = \sum_{K \in \mathcal{T}_h} \eta_K^{(c)}(u_h, z) \; + \; a_h(e_h^{(p)}, z) \; - \; J(e_h^{(p)}) \, ,$$

which proves (69). ■

Note that the terms of $\eta_K^{(c)}(u_h, z)$ in (70) are the same as for the error representation in the case of a conforming space V_h (see [3], [20]). The additional term $\eta^{(p)}(u_h, z)$, which depends on the computable post-processing error $e_h^{(p)} = u_h - R_h u_h$, can be regarded as a consistency

error in the dual problem (68) for the nonconforming test function $\varphi = e_h^{(p)} \in V_h + V$. In the conforming case, where $V_h \subset V$ would imply $e_h^{(p)} \in V$, this consistency error would be zero.

The problem with the error representation (69) is that the solution $z \in V$ of the dual problem (68) is not known exactly. Therefore, in order to estimate the error $J(u) - J(u_h)$, we need an approximation $\widetilde{z}_{h'} \in \widetilde{V}_{h'}$ of z in an appropriate conforming space $\widetilde{V}_{h'} \subset V$. The notation h' in $\widetilde{V}_{h'}$ indicates that the space $\widetilde{V}_{h'}$ can be associated with a finer grid $\mathcal{T}_{h'}$ instead of $\mathcal{T}_h$. Some choices for the space $\widetilde{V}_{h'}$ and the approximation $\widetilde{z}_{h'} \approx z$ have been discussed in [3], [19], [20]. One possibility is to compute $\widetilde{z}_{h'} \in \widetilde{V}_{h'}$ as the solution of the *discrete dual problem*

$$a(\varphi_{h'}, \widetilde{z}_{h'}) = J(\varphi_{h'}) \qquad \forall\, \varphi_{h'} \in \widetilde{V}_{h'}\,. \tag{73}$$

Once the approximation $\widetilde{z}_{h'}$ of z has been calculated, an estimate for the error with respect to our quantity of interest is

$$J(u) - J(u_h) \;\approx\; \eta(u_h, \widetilde{z}_{h'}) := \sum_{K \in \mathcal{T}_h} \eta_K^{(c)}(u_h, \widetilde{z}_{h'}) \quad + \quad \eta^{(p)}(u_h, \widetilde{z}_{h'})\,. \tag{74}$$

The quantity $\eta(u_h, \widetilde{z}_{h'})$, which is derived by numerical evaluation of the error representation (69) without separating its additive components by means of the triangle inequality, in general gives a better estimate than the a posteriori error bound

$$|J(u) - J(u_h)| \;\lesssim\; \widetilde{\eta}(u_h, \widetilde{z}_{h'}) := \sum_{K \in \mathcal{T}_h} \widetilde{\eta}_K^{(c)}(u_h, \widetilde{z}_{h'}) \quad + \quad |\eta^{(p)}(u_h, \widetilde{z}_{h'})| \tag{75}$$

using computable bounds $\widetilde{\eta}_K^{(c)}(u_h, \widetilde{z}_{h'}) \geq |\eta_K^{(c)}(u_h, \widetilde{z}_{h'})|$. For example, we can take (see [20])

$$\widetilde{\eta}_K^{(c)}(u_h, \widetilde{z}_{h'}) := \sum_{i=1}^{2} \varrho_K^{(c,i)}(u_h)\, \omega_K^{(c,i)}(\widetilde{z}_{h'}) \tag{76}$$

where

$$\begin{aligned} &\varrho_K^{(c,1)}(u_h) := \|f + \triangle u_h\|_{0,K}\,, && \omega_K^{(c,1)}(\widetilde{z}_{h'}) := \|\widetilde{z}_{h'} - I_h^{C,1}\widetilde{z}_{h'}\|_{0,K}\,, \\ &\varrho_K^{(c,2)}(u_h) := \tfrac{1}{2}\Big\{ \sum_{E \in \mathcal{E}(K) \cap \mathcal{E}_0} \| \left[\nabla u_h \cdot n_E\right]_E \|_{0,E}^2 \Big\}^{1/2}, && \omega_K^{(c,2)}(\widetilde{z}_{h'}) := \|\widetilde{z}_{h'} - I_h^{C,1}\widetilde{z}_{h'}\|_{0,\partial K}\,. \end{aligned} \tag{77}$$

These factors can be interpreted as follows. The term $\varrho_K^{(c,1)}(u_h)$ describes an "equation residual", $\varrho_K^{(c,2)}(u_h)$ a "flux residual" and the terms $\omega_K^{(c,\cdot)}(\widetilde{z}_{h'})$ can be regarded as weighting factors which provide quantitative information about the impact of the cell-residuals $\varrho_K^{(c,\cdot)}(u_h)$ on the error $J(u) - J(u_h)$.

For deriving mesh refinement criteria, we use element-wise error indicators $\widetilde{\eta}_K$ for all $K \in \mathcal{T}_h$. These indicators are chosen as local contributions in the global a posteriori error

bound (75). A first variant is to neglect the term $|\eta^{(p)}(u_h, \widetilde{z}_{h'})|$ and to use $\widetilde{\eta}_K := \widetilde{\eta}_K^{(c)}(u_h, \widetilde{z}_{h'})$. Alternatively we can try to split also $\eta^{(p)}(u_h, \widetilde{z}_{h'})$ into element-wise components $\widetilde{\eta}_K^{(p)}(u_h, \widetilde{z}_{h'})$ and use the error indicator

$$\widetilde{\eta}_K := \widetilde{\eta}_K^{(c)}(u_h, \widetilde{z}_{h'}) + \widetilde{\eta}_K^{(p)}(u_h, \widetilde{z}_{h'}) . \tag{78}$$

The question which variant is the better one has to be investigated by numerical experiments. One way to derive element-wise components $\widetilde{\eta}_K^{(p)}(u_h, \widetilde{z}_{h'})$ from the post-processing term $\eta^{(p)}(u_h, \widetilde{z}_{h'})$ defined in (71) is based on the splitting

$$\eta^{(p)}(u_h, \widetilde{z}_{h'}) = \sum_{K \in \mathcal{T}_h} a_{h,K}(u_h - R_h u_h, \widetilde{z}_{h'}) \quad + \quad J(u_h - R_h u_h) ,$$

where $a_{h,K}(v, w) := (\nabla v, \nabla w)_K$ in the case of our model problem (1). Here local effects in the term $J(u_h - R_h u_h)$ are neglected and the local indicator corresponding to $\eta^{(p)}(u_h, \widetilde{z}_{h'})$ is defined by

$$\widetilde{\eta}_K^{(p)}(u_h, \widetilde{z}_{h'}) := |a_{h,K}(u_h - R_h u_h, \widetilde{z}_{h'})| . \tag{79}$$

Another way to define the values $\widetilde{\eta}_K^{(p)}(u_h, \widetilde{z}_{h'})$ would be to decompose also the term $J(u_h - R_h u_h)$ into element-wise contributions. In many cases, the functional $J(\cdot)$ can be written as a sum of local functionals $J_K(\cdot)$, i.e.

$$J(\varphi) = \sum_{K \in \mathcal{T}_h} J_K(\varphi) \qquad \forall\, \varphi \in V + V_h .$$

Then, we can define the local indicator as

$$\widetilde{\eta}_K^{(p)}(u_h, \widetilde{z}_{h'}) := |a_{h,K}(u_h - R_h u_h, \widetilde{z}_{h'})| + |J_K(u_h - R_h u_h)| . \tag{80}$$

Let us consider an example for the splitting of the functional $J(\cdot)$. Assume that $\Omega \subset \mathbb{R}^2$ and

$$J(\varphi) := \int_\Gamma \varphi \, ds \qquad \forall\, \varphi \in V + V_h ,$$

where $\Gamma \subset \overline{\Omega}$ is a given contour. Then, we can define a disjoint partition of the contour $\Gamma = \bigcup_{K \in \mathcal{T}_h} \Gamma_K$ with $\Gamma_K \subset \Gamma \cap \overline{K}$ and $\Gamma_K \cap \Gamma_{K'} = \emptyset$ for all pairs of different elements $K, K' \in \mathcal{T}_h$. The partition of Γ generates a splitting of $J(\cdot)$ into the following element-wise functionals

$$J_K(\varphi) := \int_{\Gamma_K} \varphi \, ds \qquad \forall\, \varphi \in V + V_h \,, K \in \mathcal{T}_h .$$

Note that the functional $J_K(\cdot)$ is zero if $\Gamma_K = \emptyset$ or $\mathrm{meas}(\Gamma_K) = 0$.

Based on computed error indicators $\{\widetilde{\eta}_K : K \in \mathcal{T}_h\}$, there are different strategies for controlling the mesh refinement. In [3], [19] and [20], an *error balancing strategy*, a *fixed*

fraction strategy and a *mesh optimization strategy* are discussed for the case of a conforming finite element discretization. The same strategies can be applied to the case of nonconforming finite elements. The only difference concerns the choice of the local error indicators $\widetilde{\eta}_K$ as mentioned above.

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