Analysis of Post-Processing for Nonconforming Finite Element Solutions

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Abstract

For a general pair of a given nonconforming and a desired conforming finite element space, we construct and analyze a transfer operator that can be used for a post-processing procedure where a conforming approximation is computed from a nonconforming finite element solution. For this post-processing, we prove local and global estimates of the error between the exact solution and the conforming approximation of the nonconforming discrete solution. Our analysis is independent of the concrete problem that has to be solved. Applications of this analysis are discussed, in particular in the field of a posteriori error control.

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1 Introduction

For mathematical reasons, it is sometimes useful to weaken the continuity requirement of finite element approximations and to work with nonconforming finite elements. An example is the Stokes or Navier-Stokes problem where standard low-order conforming element pairs do not satisfy the Babuška-Brezzi stability condition (see [2]). Here the nonconforming P_1/P_0 triangular element pair of Crouzeix and Raviart (see [4]) satisfies that condition and provides an optimal order of convergence. The same is true for the analogue of this element pair in the case of quadrilateral elements which has been proposed by Rannacher and Turek (see [7]) and which has proved to be attractive for the numerical solution of the Navier-Stokes equations (see [8], [9], [10]). Moreover, an advantage of these nonconforming finite

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elements is that each degree of freedom belongs to at most two elements which simplifies the local communication for a parallelization of the method, particularly in the 3D case.
However, for practical reasons (e.g. graphical post-processing), one wants to have a finite element approximation of the solution which is "nice" in some sense , i.e. which is at least continuous. The obvious idea used in practice is to compute in a post-processing step from a solution u_h^N in a given nonconforming finite element space V_h^N an approximation u_h^C in some desired conforming finite element space V_h^C.
In this paper, we define a general transfer operator $R_h : V_h^N \to V_h^C$ for an arbitrary pair of finite element spaces V_h^N and V_h^C. This operator is based on averaging of local nodal functionals. The computation of the "smoothed function" $u_h^C = R_h u_h^N$ can be implemented efficiently by one loop over the elements followed by one loop over the nodes corresponding to the space V_h^C.
We prove local and global estimates for the error between the exact solution u and the conforming approximation $R_h u_h^N$ in the L^2-norm and the H^1-semi-norm. These estimates show that this error is controlled by the error of the used nonconforming finite element method $u - u_h^N$ and the interpolation error of the exact solution u by means of a conforming finite element function $v_h^C \in V_h^C$.
One "nice" feature of our analysis is that it is independent of the concrete problem which should be solved by the finite element method. The only fact that we have to assume is that our computed nonconforming finite element solution $u_h^N \in V_h^N$ is close to a sufficiently smooth solution u of the continuous problem.
The computable quantities η_K^{post}, being defined as the local norms of the "post-processing error" $u_h^N - R_h u_h^N$ on the elements K, can be used for a posteriori error control. For a model problem, we claim an a posteriori error estimate for the error $u - R_h u_h^N$ in a discrete H^1-semi-norm which contains the quantities η_K^{post}. This assertion will be proved in a forthcoming paper.

2 Preliminaries and Notations

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded polyhedral domain and $\mathcal{T}_h$ a regular decomposition of Ω into elements $K \in \mathcal{T}_h$. The diameter of an element K is denoted by h_K and the elements are assumed to be shape regular in the usual sense (see e.g. [1]). Let us note that this assumption includes the case of a locally refined grid where the ratio h/h_K with $h := \max_{K \in \mathcal{T}_h} h_K$ may be very large for elements K in some critical subregions of Ω.
We consider three finite element spaces defined on the decomposition $\mathcal{T}_h$, a given nonconforming space V_h^N, some desired conforming space V_h^C and an auxiliary space V_h^A. For the definition of these spaces, we use the reference transformation $F_K : \widehat{K} \to K$ that maps a fixed reference element $\widehat{K}$ onto the actual element $K \in \mathcal{T}_h$. We assume, for simplicity, that we have only one reference element $\widehat{K}$. However, our analysis can be generalized to the situation that we have a fixed finite set of reference elements which occurs for example if the decomposition of Ω contains both triangular and quadrilateral elements.

In the following, for a given domain $G \subset \mathbb{R}^d$, we denote by $\mathbb{P}_m(G)$ the set of all polynomials on G with degree less or equal to m.
Let us characterize the nonconforming space V_h^N as a subspace of a corresponding discontinuous space $\widetilde{V}_h^N$, i.e.

$$V_h^N \subset \widetilde{V}_h^N := \left\{ v \in L^2(\Omega) : \ v\big|_K \in \mathcal{P}_K^N \quad \forall\, K \in \mathcal{T}_h \right\}, \tag{1}$$

where

$$\mathcal{P}_K^N := \left\{ p = \widehat{p} \circ F_K^{-1} : \ \widehat{p} \in \widehat{\mathcal{P}}^N \right\} \qquad \text{with} \qquad \mathbb{P}_0(\widehat{K}) \subset \widehat{\mathcal{P}}^N \subset \mathbb{P}_{r_1}(\widehat{K}) \tag{2}$$

is the space of local shape functions of V_h^N on element K defined by some given reference space $\widehat{\mathcal{P}}^N$. A function $v_h \in \widetilde{V}_h^N$ belongs to V_h^N if it is continuous with respect to those nodal functionals that are associated with V_h^N. Similarly, let the conforming space V_h^C be characterized as

$$V_h^C \subset \widetilde{V}_h^C := \left\{ v \in C^0(\Omega) : \ v\big|_K \in \mathcal{P}_K^C \quad \forall\, K \in \mathcal{T}_h \right\}, \tag{3}$$

where $\mathcal{P}_K^C$ is again defined via transformation of some given reference space $\widehat{\mathcal{P}}^C$, i.e.

$$\mathcal{P}_K^C := \left\{ p = \widehat{p} \circ F_K^{-1} : \ \widehat{p} \in \widehat{\mathcal{P}}^C \right\} \qquad \text{with} \qquad \mathbb{P}_0(\widehat{K}) \subset \widehat{\mathcal{P}}^C \subset \mathbb{P}_{r_2}(\widehat{K}) \,. \tag{4}$$

A function $v_h \in \widetilde{V}_h^C$ belongs to V_h^C if it satisfies some desired boundary conditions. The continuity of the piecewise smooth functions in V_h^C implies $V_h^C \subset H^1(\Omega)$.
Moreover, in this paper, we use as an auxiliary space V_h^A the following discontinuous finite element space

$$V_h^A := \left\{ v \in L^2(\Omega) : \ v\big|_K \in \mathcal{P}_K^A \quad \forall\, K \in \mathcal{T}_h \right\}, \tag{5}$$

where

$$\mathcal{P}_K^A := \left\{ p = \widehat{p} \circ F_K^{-1} : \ \widehat{p} \in \widehat{\mathcal{P}}^A \right\} \qquad \text{with} \qquad \widehat{\mathcal{P}}^N + \widehat{\mathcal{P}}^C \subset \widehat{\mathcal{P}}^A \,. \tag{6}$$

A possible choice for $\widehat{\mathcal{P}}^A$ would be for instance $\widehat{\mathcal{P}}^A = \mathbb{P}_{r_3}(\widehat{K})$ with $r_3 = \max(r_1, r_2)$. By construction this auxiliary space has the property

$$V_h^N + V_h^C \subset V_h^A \tag{7}$$

which will be exploited in the subsequent analysis.
Finally, let us introduce some general notations. We denote by $\mathrm{card}(\mathcal{M})$ the number of elements of a finite set $\mathcal{M}$. Let Y' be the dual space of a given space Y. For a set $G \subset \mathbb{R}^d$, we denote by $\mathrm{int}(G)$ and $\overline{G}$ the interior and closure of G, respectively. Further, let $\|\cdot\|_{m,p,G}$ and $|\cdot|_{m,p,G}$ denote the usual norm and semi-norm in the Sobolev space $W^{m,p}(G)$ where we omit the index p if $p = 2$. We use the notation C for a general constant which may have different values at different occurrences but which is independent of the mesh parameters h_K of the elements $K \in \mathcal{T}_h$.

3 A General Transfer Operator

For the definition of the transfer operator we need a more detailed characterization of the conforming finite element space V_h^C by means of its basis functions and nodal functionals. Let $\{\varphi_j^C : j \in J^C\}$ be the finite element basis of V_h^C. The indices $j \in J^C$ are called "nodes". For a given element $K \in \mathcal{T}_h$, we denote by $J^C(K)$ the set of all nodes $j \in J^C$ that are associated with K, i.e.

$$J^C(K) := \left\{ j \in J^C : \operatorname{supp}(\varphi_j^C) \cap \operatorname{int}(K) \neq \emptyset \right\}. \tag{8}$$

The element nodes $j \in J^C(K)$ are associated with a set of local basis functions $\varphi_j^C\big|_K$ and a set of local nodal functionals $N_{j,K} \in (C^s(\overline{K}))'$ which are "dual" (see [1]) in the sense that

$$N_{i,K}(\varphi_j^C\big|_K) = \delta_{ij} \qquad \forall\, i,j \in J^C(K), \tag{9}$$

where δ_{ij} denotes the Kronecker delta. Note that we simply write $N_{i,K}$ instead of $N_{i,K}^C$ since we do not consider nodal functionals for the finite element spaces V_h^N and V_h^A. The parameter s depends on the type of the nodal functionals and corresponds to the highest order of derivatives that have to be evaluated for a function v in order to define $N_{j,K}(v)$. Typical examples for nodal functionals in the case $s = 0$ are point values $N_{j,K}(v) := v_{|K}(a_j)$, where $a_j \in \overline{K}$ is a given point, or integral mean values $N_{j,K}(v) := |\Gamma_j|^{-1} \int_{\Gamma_j} v\big|_K \omega_j \, d\gamma$, where Γ_j is a face of element K and ω_j a given polynomial weighting function.
Now, by means of the local nodal functionals $N_{j,K}$, we define global functionals N_j on a general space $Y(\mathcal{T}_h, s)$ of piecewise smooth functions given by

$$Y(\mathcal{T}_h, s) := \left\{ v \in L^2(\Omega) : \; v\big|_K \in C^s(\overline{K}) \quad \forall\, K \in \mathcal{T}_h \right\}. \tag{10}$$

For a node $j \in J^C$, let

$$\mathcal{T}_{h,j} := \left\{ K \in \mathcal{T}_h : \; j \in J^C(K) \right\} \tag{11}$$

be the set of all elements that are associated with this node. Then, as a natural choice for $N_j \in (Y(\mathcal{T}_h, s))'$, we define the value $N_j(v)$ as the arithmetic mean of all local values $N_{j,K}(v_{|K})$ that are associated with the node $j \in J^C$, i.e.

$$N_j(v) := (\operatorname{card}(\mathcal{T}_{h,j}))^{-1} \sum_{K \in \mathcal{T}_{h,j}} N_{j,K}(v\big|_K) \qquad \forall\, v \in Y(\mathcal{T}_h, s). \tag{12}$$

By means of these global nodal functionals we define a general transfer operator $R_h : Y(\mathcal{T}_h, s) \to V_h^C$ as

$$R_h v := \sum_{j \in J^C} N_j(v)\, \varphi_j^C. \tag{13}$$

Note that, by (13), $R_h v$ is defined in particular for all $v \in V_h^A$ since by definition V_h^A is a subspace of $Y(\mathcal{T}_h, s)$. In the following, we only need the restriction of R_h to V_h^A and denote this operator again by $R_h : V_h^A \to V_h^C$.

Remark 1 *The computation of the function $R_h v \in V_h^C$ for a given nonconforming finite element function $v \in V_h^N$ can be implemented efficiently. One has to compute the nodal values v_j^C, $j \in J^C$, for the basis representation $R_h v = \sum_{j \in J^C} v_j^C \varphi_j^C$. At the beginning two vectors $\mathbf{c} = (c_j)_{j \in J^C}$ and $\mathbf{v^C} = (v_j^C)_{j \in J^C}$ are initialized by zero. Then, in a first loop over the elements, for each element $K \in \mathcal{T}_h$ all the contributions $N_{j,K}(v\big|_K)$ are computed and added to v_j^C for $j \in J^C(K)$, i.e. for those nodes $j \in J^C$ that are associated with element K. Furthermore, if $j \in J^C(K)$, the value of c_j is increased by one. In a final loop over the nodes $j \in J^C$, the right values $v_j^C = N_j(v)$ are computed by $v_j^C := v_j^C / c_j$.*

The next lemma shows that R_h is the identity on the subspace V_h^C of V_h^A which is an essential property for our analysis.

Lemma 2 *For the transfer operator R_h defined in (13) it holds*

$$R_h v = v \qquad \forall\, v \in V_h^C. \tag{14}$$

Proof. Suppose that $v \in V_h^C$ has the basis representation $v = \sum_{j \in J^C} \underline{v}_j \, \varphi_j^C$ with $\underline{v}_j \in \mathbb{R}$. The property $\varphi_j^C\big|_K \equiv 0$ for all $j \in J^C \setminus J^C(K)$ implies the following local representation of v

$$v\big|_K = \sum_{j \in J^C(K)} \underline{v}_j \, \varphi_j^C\big|_K \qquad \forall\, K \in \mathcal{T}_h.$$

Applying the functional $N_{j,K}$ to both sides of this equation and using (9) leads to

$$N_{j,K}(v\big|_K) = \underline{v}_j \qquad \forall\, K \in \mathcal{T}_{h,j}.$$

Thus, by the definition (12) of N_j we obtain $N_j(v) = \underline{v}_j$. Using the local representation of $R_h v$, we get for arbitrary $K \in \mathcal{T}_h$

$$R_h v\big|_K = \sum_{j \in J^C(K)} N_j(v) \, \varphi_j^C\big|_K = \sum_{j \in J^C(K)} \underline{v}_j \, \varphi_j^C\big|_K = v\big|_K$$

which proves (14). ■

4 Local L^2-Stability

In this section, we present a local L^2-stability estimate of the transfer operator $R_h : V_h^A \to V_h^C$ which plays an essential role in the subsequent analysis. Let us first introduce some notation. We denote by $\{\varphi_j^A : j \in J^A\}$ the finite element basis of V_h^A and by $J^A(K)$ the

set of all nodes $j \in J^A$ that are associated with element $K \in \mathcal{T}_h$ in an analogous manner as in (8). For a given element K, let

$$\delta(K) := \bigcup_{j \in J^C(K)} \bigcup_{\widetilde{K} \in \mathcal{T}_{h,j}} \widetilde{K} \tag{15}$$

be the vicinity of K consisting of all elements $\widetilde{K} \in \mathcal{T}_h$ that have a V_h^C-node in common with K, i.e. $J^C(K) \cap J^C(\widetilde{K}) \neq \emptyset$.
In the following, we list some basic assumptions that are needed for the proof of our stability estimate. For the basis functions of V_h^C, we assume that there is a constant C_1 such that

$$\textbf{(A1)} \qquad \|\varphi_j^C\|_{0,K} \leq C_1 h_K^{d/2} \qquad \forall\, j \in J^C(K),\ K \in \mathcal{T}_h.$$

Furthermore, we suppose that the values of the local nodal functionals of the space V_h^C applied to the basis functions of V_h^A are uniformly bounded, i.e. that there exists a constant C_2 such that

$$\textbf{(A2)} \qquad \left|N_{j,K}(\varphi_i^A\big|_K)\right| \leq C_2 \qquad \forall\, j \in J^C(K),\ i \in J^A(K),\ K \in \mathcal{T}_h.$$

Finally, we require that changes of the mesh size within the vicinity $\delta(K)$ of an element K are uniformly bounded by a constant factor C_3, i.e.

$$C_3^{-1}\, h_{\widetilde{K}} \leq h_K \leq C_3\, h_{\widetilde{K}} \qquad \forall\, \widetilde{K} \subset \delta(K),\ \ \widetilde{K}, K \in \mathcal{T}_h. \tag{16}$$

These assumptions are not restrictive since (A1) and (A2) are fulfilled for the usual finite element spaces and (16) is satisfied if the mesh $\mathcal{T}_h$ consists of shape regular elements.
Now we can prove the following local L^2-stability estimate of the operator R_h.

Lemma 3 *Suppose that the spaces V_h^C and V_h^A fulfill the assumptions (A1) and (A2) and that the mesh $\mathcal{T}_h$ consists of shape regular elements such that (16) is satisfied. Then, for the transfer operator $R_h : V_h^A \to V_h^C$ defined in (13), there exists a constant C_S such that*

$$\|R_h v\|_{0,K} \leq C_S\, \|v\|_{0,\delta(K)} \qquad \forall\, v \in V_h^A, \quad K \in \mathcal{T}_h\,. \tag{17}$$

Proof. Let $v \in V_h^A$ and $K \in \mathcal{T}_h$ be given. From the local representation of $R_h v$ on K and the estimate (A1) we obtain

$$\|R_h v\|_{0,K} = \|\sum_{j \in J^C(K)} N_j(v)\, \varphi_j^C\, \|_{0,K} \ \leq\ C_1 h_K^{d/2} \sum_{j \in J^C(K)} |N_j(v)|. \tag{18}$$

Let $v = \sum_{i \in J^A} \underline{v}_i\, \varphi_i^A$ be the basis representation of v and $\underline{v} = (\underline{v}_i)_{i \in J^A} \in \mathbb{R}^{J^A}$ the corresponding coefficient vector. Then, from the local representation of v we get

$$N_{j,\widetilde{K}}(v\big|_{\widetilde{K}}) = \sum_{i \in J^A(\widetilde{K})} \underline{v}_i\, N_{j,\widetilde{K}}(\varphi_i^A\big|_{\widetilde{K}}) \qquad \forall\, \widetilde{K} \in \mathcal{T}_{h,j}.$$

For each $\widetilde{K} \in \mathcal{T}_h$, let

$$\|\underline{v}\|_{\ell^2(\widetilde{K})} := \Big(\sum_{i \in J^A(\widetilde{K})} \underline{v}_i^2 \Big)^{1/2}$$

be the Euclidean norm of the local coefficient vector associated with the element $\widetilde{K}$. Using (A2) and the fact that $\mathrm{card}(J^A(\widetilde{K})) \le d_A := \dim(\widehat{\mathcal{P}}^A)$ for all $\widetilde{K} \in \mathcal{T}_h$, we get

$$\big| N_{j,\widetilde{K}}(v\big|_{\widetilde{K}}) \big| \le C_2 \, d_A^{1/2} \, \|\underline{v}\|_{\ell^2(\widetilde{K})} \qquad \forall \, \widetilde{K} \in \mathcal{T}_{h,j}. \tag{19}$$

By similar arguments as for the proof of inverse inequalities (transformation onto reference element and equivalence of norms on finite dimensional spaces, see e.g. [3]) we can show that there exists a constant C_4 such that

$$\|\underline{v}\|_{\ell^2(\widetilde{K})} \le C_4 \, h_{\widetilde{K}}^{-d/2} \|v\|_{0,\widetilde{K}} \qquad \forall \, \widetilde{K} \in \mathcal{T}_h. \tag{20}$$

Combining (20) with (19) and (16), we obtain for each $j \in J^C(K)$ the estimate

$$\big| N_{j,\widetilde{K}}(v\big|_{\widetilde{K}}) \big| \le C \, h_K^{-d/2} \|v\|_{0,\widetilde{K}} \qquad \forall \, \widetilde{K} \in \mathcal{T}_{h,j}$$

with the constant $C = C_2 \, d_A^{1/2} C_4 \, C_3^{d/2}$. The fact that $\widetilde{K} \subset \delta(K)$ for all $\widetilde{K} \in \mathcal{T}_{h,j}$, where $j \in J^C(K)$, and the definition (12) of $N_j(v)$ imply

$$|N_j(v)| \le C \, h_K^{-d/2} \|v\|_{0,\delta(K)} \qquad \forall \, j \in J^C(K).$$

Together with (18) this proves (17). ■

5 Error Estimates

5.1 L^2-Estimates

Using the local L^2-stability of R_h and the fact that this operator is the identity on the subspace V_h^C of V_h^A, we can prove the following local error estimate.

Theorem 4 *Assume that the elements $K \in \mathcal{T}_h$ are shape regular such that (16) is satisfied. Furthermore, suppose that the spaces V_h^C and V_h^A fulfill the assumptions (A1) and (A2). Let $u \in L^2(\Omega)$ be the exact solution of some problem and $u_h^N \in V_h^N$ the nonconforming finite element solution of a corresponding discrete problem. Then, for the conforming approximation $R_h u_h^N \in V_h^C$ of u_h^N, it holds the error estimate*

$$\|u - R_h u_h^N\|_{0,K} \le C \left\{ \|u - u_h^N\|_{0,\delta(K)} + \inf_{v_h^C \in V_h^C} \|u - v_h^C\|_{0,\delta(K)} \right\} \tag{21}$$

for each element $K \in \mathcal{T}_h$ with a constant C independent of K.

Proof. Let v_h^C be an arbitrary element of V_h^C and $K \in \mathcal{T}_h$. Using (7) we have $u_h^N - v_h^C \in V_h^A$ and the stability estimate (17) yields

$$\|R_h(u_h^N - v_h^C)\|_{0,K} \leq C_S \|u_h^N - v_h^C\|_{0,\delta(K)} .$$

With $R_h v_h^C = v_h^C$ we obtain

$$\begin{aligned} \|u - R_h u_h^N\|_{0,K} &= \|(u - v_h^C) - R_h(u_h^N - v_h^C)\|_{0,K} \\ &\leq \|u - v_h^C\|_{0,K} + C_S \|u_h^N - v_h^C\|_{0,\delta(K)} \\ &\leq (1 + C_S)\|u - v_h^C\|_{0,\delta(K)} + C_S \|u - u_h^N\|_{0,\delta(K)} \end{aligned}$$

which implies the estimate (21). ■

From this local L^2-estimate we immediately get the following global one.

Theorem 5 *Under the assumptions of Theorem 4, it holds the error estimate*

$$\|u - R_h u_h^N\|_{0,\Omega} \leq C \left\{ \|u - u_h^N\|_{0,\Omega} + \inf_{v_h^C \in V_h^C} \|u - v_h^C\|_{0,\Omega} \right\} . \tag{22}$$

Proof. Let v_h^C be a fixed element of V_h^C. Then, from (21) we obtain for each $K \in \mathcal{T}_h$

$$\|u - R_h u_h^N\|_{0,K}^2 \leq C \sum_{\widetilde{K} \subset \delta(K)} \left\{ \|u - u_h^N\|_{0,\widetilde{K}}^2 + \|u - v_h^C\|_{0,\widetilde{K}}^2 \right\} \tag{23}$$

Using the shape regularity of the elements, it follows by geometrical arguments that there is a constant C_5 such that

$$\max_{\widetilde{K} \in \mathcal{T}_h} \text{card} \left(\{K \in \mathcal{T}_h : \widetilde{K} \subset \delta(K)\} \right) \leq C_5 . \tag{24}$$

Then, summation of (23) for all $K \in \mathcal{T}_h$ yields

$$\|u - R_h u_h^N\|_{0,\Omega}^2 \leq C C_5 \left\{ \|u - u_h^N\|_{0,\Omega}^2 + \|u - v_h^C\|_{0,\Omega}^2 \right\} .$$

Since v_h^C can be any element of V_h^C, we obtain (22). ■

5.2 H^1-Estimates

In this section, we want to present error estimates in a discrete H^1-semi-norm which is a natural generalization of the usual H^1-semi-norm to the case of element-wise smooth functions. For a subdomain $G \subset \Omega$, let

$$\mathcal{T}_h(G) := \{K \in \mathcal{T}_h : K \subset G\} \tag{25}$$

be the set of all elements belonging to G. Then, for $G \subset \Omega$ with $\overline{G} = \bigcup_{K \in \mathcal{T}_h(G)} \overline{K}$ and for functions $v \in V_h^N + H^1(\Omega)$, we define the discrete H^1-semi-norm by

$$|v|_{1,h,G} := \Big(\sum_{K \in \mathcal{T}_h(G)} |v|_{1,K}^2 \Big)^{1/2}. \tag{26}$$

In contrast to L^2-estimates, here we have to assume at least the following weak continuity condition for functions of the nonconforming finite element space V_h^N

$$\textbf{(A3)} \qquad \int_{\partial K \cap \partial K'} \big(v_h\big|_K - v_h\big|_{K'} \big)\, ds = 0 \qquad \forall\, v_h \in V_h^N, \quad K, K' \in \mathcal{T}_h\,,$$

where the integral in (A3) has to be understood as a $(d-1)$-dimensional integral. On the other hand, this condition is a minimal requirement to ensure consistency of a nonconforming finite element discretization, i.e. it is a very natural assumption. Furthermore, for our proof of H^1-estimates, we have to take into account the boundary conditions that are imposed on the discrete spaces V_h^C and V_h^N. For simplicity, we consider here the case of homogeneous Dirichlet boundary conditions, i.e. we suppose $V_h^C = \widetilde{V}_h^C \cap H_0^1(\Omega)$ and the following condition for the nonconforming space V_h^N

$$\textbf{(A4)} \qquad \int_{\partial K \cap \partial \Omega} v_h\, ds = 0 \qquad \forall\, v_h \in V_h^N, \quad K \in \mathcal{T}_h.$$

The basic tool for deriving H^1-error estimates for the conforming approximation $R_h u_h^N \in V_h^C$ of a nonconforming solution $u_h^N \in V_h^N$ is again a local stability estimate of the transfer operator R_h which is presented in the next lemma.
At first, we have to define a new local vicinity $\delta'(K)$ of a given element K which differs from the vicinity $\delta(K)$ defined in (15) only if K is located near the boundary $\partial\Omega$. Two different elements $K, K' \in \mathcal{T}_h$ are called "face-neighbors" if $\partial K \cap \partial K'$ is a common $(d-1)$-dimensional face of K and K'. For $K \in \mathcal{T}_h$, let $\gamma^* := \partial\Omega \cap \overline{\delta(K)}$ be a non-empty set of points with a dimension less than $d-1$, i.e. a vertex or, in the 3D case, an edge of the element K. Furthermore, in this case, let $K^* \in \mathcal{T}_h(\delta(K))$ be an element in $\delta(K)$ with $\gamma^* \subset \overline{K^*}$. Then, there exists a chain of elements $\kappa_i \in \mathcal{T}_h$, $i = 1, \ldots, m_K$, such that $\kappa_1 := K^*$, $\gamma^* \subset \overline{\kappa_i}$ for all $1 \le i \le m_K$, κ_i and κ_{i+1} are face-neighbors for all $1 \le i < m_K$ if $m_K > 1$ and the "final chain element" $K^b := \kappa_{m_K}$ has a $(d-1)$-dimensional boundary face $\Gamma^b \subset \partial K^b \cap \partial\Omega$ with $\gamma^* \subset \Gamma^b$. Now, we define

$$\delta'(K) := \delta(K) \cup \bigcup_{i=1}^{m_K} \kappa_i \tag{27}$$

as the new vicinity of K in the above defined case where K is in some sense near to the boundary $\partial\Omega$. In all other cases concerning the location of K, we define $\delta'(K) := \delta(K)$. Due to the shape-regularity of the grid $\mathcal{T}_h$, all the numbers m_K are uniformly bounded, i.e. $m_K \le C$ for all $K \in \mathcal{T}_h$, and furthermore we have, similarly to (16), that there exists a constant C_3' such that

$$(C_3')^{-1}\, h_{\widetilde{K}} \le h_K \le C_3'\, h_{\widetilde{K}} \qquad \forall\, \widetilde{K} \subset \delta'(K), \;\; \widetilde{K}, K \in \mathcal{T}_h. \tag{28}$$

Lemma 6 *Let $V_h^C = \widetilde{V}_h^C \cap H_0^1(\Omega)$ with $\widetilde{V}_h^C$ from (3). Assume that the spaces V_h^C and V_h^A fulfill the conditions (A1) and (A2) and that V_h^N satisfies (A3) and (A4). The mesh $\mathcal{T}_h$ is supposed to consist of shape regular elements such that (28) is satisfied. Then, for the transfer operator R_h defined in (13), there exists a constant C_S' such that*

$$|R_h v|_{1,K} \leq C_S' \, |v|_{1,h,\delta'(K)} \qquad \forall\, v \in V_h^N + V_h^C, \quad K \in \mathcal{T}_h \,. \tag{29}$$

Proof. Let $v \in V_h^N + V_h^C$ and $K \in \mathcal{T}_h$ be fixed. The proof of (29) consists of 4 steps.
STEP 1. We construct a function $c_K \in V_h^C$ which is a constant approximation of v in the vicinity $\delta'(K)$ of the element K (defined as above). More precisely, let

$$c_K(x) = v\big|_{K^*}(z_{K^*}) \qquad \forall\, x \in \overline{\delta'(K)}, \tag{30}$$

where $K^* \in \mathcal{T}_h(\delta'(K))$ is a properly chosen element and $z_{K^*} \in \overline{K^*}$ a suitable point defined as follows.
In the case, when $\delta'(K)$ has a positive distance to the boundary $\partial\Omega$, i.e. when $\overline{\delta'(K)} \subset \mathrm{int}(\Omega)$, we choose $K^* = K$ and z_{K^*} as the barycentre of K. It is clear that in this case there exists a continuous extension of the constant function $c_K\big|_{\delta'(K)}$ to the whole domain Ω such that $c_K \in V_h^C$, that means in particular that the homogeneous Dirichlet boundary conditions due to $V_h^C \subset H_0^1(\Omega)$ are satisfied.
In the remaining case, i.e. when $\overline{\delta'(K)} \cap \partial\Omega \neq \emptyset$, we choose $c_K(x) := 0$ for all $x \in \overline{\Omega}$ in order to guarantee that $c_K \in V_h^C \subset H_0^1(\Omega)$. Therefore, in order to satisfy (30), we choose $K^* \in \mathcal{T}_h(\delta'(K))$ as a boundary element, that has a $(d-1)$-dimensional boundary face $\Gamma^* \subset \partial K^* \cap \partial\Omega$, and we choose $z_{K^*} \in \Gamma^*$ as a point with $v\big|_{K^*}(z_{K^*}) = 0$. The existence of such an element K^* follows from the definition of the vicinity $\delta'(K)$ and the existence of such point z_{K^*} is a consequence of $v \in V_h^N + V_h^C$ and the assumption (A4).
STEP 2. We will prove the following estimate

$$\|v - c_K\|_{0.\infty,\widetilde{K}} \leq C_6 \, h_K \sum_{T \in \mathcal{T}_h(\delta'(K))} |v|_{1,\infty,T} \qquad \forall\, \widetilde{K} \in \mathcal{T}_h(\delta'(K)) \,. \tag{31}$$

In the case $\widetilde{K} = K^*$, the standard interpolation estimate

$$\|v - c_K\|_{0.\infty,K^*} = \|v - v(z_{K^*})\|_{0.\infty,K^*} \leq h_{K^*} |v|_{1,\infty,K^*} \tag{32}$$

and relation (28) imply the estimate (31) with $C_6 = C_3'$. In the case $\widetilde{K} \in \mathcal{T}_h(\delta'(K))$ with $\widetilde{K} \neq K^*$, there exists a chain of face-neighbored elements

$$K^* =: K_0 \to \cdots \to K_n := \widetilde{K}$$

such that for all $\mu = 1, \ldots, n$, the elements $K_{\mu-1}$ and K_μ have a common $(d-1)$-dimensional face $\Gamma_\mu := \partial K_{\mu-1} \cap \partial K_\mu$. From $v \in V_h^N + V_h^C$ and the consistency condition (A3), it follows that there exists a point $x_\mu \in \Gamma_\mu$ such that

$$v\big|_{K_{\mu-1}}(x_\mu) = v\big|_{K_\mu}(x_\mu) \qquad \forall\, \mu = 1, \ldots, n \,. \tag{33}$$

Using a standard interpolation estimate and the relation (28) we get for the point $x_n \in \overline{K_n}$

$$\|v - v(x_n)\|_{0.\infty,K_n} \leq h_{K_n} |v|_{1,\infty,K_n} \leq C_3' \, h_K |v|_{1,\infty,K_n} \, .$$

Applying this estimate and using $K_n = \widetilde{K}$ and (30) with $v^* := v\big|_{K^*}(z_{K^*})$ we obtain

$$\|v - c_K\|_{0.\infty,\widetilde{K}} \leq \|v - v(x_n)\|_{0.\infty,\widetilde{K}} + |\, v(x_n) - v^*| \, \leq \, C_3' \, h_K |v|_{1,\infty,\widetilde{K}} + |\, v(x_n) - v^*| \, . \quad (34)$$

Furthermore, we have $x_1 \in \overline{K_0} = \overline{K^*}$, i.e. x_1 and z_{K^*} belong to the same element K^*, which implies

$$|\, v(x_1) - v^*| \, \leq \, C_3' \, h_K |v|_{1,\infty,K^*} \, . \quad (35)$$

In the case $n = 1$, from (34) and (35) we obtain the estimate (31) with $C_6 = C_3'$. In the case $n \geq 2$, using (35), (28) and the fact that $x_{\mu+1}, x_\mu \in \overline{K_\mu}$ for all $\mu = 1, \ldots, n-1$, we can estimate

$$\begin{aligned} |\, v(x_n) - v^*| \;\; &\leq \;\; \sum_{\mu=1}^{n-1} |\, v(x_{\mu+1}) - v(x_\mu)| \;\; + \;\; |\, v(x_1) - v^*| \\ &\leq \;\; \sum_{\mu=1}^{n-1} C_3' \, h_K \, |v|_{1,\infty,K_\mu} \;\; + \;\; C_3' \, h_K \, |v|_{1,\infty,K^*} \, . \end{aligned}$$

Together with (34) this implies the estimate (31) with $C_6 = C_3'$.
STEP 3. From (31) we will now derive an estimate of $v - c_K$ in the L^2-norm. Let $\widetilde{K} \in \mathcal{T}_h(\delta'(K))$ be a given element in the vicinity of K. Then, using (31), inverse inequalities (see e.g. [3]) and the local mesh-size relation (28) we get

$$\begin{aligned} \|v - c_K\|_{0,\widetilde{K}}^2 \;\; &\leq \;\; C \, h_{\widetilde{K}}^d \|v - c_K\|_{0,\infty,\widetilde{K}}^2 \\ &\leq \;\; C \, h_{\widetilde{K}}^d h_K^2 \left(\sum_{T \in \mathcal{T}_h(\delta'(K))} |v|_{1,\infty,T} \right)^2 \\ &\leq \;\; C \, h_K^2 \, \mathrm{card}(\mathcal{T}_h(\delta'(K))) \sum_{T \in \mathcal{T}_h(\delta'(K))} (h_{\widetilde{K}}/h_T)^d \, |v|_{1,T}^2 \, . \end{aligned}$$

Since $h_{\widetilde{K}}/h_T \leq (C_3')^2$ for all $\widetilde{K}, T \in \mathcal{T}_h(\delta'(K))$ and $\mathrm{card}(\mathcal{T}_h(\delta'(K))) \leq C$ due to the shape-regularity of $\mathcal{T}_h$, it follows the L^2-norm estimate

$$\|v - c_K\|_{0,\widetilde{K}} \, \leq \, C \, h_K \, |v|_{1,h,\delta'(K)} \qquad \forall \, \widetilde{K} \in \mathcal{T}_h(\delta'(K)) \, . \quad (36)$$

STEP 4. Now we are able to prove (29). Since $c_K \in V_h^C$, it follows from Lemma 2 that $c_K = R_h c_K$. Using the fact that c_K is constant on K and applying an inverse inequality for functions of the space V_h^C we get

$$|R_h v|_{1,K} = |R_h v - c_K|_{1,K} = |R_h(v - c_K)|_{1,K} \leq C \, h_K^{-1} \|R_h(v - c_K)\|_{0,K} \, .$$

The local L^2-stability (17) of the operator R_h and the estimate (36) for all elements $\widetilde{K} \subset \delta(K) \subset \delta'(K)$ yield

$$\|R_h(v - c_K)\|_{0,K} \leq C_S \, \|v - c_K\|_{0,\delta(K)} \leq C \, C_S \, h_K \, |v|_{1,h,\delta'(K)} \, ,$$

which finally proves the estimate (29). ■

Using the local H^1-stability of R_h proved in Lemma 6 and the fact that R_h is the identity operator on the subspace V_h^C of V_h^A, we can prove the following local and global error estimates.

Theorem 7 *Suppose that the assumptions of Lemma 6 are fulfilled. Let $u \in H_0^1(\Omega)$ be the exact solution of some problem and $u_h^N \in V_h^N$ the nonconforming finite element solution of a corresponding discrete problem. Then, for the conforming approximation $R_h u_h^N \in V_h^C$ of u_h^N, it holds for each element $K \in \mathcal{T}_h$ the local error estimate*

$$|u - R_h u_h^N|_{1,K} \leq C \left\{ |u - u_h^N|_{1,h,\delta'(K)} \, + \, \inf_{v_h^C \in V_h^C} |u - v_h^C|_{1,\delta'(K)} \right\} \tag{37}$$

with a constant C independent of K. Furthermore, it holds the global estimate

$$|u - R_h u_h^N|_{1,h,\Omega} \leq C \left\{ |u - u_h^N|_{1,h,\Omega} \, + \, \inf_{v_h^C \in V_h^C} |u - v_h^C|_{1,\Omega} \right\} \, . \tag{38}$$

Proof. The proof of the error estimates (37) and (38) is completely analogous to the proof of the corresponding L^2-norm estimates (21) and (22) in Theorem 4 and Theorem 5. The only modification is to replace the L^2-norm by the discrete H^1-semi-norm. ■

Remark 8 *The presented theory can be generalized to the case where the desired continuous finite element space V_h^C corresponds to a grid $\mathcal{T}_h^C$ which is generated from the original grid $\mathcal{T}_h$ by applying a regular mesh-refinement procedure on $\mathcal{T}_h$, where "regular" means that each cell $K \in \mathcal{T}_h$ is divided into a number of $s_K \geq 1$ "son-elements" $T_i(K) \in \mathcal{T}_h^C$, $i = 1, \ldots, s_K$, and that all these son-elements form a regular finite element grid.*

6 Applications

6.1 Graphical Output

The basic element of any graphical output consists in drawing a straight line. For example, the visualization of any curved line is realized by drawing a suitable sequence of straight

lines. One basic task for the visualization of a scalar finite element function u_h is to plot a set of "contour-lines" $\mathcal{L}_i$, $i = 1, \ldots, M$, where $\mathcal{L}_i$ is the set of those points x in the domain Ω where the function u_h has a constant value, i.e. $u_h(x) = c_i$ for all $x \in \mathcal{L}_i$.
If for some element $K \in \mathcal{T}_h$, $u_h\big|_K$ is a linear function the "K-part" of the i-th contour-line $\mathcal{L}_i^K := \mathcal{L}_i \cap K$ is a straight line.
That means, a piecewise linear finite element function would be desirable for graphical output. This leads to the idea that we take for our desired conforming finite element space V_h^C the space of piecewise linear continuous functions on a triangular or tetrahedral grid $\mathcal{T}_h^C$, where $\mathcal{T}_h^C$ is derived from the original grid $\mathcal{T}_h$ by a regular refinement process, which means that $\mathcal{T}_h^C$ is a regular decomposition of the domain Ω and that for each element $K^C \in \mathcal{T}_h^C$, there exists exactly one "father-element" $K \in \mathcal{T}_h$ with $K^C \subset K$. Thus, the definition of the transfer operator $R_h : V_h^N \to V_h^C$ in Section 3 can be generalized to the situation where V_h^N is associated with the grid $\mathcal{T}_h$ and V_h^C with the grid $\mathcal{T}_h^C$. We simply define, as in (13), $R_h v := \sum_{j \in J^C} N_j(v)\, \varphi_j^C$ with

$$N_j(v) := \left(\mathrm{card}(\mathcal{T}_{h,j}^C)\right)^{-1} \sum_{K \in \mathcal{T}_{h,j}^C} N_{j,K}(v\big|_K) \tag{39}$$

and

$$\mathcal{T}_{h,j}^C := \left\{K \in \mathcal{T}_h^C : \; j \in J^C(K)\right\} . \tag{40}$$

Here again $N_{j,K}(\cdot)$ denotes the local (w.r.t. $K \in \mathcal{T}_h^C$) nodal functional associated with the node $j \in J^C$ of the space V_h^C, i.e. in our context of a piecewise linear finite element space V_h^C that $N_{j,K}(\cdot)$ corresponds to a point value $N_{j,K}(v) := v\big|_K(a_j)$ where a_j is the j-th vertex-node.
Let us note that the above ideas can be applied also to the case where V_h^N is a conforming finite element space constructed with polynomials of higher degree.

6.2 A Posteriori Error Estimation

Let us consider the following model problem: *Find a function u such that*

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \Omega\,, \\ u &= 0 \quad \text{on} \quad \partial\Omega\,, \end{aligned} \tag{41}$$

where Ω is a bounded polygonal domain and $f \in L^2(\Omega)$ a given function. If $u_h \in V_h \subset H_0^1(\Omega)$ is a conforming finite element approximation for problem (41), then it is well-known (see e.g. [11]) that

$$\eta := \left(\sum_{K \in \mathcal{T}_h} \eta_K^2\right)^{1/2} \tag{42}$$

with

$$\eta_K := \left\{ h_K^2 \| f + \triangle u_h \|_{0,K}^2 + \sum_{E \in \mathcal{E}(K)} h_E \, \| \, [n_E \cdot \nabla u_h]_E \, \|_{L^2(E)}^2 \right\}^{1/2} \tag{43}$$

is a suitable a posteriori error estimator, where $\mathcal{E}(K)$ denotes the set of the $(d-1)$-dimensional faces of the element K, h_E denotes the diameter of the face $E \in \mathcal{E}(K)$, n_E is a unit vector normal to E and $[v]_E$ denotes the jump of the function $v(\cdot)$ along the element face E. For this estimator, it holds

$$|u - u_h|_{1,\Omega} \leq C_7 \, \eta \, . \tag{44}$$

However, if $u_h^N \in V_h^N \not\subset H_0^1(\Omega)$ is a nonconforming finite element approximation of the weak solution $u \in H_0^1(\Omega)$ of problem (41), then it is also well-known from the literature (see [6], [5]) that the estimators η_K have to be modified into $\widetilde{\eta}_K$ by adding some extra terms containing the jumps of the derivatives of u_h^N in the tangential directions t_E of the element faces $E \in \mathcal{E}(K)$. For the corresponding modified estimator $\widetilde{\eta} := \left(\sum_{K \in \mathcal{T}_h} \widetilde{\eta}_K^2 \right)^{1/2}$ the a posteriori error estimate

$$|u - u_h|_{1,h,\Omega} \leq C \, \widetilde{\eta} \tag{45}$$

has been proved for the two-dimensional case. However, it seems that the analysis does not work in the three-dimensional case.
Using our post-processing operator $R_h : V_h^N \to V_h^C \subset H_0^1(\Omega)$ to compute a conforming approximation $R_h u_h^N$ of the solution, we can define the following computable "post-processing error"

$$\eta^{post} := \left(\sum_{K \in \mathcal{T}_h} (\eta_K^{post})^2 \right)^{1/2} \qquad \text{with} \qquad \eta_K^{post} := |u_h^N - R_h u_h^N|_{1,K} \tag{46}$$

We claim that we can prove the following a posteriori error estimate

$$|u - R_h u_h^N|_{1,\Omega} \leq C_7 \, \eta + \eta^{post} \, , \tag{47}$$

where C_7 and η are the quantities in (44) and (42), (43). The proof of (47) will be presented in a forthcoming paper. Note that from (47) we directly get the a posteriori error estimate for the nonconforming finite element solution $u_h^N \in V_h^N$

$$|u - u_h^N|_{1,h,\Omega} \leq C_7 \, \eta + 2\eta^{post} \tag{48}$$

and let us emphasize that this error bound can be proved for the general d-dimensional case $\Omega \subset \mathbb{R}^d$.

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