Nonconforming finite elements of higher order satisfying a modified compatibility condition

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Abstract

For a general diffusion-convection-reaction equation, we analyze families of nonconforming finite elements of arbitrary order on a sequence of multilevel grids consisting of quadrilaterals or hexahedra. We prove existence and uniqueness of the discrete solution and optimal order of convergence in the broken H^1-seminorm and the L^2-norm. The novelty of our approach is that the usual integral compatibility condition of the discrete functions across the element faces is modified such that it can be solely treated on the reference element once for all faces of the grid. A numerical comparison between conforming and nonconforming discretizations will be given in the three-dimensional case.

Keywords: nonconforming quadrilateral and hexahedral finite elements, multilevel grids, diffusion-convection-reaction equation, error estimates

Mathematics Subject Classification (2000): 65N15, 65N30, 65N50

1 Introduction

The Q_r-elements of continuous, piecewise polynomials of degree $r \geq 1$ in each variable on quadrilaterals and hexahedra form families of finite elements of optimal order r in the H^1-norm if scalar elliptic problems are considered. The situation is much less clear if we look for families of nonconforming elements of order r. The use of nonconforming approximations is attractive since their degrees of freedom are face-oriented. This results in cheap local communication when these methods are implemented on a parallel computer [21].

Nonconforming discretizations of higher order on quadrilaterals and hexahedra have less degrees of freedom and much less non-zero matrix entries than the corresponding conforming discretizations. Indeed, we have for the conforming Q_r-element on regular $N \times N$ decompositions of the unit square approximately $N^2\, r^2$ degrees of freedom and $N^2\, r^4$ non-zero matrix entries. The nonconforming elements of order r proposed in [14] result in approximately $N^2\, r^2/2$ degrees of freedom and $N^2\, r^4/4$ non-zero matrix entries. The behavior will be more dramatic in three space dimensions. We have for the conforming Q_r-element on regular $N \times N \times N$ decompositions of the unit cube approximately $N^3\, r^3$ degrees of freedom and $N^3\, r^6$ non-zero matrix entries. The nonconforming element of order r considered in this

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paper will result in approximately $N^3\,r^6/6$ degrees of freedom and $N^3\,r^3/36$ non-zero matrix entries. This makes nonconforming discretizations attractive form the practical point of view.

There are also advantages of nonconforming elements in the context of *hp*-FEM. Since each degree of freedom for nonconforming finite elements belongs to at most two elements, the matching of finite elements with different orders is much easier to handle compared to the conforming case. Also a local mesh refinement by means of hanging nodes can be done in a natural way. Moreover, the matching of hexahedra and tetrahedra is easily possible without any special elements like prisms and pyramids.

Nonconforming finite element methods can be regarded as mortar finite element methods on element level where the additional Lagrangian multipliers have been eliminated. Moreover, nonconforming finite element methods can be seen somehow between conforming finite element methods and discontinuous Galerkin methods. The continuity requirement of conforming finite element methods is weakened in nonconforming finite element methods but not removed completely from the approximation spaces as it is done in discontinuous Galerkin methods. Furthermore, the use of nonconforming finite element methods avoids the implementation of jumps terms which are essential for discontinuous Galerkin methods.

In the literature, mostly lowest order nonconforming methods on quadrilaterals and hexahedra have been considered so far, see [8, 19, 3, 4, 6]. One reason could be that there seems to be no generic construction for a family of nonconforming discretizations. However, in [9], algorithms are investigated which allow for a given set of nodal functionals to construct polynomial spaces being unisolvent with respect to the nodal functionals. For a proper choice of these nodal functionals, polynomial spaces with optimal approximation properties have been constructed in [9, 14]. The resulting nonconforming elements of higher order were successfully applied to the Stokes problem in [14]. The use of higher order nonconforming elements on anisotropic meshes has been investigated in [1].

Nonconforming finite elements of higher order on triangles were considered in [18, 17] while only [17] focuses on their application to the Stokes problem. On tetrahedra, a lowest order nonconforming finite element has been already given in [5].

In this paper, we consider the following model problem

$$\begin{aligned} -\mathrm{div}(A\nabla u) + b\cdot\nabla u + cu &= f \qquad \text{in } \Omega,\\ u &= g \qquad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a domain. The symmetric matrix $A \in \mathbb{R}^{d\times d}$, the vector $b \in \mathbb{R}^d$ as well as c, f, and g are given sufficiently smooth functions satisfying the conditions

$$\begin{aligned} a_0\|\xi\|_2^2 \le \xi^T A(x)\xi \le a_1\|\xi\|_2^2 &\qquad \forall\, x \in \Omega,\ \forall\, \xi \in \mathbb{R}^d,\\ c(x) - \frac{1}{2}\mathrm{div}\, b(x) \ge 0 &\qquad \forall\, x \in \Omega, \end{aligned} \tag{2}$$

where a_0, a_1 are positive constants and $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^d$.

We will consider families of scalar nonconforming finite elements on quadrilaterals and hexahedra which have been proposed in [9, 14]. The aim of this paper is to prove optimal estimates of the discretization error in the broken H^1-seminorm and the L^2-norm for these nonconforming elements applied to model problem (1). The new contribution with respect to [5] and [14] is that we treat a diffusion-convection-reaction equation and the general case of non-affine equivalent finite elements. However, we will not address here the problems arising from small diffusion. In this case, a stabilized discretization is necessary which will be investigated in a forthcoming study. It has been shown in [14] that the vector-valued versions of the nonconforming elements of order r for the velocity approximation are inf-sup stable together with a discontinuous piecewise P_{r-1} approximation for the pressure such that they are well suited for Stokes and Navier–Stokes problems. Moreover, the general construction principles and the proof of the unisolvence with respect to the nodal functionals can be found in [14]. The main points of our analysis can be also used for an analysis of the Stokes or Oseen problem in the case of non-affine equivalent elements.

The main part of this paper is devoted to the analysis of the consistency error. In the classical approach of Crouzeix and Raviart [5], one needs integral compatibility conditions of the finite element functions across the $(d-1)$-dimensional element faces which involve, in the case $d = 3$, surface integrals over the actual element faces. These faces can be curved for general hexahedral elements which causes additional difficulties for the construction and the implementation of nonconforming finite elements. The novelty of our approach is that we have modified these compatibility conditions such that they can be solely treated on the reference element once for all element faces. For the two-dimensional case and for the special case of affine equivalent three-dimensional elements, our modified compatibility condition is equivalent to the original one given in [5].

The linear system which arises from the nonconforming discretization can be solved efficiently by iterative solvers based on multigrid methods. Details on multigrid methods for nonconforming discretizations can be found in [2, 11]. The required transfer operators within the multigrid method are derived from [22].

This paper is organized as follows. In Section 2, we introduce the notations and the used multilevel grids. The finite element spaces are defined in Section 3. In Section 4, we analyze the consistency error and derive optimal error estimates for the discretization error. Finally, a comparison between our nonconforming elements and the well-known conforming Q_r-elements with respect to the number of degrees of freedom, the number of non-zero matrix entries, the accuracy, the orders of convergence and the computing time will be given in Section 5.

2 Preliminaries and Notations

2.1 General Notations

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with a polygonal or polyhedral boundary. For a domain $G \subset \Omega$, let $|\cdot|_{m,p,G}$ and $\|\cdot\|_{m,p,G}$ denote the usual seminorm and norm in the Sobolev spaces $W^{m,p}(G)$ and $(W^{m,p}(G))^d$, respectively. For the Hilbert spaces $H^m(G)$ and $(H^m(G))^d$, we omit the index p and denote the seminorm and norm by $|\cdot|_{m,G}$ and $\|\cdot\|_{m,G}$. The inner product in $L^2(G)$ and $(L^2(G))^d$ will be denoted by $(\cdot,\cdot)_G$, whereas for a sufficiently smooth $(d-1)$-dimensional face E of G, the inner product in $L^2(E)$ is denoted by $\langle \cdot,\cdot \rangle_E$. By $\mathbb{P}_m(D)$ we denote the space of all polynomials on the domain $D \subset \mathbb{R}^n$, $1 \le n \le d$, with total degree less than or equal to m and by $\mathbb{Q}_m(D)$ the space of those polynomials where the maximum power in each coordinate is less than or equal to m. For $p \in [1,\infty]$, both the usual p-norm of a vector in $\mathbb{R}^d$ and the induced p-norm of a $d \times d$-matrix are denoted by $\|\cdot\|_p$. For a set $G \subset \mathbb{R}^d$, we denote by $\mathrm{int}(G)$ and $\overline{G}$ the interior and closure of G, respectively. If G is a domain in $\mathbb{R}^d$ we denote by $|G|$ the d-measure of G and if E is a $(d-1)$-dimensional manifold we denote by $|E|$ the $(d-1)$-measure of E.

For a non-negative integer i, let $L_i(t)$ denote the i-th one-dimensional Legendre polynomial normalized such that $L_i(1) = 1$ for all $i \ge 0$. Let n with $1 \le n \le d$ denote a given dimension and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ a corresponding multi-index where $\mathbb{N}_0$ is the set of the non-negative integers. Then, we define

$$L_\alpha^{(n)}(\xi) := \prod_{k=1}^{n} L_{\alpha_k}(\xi_k) \qquad \forall\, \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,$$

i.e., in particular, we have $L_\alpha^{(1)}(t) = L_\alpha(t)$ for $\alpha \in \mathbb{N}_0$, $t \in \mathbb{R}$. Furthermore, for multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ we denote by $\delta_{\alpha,\beta}$ the Kronecker Delta which is 1 if $\alpha = \beta$ and 0 in all other cases.

Throughout this paper, C will denote a generic constant which may have different values at different places. All these constants occurring inside of any estimate will be independent of the local and global mesh parameters h_K and h defined below.

2.2 Multilevel Grids

Let the bounded domain $\Omega \subset \mathbb{R}^d$ be partitioned by a *grid* $\mathcal{T}_h$ consisting of elements $K \in \mathcal{T}_h$ which are assumed to be open quadrilaterals or hexahedra such that $\Omega = \mathrm{int}\big(\bigcup_{K\in\mathcal{T}_h} \overline{K}\big)$. For an element $K \in \mathcal{T}_h$, we denote by h_K the diameter of the element K. The *mesh size* h of $\mathcal{T}_h$ is given by $h := \max_{K\in\mathcal{T}_h} h_K$. We denote by $F_K : \widehat{K} \to K$ the mapping between the reference element $\widehat{K} := (-1,+1)^d$ and the original element K. In the sequel of this paper, the mapping F_K is assumed to be multi-linear, i.e. $F_K \in (\mathbb{Q}_1(\widehat{K}))^d$. Therefore, the two dimensional faces of 3D hexahedral elements $K \in \mathcal{T}_h$ can be curved in general. For our analysis, we assume that the mesh family $\{\mathcal{T}_h\}$ is *shape regular* in the sense presented in [16]. This shape regularity assumption guarantees that the reference mapping $F_K : \widehat{K} \to K$ is bijective and that the following properties

$$\|DF_K(\hat{x})\|_\infty \le C h_K \qquad \forall \hat{x} \in \widehat{K}, \tag{3}$$

$$C h_K^d \le |\det(DF_K(\hat{x}))| \le C' h_K^d \qquad \forall \hat{x} \in \widehat{K}. \tag{4}$$

are satisfied for all $K \in \mathcal{T}_h$, see [10] for the proof.

We assume that $\{\mathcal{T}_h\}$ is a family of *multilevel grids* generated by a refinement process in the following way. We start with an initial partition $\mathcal{T}^0$ of the domain Ω into elements $K \in \mathcal{T}^0$ of grid-level 0, i.e. $\Omega = \mathrm{int}\big(\bigcup_{K\in\mathcal{T}^0} \overline{K}\big)$. The grid $\mathcal{T}^0$ is assumed to be *regular* in the usual sense, i.e., the intersection $\overline{K_1} \cap \overline{K_2}$ of any two different elements $K_1, K_2 \in \mathcal{T}^0$ is either empty or a common $(d-m)$-dimensional face of K_1 and K_2 where $m \in \{1, \ldots, d\}$. In order to generate the elements of grid level 1, each element $K \in \mathcal{T}^0$ is refined, i.e., it is splitted into 2^d new elements of the grid $\mathcal{T}^1$ called *son-elements* of K which are denoted by $\sigma_i(K)$, $i = 1, \ldots, 2^d$, see Figure 1, right. These son-elements are defined as follows. Let $\widehat{K}_i$, $i \in \{1, \ldots, 2^d\}$, be the son-elements of the reference element $\widehat{K} = (-1,1)^d$ which are congruent d-cubes with $|\widehat{K}_i| = 1$, see Figure 1, left. The i-th son-element $\sigma_i(K)$ of the original element K is defined as $\sigma_i(K) := F_K(\widehat{K}_i)$, $i = 1, \ldots, 2^d$. Note that it can happen in the three-dimensional case that son-elements $\sigma_i(K)$ have curved faces even if the element K has only planar faces, see [13, Remark 3.3] for an example. Having created the elements of the grid $\mathcal{T}^1$, we continue this refinement process to generate further grids $\mathcal{T}^2, \ \ldots\ , \mathcal{T}^\ell$. Then, our actual grid $\mathcal{T}_h$ is the grid $\mathcal{T}^\ell$ on the finest grid level ℓ.

An advantage of multilevel grids is that the reference mapping F_K behaves – up to a power of the mesh size h – like an affine mapping. This is one of the key points of the analysis in this paper. We have the following result.

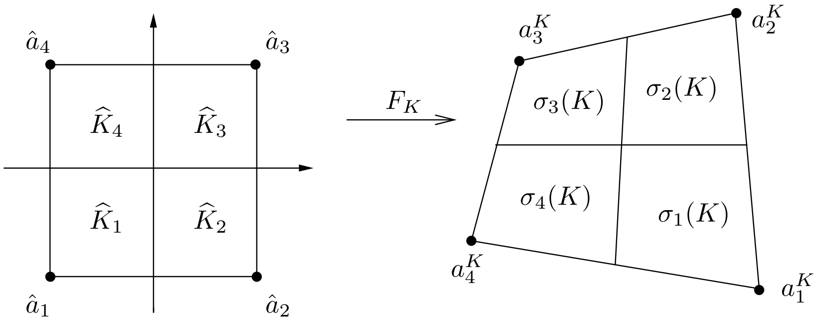

Figure 1: *Refinement of element K into son-elements $\sigma_i(K)$, $i = 1, \ldots, 2^d$.*

Lemma 1. *Let $\{\mathcal{T}_h\}$ be a family of multilevel grids generated from a shape regular coarse grid $\mathcal{T}^0$ as described above. Then, there exist constants $C_1(m, \mathcal{T}^0)$, $0 \le m \le d$, such that the estimates*

$$\begin{aligned} |F_K|_{m,\infty,\widehat{K}} &\le C_1(m, \mathcal{T}^0)\, h_K^m \qquad && \forall\, 0 \le m \le d, \\ |F_K|_{m,\infty,\widehat{K}} &= 0 && \forall\, m \ge d+1, \end{aligned} \tag{5}$$

are satisfied for all $K \in \mathcal{T}_h$.

Proof. See Lemma 5 in [15]. □

Finally, we introduce some notation to describe the element faces. For an element $K \in \mathcal{T}_h$, we denote by $\mathcal{E}(K)$ the set of all $(d-1)$-dimensional faces of K. Let $\mathcal{E}_h := \bigcup_{K \in \mathcal{T}_h} \mathcal{E}(K)$ be the set of all element faces of the grid $\mathcal{T}_h$. We split $\mathcal{E}_h$ as $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$ where $\mathcal{E}_h^i$ denotes the set of all inner faces and $\mathcal{E}_h^b$ the set of all faces located at the boundary of Ω. For each inner face $E \in \mathcal{E}_h^i$, there exist exactly two different elements denoted by $K(E)$ and $K'(E)$ such that E is one of their faces. For the boundary faces $E \in \mathcal{E}_h^b$, there is only one element denoted by $K(E)$ which has E as one of its faces and we set formally $K'(E) := \emptyset$. For an element $K \in \mathcal{T}_h$ and a face $E \in \mathcal{E}(K)$, we denote by n_E^K the unit normal on E pointing outward with respect to K. We assign to each face $E \in \mathcal{E}_h$ the unit normal $n_E := n_E^{K(E)}$.

3 Finite Element Spaces

3.1 General definition

Let $r \ge 1$ denote the fixed degree of our finite element space and let us suppose that we have already defined the finite element $\big(\widehat{K}, \widehat{V}_r, \widehat{\mathcal{N}}_r\big)$ on the reference cell $\widehat{K} = (-1,1)^d$ with the function space $\widehat{V}_r$ and the set $\widehat{\mathcal{N}}_r$ of nodal functionals. Then, we define the function space $V_r(K)$ on the original element $K \in \mathcal{T}_h$ as

$$V_r(K) := \big\{ v = \hat{v} \circ F_K^{-1} \;:\; \hat{v} \in \widehat{V}_r \big\}. \tag{6}$$

Let $H^{1,h}(\Omega)$ denote the space of elementwise H^1-functions with respect to $\mathcal{T}_h$, i.e.,

$$H^{1,h}(\Omega) := \{ v \in L^2(\Omega) \;:\; v|_K \in H^1(K)\, \forall K \in \mathcal{T}_h \}. \tag{7}$$

Furthermore, we will introduce the *jump* $[v]_E$ of a function v across a $(d-1)$-dimensional face $E \in \mathcal{E}_h^i$ as

$$[v]_E := \big(v|_{K(E)}\big)\big|_E - \big(v|_{K'(E)}\big)\big|_E \qquad \forall v \in H^{1,h}(\Omega). \tag{8}$$

The usual global nonconforming finite element space is defined by

$$\widetilde{V}_{g,h} := \bigg\{ v_h \in H^{1,h}(\Omega) : v_h|_K \in V_r(K)\ \forall K \in \mathcal{T}_h,\ \int_E q\,[v_h]_E\, d\gamma = 0\ \ \forall E \in \mathcal{E}_h^i\ \ \forall q \in P_{r-1}(E),$$
$$\int_E q(v_h - g)\, d\gamma = 0\ \ \forall E \in \mathcal{E}_h^b\ \ \forall q \in P_{r-1}(E) \bigg\}, \tag{9}$$

see [5]. However, it is not really clear what the polynomial space $P_{r-1}(E)$ means in the three-dimensional case where the element faces $E \in \mathcal{E}_h$ are curved in general. Moreover, in this case, the finite element basis functions have to be computed separately for each element by evaluating surface integrals and the solution of a linear system of equations which leads to higher computing costs and memory requirements to store these solutions.

Due to this drawbacks, we introduce the global finite element space in a different way. Let $\mathcal{A}_s^{(n)}$, $n \geq 1$, be the following set of multi-indices

$$\mathcal{A}_s^{(n)} := \Big\{ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n : \sum\nolimits_{k=1}^n \alpha_k \leq s \Big\}.$$

For an element $K \in \mathcal{T}_h$, an associated $(d-1)$-face $E \in \mathcal{E}(K)$ and a multi-index $\alpha \in \mathcal{A}_{r-1}^{(d-1)}$, we denote by $N_{E,\alpha}^K$ the nodal functional

$$N_{E,\alpha}^K(v) := \int_{U_{d-1}} \big(v\big|_K \circ T_E\big)(\xi) L_\alpha^{(d-1)}(\xi)\, d\xi \qquad \forall v \in H^{1,h}(\Omega), \tag{10}$$

where $U_m := (-1,1)^m$ denotes the m-dimensional unit cube and $T_E : U_{d-1} \to E$ the multilinear transformation

$$T_E(\xi) := F_{K(E)}\big(\widehat{T}_{\widehat{E}}(\xi)\big) \qquad \forall \xi \in U_{d-1} \tag{11}$$

based on an affine mapping $\widehat{T}_{\widehat{E}} : U_{d-1} \to \widehat{E} := F_{K(E)}^{-1}(E)$ which will be defined below. The compatibility conditions of a discrete function $v_h \in V_h$ on inner element faces are

$$N_{E,\alpha}^{K(E)}(v_h) = N_{E,\alpha}^{K'(E)}(v_h) \qquad \forall E \in \mathcal{E}_h^i,\ \alpha \in \mathcal{A}_{r-1}^{(d-1)}. \tag{12}$$

These conditions are equivalent to

$$\int_{U_{d-1}} q(\xi)\, \big[v_h\big(T_E(\xi)\big)\big]_E\, d\xi = 0 \qquad \forall q \in \mathbb{P}_k(U_{d-1}),\ \ E \in \mathcal{E}_h^i, \tag{13}$$

where $k := r-1$. In fact, let $q \in \mathbb{P}_{r-1}(U_{d-1})$ with the representation $q = \sum\limits_{\alpha \in \mathcal{A}_{r-1}^{(d-1)}} c_\alpha L_\alpha^{(d-1)}$ be an arbitrary polynomial. Then, due to (12), it holds

$$\int\limits_{U_{d-1}} q(\xi) \left(v_h\big|_{K(E)}(T_E(\xi)) - v_h\big|_{K'(E)}(T_E(\xi)) \right) d\xi = \sum_{\alpha \in \mathcal{A}_{r-1}^{(d-1)}} c_\alpha \left(N_{E,\alpha}^{K(E)}(v_h) - N_{E,\alpha}^{K'(E)}(v_h) \right) = 0.$$

Now, we define our global nonconforming finite element space V_h as

$$V_h := \Big\{ v_h \in H^{1,h}(\Omega) : v_h|_K \in V_r(K)\ \forall K \in \mathcal{T}_h,\ \ v_h \text{ satisfies (12)} \Big\}. \tag{14}$$

The advantage of the modified compatibility condition (12) compared to the usual one in (9) is that in the 3D case only volume integrals over the unit cube U_2 have to be evaluated instead of surface integrals over curved element faces. The space to approximate the solution u of (1) is

$$V_{g,h} := \Big\{ v_h \in V_h : N_{E,\alpha}^{K(E)}(v_h) = N_{E,\alpha}^{K(E)}(g)\ \ \forall E \in \mathcal{E}_h^b\ \forall \alpha \in \mathcal{A}_{r-1}^{(d-1)} \Big\}. \tag{15}$$

Here, g formally has to be understood as function in $H^1(\Omega)$ which is an extension of the boundary function g in (1). Note that the value $N_{E,\alpha}^{K(E)}(g)$ is the same for all possible extensions. Furthermore, let $V_{0,h}$ denote the space $V_{g,h}$ with $g = 0$. In $V_{0,h}$ the conditions $N_{E,\alpha}^{K(E)}(v) = 0$ from (15) for all $E \in \mathcal{E}_h^b$ and $\alpha \in \mathcal{A}_{r-1}^{(d-1)}$ are equivalent to

$$\int_{U_{d-1}} q(\xi) v_h\big(T_E(\xi)\big)\, d\xi = 0 \qquad \forall q \in \mathbb{P}_k(U_{d-1}),\ \ E \in \mathcal{E}_h^b, \tag{16}$$

with $k := r - 1$.

It is easy to show that $\widetilde{V}_{g,h} = V_{g,h}$ holds in the two-dimensional case for all grids $\mathcal{T}_h$ and in the three-dimensional case for grids $\mathcal{T}_h$ consisting of affine equivalent elements (parallelepipeds). This is mainly caused by the fact that in these cases the mapping T_E is affine.

3.2 Construction on the reference element

Let $\widehat{\mathcal{E}}$ denote the set of all $(d-1)$-faces of the reference element $\widehat{K} = (-1,1)^d$. In the following, we will define the set $\widehat{\mathcal{N}}_r$ of nodal functionals on the reference element which consists of two subsets. In the first subset, we assign to each face $\widehat{E} \in \widehat{\mathcal{E}}$ and each multi-index $\alpha \in \mathcal{A}_{r-1}^{(d-1)}$ the functional

$$\widehat{N}_{\widehat{E},\alpha}(\widehat{v}) := \int_{U_{d-1}} \big(\widehat{v} \circ \widehat{T}_{\widehat{E}}\big)(\xi) L_\alpha^{(d-1)}(\xi)\, d\xi\,, \tag{17}$$

where $\widehat{T}_{\widehat{E}} : U_{d-1} \to \widehat{E}$ is an affine mapping defined as follows. Let $\widehat{a}_{\widehat{E},i}$, $i = 1, \ldots, 2^{d-1}$, denote the vertices of the face $\widehat{E}$ and $b_{\widehat{E}} := 2^{1-d} \sum_{i=1}^{2^{d-1}} \widehat{a}_{\widehat{E},i}$ the barycenter of $\widehat{E}$. We assume that the vertices are numbered in a counterclockwise sense looking from inside of $\widehat{K}$ at $\widehat{E}$. Then, we define

$$\widehat{T}_{\widehat{E}}(\xi) := b_{\widehat{E}} + \sum_{k=1}^{d-1} \xi_k \widehat{r}_{\widehat{E}}^{(k)} \qquad \forall\, \xi = (\xi_1, \ldots, \xi_{d-1}) \in U_{d-1}, \tag{18}$$

where $\widehat{r}_{\widehat{E}}^{(1)} := \frac{1}{2}\big(\widehat{a}_{\widehat{E},2} - \widehat{a}_{\widehat{E},1}\big)$ and, in the case $d = 3$, $\widehat{r}_{\widehat{E}}^{(2)} := \frac{1}{2}\big(\widehat{a}_{\widehat{E},4} - \widehat{a}_{\widehat{E},1}\big)$. In the second subset of nodal functionals, we assign to each multi-index $\beta \in \mathcal{B}_r^{(d)}$ the functional

$$\widehat{N}_{\widehat{K},\beta}(\widehat{v}) := \int_{\widehat{K}} \widehat{v}(\widehat{x}) L_\beta^{(d)}(\widehat{x})\, d\widehat{x}\,, \tag{19}$$

where the set $\mathcal{B}_r^{(d)}$ is defined below. Now, the set $\widehat{\mathcal{N}}_r$ is given by

$$\widehat{\mathcal{N}}_r := \big\{ \widehat{N}_{\widehat{E},\alpha} : \widehat{E} \in \widehat{\mathcal{E}}, \alpha \in \mathcal{A}_{r-1}^{(d-1)} \big\} \cup \big\{ \widehat{N}_{\widehat{K},\beta} : \beta \in \mathcal{B}_r^{(d)} \big\}. \tag{20}$$

After choosing the set $\widehat{\mathcal{N}}_r$ of nodal functionals, we have to look for suitable function spaces $\widehat{V}_r$. A first requirement is that its dimension is equal to the number of nodal functionals in $\widehat{\mathcal{N}}_r$. A second demand is that the polynomial space $\mathbb{P}_r$ should be a subspace of $\widehat{V}_r$ since we want to have finite elements of order r. Moreover, the set $\widehat{\mathcal{N}}_r$ has to be unisolvent on $\widehat{V}_r$.

In [14], several families were considered for the choice $\mathcal{B}_r^{(d)} = \mathcal{A}_{r-2}^{(d)}$. We will give them here for completeness. We use in the following

$$L_{a,b} := L^{(2)}_{(a,b)}, \qquad L_{a,b,c} := L^{(3)}_{(a,b,c)},$$

as abbreviations.

We start with the two-dimensional case. A first choice $\widehat{V}_r^1$ was already derived in [9, Example 5]. We have

$$\widehat{V}_r^1 := \mathbb{P}_r(\widehat{K}) \oplus \widehat{R}_r \oplus \widehat{S}_r$$

with

$$\widehat{S}_r := \operatorname{span}\Big\{ L_{i,i}, L_{i+1,i}, L_{i,i+1}, L_{i+2,i} - L_{i,i+2} \ : \ r/2 < i \le r-1 \Big\}$$

and

$$\widehat{R}_r := \begin{cases} \operatorname{span}\Big\{ L_{i+1,i}, L_{i,i+1}, L_{i+2,i} - L_{i,i+2}, \ : \ i = r/2 \Big\}, & r \text{ even}, \\ \operatorname{span}\Big\{ L_{i+2,i} - L_{i,i+2} \ : \ i = (r-1)/2 \Big\}, & r \text{ odd}. \end{cases}$$

The space $\widehat{V}_r^1$ is the output of Algorithm 1 in [9] which ensures the unisolvence, see [9, Lemma 1]. The two choices $\widehat{V}_r^2$ and $\widehat{V}_r^3$ for $\widehat{V}_r$ are given by

$$\begin{aligned} \widehat{V}_r^2 := \mathbb{P}_r(\widehat{K}) \ &\oplus \operatorname{span}\Big\{ L_{i+2,r-1-i} - L_{i,r+1-i} \, , \ i = 0, \ldots, r-1 \Big\} \\ &\oplus \operatorname{span}\Big\{ L_{i+2,r-i} - L_{i,r+2-i} \, , \ i = 1, \ldots, r-1 \Big\} \end{aligned} \tag{21}$$

and

$$\begin{aligned} \widehat{V}_r^3 := \mathbb{P}_r(\widehat{K}) \ &\oplus \operatorname{span}\Big\{ L_{i,r+1-i}, L_{r+1-i,i} \, , \ i = 0, \ldots, \lfloor r/2 \rfloor - 1 \Big\} \\ &\oplus \operatorname{span}\Big\{ L_{i,r+2-i}, L_{r+2-i,i} \, , \ i = 1, \ldots, \lfloor (r+1)/2 \rfloor - 1 \Big\} \\ &\oplus \operatorname{span}\Big\{ L_{\lfloor r/2 \rfloor + 2, \lfloor r/2 \rfloor} - L_{\lfloor r/2 \rfloor, \lfloor r/2 \rfloor + 2} \Big\} \end{aligned} \tag{22}$$

where $\lfloor x \rfloor$ is the largest integer which is less than or equal to x. The proof of the unisolvence of $\widehat{\mathcal{N}}_r$ on $\widehat{V}_r^2$ and $\widehat{V}_r^3$ can be found in [14]. Note that the case $r = 1$ recovers in all three cases the mean value oriented rotated bilinear element by Rannacher and Turek [19].

All spaces $\widehat{V}_r^1$, $\widehat{V}_r^2$, $\widehat{V}_r^3$ are subspaces of $\mathbb{Q}_{r+1}(\widehat{K})$. The advantage of the spaces $\widehat{V}_r^2$, $\widehat{V}_r^3$ is that both are contained in $\mathbb{P}_{r+2}(\widehat{K})$, in contrast to $\widehat{V}_r^1$ which contains functions from $\mathbb{P}_{2r}(\widehat{K})$. We recommend the use of $\widehat{V}_r^3$ for two reasons. It contains less differences of Legendre polynomials than $\widehat{V}_r^2$. Moreover, among $\widehat{V}_r^1$, $\widehat{V}_r^2$, $\widehat{V}_r^3$ only $\widehat{V}_r^3$ works on anisotropic meshes [1].

The three-dimensional family was already given in [9, Example 10]. The space $\widehat{V}_r$ is defined as

$$\widehat{V}_r := \mathbb{P}_{r-2}(\widehat{K}) \oplus \bigoplus_{i=1}^{3} \widehat{A}_i \oplus \bigoplus_{i=1}^{3} \widehat{B}_i \oplus \widehat{C}$$

with

$$\widehat{A}_1 := \operatorname{span}\Big\{L_{i,j,k}, L_{i+1,j,k} \;:\; i > \max(j,k), i+j+k = r-1\Big\},$$
$$\widehat{A}_2 := \operatorname{span}\Big\{L_{i,j,k}, L_{i,j+1,k} \;:\; j > \max(i,k), i+j+k = r-1\Big\},$$
$$\widehat{A}_3 := \operatorname{span}\Big\{L_{i,j,k}, L_{i,j,k+1} \;:\; k > \max(i,j), i+j+k = r-1\Big\},$$

and

$$\widehat{B}_1 := \operatorname{span}\Big\{L_{j,i,i}, L_{j,i+1,i}, L_{j,i,i+1}, L_{j,i,i+2} - L_{j,i+2,i} \;:\; i > j, r-1 \leq 2i+j \leq 2(r-1)-j\Big\},$$
$$\widehat{B}_2 := \operatorname{span}\Big\{L_{iji}, L_{i+1,j,i}, L_{i,j,i+1}, L_{i,j,i+2} - L_{i+2,j,i} \;:\; i > j, r-1 \leq 2i+j \leq 2(r-1)-j\Big\},$$
$$\widehat{B}_3 := \operatorname{span}\Big\{L_{i,i,j}, L_{i+1,i,j}, L_{i,i+1,j}, L_{i,i+2,j} - L_{i+2,i,j} \;:\; i > j, r-1 \leq 2i+j \leq 2(r-1)-j\Big\}.$$

Furthermore, we set

$$\begin{aligned}\widehat{C} := \quad & \operatorname{span}\Big\{L_{i,i,i}, L_{i+1,i,i}, L_{i,i+1,i}, L_{i,i,i+1}, \;:\; (r-1)/3 \leq i \leq (r-1)/2\Big\} \\ & \oplus \operatorname{span}\Big\{\alpha L_{i+2,i,i} + \beta L_{i,i+2,i} + \gamma L_{i,i,i+2} \text{ with } \alpha+\beta+\gamma = 0 \;:\; (r-1)/3 \leq i \leq (r-1)/2\Big\}.\end{aligned}$$

The unisolvence follows directly from [9, Lemma 2]. It is also shown in [9] that $\mathbb{P}_r(\widehat{K}) \subset \widehat{V}_r$. Note that the well-known mean value oriented element by Rannacher and Turek [19] is obtained for the case $r = 1$.

Due to the unisolvence of $\widehat{\mathcal{N}}_r$ with respect to the space $\widehat{V}_r$, there exists a unique dual basis of $\widehat{V}_r$, i.e.

$$\widehat{V}_r = \operatorname{span}\{\widehat{\phi}_{G,\alpha} : (G,\alpha) \in \widehat{\mathcal{E}} \times \mathcal{A}_{r-1}^{(d-1)} \cup \{\widehat{K}\} \times \mathcal{B}_r^{(d)}\} \tag{23}$$

with

$$\widehat{N}_{G,\alpha}(\widehat{\phi}_{H,\beta}) = \begin{cases} 1, & \text{if } (G,\alpha) = (H,\beta), \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

Using this basis on the reference element, we can define the following global basis functions:

$$\phi_{K,\beta}(x) := \widehat{\phi}_{\widehat{K},\beta}(F_K^{-1}(x)) \qquad \forall x \in K \tag{25}$$

with $\operatorname{supp}(\phi_{K,\beta}) = \overline{K}$ and

$$(\phi_{E,\alpha})\big|_K(x) := \begin{cases} \widehat{\phi}_{\widehat{E},\alpha}(F_K^{-1}(x)), & \text{if } K = K(E),\ \widehat{E} := F_{K(E)}^{-1}(E), \\ s(\alpha')\widehat{\phi}_{\widehat{E}',\alpha'}(F_K^{-1}(x)), & \text{if } K = K'(E),\ \widehat{E}' := F_{K'(E)}^{-1}(E). \end{cases} \tag{26}$$

with $\operatorname{supp}(\phi_{E,\alpha}) = \overline{K(E)} \cup \overline{K'(E)}$ for $E \in \mathcal{E}_h^i$ and $\operatorname{supp}(\phi_{E,\alpha}) = \overline{K(E)}$ for $E \in \mathcal{E}_h^b$. Here, for a given inner face $E \in \mathcal{E}_h^i$ and a given multi-index $\alpha \in \mathcal{A}_{r-1}^{(d-1)}$, the multi-index $\alpha' \in \mathcal{A}_{r-1}^{(d-1)}$ and the sign $s(\alpha') \in \{-1,1\}$ are defined as follows. In the element $K(E)$, the face E corresponds to the reference face $\widehat{E} := F_{K(E)}^{-1}(E)$ and in the element $K'(E)$ to $\widehat{E}' := F_{K'(E)}^{-1}(E)$. One can show for this situation that there exist a bijective mapping $\mu : \{1,\ldots,d-1\} \to \{1,\ldots,d-1\}$ and signs $s_k \in \{-1,1\}$, such that

$$F_{K(E)}\big(\widehat{T}_{\widehat{E}}(\xi)\big) = F_{K'(E)}\big(\widehat{T}_{\widehat{E}'}(\xi')\big) \qquad \forall \xi \in U_{d-1}, \tag{27}$$

where $\xi' = (\xi'_1, \ldots, \xi'_{d-1}) \in U_{d-1}$ with $\xi'_k := s_k\, \xi_{\mu(k)}$ for $k = 1, \ldots, d-1$. Then, we define

$$\alpha' = (\alpha'_k) := (\alpha_{\mu(k)}) \in \mathcal{A}^{(d-1)}_{r-1}, \qquad s(\alpha') := \prod_{k=1}^{d-1} (s_k)^{\alpha'_k} \in \{-1, +1\}. \tag{28}$$

In the implementation of the corresponding finite element method, the mapping μ and the signs s_k can be easily computed.

Theorem 2. *For the space V_h defined in* (14) *it holds*

$$V_h = \operatorname{span}\big\{\phi_{E,\alpha} : E \in \mathcal{E}_h, \alpha \in \mathcal{A}^{(d-1)}_{r-1}\big\} \oplus \operatorname{span}\big\{\phi_{K,\beta} : K \in \mathcal{T}_h, \beta \in \mathcal{B}^{(d)}_r\big\}. \tag{29}$$

Proof. We only have to show that the basis functions defined in (25) and (26) are linearly independent and satisfy the compatibility conditions (12). By definition (25), the functions $\phi_{K,\beta}$ with $K \in \mathcal{T}_h$ and $\beta \in \mathcal{B}^{(d)}_r$ are linearly independent. Let us fix $E \in \mathcal{E}^i_h$ and $\alpha \in \mathcal{A}^{(d-1)}_{r-1}$. Since

$$N^{K(E)}_{E,\alpha}(v) = \int_{U_{d-1}} v\big|_{K(E)} \Big(F_{K(E)}\big(\widehat{T}_{\widehat{E}}(\xi)\big)\Big)\, L^{(d-1)}_\alpha(\xi)\, d\xi \;=\; \widehat{N}_{\widehat{E},\alpha}(\widehat{v}),$$

where $\widehat{E} := F^{-1}_{K(E)}(E)$ and $\widehat{v} := v \circ F_{K(E)}$, and due to (24), we have

$$\begin{aligned} N^{K(E)}_{E,\alpha}(\phi_{G,\beta}) &= 0 && \forall (G,\beta) \in \mathcal{T}_h \times \mathcal{B}^{(d)}_r \cup \big(\mathcal{E}_h \setminus \{E\}\big) \times \mathcal{A}^{(d-1)}_{r-1}, \\ N^{K(E)}_{E,\alpha}(\phi_{E,\beta}) &= \delta_{\alpha,\beta} && \forall \beta \in \mathcal{A}^{(d-1)}_{r-1}. \end{aligned}$$

This shows that all functions $\phi_{E,\alpha}$ and $\phi_{K,\beta}$ are linearly independent. Using the property

$$L^{(d-1)}_\alpha(\xi) = s(\alpha') L^{(d-1)}_{\alpha'}(\xi'),$$

we get

$$N^{K'(E)}_{E,\alpha}(v) = \int_{U_{d-1}} v\big|_{K'(E)} \Big(F_{K'(E)}\big(\widehat{T}_{\widehat{E}'}(\xi')\big)\Big)\, s(\alpha') L^{(d-1)}_{\alpha'}(\xi')\, d\xi' \;=\; s(\alpha')\widehat{N}_{\widehat{E}',\alpha'}(\widehat{v}),$$

where $\widehat{E}' := F^{-1}_{K'(E)}(E)$ and $\widehat{v} := v \circ F_{K'(E)}$. Due to (24) this implies

$$\begin{aligned} N^{K'(E)}_{E,\alpha}(\phi_{G,\beta}) &= 0 && \forall (G,\beta) \in \mathcal{T}_h \times \mathcal{B}^{(d)}_r \cup \big(\mathcal{E}_h \setminus \{E\}\big) \times \mathcal{A}^{(d-1)}_{r-1}, \\ N^{K'(E)}_{E,\alpha}(\phi_{E,\beta}) &= s(\alpha')s(\beta')\delta_{\alpha',\beta'} = \delta_{\alpha,\beta} && \forall \beta \in \mathcal{A}^{(d-1)}_{r-1}. \end{aligned}$$

Thus, we have proved (12) for all basis functions of V_h. □

The representation (29) together with (25) and (26) shows that it is sufficient to compute the basis functions on the reference element and to use the reference mapping for the definition of the basis functions on the elements $K \in \mathcal{T}_h$.

4 Error analysis in the 3D case

We present the analysis of our finite element method only for the more interesting three-dimensional case $d = 3$. The final results in Theorem 6 and 7 also hold for the 2D case and can be obtained in a similar but often more simpler way as in the 3D case. This is caused by the fact that in the two-dimensional case the element faces can be parameterized by an affine mapping and the unit normal vector is constant.

For the boundary data $g \in H^{1/2}(\partial\Omega)$, we define $V_g := \{v \in H^1(\Omega) : v\big|_{\partial\Omega} = g\}$. Then, a weak formulation of problem (1) reads:

Find $u \in V_g$ such that

$$a(u,v) := (A\nabla u, \nabla v)_\Omega + (b \cdot \nabla u + cu, v)_\Omega = (f, v)_\Omega \qquad \forall v \in H_0^1(\Omega). \tag{30}$$

If the data is sufficiently smooth and the assumptions (2) are satisfied, one can prove that there exists a unique solution of problem (30), see Theorem 6 below.

On the space $H^{1,h}(\Omega)$ defined in (7), we introduce the seminorms

$$|v|_{1,h} := \Big(\sum_{K\in\mathcal{T}_h} |v|_{1,K}^2 \Big)^{1/2} \qquad \text{and} \qquad \|v\|_h := \Big(|v|_{1,h}^2 + \big(|c|v, v\big)_\Omega \Big)^{1/2}. \tag{31}$$

Note that there may exist patches of elements where $c = 0$ such that $\|\cdot\|_h$ in general is not a norm on $H^{1,h}(\Omega)$. Using the compatibility condition (12) for an arbitrary fixed $v_h \in V_h$ and $\alpha = (0,\ldots,0) \in \mathcal{A}_{r-1}^{(d-1)}$, we obtain that for each inner face $E \in \mathcal{E}_h^i$ there exists a point $x_E \in \overline{E}$ such that $v_h\big|_{K(E)}(x_E) = v_h\big|_{K'(E)}(x_E)$. Furthermore, we get that for an arbitrary fixed $v_h \in V_{0,h}$ and each boundary face $E \in \mathcal{E}_h^b$, there exists a point $x_E \in \overline{E}$ such that $v_h\big|_{K(E)}(x_E) = 0$. With these properties we can prove the following discrete Poincaré-Friedrichs inequality

$$\|v_h\|_{0,\Omega} \le C_2 |v_h|_{1,h} \qquad \forall v_h \in V_{0,h} + H_0^1(\Omega), \tag{32}$$

which shows that $|\cdot|_{1,h}$ is a norm on the space $V_{0,h}$. The idea of the proof is that a conforming $\mathbb{Q}_1$-interpolate of v_h is constructed by means of the Scott-Zhang operator and that the continuous version of the Poincaré-Friedrichs inequality is applied to that interpolate. Moreover, we get for $c \in L^\infty(\Omega)$ that

$$|v_h|_{1,h} \le \|v_h\|_h \le C_3 |v_h|_{1,h} \qquad \forall v_h \in V_{0,h} + H_0^1(\Omega) \tag{33}$$

holds true. Using similar arguments as for the proof of (32), we can also prove that

$$\|u - v_h\|_{0,\Omega} \le C |u - v_h|_{1,h} \qquad \forall u \in V_g, \ \forall v_h \in V_{g,h}. \tag{34}$$

This shows that the convergence of the finite element method in the seminorm $\|\cdot\|_h$ implies its convergence in the L^2-norm.

Let us define the bilinear form

$$a_h(u,v) := \sum_{K\in\mathcal{T}_h} \Big\{ (A\nabla u, \nabla v)_K + (b\cdot\nabla u + cu, v)_K \Big\} \qquad \forall u, v \in H^{1,h}(\Omega). \tag{35}$$

Then, the discrete problem reads:

Find $u_h \in V_{g,h}$ such that

$$a_h(u_h, v_h) = (f, v)_\Omega \qquad \forall v_h \in V_{0,h}. \tag{36}$$

Existence and uniqueness of a discrete solution u_h will be proved in Theorem 6. The estimate of the discretization error in Lemma 5 contains a term with the consistency error functional

$$e(u, v_h) := a_h(u, v_h) - (f, v_h)_\Omega.$$

In the following, we will estimate this functional. Under the assumptions that A is sufficiently smooth and $u \in V_g \cap H^2(\Omega)$, we obtain for each $v_h \in V_{0,h} + H_0^1(\Omega)$

$$\begin{aligned} e(u, v_h) &= \sum_{K \in \mathcal{T}_h} \Big\{ (A\nabla u, \nabla v_h)_K + (b \cdot \nabla u + cu, v_h)_K - \big(- \operatorname{div}(A\nabla u) + b \cdot \nabla u + cu, v_h \big)_K \Big\} \\ &= \sum_{K \in \mathcal{T}_h} \sum_{E \in \mathcal{E}(K)} \Big\langle \left(n_E^K\right)^T A\nabla u, v_h \Big\rangle_E \\ &= \sum_{E \in \mathcal{E}_h^i} \underbrace{\big\langle (n_E)^T A\nabla u, [v_h]_E \big\rangle_E}_{=:\, e_E^{(i)}(u,v_h)} + \sum_{E \in \mathcal{E}_h^b} \underbrace{\Big\langle (n_E)^T A\nabla u, v_h\big|_{K(E)} \Big\rangle_E}_{=:\, e_E^{(b)}(u,v_h)}. \end{aligned} \tag{37}$$

In order to estimate $e_E^{(i)}(u, v_h)$ and $e_E^{(b)}(u, v_h)$, we have to compute surface integrals. For a fixed face $E \in \mathcal{E}_h$, the parameterization $T_E : U_2 \to E$ is given by (11) with $\widehat{T}_{\widehat{E}} : U_2 \to \widehat{E}$ defined in (18). Then, the unit normal n_E which points outward with respect to $K(E)$ is given by

$$n_E(\xi) = \frac{\widetilde{n}_E(\xi)}{\|\widetilde{n}_E(\xi)\|} \qquad \text{with} \qquad \widetilde{n}_E(\xi) := -\left(\frac{\partial T_E}{\partial \xi_1} \times \frac{\partial T_E}{\partial \xi_2} \right)(\xi). \tag{38}$$

An important ingredient of our analysis is the definition of a smooth function $\widehat{n}_{\widehat{E}} : \overline{\widehat{K}} \to \mathbb{R}^3$ such that

$$\widehat{n}_{\widehat{E}}\big(\widehat{T}_{\widehat{E}}(\xi)\big) = \widetilde{n}_E(\xi) \qquad \forall \xi \in U_2.$$

To this end, let $\xi_{\widehat{E}} : \overline{\widehat{K}} \to U_2$ be a smooth mapping with the property

$$\xi_{\widehat{E}}\big(\widehat{T}_{\widehat{E}}(\xi)\big) = \xi \qquad \forall \xi \in U_2.$$

As an example consider the face $\widehat{E} = \{\widehat{x} \in \overline{\widehat{K}} : \widehat{x}_2 = 1\}$. Here, we have $\widehat{T}_{\widehat{E}}(\xi_1, \xi_2) = (\xi_1, 1, \xi_2)^T \in \widehat{E}$ for all $\xi \in U_2$ and can choose

$$\xi_{\widehat{E}}(\widehat{x}) = \xi_{\widehat{E}}(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) := (\widehat{x}_1, \widehat{x}_3)^T \in U_2 \qquad \forall \widehat{x} \in \overline{\widehat{K}}.$$

The mappings $\xi_{\widehat{E}}$ for the other faces $\widehat{E}$ are constructed analogously. Now, with a given mapping $\xi_{\widehat{E}}$, we define the smooth function $\widehat{n}_{\widehat{E}}$ by

$$\widehat{n}_{\widehat{E}}(\widehat{x}) := \widetilde{n}_E\big(\xi_{\widehat{E}}(\widehat{x})\big) \qquad \forall \widehat{x} \in \overline{\widehat{K}}. \tag{39}$$

In order to estimate $e_E^{(i)}(u, v_h)$ and $e_E^{(b)}(u, v_h)$ in (37), we need the following two lemmata.

Lemma 3. *Let $\{\mathcal{T}_h\}$ be a family of multilevel grids generated from a coarse grid $\mathcal{T}^0$ and let $E \in \mathcal{E}_h$ be an arbitrary face, $K := K(E)$ the corresponding element and $\widehat{E} := F_K^{-1}(E)$. Then, the estimates*

$$\begin{aligned} |\,\widehat{n}_{\widehat{E}}\,|_{m,\infty,\widehat{K}} &\le C(\mathcal{T}^0) h_K^{2+m} && \text{for } m \in \{0,1\}, \\ |\,\widehat{n}_{\widehat{E}}\,|_{m,\infty,\widehat{K}} &= 0 && \text{for } m \ge 2, \end{aligned} \tag{40}$$

hold for the vector-valued function $\widehat{n}_{\widehat{E}}$ defined in (39).

Proof. We will restrict the proof to the case where the face $\widehat{E}$ is given by $\widehat{E} = \{\widehat{x} \in \overline{\widehat{K}} : \widehat{x}_2 = 1\}$. The other cases are treated analogously. In our case, we have $\widehat{T}_{\widehat{E}}(\xi) = (\xi_1, 1, \xi_2)^T$ for all $\xi \in U_2$, $\xi_{\widehat{E}}(\widehat{x}) = (\widehat{x}_1, \widehat{x}_3)^T$ for all $\widehat{x} \in \overline{\widehat{K}}$ and

$$\widetilde{n}_E(\xi) = -\frac{\partial F_K\big(\widehat{T}_{\widehat{E}}(\xi)\big)}{\partial \xi_1} \times \frac{\partial F_K\big(\widehat{T}_{\widehat{E}}(\xi)\big)}{\partial \xi_2} = -\frac{\partial G_K(\xi)}{\partial \xi_1} \times \frac{\partial G_K(\xi)}{\partial \xi_2},$$

where $G_K(\xi) := F_K(\xi_1, 1, \xi_2)$. Let $a_1 := F_K(-1,1,-1)$, $a_2 := F_K(1,1,-1)$, $a_3 := F_K(1,1,1)$ and $a_4 := F_K(-1,1,1)$ be the vertices of the face E of K and let

$$\begin{aligned} b_E &:= \tfrac{1}{4}(a_1 + a_2 + a_3 + a_4), \qquad & r_1 &:= \tfrac{1}{4}(-a_1 + a_2 + a_3 - a_4), \\ r_2 &:= \tfrac{1}{4}(-a_1 - a_2 + a_3 + a_4), \qquad & d_E &:= \tfrac{1}{4}(a_1 - a_2 + a_3 - a_4). \end{aligned}$$

Then, we have

$$G_K(\xi_1, \xi_2) = b_E + r_1\xi_1 + r_2\xi_2 + d_E\xi_1\xi_2$$

and

$$\widetilde{n}_E(\xi) = -\,(r_1 + d_E\xi_2) \times (r_2 + d_E\xi_1) = -(r_1 \times r_2) - (r_1 \times d_E)\xi_1 + (r_2 \times d_E)\xi_2.$$

This yields

$$\widehat{n}_{\widehat{E}}(\widehat{x}) = \widetilde{n}_E\big(\xi_{\widehat{E}}(\widehat{x})\big) = \widetilde{n}_E(\widehat{x}_1, \widehat{x}_3) = -(r_1 \times r_2) - (r_1 \times d_E)\widehat{x}_1 + (r_2 \times d_E)\widehat{x}_3, \tag{41}$$

which implies (40) for $m \geq 2$. By the triangle inequality and $\|a_i - a_j\| \leq h_K$ for $i, j \in \{1, \ldots, 4\}$, we obtain

$$\|r_i\| \leq \frac{1}{2} h_K \qquad i = 1, 2.$$

Furthermore, using Lemma 1 we get

$$\|d_E\| = \left\| \frac{\partial^2 G_K}{\partial \xi_1 \partial \xi_2} \right\| = \left\| \frac{\partial^2 F_K(\xi_1, 1, \xi_2)}{\partial \xi_1 \partial \xi_2} \right\| \leq |F_K|_{2,\infty,\widehat{K}} \leq C_1(2, \mathcal{T}^0) h_K^2.$$

Now, using the estimates for r_i and d_E, the estimate (40) for $m \in \{0, 1\}$ follows from (41). $\square$

Lemma 4. *Assume that $\{\mathcal{T}_h\}$ is a family of multilevel grids generated from a coarse grid $\mathcal{T}^0$. Let $v_h \in V_h + H^1(\Omega)$ be a function which satisfies the compatibility conditions* (13) *and* (16) *of order k. Furthermore, let μ be an integer with $1 \leq \mu \leq k+1$ and, for a face $E \in \mathcal{E}_h$, let $K \in \mathcal{T}_h$ denote an element such that $E \in \mathcal{E}(K)$. Then, there hold the estimates*

$$\big| \big\langle\, n_E^T \psi, [v_h]_E \,\big\rangle_E \big| \leq C(\mathcal{T}^0) h_K^\mu \|\psi\|_{\mu,K} \left\{ |v_h|_{1,K(E)} + |v_h|_{1,K'(E)} \right\} \quad \forall E \in \mathcal{E}_h^i, \tag{42}$$

$$\big| \big\langle\, n_E^T \psi, v_h \,\big\rangle_E \big| \leq C(\mathcal{T}^0) h_K^\mu \|\psi\|_{\mu,K} |v_h|_{1,K} \qquad \forall E \in \mathcal{E}_h^b \tag{43}$$

for all vector-valued functions $\psi \in (H^\mu(K))^3$.

Proof. At first we prove the estimate (42). A parameterization of E yields

$$T := \big\langle\, n_E^T \psi, [v_h]_E \,\big\rangle_E = \int_{U_2} \widetilde{n}_E(\xi)^T\, \psi\big(T_E(\xi)\big)\, \big[v_h\big(T_E(\xi)\big)\big]_E \, d\xi. \tag{44}$$

We introduce the function $w : U_2 \to \mathbb{R}$ as

$$w(\xi) := \big(\widetilde{n}_E(\xi)\big)^T \, \psi(T_E(\xi)) \tag{45}$$

and the function $\widehat{w} : \widehat{K} \to \mathbb{R}$ as

$$\widehat{w}(\widehat{x}) := \big(\widehat{n}_{\widehat{E}}(\widehat{x})\big)^T \, \widehat{\psi}(\widehat{x}), \tag{46}$$

where the function $\widehat{\psi}$ is defined by $\widehat{\psi}(\widehat{x}) := \psi\big(F_K(\widehat{x})\big)$. Due to the above definitions we have

$$\widehat{w}\big(\widehat{T}_{\widehat{E}}(\xi)\big) = w(\xi) \qquad \forall \xi \in U_2. \tag{47}$$

In the following, let $\widehat{q}$ denote the $L^2(\widehat{K})$-projection of $\widehat{w}$ in the space $\mathbb{P}_{\mu-1}(\widehat{K})$. Then, the polynomial q, defined as $q(\xi) := \widehat{q}\big(\widehat{T}_{\widehat{E}}(\xi)\big)$ for $\xi \in U_2$, is contained in the space $\mathbb{P}_{\mu-1}(U_2)$ since $\widehat{T}_{\widehat{E}}$ is an affine mapping. Furthermore, let us introduce the notation

$$v_E(\xi) := v_h\big|_{K(E)}(T_E(\xi)), \qquad v'_E(\xi) := v_h\big|_{K'(E)}(T_E(\xi)), \tag{48}$$

$$c_E := \frac{1}{4}\int_{U_2} v_E(\xi)\, d\xi = \frac{1}{4}\int_{U_2} v'_E(\xi)\, d\xi, \tag{49}$$

where the last equation is a consequence of (13). Then, from (44) and the compatibility condition (13) of order $k \geq \mu - 1$, we obtain

$$\begin{aligned} |T| &= \left| \int_{U_2} \big(w(\xi) - q(\xi)\big)\, \big(v_E(\xi) - v'_E(\xi)\big)\, d\xi \right| \\ &\leq \|w - q\|_{0,U_2} \left\{ \|v_E - c_E\|_{0,U_2} + \left\|v'_E - c_E\right\|_{0,U_2} \right\}. \end{aligned} \tag{50}$$

Let $\widehat{v}$ be the function $\widehat{v}(\widehat{x}) := v_h\big|_{K(E)}(F_{K(E)}(\widehat{x}))$. Using the special structure of the parameterization $\widehat{T}_{\widehat{E}} : U_2 \to \widehat{E}$ of the face $\widehat{E}$ (see (18)), we get

$$\|v_E - c_E\|^2_{0,U_2} = \int_{U_2} \big(v_E(\xi) - c_E\big)^2\, d\xi = \int_{U_2} \big(\widehat{v}(\widehat{T}_{\widehat{E}}(\xi)) - c_E\big)^2\, d\xi = \|\widehat{v} - c_E\|^2_{0,\widehat{E}}.$$

Applying the trace theorem for the face $\widehat{E}$ of $\widehat{K}$ and using the fact that $\widehat{v} - c_E = 0$ for a constant function $\widehat{v}$, we obtain with the Bramble–Hilbert Lemma

$$\|v_E - c_E\|_{0,U_2} \leq C\left\{ \|\widehat{v} - c_E\|_{0,\widehat{K}} + |\widehat{v} - c_E|_{1,\widehat{K}} \right\} \leq C|\widehat{v}|_{1,\widehat{K}} \leq C h^{1-3/2}_{K(E)} |v_h|_{1,K(E)}. \tag{51}$$

If we take $\widehat{v} := v_h\big|_{K'(E)} \circ F_{K'(E)}$ and apply (27), we have

$$v'_E(\xi) = \widehat{v}\big(\widehat{T}_{\widehat{E}'}(\xi')\big) \qquad \forall \xi \in U_2,$$

where $\widehat{E}' := F^{-1}_{K'(E)}(E)$ is the reference face of $\widehat{K}$, which corresponds to the face E in $K'(E)$, and ξ' is defined in dependence of ξ like in (27). Then, we obtain in an analogous way as above

$$\|v'_E - c_E\|_{0,U_2} \leq C h^{1-3/2}_{K'(E)} |v_h|_{1,K'(E)}. \tag{52}$$

It remains to estimate the difference $w-q$ in (50). We get

$$\|w-q\|_{0,U_2}^2 = \int_{U_2} \big(w(\xi)-q(\xi)\big)^2\,d\xi = \|\widehat{w}-\widehat{q}\|_{0,\widehat{E}}^2,$$

which implies by means of the trace theorem and the Bramble–Hilbert Lemma

$$\|w-q\|_{0,U_2} \le C\left\{\|\widehat{w}-\widehat{q}\|_{0,\widehat{K}} + |\widehat{w}-\widehat{q}|_{1,\widehat{K}}\right\} \le C|\widehat{w}|_{\mu,\widehat{K}}. \tag{53}$$

Summarizing (50), (51), (52) and (53) and using the fact that, due to the shape-regularity of the grid, it holds

$$Ch_{K(E)} \le h_{K'(E)} \le C'h_{K(E)} \qquad \forall E\in\mathcal{E}_h^i,$$

we obtain

$$|T| \le Ch_K^{1-3/2}|\widehat{w}|_{\mu,\widehat{K}}\left\{|v_h|_{1,K(E)} + |v_h|_{1,K'(E)}\right\}. \tag{54}$$

Now, from the definition of $\widehat{w}$ in (46) and Lemma 3, it follows

$$\begin{aligned}|\widehat{w}|_{\mu,\widehat{K}} &\le C\|\widehat{n}_{\widehat{E}}\|_{0,\infty,\widehat{K}}|\widehat{\psi}|_{\mu,\widehat{K}} + C|\widehat{n}_{\widehat{E}}|_{1,\infty,\widehat{K}}|\widehat{\psi}|_{\mu-1,\widehat{K}}\\ &\le C(\mathcal{T}^0)\left\{h_K^2|\widehat{\psi}|_{\mu,\widehat{K}} + h_K^3|\widehat{\psi}|_{\mu-1,\widehat{K}}\right\}.\end{aligned} \tag{55}$$

Since the reference mapping F_K is smooth, the vector function $\widehat{\psi} := \psi\circ F_K$ is in the space $(H^m(\widehat{K}))^3$ iff the vector function ψ is in $(H^m(K))^3$. Moreover, since $|F_K|_{k,\infty,\widehat{K}} = 0$ for $k\ge 4$, it holds

$$|\widehat{\psi}|_{m,\widehat{K}} \le C\|\det D(F_K^{-1})\|_{0,\infty,K}^{1/2}\|\psi\|_{m,K}\sum_{\substack{\alpha+2\beta+3\gamma\\=m}}|F_K|_{1,\infty,\widehat{K}}^{\alpha}|F_K|_{2,\infty,\widehat{K}}^{\beta}|F_K|_{3,\infty,\widehat{K}}^{\gamma},$$

see [7, p. 222], where α,β,γ denote nonnegative integers. Using (4) and Lemma 1, we get

$$|\widehat{\psi}|_{m,\widehat{K}} \le Ch_K^{m-3/2}\|\psi\|_{m,K} \tag{56}$$

for each nonnegative integer m. Together with (55) this implies the estimate

$$|\widehat{w}|_{\mu,\widehat{K}} \le C(\mathcal{T}^0)h_K^{\mu+2-3/2}\|\psi\|_{\mu,K}.$$

Therefore, using (44) and (54), we obtain (42).

The estimate (43) can be proved in an analogous way as (42). The only modification is that there is no element $K'(E)$ which can formally be handled by $v_h\big|_{K'(E)} = 0$ and $c_E = 0$. $\square$

Now we come back to the estimation of $e_E^{(i)}(u,v_h)$ in (37). We apply Lemma 4 with $\psi := A\nabla u$, $k = r-1$, $\mu = r$ and obtain for sufficiently smooth A and for each inner face $E\in\mathcal{E}_h^i$

$$\left|e_E^{(i)}(u,v_h)\right| \le C(\mathcal{T}^0)h_{K(E)}^r\|u\|_{r+1,K(E)}\left\{|v_h|_{1,K(E)} + |v_h|_{1,K'(E)}\right\}. \tag{57}$$

For each boundary face $E\in\mathcal{E}_h^b$, we get with the same setting the estimate

$$\left|e_E^{(b)}(u,v_h)\right| \le C(\mathcal{T}^0)h_{K(E)}^r\|u\|_{r+1,K(E)}|v_h|_{1,K(E)}. \tag{58}$$

Finally, we combine (37) with (57) and (58), apply the Cauchy-Schwarz inequality and obtain the following estimate for all $v_h \in V_{0,h} + H_0^1(\Omega)$

$$\begin{aligned} e(u, v_h) &\le C(\mathcal{T}^0)\Big\{ \sum_{K\in\mathcal{T}_h} h_K^{2r} \|u\|_{r+1,K}^2 \Big\}^{1/2} \Big\{ \sum_{K\in\mathcal{T}_h} |v_h|_{1,K}^2 \Big\}^{1/2} \\ &\le C(\mathcal{T}^0) h^r \|u\|_{r+1,\Omega}\, \|v_h\|_h. \end{aligned} \tag{59}$$

For the analysis of the discrete problem, we need the following statements.

Lemma 5. *Assume that the coefficients A, b and c in problem* (1) *are sufficiently smooth and satisfy the conditions* (2). *Let $\{\mathcal{T}_h\}$ be a family of multilevel grids generated from a shape regular coarse grid $\mathcal{T}^0$ and let $V_{g,h}$, $V_{0,h}$ be the finite element spaces defined in Section 3. Then, there exist a sufficiently small mesh size h_0 and constants C_4, C_5 such that*

$$a_h(v_h, v_h) \ge C_4 \|v_h\|_h^2 \qquad \forall v_h \in V_{0,h},\ h \le h_0, \tag{60}$$

$$a_h(w, v_h) \le C_5\, \|w\|_h\, \|v_h\|_h \qquad \forall w \in H^{1,h}(\Omega),\ \ v_h \in V_{0,h} + H_0^1(\Omega), \tag{61}$$

and

$$\|u - u_h\|_h \le C \left\{ \inf_{v_h \in V_{g,h}} \|u - v_h\|_h + \sup_{v_h \in V_{0,h}} \frac{a_h(u, v_h) - (f, v_h)_\Omega}{\|v_h\|_h} \right\}, \tag{62}$$

where u and u_h denote the solutions of problem (30) *and* (36), *respectively.*

Proof. To show (60), we choose an arbitrary $v_h \in V_{0,h}$. Then, we have

$$\begin{aligned} a_h(v_h, v_h) &= \sum_{K\in\mathcal{T}_h} \Big\{ (A\nabla v_h, \nabla v_h)_K + \Big(c - \frac{1}{2}\mathrm{div}\, b, v_h^2\Big)_K + \frac{1}{2} \sum_{E\in\mathcal{E}(K)} \big\langle\, b\cdot n_E^K, v_h^2 \,\big\rangle_E \Big\} \\ &\ge a_0 |v_h|_{1,h}^2 + \frac{1}{2} \sum_{E\in\mathcal{E}_h^i} \big\langle\, b\cdot n_E, [v_h^2]_E \,\big\rangle_E + \frac{1}{2} \sum_{E\in\mathcal{E}_h^b} \big\langle\, b\cdot n_E, v_h^2 \,\big\rangle_E . \end{aligned} \tag{63}$$

In the following, we will estimate the term

$$\beta_E(v_h) := \big\langle\, b\cdot n_E, [v_h^2]_E \,\big\rangle_E$$

for an inner face $E \in \mathcal{E}_h^i$. We have

$$\beta_E(v_h) = \underbrace{\big\langle\, n_E^T\, b\, v_h|_{K(E)}, [v_h]_E \,\big\rangle_E}_{=:\ \gamma_E(v_h)} + \underbrace{\big\langle\, n_E^T\, b\, v_h|_{K'(E)}, [v_h]_E \,\big\rangle_E}_{=:\ \gamma'_E(v_h)} \tag{64}$$

In order to estimate the term $\gamma_E(v_h)$, we apply Lemma 4 with $\psi := b\, v_h|_{K(E)}$, $K := K(E)$, $k = r-1 \ge 0$, $\mu = 1$ and obtain for sufficiently smooth b

$$|\gamma_E(v_h)| \le C(\mathcal{T}^0) h_{K(E)} \big\{ \|v_h\|_{0,K(E)} + |v_h|_{1,K(E)} \big\} \big\{ |v_h|_{1,K(E)} + |v_h|_{1,K'(E)} \big\}. \tag{65}$$

We obtain the estimate

$$\big|\gamma'_E(v_h)\big| \le C(\mathcal{T}^0) h_{K(E)} \big\{ \|v_h\|_{0,K'(E)} + |v_h|_{1,K'(E)} \big\} \big\{ |v_h|_{1,K(E)} + |v_h|_{1,K'(E)} \big\} \tag{66}$$

in an analogous way. Now, we apply the estimates (65) and (66) in (64), the Cauchy-Schwarz inequality and the Poincaré-Friedrichs inequality (32) to get

$$\sum_{E\in\mathcal{E}_h^i} |\beta_E(v_h)| \leq C(\mathcal{T}^0)h\big\{\|v_h\|_{0,\Omega}|v_h|_{1,h} + |v_h|_{1,h}^2\big\} \leq C(\mathcal{T}^0)h|v_h|_{1,h}^2. \tag{67}$$

We can also prove with nearly the same arguments

$$\sum_{E\in\mathcal{E}_h^b} \big|\big\langle\, b\cdot n_E, v_h^2\,\big\rangle_E\big| \leq C(\mathcal{T}^0)h|v_h|_{1,h}^2. \tag{68}$$

If we apply the estimates (67) and (68) in (63), we obtain

$$a_h(v_h,v_h) \geq \big(a_0 - C(\mathcal{T}^0)h\big)|v_h|_{1,h}^2 \geq \big(a_0 - C(\mathcal{T}^0)h\big)C_3^{-2}\|v_h\|_h^2.$$

This means that the estimate (60) is satisfied with $C_4 = a_0C_3^{-2}/2$ for all $h \leq h_0 := a_0/(2C(\mathcal{T}^0))$.

The continuity (61) of $a_h(\cdot,\cdot)$ is obtained as follows

$$a_h(w,v_h) \leq a_1|w|_{1,h}|v_h|_{1,h} + \|b\|_{0,\infty,\Omega}|w|_{1,h}\|v_h\|_{0,\Omega} + \|w\|_h\|v_h\|_h \leq C_5\|w\|_h\|v_h\|_h,$$

where the last inequality is a consequence of (33) and the discrete Poincaré-Friedrichs inequality (32).

Using (60), (61) and the second Lemma of Strang, we get the error estimate (62). □

Now, we can state our main result.

Theorem 6. *Assume that A, b, c, f, and g in problem* (1) *are sufficiently smooth and satisfy the conditions* (2)*. Then, there exists a unique solution $u \in V_g$ of problem* (30)*. Assume that this solution satisfies $u \in H^{r+1}(\Omega)$. Let the sequence of grids $\{\mathcal{T}_h\}$ be a family of multilevel grids generated from a shape-regular coarse grid $\mathcal{T}^0$ and let $V_{g,h}$, $V_{0,h}$ be the finite element spaces of order r defined in Section 3. Then, there exists a sufficiently small mesh size h_0 such that for all $h \leq h_0$ there exists a unique solution u_h of the discrete problem* (36) *which satisfies the error estimate*

$$|u-u_h|_{1,h} \leq \|u-u_h\|_h \leq C(\mathcal{T}^0)h^r\|u\|_{r+1,\Omega}. \tag{69}$$

Proof. Let $\widetilde{g} \in V_g$ be an extension of the boundary function g and let $u_0 := u - \widetilde{g}$. Then, the problem (30) for u is equivalent to the following problem:

Find $u_0 \in H_0^1(\Omega)$ such that

$$a(u_0,v) = f_0(v) := (f,v)_\Omega - a(\widetilde{g},v) \qquad \forall v \in H_0^1(\Omega).$$

The bilinear form $a(\cdot,\cdot)$ is coercive since

$$a(v,v) = (A\nabla v,\nabla v)_\Omega + \left(c - \frac{1}{2}\operatorname{div} b, v^2\right)_\Omega \geq a_0|v|_{1,\Omega}^2 \qquad \forall v \in H_0^1(\Omega)$$

and $|\cdot|_{1,\Omega}$ is a norm on $H_0^1(\Omega)$. Moreover, $a(\cdot,\cdot)$ is continuous since

$$a(w,v) \leq a_1|w|_{1,\Omega}|v|_{1,\Omega} + \|b\|_{0,\infty,\Omega}|w|_{1,\Omega}\|v\|_{0,\Omega} + \|c\|_{0,\infty,\Omega}\|w\|_{0,\Omega}\|v\|_{0,\Omega} \leq C|w|_{1,\Omega}|v|_{1,\Omega}.$$

holds for all $w, v \in H_0^1(\Omega)$. The functional $f_0(\cdot)$ is continuous on $H_0^1(\Omega)$. Thus, there exists by the Lax–Milgram Theorem a unique solution u_0 which implies that there is also a unique solution u of (30).

The arguments for the discrete problem are nearly the same. The bilinear form $a_h(\cdot,\cdot)$ is coercive and continuous on $V_{0,h}$ which has been shown in Lemma 5. We get by the Lax–Milgram Theorem the existence and uniqueness of a solution $u_{0,h}$ of a corresponding discrete problem with homogeneous boundary conditions which implies the existence and uniqueness of a solution u_h of the original discrete problem.

The error estimate (69) is a consequence of the estimates (62), (59) and the interpolation error estimate given in [15, Lemma 9]. □

At the end of this section, we will prove an optimal L^2 error estimate using a duality argument. For a given $\eta \in L^2(\Omega)$, let φ denote the solution of

$$\begin{aligned} -\mathrm{div}(A\nabla\varphi) + b^* \cdot \nabla\varphi + c^*\varphi &= \eta \qquad && \text{in } \Omega, \\ \varphi &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{70}$$

which is the adjoint problem to (1) where

$$b^* := -b, \qquad c^* := c - \operatorname{div} b \tag{71}$$

are sufficiently smooth coefficient functions. We assume that the problem (70) is H^2-regular, i.e., there exists a positive constant C^* such that

$$\|\varphi\|_{2,\Omega} \le C^* \|\eta\|_{0,\Omega} \tag{72}$$

holds for all associated pairs $\eta \in L^2(\Omega)$ and φ. In the case that the domain Ω is convex, this regularity condition is satisfied.

Theorem 7. *Let the assumptions of Theorem 6 be satisfied. Assume that the dual problem* (70) *is H^2-regular in the sense of* (72)*. Then, the solution u_h of the discrete problem* (36) *satisfies the error estimate*

$$\|u - u_h\|_{0,\Omega} \le C(\mathcal{T}^0) h^{r+1} \|u\|_{r+1,\Omega} \tag{73}$$

for $h \le h_0$ with h_0 from Theorem 6.

Proof. In the following, we do not mention the dependency of the constants on the coarsest grid $\mathcal{T}^0$. Let $\eta \in L^2(\Omega)$ be defined as $\eta := u - u_h$ and $\varphi \in H^2(\Omega)$ the solution of (70). We obtain from (70) and (71) after integration by parts

$$\begin{aligned} (\eta,\eta)_\Omega &= (\eta, -\mathrm{div}(A\nabla\varphi) + b^*\nabla\varphi + c^*\varphi)_\Omega \\ &= \sum_{K\in\mathcal{T}_h} \Big\{ (A\nabla\eta, \nabla\varphi)_K + (b\cdot\nabla\eta + c\eta, \varphi)_K - \big\langle n_K^T A\nabla\varphi, \eta \big\rangle_{\partial K} - \big\langle n_K^T\, b\,\varphi, \eta \big\rangle_{\partial K} \Big\} \\ &= a_h(\eta,\varphi) - \sum_{E\in\mathcal{E}_h^b} \langle n_E^T A\nabla\varphi, \eta\rangle_E - \sum_{E\in\mathcal{E}_h^b} \langle n_E^T\, b\,\varphi, \eta\rangle_E \\ &\qquad - \sum_{E\in\mathcal{E}_h^i} \langle n_E^T A\nabla\varphi, [\eta]_E\rangle_E - \sum_{E\in\mathcal{E}_h^i} \langle n_E^T\, b\,\varphi, [\eta]_E\rangle_E \end{aligned} \tag{74}$$

where the symmetry of A was exploited. The five terms in (74) will be estimated separately. We obtain for the first term

$$a_h(\eta, \varphi) = a_h(\eta, \varphi - \varphi_h) + a_h(\eta, \varphi_h) \tag{75}$$

where $\varphi_h \in V_{0,h}$ is a finite element interpolant of $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying

$$\|\varphi - \varphi_h\|_h \leq C\, h \|\varphi\|_{2,\Omega}. \tag{76}$$

Using the continuity of a_h, Theorem 6 and (76), we get

$$|a_h(\eta, \varphi - \varphi_h)| \leq C_5 \|\eta\|_h \,\|\varphi - \varphi_h\|_h \leq C\, h^{r+1} \|u\|_{r+1,\Omega} \,\|\varphi\|_{2,\Omega}$$

The second term in (75) is just $e(u, \varphi_h)$. Due to the homogeneous boundary conditions of φ, we have

$$e(u, \varphi_h) = e(u, \varphi_h - \varphi)$$

which can be estimated by (59) and (76) to obtain

$$|e(u, \varphi_h)| \leq C\, h^r \|u\|_{r+1,\Omega} \,\|\varphi_h - \varphi\|_h \leq C\, h^{r+1} \|u\|_{r+1,\Omega} \,\|\varphi\|_{2,\Omega}$$

Hence, the expression in (75) can be estimated as follows

$$|a_h(\eta, \varphi)| \leq C\, h^{r+1} \|u\|_{r+1,\Omega} \|\varphi\|_{2,\Omega}. \tag{77}$$

Each term in the four sums in (74) can be estimated by Lemma 4 with $\mu = 1$, $k = r - 1 \geq 0$, $\psi = A\nabla\varphi$ or $\psi = b\varphi$ using the fact that $\eta \in V_h + H^1(\Omega)$ satisfies the compatibility conditions (13) and (16) of order k. If we combine these estimates and use

$$\|A\nabla\varphi\|_{1,K} \leq C\|\varphi\|_{2,K}, \qquad \|b\varphi\|_{1,K} \leq C\|\varphi\|_{2,K},$$

we obtain

$$\begin{aligned}
&\left| \sum_{E\in\mathcal{E}_h^i} \langle n_E^T A\nabla\varphi, [\eta]_E\rangle_E + \sum_{E\in\mathcal{E}_h^i} \langle n_E^T\, b\,\varphi, [\eta]_E\rangle_E + \sum_{E\in\mathcal{E}_h^b} \langle n_E^T A\nabla\varphi, \eta\rangle_E + \sum_{E\in\mathcal{E}_h^b} \langle n_E^T\, b\,\varphi, \eta\rangle_E \right| \\
&\leq C\, h \|\varphi\|_{2,\Omega} |\eta|_{1,h} \leq C\, h^{r+1} \|u\|_{r+1,\Omega} \|\varphi\|_{2,\Omega}.
\end{aligned} \tag{78}$$

Using (74), (77), (78), and the H^2-regularity (72), we get

$$(\eta, \eta)_\Omega \leq C\, h^{r+1} \|u\|_{r+1,\Omega} \|\eta\|_{0,\Omega}$$

which results in the L^2-error estimate (73). □

5 Numerical results

We will present in this section numerical results for a three-dimensional test problem. All calculations were carried out on one processor of the HP 9000 Superdome Server (3 GFlops) by using the program package MooNMD [12].

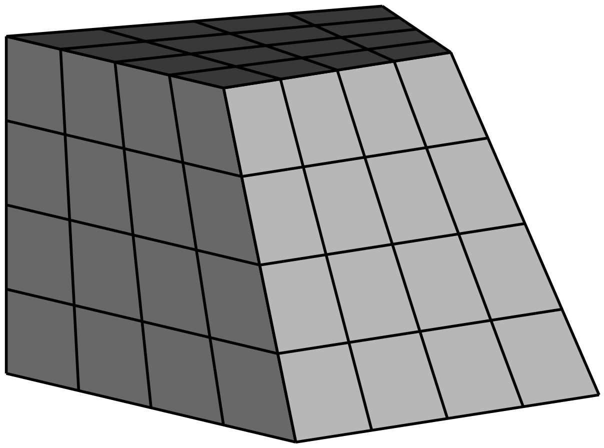

Figure 2: *Multilevel grid of the computational domain on grid level 2.*

Let $\Omega \subset \mathbb{R}^3$ be the domain which is bounded by the following six planes

$$\begin{aligned} z &= 0, \qquad & y &= 0, \qquad & 3x + y + z &= 8, \\ z &= 2, \qquad & x &= 0, \qquad & x + 3y + z &= 8. \end{aligned}$$

Note that none of the faces is a parallelogram such that the elements on all grids are not affine equivalent. The computational domain Ω together with the grid on level 2 is shown in Fig. 2.

We consider problem (1) where A is the identity matrix,

$$b = \big(z - 1, 1 - x, y - 1\big)^T, \qquad c = 0$$

and the right-hand side f as well as the Dirichlet boundary data g are chosen such that

$$u(x, y, z) = \cos x \, \sin 2y + \cos 3z$$

is the solution of (1). The assumptions in (2) are fulfilled with $a_0 = a_1 = 1$.

The discrete system (36) was solved by a flexible GMRES method [20] with a multigrid preconditioner. The multigrid hierarchy is based on the grid sequence $\mathcal{T}^0, \ldots, \mathcal{T}^\ell = \mathcal{T}_h$ which was described in Sect. 2.2. The initial grid $\mathcal{T}^0$ consists of a single hexahedral element which coincides with the domain Ω. For details on multigrid methods for nonconforming finite elements, we refer to [2, 11]. The used prolongation and restriction operators are based on the general interpolation operator described in [22].

We will compare the results for conforming and nonconforming finite elements of the same order with respect to the number of degrees of freedom, the number of non-zero matrix entries, the obtained accuracy, and the used computing time. Moreover, we will show that the use of higher order finite elements allows to obtain the same accuracy in a shorter time and with less memory compared to lower order finite elements.

Table 1 shows for conforming and nonconforming finite elements ($\mathbb{Q}_r$ and $\widetilde{\mathbb{Q}}_r$) of different orders r on different grid levels the number of degrees of freedom, the number of non-zero matrix entries, the obtained accuracy in the broken H^1-seminorm $|\cdot|_{1,h}$ and the L^2-norm,

Table 1: *Number of degrees of freedom and non-zero matrix entries, error in the broken H^1-seminorm $|\cdot|_{1,h}$ and the L^2-norm, and total time in seconds for conforming and nonconforming discretizations on different grid levels.*

f.e.	level	d.o.f.	non-zeros	H^1-error	L^2-error	total time
$\mathbb{Q}_1$	8	16,974,593	448,090,343	4.31-02	1.07-04	3329.3
$\widetilde{\mathbb{Q}}_1$	8	50,528,256	551,878,656	3.26-02	4.83-05	5210.6
$\mathbb{Q}_2$	4	35,937	1,867,013	3.35-02	6.49-04	9.5
$\widetilde{\mathbb{Q}}_2$	4	43,264	1,292,032	3.42-02	6.58-04	11.1
$\mathbb{Q}_2$	7	16,974,593	1,058,483,077	5.24-04	1.27-06	5231.3
$\widetilde{\mathbb{Q}}_2$	7	21,118,976	695,582,720	5.27-04	1.28-06	6964.4
$\mathbb{Q}_4$	3	35,937	6,134,633	3.94-04	7.81-06	95.6
$\widetilde{\mathbb{Q}}_4$	3	22,400	2,109,440	4.72-04	9.55-06	47.6
$\mathbb{Q}_4$	5	2,146,689	437,343,785	1.54-06	7.72-09	6435.5
$\widetilde{\mathbb{Q}}_4$	5	1,341,440	146,800,640	1.81-06	9.28-09	3105.0

and also the total computing time in seconds which splits into the time for assembling the matrices on all multigrid levels and the time for solving the discrete system. For the finite elements of first order, we present the results on the finest possible grid $\mathcal{T}^8$. We see that the results of the conforming and the nonconforming discretization are comparable for both considered norms and seminorms. The next two lines show that nearly the same accuracy can be obtained by finite elements of second order on a much coarser grid, in a much shorter time and with much less memory. The results for the second order elements on the finest possible grid $\mathcal{T}^7$ are also given. We see that the nonconforming discretization of second order has less non-zero matrix entries than the conforming one. Finally, we give the results of the discretizations with fourth order elements. The accuracy in the H^1-seminorm of the discretization on grid $\mathcal{T}^3$ is of the same size as for second order elements on the much finer grid $\mathcal{T}^7$ but obtained in a much shorter time and with much less memory. Note that the fourth order discretizations on the finest possible grid $\mathcal{T}^5$ obtain in a comparable time as first order discretizations on $\mathcal{T}^8$ an accuracy which is four orders of magnitude better. Moreover, we see that for fourth order elements the number of degrees of freedom for nonconforming discretizations is much less than for conforming ones. The difference is much more obvious if one looks at the number of non-zero matrix entries. Furthermore, note that conforming and nonconforming discretizations of the same order generate solutions which are nearly of the same accuracy.

Table 2 shows the numerical order of convergence in the broken H^1-seminorm $|\cdot|_{1,h}$ and the L^2-norm for conforming and nonconforming discretizations of first to fourth order. The numerical order of convergence was calculated from the results on the grids $\mathcal{T}^4$ and $\mathcal{T}^5$. It is clearly to see that the order of convergence is optimal for all considered discretizations. Furthermore, Table 2 presents for discretizations on the grid $\mathcal{T}^5$ the time in seconds for solving the discrete problems by the flexible GMRES method with a multigrid preconditioner, the time for assembling all matrices, and the sum of both times. It becomes obvious that the time

Table 2: *Numerical order of convergence in the broken H^1-seminorm $|\cdot|_{1,h}$ and the L^2-norm as well as the times in seconds for assembling, solving and their sum for conforming and nonconforming discretizations on grid level $\ell = 5$.*

	$\mathbb{Q}_1$	$\widetilde{\mathbb{Q}}_1$	$\mathbb{Q}_2$	$\widetilde{\mathbb{Q}}_2$	$\mathbb{Q}_3$	$\widetilde{\mathbb{Q}}_3$	$\mathbb{Q}_4$	$\widetilde{\mathbb{Q}}_4$
order H^1	1.00	0.99	2.00	2.01	3.00	3.00	4.00	4.00
order L^2	2.00	2.00	3.00	3.00	3.99	3.99	4.99	5.00
assemble	4.1	6.7	53.6	55.3	511.3	583.7	5087.4	2553.1
solving	1.9	2.8	26.2	46.7	195.5	183.4	1348.1	551.9
total	6.0	9.5	79.8	102.0	706.8	767.0	6435.5	3105.0

for assembling is larger (sometimes even much larger) than the time for solving. Moreover, the total time for the nonconforming discretization of fourth order is half of time of the corresponding conforming discretization.

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