Uniformly stable mixed hp-finite elements on multilevel adaptive grids with hanging nodes

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Abstract

We consider a family of quadrilateral or hexahedral mixed hp-finite elements for an incompressible flow problem with Q_r-elements for the velocity and discontinuous P_{r-1}-elements for the pressure where the order r can vary from element to element between 2 and an arbitrary bound. For multilevel adaptive grids with hanging nodes and a sufficiently small mesh size, we prove the inf-sup condition uniformly with respect to the mesh size and the polynomial degree.

Keywords: Stokes problem, inf-sup condition, mixed hp-FEM, quadrilateral and hexahedral finite elements, multilevel adaptive grids, hanging nodes

Mathematics Subject Classification (2000): 65N30, 65N35, 76D07

1 Introduction

For the numerical solution of incompressible flow problems, the mixed hp-finite element method is a well established efficient technique (see [21] and the references therein). It is well known that the inf-sup condition has to be satisfied in order to guarantee stability and accuracy of the discrete solution [8]. The inf-sup condition has been proven for many pairs of finite element spaces for the h-version (see e.g. [10, 8, 7, 22, 4, 19, 14]) as well as for the p- and hp-version (see e.g. [3, 9, 23, 20, 24, 1]). In all cases of the h- and hp-version, it

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has been shown that the inf-sup constant is independent of the mesh size h. However, in the literature on the h-version (see e.g. [10, 7, 22, 19, 14]), the dependency of the inf-sup constant on the polynomial degree has not been investigated.

For the hp-version, the inf-sup condition was proved in [23] for different families of finite element pairs, in particular for the pairs $(\mathbb{Q}_r, \mathbb{P}_{r-1}^{disc})$ and $(\mathbb{Q}_{r+1}, \mathbb{Q}_{r-1}^{disc})$ where "$disc$" means that the space consists of discontinuous functions which are elementwise defined by polynomials. It was proved that the inf-sup constant decreases as $Cr^{-(d-1)/2}$ where d denotes the dimension of the domain and C a constant independent of the mesh size h and the polynomial degree r. In [20] and [24], for the two and three dimensional case, respectively, the inf-sup condition was proved for the family of finite element pairs $(\mathbb{Q}_r, \mathbb{Q}_{r-2}^{disc})$ on so-called anisotropically and geometrically refined meshes with hanging nodes. It has been shown that the inf-sup constant can be estimated from below by $Cr^{-1/2}$ in the 2D-case and $Cr^{-3/2}$ in the 3D-case where the constant C is independent of the mesh size h, the polynomial degree r and the element aspect ratio. In [23, 20, 24], the cells of the mesh are assumed to be affine equivalent to a reference element. Therefore, the analysis covers only the case where the quadrilateral or hexahedral elements are parallelograms or parallelepipeds, respectively, which seems to be too restrictive from the practical point of view.

For two dimensional non-affine equivalent quadrilateral meshes with hanging nodes, Ainsworth and Coggins [1] proposed a family of finite element pairs $(\widetilde{\mathbb{Q}}_r, \mathbb{Q}'_{r-1})$ which is inf-sup stable uniformly with respect to the mesh size h and the polynomial degree r. Here $\mathbb{Q}'_{r-1}$ denotes a continuous pressure space constructed from the discontinuous space $\mathbb{P}_r^{disc}$ by some additional $\mathbb{Q}_{r-1}$-functions. The velocity space $\widetilde{\mathbb{Q}}_r$ denotes a modification of the usual space $\mathbb{Q}_r$ where the basis functions associated with nodes at the element boundary are defined by means of the usual reference transformation but the functions corresponding to interior element nodes are based on the *Piola transformation* (see [9] for a similar construction). This causes some extra difficulties in the implementation, in particular for the derivatives of the velocity functions.

In this paper, we prove, for the d-dimensional case, $d \in \{2, 3\}$, a uniform inf-sup condition for the family of finite element pairs $(\mathbb{Q}_r, \mathbb{P}_{r-1}^{disc})$ on so called *multilevel adaptive grids* of quadrilateral or hexahedral elements with hanging nodes. By $\mathbb{P}_{r-1}^{disc}$ we denote the space of mapped discontinuous functions which are defined on the reference element by polynomials of total degree not greater than $r-1$. Note that this mapped space may have non-optimal approximation properties on general quadrilateral or hexahedral grids [4, 2]. However, one can show that it has optimal approximation properties on multilevel adaptive grids [16, 18]. Multilevel adaptive grids are created from a regular coarse grid by means of recursive refinement of existing elements into 2^d many son-elements combined with the introduction of hanging nodes. From the practical point of view, such grids occur very naturally in applications with adaptive refinement based on a posteriori error estimators (see e.g. [13, 12]). Typical grids are depicted in Figure 1. Note that we do not assume that the elements are affine equivalent.

The new contribution of this paper compared to the results of [23, 19, 14] on the element pair $(\mathbb{Q}_r, \mathbb{P}_{r-1}^{disc})$ is that we prove the inf-sup constant to be independent of the polynomial degree r if the mesh size h is sufficiently small. We use the result of Bernardi and Maday [3] that, for the situation of just one element $K = (-1,1)^d$, the inf-sup constant of this element pair is independent of the polynomial degree r. Then, the main idea is to transform the inf-sup condition from the reference element to the original element by means of a *modified Piola transformation*. This proves the local inf-sup condition. The proof of the global inf-sup condition is based on the technique of Boland and Nicolaides [5].

As a model for an incompressible flow problem we consider the Stokes equations with homogeneous Dirichlet conditions: *Find a velocity u and a pressure p such that*

$$\begin{aligned} -\nu\triangle u + \nabla p &= f && \text{in } \Omega\,, \\ \operatorname{div} u &= 0 && \text{in } \Omega\,, \\ u &= 0 && \text{on } \Gamma\,. \end{aligned} \tag{1}$$

Here, Ω denotes a bounded domain in $\mathbb{R}^d$, $d \in \{2,3\}$, with a polygonal or polyhedral boundary Γ, $\nu > 0$ the kinematic viscosity and f a given body force. Then, a pair (X^N, M^N) of conforming finite element spaces for the approximation of the velocity $u \in (H_0^1(\Omega))^d$ and the pressure $p \in L_0^2(\Omega)$ is said to fulfill the *discrete inf-sup condition* if the following estimate holds:

$$\exists \beta > 0 : \qquad \inf_{q^N \in M^N\setminus\{0\}} \sup_{v^N \in X^N\setminus\{0\}} \frac{(\operatorname{div} v^N, q^N)}{|v^N|_{1,\Omega} \|q^N\|_{0,\Omega}} \geq \beta. \tag{2}$$

For the stability and accuracy of the solution $(u^N, p^N) \in X^N \times M^N$ of the discrete Stokes problem, it is desirable that the inf-sup constant β in (2) is independent of the mesh size and the polynomial degree.

The outline of this paper is as follows. In Section 2, we introduce the notations, the concept of multilevel adaptive grids and the used finite element spaces. In Section 3.1, we present the global and local inf-sup conditions that are used for the technique of Boland and Nicolaides. The modified Piola transformation is introduced and studied in Section 3.2. In Section 3.3, we prove the local inf-sup condition. Finally, we summarize our main result in Section 3.4.

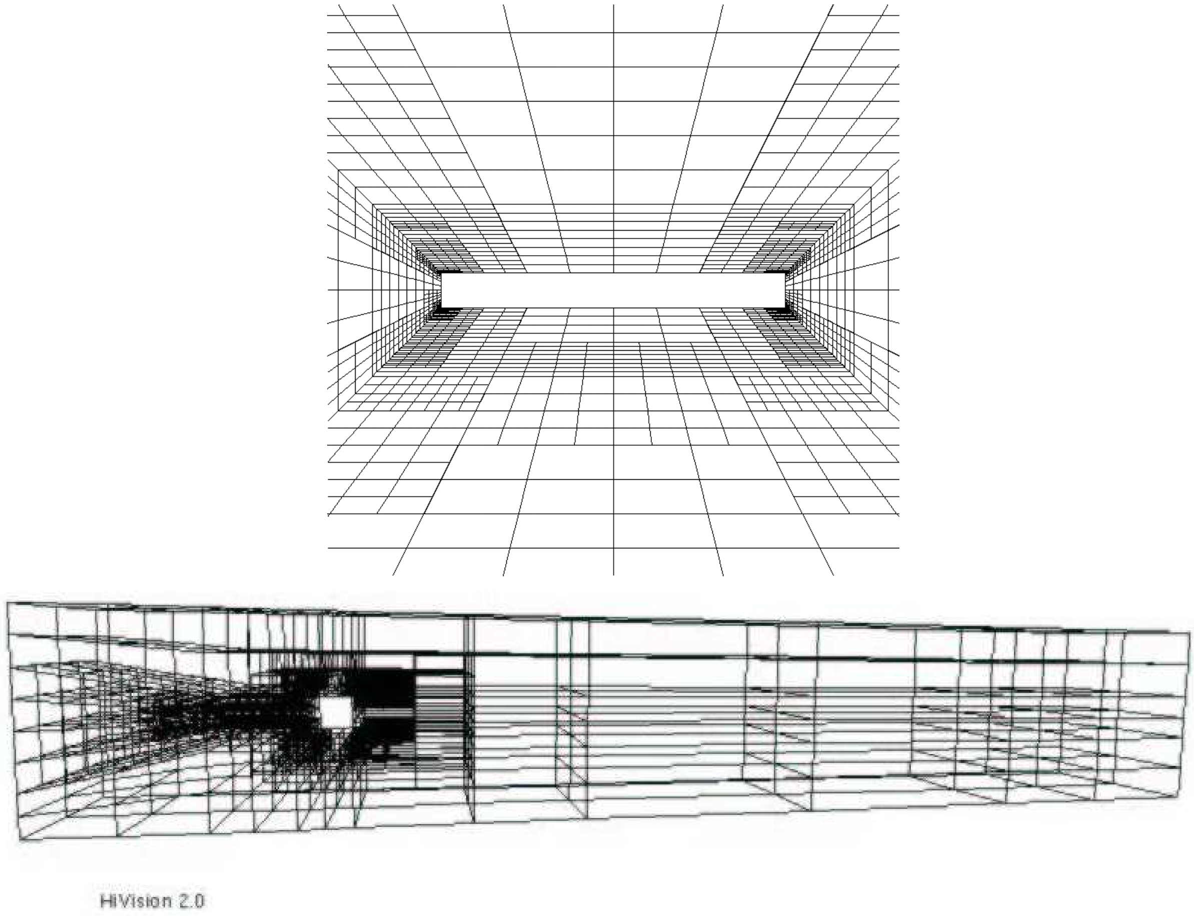

Figure 1: Typical local refined grids with hanging nodes; (top) two-dimensional grid for the computation of an exterior flow problem [6]; (bottom) three-dimensional grid for a channel flow problem [11].

2 Preliminaries and Notations

2.1 General Notations

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded domain with a polygonal or polyhedral boundary. For a set $G \subset \Omega$, let $|\cdot|_{m,p,G}$ and $\|\cdot\|_{m,p,G}$ denote the usual semi-norm and norm in the Sobolev spaces $W^{m,p}(G)$ and $(W^{m,p}(G))^d$, respectively. For the Hilbert spaces $H^m(G)$ and $(H^m(G))^d$, we omit the index p and denote the semi-norm and norm by $|\cdot|_{m,G}$ and $\|\cdot\|_{m,G}$. The scalar product in $L^2(G)$ and $(L^2(G))^d$ will be denoted by $(\cdot,\cdot)_G$. By $\mathbb{P}_m(G)$ we denote the space of all polynomials on the domain $G \subset \mathbb{R}^d$ with total degree less than or equal to m and by $\mathbb{Q}_m(G)$ the space of those polynomials where the maximum power in each coordinate is less than or equal to m. For $p \in [1, \infty]$, both the usual p-norm of a vector in $\mathbb{R}^d$ and the induced p-norm of a $d \times d$-matrix are denoted by $\|\cdot\|_p$. For a set $G \subset \mathbb{R}^d$, we denote by $\mathrm{int}(G)$ and $\overline{G}$ the interior and closure of G, respectively. Throughout this paper, C will denote a generic constant which may have different values at different places. All these constants occurring inside of any estimate will be – unless it is explicitly stated – independent of the local and global mesh parameters h_K and h and the polynomial degree vector $\underline{r}$ which will be defined below.

2.2 Multilevel Adaptive Grids

Let the bounded domain $\Omega \subset \mathbb{R}^d$ be partitioned by a *grid* $\mathcal{T}_h$ consisting of elements $K \in \mathcal{T}_h$ which are assumed to be open quadrilaterals in the 2D-case and open hexahedrons in the 3D-case such that $\overline{\Omega} = \cup_{K\in\mathcal{T}_h}\overline{K}$. For an element $K \in \mathcal{T}_h$, we denote by h_K the diameter of the element K. The *mesh size* h of $\mathcal{T}_h$ is given by $h := \max_{K\in\mathcal{T}_h} h_K$. We denote by $F_K : \widehat{K} \to K$ the mapping between the reference element $\widehat{K} := (-1, +1)^d$ and the original element K. In the sequel of this paper, the mapping F_K is assumed to be multi-linear, i.e. $F_K \in (\mathbb{Q}_1(\widehat{K}))^d$. Therefore, the two dimensional faces of 3D hexahedral elements $K \in \mathcal{T}_h$ can be curved in general. For our analysis, we assume that the mesh family $\{\mathcal{T}_h\}$ is *shape regular* in the sense presented in [19], see also [14]. This shape regularity assumption guarantees that the reference mapping $F_K : \widehat{K} \to K$ is bijective and that it satisfies for all $K \in \mathcal{T}_h$ the following properties (for the proof, see [15]):

$$\|DF_K(\hat{x})\|_\infty \le Ch_K \qquad \forall \hat{x} \in \widehat{K}, \tag{3}$$

$$Ch_K^d \le |\det(DF_K(\hat{x}))| \le C'h_K^d \qquad \forall \hat{x} \in \widehat{K}. \tag{4}$$

In this paper, we study the case of hanging nodes, i.e., the usual assumption of a *regular* grid $\mathcal{T}_h$ has to be weakened. In the following, we will describe the type of grids that are treated in this paper. We assume that $\mathcal{T}_h$ is a *multilevel adaptive grid* generated by a refinement process in the following way. We start with an initial partition $\mathcal{T}^0$ of the domain Ω into elements $K \in \mathcal{T}^0$ of grid-level 0, i.e. $\Omega = \mathrm{int}\big(\bigcup_{K\in\mathcal{T}^0} \overline{K}\big)$. The grid $\mathcal{T}^0$ is assumed to be *regular* in the usual sense, i.e., for any two different elements $K_1, K_2 \in \mathcal{T}^0$ the intersection $\overline{K_1} \cap \overline{K_2}$ is either empty or a common $(d-m)$-dimensional face of K_1 and K_2 where $m \in \{1, \dots, d\}$. Now, starting with the elements $K \in \mathcal{T}^0$, an existing element K can be refined, i.e., it can be splitted into 2^d many new elements called *son-elements* which are denoted by $\sigma_i(K)$, $i = 1, \dots, 2^d$, see Figure 2. These son-elements are defined as

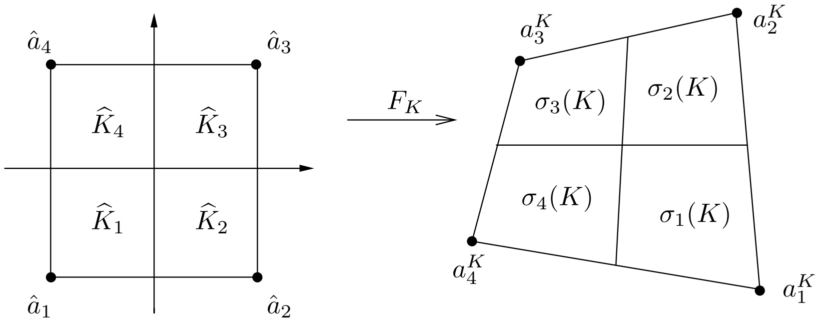

Figure 2: *Refinement of element K into son-elements $\sigma_i(K)$, $i = 1, \dots, 2^d$.*

follows. Let $\widehat{K}_i$, $i \in \{1, \dots, 2^d\}$, be the son-elements of the reference element $\widehat{K} = (-1, 1)^d$ which are congruent d-cubes with $|\widehat{K}_i| = 1$. Then, the i-th son-element $\sigma_i(K)$ of the original element K is defined as $\sigma_i(K) := F_K(\widehat{K}_i)$ for all $i = 1, \dots, 2^d$. Note that in the 3D-case, even if an element K has only planar faces, it can happen that the son-elements $\sigma_i(K)$ have curved faces in the interior of K (see [17, Remark 3.3] for an example). For a new element $K' = \sigma_i(K)$, we will say that K is the *father-element* of K' and we will write $K = \mathcal{F}(K')$. If an element K is refined then it is replaced in the partition of the domain Ω by the set of its son-elements $\sigma_i(K)$, $i = 1, \dots, 2^d$. The new elements can be refined again and again until the final partition $\mathcal{T}_h$ of Ω is created. Examples of such grids used in practical computations can be found in [12, 13].

Definition 1 *For an element $K \in \mathcal{T}_h$, generated from the initial grid $\mathcal{T}^0$ by the refinement process described above, we define the* **refinement level** *$\ell(K)$ as $\ell(K) := 0$ if $K \in \mathcal{T}^0$ and $\ell(K) := m \geq 1$ if there exists a chain of m father-elements K_i, $i = m-1, \dots, 0$, starting* *from* $K_m := K$ *and defined by* $K_{i-1} := \mathcal{F}(K_i)$ *for* $i = m, \ldots, 1$*, such that* $K_0 \in \mathcal{T}^0$*. The grid* $\mathcal{T}_h$*, generated by the above defined refinement process, is called* **1-irregular** *if the condition*

$$|\ell(K) - \ell(K')| \leq 1 \tag{5}$$

holds for any pair of different neighbored elements $K, K' \in \mathcal{T}_h$ *where* $\partial K \cap \partial K'$ *is a one- or two-dimensional manifold.*

Note that, in the 3D-case, the condition (5) has to be satisfied if the elements K and K' have a common edge or a common face. The above defined *refinement level* $\ell(K)$ is equal to the number of refinement steps that are needed to generate the element K from an element of the coarsest grid $\mathcal{T}^0$.

In many practical computations, 1-irregular grids are used. They are sufficient to describe complicated structures in a reasonable way, see e.g. [12, 13]. An example that shows the capabilities of 1-irregular quadrilateral grids to resolve an inner fine scale structure is presented in Figure 3.

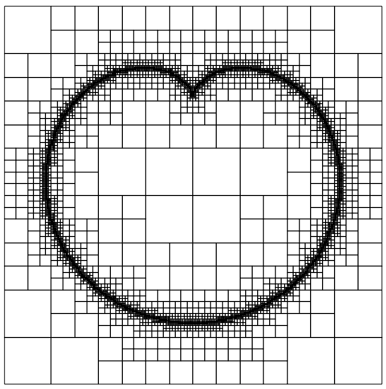

Figure 3: *Resolution of an inner fine scale structure by a multilevel adaptive 1-irregular grid.*

An advantage of multilevel adaptive grids is that the reference mapping F_K behaves – up to a power of the mesh size h – like an affine mapping. This is one of the key points of the analysis in this paper. We have the following result.

Lemma 2 *Let* $\{\mathcal{T}_h\}$ *be a family of multilevel adaptive grids generated from a shape regular coarse grid* $\mathcal{T}^0$ *as described above. Then, there exist constants* $C_1(m, \mathcal{T}^0)$ *and* $C_2(\mathcal{T}^0)$ *such* *that the following estimates are satisfied for all* $K \in \mathcal{T}_h$

$$|F_K|_{m,\infty,\widehat{K}} \;\le\; C_1(m,\mathcal{T}^0)\, h_K^m, \qquad \forall\, m \in \{0,1,2,3\}, \tag{6}$$

$$\|J_K(\widehat{x}) - B_K\|_p \;\le\; C_2(\mathcal{T}^0)\, h_K^2 \qquad \forall\, p \in [1,\infty], \quad \forall\, \widehat{x} \in \widehat{K}, \tag{7}$$

where $J_K(\widehat{x}) := DF_K(\widehat{x})$ *and* $B_K := DF_K(0)$.

Proof. See Lemma 5 in [18]. □

2.3 Finite Element Spaces

Let $\underline{r}$ denote the degree vector $\underline{r} := \{r_K : K \in \mathcal{T}_h\}$ which contains the polynomial degrees r_K of the velocity approximation on each element K. In this paper, we assume that

$$r_K \ge 2 \qquad \forall K \in \mathcal{T}_h. \tag{8}$$

The finite element space of the velocity $X^N \subset (H_0^1(\Omega))^d$ associated with the grid $\mathcal{T}_h$ and the degree vector $\underline{r}$ is defined as $X^N := (S^N)^d$ with the scalar finite element space

$$S^N := S(\Omega;\underline{r},\mathcal{T}_h) := \{\phi \in H_0^1(\Omega) :\ \phi\big|_K \circ F_K \in \mathbb{Q}_{r_K}(\widehat{K}) \quad \forall K \in \mathcal{T}_h\}\,. \tag{9}$$

The finite element space $M^N \subset L_0^2(\Omega)$ for the approximation of the pressure is chosen as the following mapped space

$$M^N := M(\Omega;\underline{r},\mathcal{T}_h) := \{q \in L_0^2(\Omega) :\ q\big|_K \circ F_K \in \mathbb{P}_{r_K-1}(\widehat{K}) \quad \forall K \in \mathcal{T}_h\}\,. \tag{10}$$

Then, the discrete Stokes problem reads: *Find* $(u^N, p^N) \in X^N \times M^N$ *such that*

$$\begin{aligned} \nu(\nabla u^N, \nabla v^N)_\Omega \;-\; (p^N, \nabla\cdot v^N)_\Omega &= (f, v^N)_\Omega \qquad \forall v^N \in X^N \\ (q^N, \nabla\cdot u^N)_\Omega &= 0 \qquad\qquad\quad\; \forall q^N \in M^N\,. \end{aligned} \tag{11}$$

For the subsequent analysis we need also the finite element space for the velocity with the polynomial order of 2, i.e. $X_h^2 := (S_h^2)^d$ based on the scalar finite element space

$$S_h^2 := \{\phi \in H_0^1(\Omega) :\ \phi\big|_K \circ F_K \in \mathbb{Q}_2(\widehat{K}) \ \ \forall K \in \mathcal{T}_h\}, \tag{12}$$

as well as the finite element space of the piecewise constant pressure functions

$$M_h^0 := \{q \in L_0^2(\Omega) :\ q\big|_K \in \mathbb{P}_0(K) \quad \forall K \in \mathcal{T}_h\}\,. \tag{13}$$

Some attention is required to ensure interelement continuity in (9) and (12) in the case of hanging nodes or if r_K is different on two adjacent cells of an inner element face. We refer to [21] for a detailed treatment of this issue. Note that, due to the assumption (8), it holds $X_h^2 \subset X^N$.

3 Proving the inf-sup condition

3.1 Global and local inf-sup conditions

Our approach to prove the inf-sup condition (2) is based on the technique proposed by Boland and Nicolaides [5] (see also [10, Sect. II.1.4]). For the partitioning of the domain Ω into non-overlapping subdomains, we take the elements $K \in \mathcal{T}_h$. According to the finite element spaces $X^N = (S^N)^d$ and M^N defined in (9) and (10), respectively, we define for each element $K \in \mathcal{T}_h$ the following local spaces

$$X^N(K) := \left\{ v \in (H_0^1(K))^d : \ v \circ F_K \in (\mathbb{Q}_{r_K}(\widehat{K}))^d \right\}, \tag{14}$$

$$M^N(K) := \left\{ q \in L_0^2(K) : \ q \circ F_K \in \mathbb{P}_{r_K - 1}(\widehat{K}) \right\}. \tag{15}$$

We say that the finite element spaces X^N and M^N fulfill the *local inf-sup condition* uniformly with respect to the mesh size h and the polynomial degree vector $\underline{r}$ if there exists a constant $\beta_0 > 0$, such that it holds

$$\inf_{q^N \in M^N(K)\setminus\{0\}} \ \sup_{v^N \in X^N(K)\setminus\{0\}} \frac{(\nabla \cdot v^N, q^N)_K}{|v^N|_{1,K} \|q^N\|_{0,K}} \geq \beta_0, \qquad \forall\, K \in \mathcal{T}_h\,. \tag{16}$$

Now, in order to prove the global inf-sup condition for our finite element spaces $X^N = (S^N)^d$ and M^N we only have to verify the local inf-sup condition (16) and the following global inf-sup condition for a suitable pair of low order subspaces $(\overline{X}^N, M_h^0)$.

Lemma 3 *Let the local inf-sup condition (16) be fulfilled uniformly with respect to the mesh size h and the degree vector $\underline{r}$. Furthermore, let there exist a subspace $\overline{X}^N \subset X^N$ such that the pair of finite element spaces $(\overline{X}^N, M_h^0)$ fulfills the global inf-sup condition*

$$\inf_{q_h \in M_h^0\setminus\{0\}} \ \sup_{v_h \in \overline{X}^N\setminus\{0\}} \frac{(\nabla \cdot v_h, q_h)_\Omega}{|v_h|_{1,\Omega} \|q_h\|_{0,\Omega}} \geq \bar{\beta}\,, \tag{17}$$

with a constant $\overline{\beta} > 0$ independent of h and $\underline{r}$. Then, the pair (X^N, M^N) satisfies the global inf-sup condition (2) with a constant $\beta > 0$ which depends on $\bar{\beta}$ and β_0 but not on h and $\underline{r}$.

Proof. See Theorem II.1.12 in [10]. $\square$

The global inf-sup condition (17) of the pair of spaces $(\overline{X}^N, M_h^0)$ can be proved for the second order subspace $\overline{X}^N = X_h^2 \subset X^N$.

Lemma 4 *Let $\mathcal{T}_h$ be a multilevel adaptive grid, which is assumed to be 1-irregular and shape-regular. Then, the pair of finite element spaces $(\overline{X}^N, M_h^0)$ with $\overline{X}^N := X_h^2$ satisfies the global inf-sup condition (17) with a constant $\overline{\beta} > 0$ independent of h and $\underline{r}$.*

Proof. See [14, Lemma 9]. $\square$

3.2 Modified Piola transformation

The Piola transformation is a well known tool to establish the local inf-sup condition [9, 1]. For an element $K \in \mathcal{T}_h$ and a given function $\widehat{v} \in (H^1(\widehat{K}))^d$, the Piola transformation $\mathcal{P}_K\widehat{v} \in (H^1(K))^d$ is defined by

$$(\mathcal{P}_K\widehat{v})\,(x) := \frac{1}{\det(J_K(\widehat{x}))}\, J_K(\widehat{x})\,\widehat{v}(\widehat{x}) \qquad \forall x \in K, \tag{18}$$

where $J_K(\widehat{x}) := DF_K(\widehat{x})$ and $\widehat{x} := F_K^{-1}(x)$. Due to the property (4) of shape-regular grids, the expression in (18) is well defined. The Piola transformation has the following essential property.

Lemma 5 *For a given element $K \in \mathcal{T}_h$ and functions $\widehat{v} \in (H_0^1(\widehat{K}))^d$ and $\widehat{q} \in H^1(\widehat{K})$, let $v \in (H_0^1(K))^d$ and $q \in H^1(K)$, be the associated functions $v := \mathcal{P}_K\widehat{v}$ and $q := \widehat{q} \circ F_K^{-1}$, respectively. Then, it holds*

$$(\operatorname{div} v, q)_K = \left(\widehat{\operatorname{div}}\,\widehat{v}, \widehat{q}\right)_{\widehat{K}}. \tag{19}$$

Proof. We easily obtain (19) if we apply partial integration and transform the integral over K into an integral over $\widehat{K}$. $\square$

For our local velocity space $X^N(K)$ defined in (14), a problem in the proof of the local inf-sup condition is the fact that for a given function $\widehat{v} \in (\mathbb{Q}_{r_K}(\widehat{K}) \cap H_0^1(\widehat{K}))^d$ the transformed function $\mathcal{P}_K\widehat{v}$ in general is not contained in $X^N(K)$. Due to this fact, the local velocity space in [9, 1] has been defined as the image of the space $(\mathbb{Q}_{r_K}(\widehat{K}) \cap H_0^1(\widehat{K}))^d$ under the operator $\mathcal{P}_K$. However, we will not use this modification of the internal finite element basis functions since it complicates the implementation, in particular for the derivatives of the velocity.

The idea to overcome this problem is to use a *modified Piola transformation* which is close to the original one. For an element $K \in \mathcal{T}_h$ and a given function $\widehat{v} \in (H^1(\widehat{K}))^d$, we define the modified Piola transformation $\widetilde{\mathcal{P}}_K\widehat{v} \in (H^1(K))^d$ by

$$\big(\widetilde{\mathcal{P}}_K\widehat{v}\big)(x) := \frac{1}{\det(B_K)}\, B_K\, \widehat{v}(F_K^{-1}(x)) \qquad \forall x \in K, \tag{20}$$

where $B_K := DF_K(0)$. Then, one easily verifies that $\widetilde{\mathcal{P}}_K\widehat{v} \in X^N(K)$ for all functions $\widehat{v} \in (\mathbb{Q}_{r_K}(\widehat{K}) \cap H_0^1(\widehat{K}))^d$. Moreover, we can prove the following results.

Lemma 6 *Let $\{\mathcal{T}_h\}$ be a family of shape-regular and 1-irregular multilevel adaptive grids generated from a coarse grid $\mathcal{T}^0$. Then, for all functions $\widehat{v} \in (H_0^1(\widehat{K}))^d$ and elements $K \in \mathcal{T}_h$, the following estimates are satisfied*

$$\begin{aligned} \big|\widetilde{\mathcal{P}}_K\widehat{v}\big|_{1,K} &\le C_3\, h_K^{-d/2}|\widehat{v}|_{1,\widehat{K}}, & (21)\\ |\widehat{v}|_{1,\widehat{K}} &\le C\, h_K^{d/2}\big|\widetilde{\mathcal{P}}_K\widehat{v}\big|_{1,K}, & (22)\\ \big|\mathcal{P}_K\widehat{v} - \widetilde{\mathcal{P}}_K\widehat{v}\big|_{1,K} &\le C_4(\mathcal{T}^0)\, h_K|\widetilde{\mathcal{P}}_K\widehat{v}|_{1,K}, & (23) \end{aligned}$$

where the constants are independent of $\widehat{v}$ and the local mesh size h_K.

Proof. From the definition (20) of the modified Piola transformation and the estimates (4) and (3) it follows

$$\big|\widetilde{\mathcal{P}}_K\widehat{v}\big|_{1,K} \le C\, h_K^{1-d}\, |\widehat{v} \circ F_K^{-1}|_{1,K}\,.$$

Furthermore, we have

$$|\widehat{v} \circ F_K^{-1}|_{1,K} \;\le\; C\,|\det(DF_K)|_{0,\infty,\widehat{K}}^{1/2}\, |F_K^{-1}|_{1,\infty,K}\, |\widehat{v}|_{1,\widehat{K}} \;\le\; C\, h_K^{d/2-1}\, |\widehat{v}|_{1,\widehat{K}}\,, \tag{24}$$

which yields the estimate (21). From

$$\widehat{v}(\widehat{x}) = \det(B_K)B_K^{-1}\big(\widetilde{\mathcal{P}}_K\widehat{v}\big)(F_K(\widehat{x}))$$

we get by means of (4) and (3) the estimates

$$|\widehat{v}|_{1,\widehat{K}} \;\le\; Ch_K^{d-1}|\big(\widetilde{\mathcal{P}}_K\widehat{v}\big) \circ F_K|_{1,\widehat{K}} \;\le\; Ch_K^{d-1}|\det(DF_K^{-1})|_{0,\infty,K}^{1/2}\, |F_K|_{1,\infty,\widehat{K}}\, |\widetilde{\mathcal{P}}_K\widehat{v}|_{1,K}\,,$$

which imply (22).

To prove (23) we start with the splitting

$$\mathcal{P}_K\widehat{v} - \widetilde{\mathcal{P}}_K\widehat{v} = \underbrace{\left\{\frac{1}{\det(J_K)} - \frac{1}{\det(B_K)}\right\} B_K(\widehat{v} \circ F_K^{-1})}_{=:T_1} \;+\; \underbrace{\frac{1}{\det(J_K)}\,\{J_K - B_K\}\,(\widehat{v} \circ F_K^{-1})}_{=:T_2}\,.$$

Let us define the function $g_K : \widehat{K} \to \mathbb{R}$ as

$$g_K := \frac{1}{\det(J_K)} - \frac{1}{\det(B_K)}.$$

Using (3) with $\widehat{x} = 0$ and (24), we get

$$|T_1|_{1,K} \le C\|g_K\|_{0,\infty,\widehat{K}} h_K^{d/2} |\widehat{v}|_{1,\widehat{K}} \;+\; C|g_K|_{1,\infty,\widehat{K}} |F_K^{-1}|_{1,\infty,K}\, h_K^{d/2+1} \|\widehat{v}\|_{0,\widehat{K}}$$

If we apply Friedrichs inequality and the estimate $|F_K^{-1}|_{1,\infty,K} \le C h_K^{-1}$, which follows from (3) and(4), we conclude

$$|T_1|_{1,K} \le C \left\{ \|g_K\|_{0,\infty,\widehat{K}} \;+\; |g_K|_{1,\infty,\widehat{K}} \right\} h_K^{d/2} |\widehat{v}|_{1,\widehat{K}}. \tag{25}$$

For the first term with g_K, we have due to (4)

$$\|g_K\|_{0,\infty,\widehat{K}} = \left\| \left(1 - \det(B_K^{-1} J_K)\right) \frac{1}{\det(J_K)} \right\|_{0,\infty,\widehat{K}} \le C h_K^{-d} \left\| 1 - \det(B_K^{-1} J_K) \right\|_{0,\infty,\widehat{K}}.$$

Separately for the case $d = 2$ and $d = 3$, one can show in a straightforward way the following estimate for an arbitrary matrix $A \in \mathbb{R}^{d\times d}$ and the matrix $M \in \mathbb{R}^{d\times d}$ defined as $M := I - A$ with the identity matrix $I \in \mathbb{R}^{d\times d}$:

$$|1 - \det(A)| \le \|M\|_1 \left\{ 1 + \sum_{k=1}^{d-1} \|A\|_1^k \right\}. \tag{26}$$

If we choose $A = B_K^{-1} J_K(\widehat{x})$ we get for $M = I - A$ the estimate

$$\|M\|_1 = \|B_K^{-1}(B_K - J_K(\widehat{x}))\|_1 \le \|B_K^{-1}\|_1 \, \|B_K - J_K(\widehat{x})\|_1 \le C(\mathcal{T}^0) h_K \,,$$

where we have used $\|B_K^{-1}\|_1 \le C h_K^{-1}$, which follows from (3) and (4), and the estimate (7). For the estimation of $\|A\|_1$, we use the estimate (3) for $J_K(\widehat{x})$ and obtain

$$\|A\|_1 \le \|B_K^{-1}\|_1 \|J_K(\widehat{x})\|_1 \le C \,.$$

Together with (26) this implies

$$\left\| 1 - \det(B_K^{-1} J_K) \right\|_{0,\infty,\widehat{K}} \le C(\mathcal{T}^0) h_K \,,$$

which yields

$$\|g_K\|_{0,\infty,\widehat{K}} \le C(\mathcal{T}^0) h_K^{1-d} \,. \tag{27}$$

For the second term with g_K in (25), we have

$$\begin{aligned}|g_K|_{1,\infty,\widehat{K}} &= \left|\frac{1}{\det(J_K)}\right|_{1,\infty,\widehat{K}} \leq C\left|\frac{1}{(\det(J_K))^2}\right|_{0,\infty,\widehat{K}} |\det(J_K)|_{1,\infty,\widehat{K}} \\ &\leq Ch_K^{-2d}|F_K|_{2,\infty,\widehat{K}}|F_K|^{d-1}_{1,\infty,\widehat{K}}\,.\end{aligned}$$

Using the estimate (6) for the multilevel adaptive grid $\mathcal{T}_h$, we obtain

$$|g_K|_{1,\infty,\widehat{K}} \leq C(\mathcal{T}^0)h_K^{1-d}\,. \tag{28}$$

If we combine (25) with (27) and (28) we get the estimate

$$|T_1|_{1,K} \leq C(\mathcal{T}^0)h_K^{1-d/2}|\widehat{v}|_{1,\widehat{K}}. \tag{29}$$

For the estimation of the term T_2, we introduce the functions $\phi_K : \widehat{K} \to \mathbb{R}$ and $G_K : \widehat{K} \to \mathbb{R}^{d\times d}$ defined as

$$\phi_K(\widehat{x}) := \frac{1}{\det(J_K(\widehat{x}))}\,, \qquad G_K(\widehat{x}) := J_K(\widehat{x}) - B_K\,.$$

Now, T_2 can be estimated as follows

$$\begin{aligned}|T_2|_{1,K} \leq\ & Ch_K^{d/2}\Big\{|\phi_K|_{1,\infty,\widehat{K}}\ \|G_K\|_{0,\infty,\widehat{K}}\|\widehat{v}\|_{0,\widehat{K}} + \|\phi_K\|_{0,\infty,\widehat{K}}\,|G_K|_{1,\infty,\widehat{K}}\,\|\widehat{v}\|_{0,\widehat{K}} \\ &+ \|\phi_K\|_{0,\infty,\widehat{K}}\,\|G_K\|_{0,\infty,\widehat{K}}\,|\widehat{v}|_{1,\widehat{K}}\Big\}|F_K^{-1}|_{1,\infty,K}\,,\end{aligned} \tag{30}$$

where the norm and semi norm of G_K are defined by

$$\|G_K\|_{0,\infty,\widehat{K}} := \sup_{\widehat{x}\in\widehat{K}}\|G_K(\widehat{x})\|_\infty\,, \qquad |G_K|_{1,\infty,\widehat{K}} := \max_{j=1,\dots,d}\ \sup_{\widehat{x}\in\widehat{K}}\left\|\frac{\partial G_K}{\partial\widehat{x}_j}(\widehat{x})\right\|_\infty .$$

For the function ϕ_K, we have by (4) and (28)

$$\|\phi_K\|_{0,\infty,\widehat{K}} \leq Ch_K^{-d}\,, \qquad |\phi_K|_{1,\infty,\widehat{K}} = |g_K|_{1,\infty,\widehat{K}} \leq C(\mathcal{T}^0)h_K^{1-d}\,.$$

Using (7) and (6), the matrix function G_K can be estimated as follows

$$\|G_K\|_{0,\infty,\widehat{K}} \leq C_2(\mathcal{T}^0)\,h_K^2\,, \qquad |G_K|_{1,\infty,\widehat{K}} \leq C|F_K|_{2,\infty,\widehat{K}} \leq C(\mathcal{T}^0)\,h_K^2\,.$$

If we apply Friedrichs inequality and the estimate $|F_K^{-1}|_{1,\infty,K} \leq Ch_K^{-1}$, we obtain from (30)

$$\begin{aligned}|T_2|_{1,K} &\leq C(\mathcal{T}^0)h_K^{d/2}\Big\{h_K^{1-d}\,h_K^2 + h_K^{-d}\,h_K^2 + h_K^{-d}\,h_K^2\Big\}h_K^{-1}|\widehat{v}|_{1,\widehat{K}}\,, \\ &\leq C(\mathcal{T}^0)h_K^{1-d/2}|\widehat{v}|_{1,\widehat{K}}\,.\end{aligned} \tag{31}$$

Now, the assertion (23) follows from (29), (31) and (22). □

3.3 Proof of the local inf-sup condition

For our pair of finite element spaces, the following local inf-sup condition is known.

Lemma 7 *Let $\mathcal{T}_h$ be a shape-regular grid. Then, for each element $K \in \mathcal{T}_h$, the pair $(X^N(K), M^N(K))$ of the mapped local finite element spaces defined in (14) and (15), satisfies the local inf-sup condition* (16) *with a constant $\beta_0 = \lambda(r_K) > 0$ which depends on the local degree r_K but which is independent of K and the mesh size h.*

Proof. See the proof of Theorem 8 in [19]. □

This result implies a local inf-sup condition (16) with a constant

$$\beta_0 = \beta_0^*(r^*) := \min_{r=1,\dots,r^*} \lambda(r) \qquad \text{where} \qquad r^* := \max_{K \in \mathcal{T}_h} r_K. \tag{32}$$

However, the dependency of $\beta_0^*(r^*)$ on the maximum polynomial degree r^* has not been investigated in [19]. For a grid $\mathcal{T}_h$ consisting of affine equivalent elements, the estimate $\beta_0^*(r^*) \geq C(r^*)^{-(d-1)/2}$ has been shown in [23]. In the following, we will prove that, for a sufficiently small mesh size, the local inf-sup condition (16) holds with a constant β_0 which is independent of the mesh size h and the polynomial degree vector $\underline{r}$.

A main ingredient of our analysis is the following result of Bernardi and Maday which we will apply on the reference element.

Lemma 8 *For $\widehat{K} = (-1,1)^d$ and an arbitrary integer $r \geq 2$, let $\widehat{X}_r := (\mathbb{Q}_r(\widehat{K}) \cap H_0^1(\widehat{K}))^d$ and $\widehat{M}_r := \mathbb{P}_{r-1}(\widehat{K}) \cap L_0^2(\widehat{K})$. Then, it holds*

$$\forall \widehat{q} \in \widehat{M}_r \quad \exists\, \widehat{v} \in \widehat{X}_r : \quad (\widehat{\operatorname{div}}\, \widehat{v}, \widehat{q})_{\widehat{K}} \geq \widehat{\beta}_0\, |\widehat{v}|_{1,\widehat{K}} \|\widehat{q}\|_{0,\widehat{K}}, \tag{33}$$

with a constant $\widehat{\beta}_0 > 0$ which is independent of the polynomial degree r.

Proof. See [3]. □

Now, using the original and modified Piola transformation, the estimate (33) can be transformed to the pair of the local finite element spaces $(X^N(K), M^N(K))$ defined in (14) and (15).

Lemma 9 *Let $\{\mathcal{T}_h\}$ be a family of shape-regular and 1-irregular multilevel adaptive grids generated from a coarse grid $\mathcal{T}^0$. Then, there exists a sufficiently small mesh size $h_0 = h_0(\mathcal{T}^0)$ such that for all grids $\mathcal{T}_h$ with $h \le h_0$ and all $K \in \mathcal{T}_h$ it holds*

$$\forall q \in M^N(K) \quad \exists\, v \in X^N(K): \quad (\operatorname{div} v, q)_K \ge \beta_0\, |v|_{1,K} \|q\|_{0,K}, \tag{34}$$

with a constant $\beta_0 > 0$ which is independent of the mesh size and the polynomial degree vector $\underline{r}$. Therefore, the local inf-sup condition (16) *is satisfied with this constant β_0.*

Proof. For a given $q \in M^N(K)$, let $\widehat{q} := q \circ F_K \in \widehat{M_r}$. Due to Lemma 8, there exists a $\widehat{v} \in \widehat{X}_r$ satisfying (33). Choose $v := h_K^d\, \mathcal{P}_K \widehat{v}$. Then, it holds

$$(\operatorname{div} v,\ q)_K = h_K^d\, (\widehat{\operatorname{div}}\, \widehat{v}, \widehat{q})_{\widehat{K}} \ge \widehat{\beta}_0\, h_K^d\, |\widehat{v}|_{1,\widehat{K}} \|\widehat{q}\|_{0,\widehat{K}}.$$

Using a standard argument for the estimation between the norms of $\widehat{q}$ and q we obtain

$$(\operatorname{div} v,\ q)_K \ge C_5\, h_K^{d/2}\, |\widehat{v}|_{1,\widehat{K}} \|q\|_{0,K}, \tag{35}$$

with a constant C_5 which is independent of the mesh size and the degree vector $\underline{r}$. Since v in general is not contained in $X^N(K)$, we define the function $\widetilde{v} := h_K^d \widetilde{\mathcal{P}}_K \widehat{v}$ which satisfies $\widetilde{v} \in X^N(K)$. From Lemma 6 we get

$$\begin{aligned} |\widehat{v}|_{1,\widehat{K}} &\ \ge\ C_3^{-1}\, h_K^{-d/2} |\widetilde{v}|_{1,K}\,, && (36)\\ \left|\left(\operatorname{div}(\widetilde{v} - v), q\right)_K\right| &\ \le\ C_4(\mathcal{T}^0)\sqrt{d}\, h_K\, |\widetilde{v}|_{1,K}\, \|q\|_{0,K}\,. && (37) \end{aligned}$$

Using (35)-(37), this implies

$$\begin{aligned} (\operatorname{div} \widetilde{v}, q)_K &= (\operatorname{div} v, q)_K + (\operatorname{div}(\widetilde{v} - v), q)_K \\ &\ge C_5\, h_K^{d/2}\, |\widehat{v}|_{1,K} \|q\|_{0,K} - C_4(\mathcal{T}^0)\sqrt{d}\, h_K\, |\widetilde{v}|_{1,K}\, \|q\|_{0,K} \\ &\ge \left(\frac{C_5}{C_3} - C_4(\mathcal{T}^0)\sqrt{d}\, h_K\right) |\widetilde{v}|_{1,K}\, \|q\|_{0,K} \\ &\ge \beta_0\, |\widetilde{v}|_{1,K}\, \|q\|_{0,K} \end{aligned}$$

for all $K \in \mathcal{T}_h$ and a grid $\mathcal{T}_h$ with $h \le h_0 := C_5/(2C_3C_4(\mathcal{T}^0)\sqrt{d})$ where $\beta_0 = C_5/(2C_3)$. Therefore, (34) is proved since $\widetilde{v} \in X^N(K)$. $\square$

3.4 Summary

Now, we can state the main result on the discrete inf-sup condition (2) for our finite element spaces.

Theorem 10 *Let $\{\mathcal{T}_h\}$ be a family of shape-regular and 1-irregular multilevel adaptive grids generated from a coarse grid $\mathcal{T}^0$. Then, there exists a sufficiently small mesh size $h_0 = h_0(\mathcal{T}^0)$ such that the following result holds. For the pair of finite element spaces (X^N, M^N), defined in (9) and (10) with $X^N = (S^N)^d$ and the degree vector $\underline{r}$, the global inf-sup condition (2) is fulfilled with a constant $\beta > 0$ that satisfies the following estimates*

$$\beta \geq \begin{cases} \beta^*(r^*)\,, & \forall\, \mathcal{T}_h, \underline{r} \quad \textit{with} \quad \max_{K\in\mathcal{T}_h} r_K \leq r^*\,, \\ \beta_1\,, & \forall\, \mathcal{T}_h, \underline{r} \quad \textit{with} \quad h \leq h_0\,, \end{cases} \tag{38}$$

where β_1 is a positive constant which is independent of the mesh size h and the degree vector $\underline{r}$.

Proof. Due to Lemma 3, it is sufficient to verify the global and local inf-sup condition. The global inf-sup condition is guaranteed by Lemma 4. Then, the first estimate in (38) is a consequence of Lemma 7 and (32). The second estimate in (38) follows from Lemma 9. □

By Theorem 10, the uniform inf-sup stability of our hp-finite element pair on multilevel adaptive grids is guaranteed under the following hp-regime. For coarser grids $\mathcal{T}_h$ with $h > h_0$, we do not allow the polynomial degree to be beyond a moderate degree r^*, i.e. $\max_{K\in\mathcal{T}_h} r_K \leq r^*$, but for finer grids $\mathcal{T}_h$ with $h \leq h_0$ the polynomial degree can be chosen arbitrarily.

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