Musterlösungen Blatt 10

1.) $$(A|E_3) = \left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -2z_1 \\ -z_1 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{-1} & -2 & -2 & 1 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ -z_2 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right) \begin{array}{l} \\ \cdot(-1) \\ \cdot(-1) \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & \boxed{1} & -1 & 1 & -1 \end{array}\right) \begin{array}{l} -3\cdot z_3 \\ -2\cdot z_3 \\ \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -3 & 3 \\ 0 & \boxed{1} & 0 & 4 & -3 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right) \begin{array}{l} -2\cdot z_2 \\ \\ \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} -4 & 3 & -1 \\ 4 & -3 & 2 \\ -1 & 1 & -1 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right)$$

$$\Rightarrow A^{-1} = \underline{\underline{\begin{pmatrix} -4 & 3 & -1 \\ 4 & -3 & 2 \\ -1 & 1 & -1 \end{pmatrix}}}$$

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2) $$(A|b) = \left(\begin{array}{ccc|c} \boxed{2} & -1 & 3 & 2 \\ -4 & 4 & a & -6 \\ 8 & -2 & 14 & 2 \end{array}\right) \begin{array}{l} \\ +2\cdot z_1 \\ -4\cdot z_1 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & \boxed{2} & 6+a & -2 \\ 0 & 2 & 2 & -6 \end{array}\right) \begin{array}{l} \\ \\ -z_2 \end{array}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2 & 6+a & -2 \\ 0 & 0 & -4-a & -4 \end{array}\right)$$

- im Fall $\boxed{-4-a=0}$, also $\boxed{a=-4}$ gibt es <u>keine Lösung</u>, da dann $rg(A) = 2$, aber $rg(A|b) = 3$ wäre
- in allen anderen Fällen, also $a \neq -4$, gibt es eine eindeutige Lösung
- <u>Lösung im Fall $a=-2$:</u>

Gleichung (3) ist dann: $-2\cdot x_3 = -4 \Leftrightarrow \boxed{x_3 = 2}$

Gl. (2): $2\cdot x_2 + \underbrace{4\cdot x_3}_{=8} = -2 \Rightarrow \boxed{x_2 = -5}$

Gl. (1): $2x_1 \underbrace{-x_2}_{5} + \underbrace{3x_3}_{6} = 2 \Rightarrow \boxed{x_1 = -\frac{9}{2}}$

$\Rightarrow$ Lösung $\underline{\underline{x = \left(-\frac{9}{2}, -5, 2\right)}}$

3) $\mathcal{A} = \{p_1, \dots, p_4\}$, $p_i(x) = x^{i-1}$, Basis von $\mathbb{P}_3$

Abbildung: $F: \mathbb{P}_3 \to \mathbb{R}^3$ mit $F(p) := \begin{pmatrix} \int_{-1}^{1} p(x)\,dx \\ p(0) \\ p'(0) \end{pmatrix}$

$$\Rightarrow \quad w_1 = F(p_1) = \begin{pmatrix} \int_{-1}^{1} 1\,dx \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$w_2 = F(p_2) = \begin{pmatrix} \int_{-1}^{1} x\,dx \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$w_3 = F(p_3) = \begin{pmatrix} \int_{-1}^{1} x^2\,dx \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 0 \\ 0 \end{pmatrix}$$

$$w_4 = F(p_4) = \begin{pmatrix} \int_{-1}^{1} x^3\,dx \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- sei $\tilde{e}_j \in \mathbb{R}^4$, $j = 1, \dots, 4$, der j-te Einheitsvektor

$$\Rightarrow \phi_{\mathcal{A}}(\tilde{e}_j) = p_j \quad \Rightarrow \quad F \circ \phi_{\mathcal{A}}(\tilde{e}_j) = F(p_j) = w_j = \begin{pmatrix} w_{1j} \\ w_{2j} \\ w_{3j} \end{pmatrix}$$

- es gilt: $w_j = w_{1j}\, e_1 + w_{2j}\, e_2 + w_{3j}\, e_3$

$$w_j = \sum_{i=1}^{3} \beta_i e_i \quad \text{wobei } \beta_i = w_{ij},\ i = 1,2,3$$

$$\Rightarrow \quad w_j = \phi_{\mathcal{B}}(\beta) \qquad e_i \in \mathbb{R}^3 \text{ Einheits-vektor}$$

$$\Rightarrow \quad \beta = \phi_{\mathcal{B}}^{-1}(w_j) = \phi_{\mathcal{B}}^{-1}\left(F \circ \phi_{\mathcal{A}}(\tilde{e}_j)\right)$$

$$\Rightarrow \quad \beta = \phi_{\mathcal{B}}^{-1} \circ F \circ \phi_{\mathcal{A}}(\tilde{e}_j) = f(\tilde{e}_j) = A\tilde{e}_j$$

$\Rightarrow$ β mit $\underline{\beta_i = w_{ij},\ i = 1,2,3,\ \text{ist } j\text{-te Spalte von } A}$

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$$\Rightarrow \quad A = \begin{pmatrix} | & & | \\ A\tilde{e}_1 & \dots & A\tilde{e}_4 \\ | & & | \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 & 0 & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}}$$

- rang-erhaltende Umformungen von A:

  ~ tausche: $Z1 \to Z3$
  $Z2 \to Z1$
  $Z3 \to Z2$

$$\Rightarrow \quad A \Leftrightarrow \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & \frac{2}{3} & 0 \end{pmatrix} \begin{matrix} \\ \\ -2\cdot Z_1 \end{matrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \end{pmatrix}$$

$$\Rightarrow \quad \underline{\underline{rg(A) = 3 = rg(F)}}$$