Musterlösungen Blatt 9

1a) • sei $d := a_{11}a_{22} - a_{12}a_{21} \neq 0$ und

$$B_1 = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

• dann gilt $B = \frac{1}{d} \cdot B_1$ und

$$B_1 \cdot A = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$$

$$\Rightarrow B \cdot A = \frac{1}{d}(B_1 \cdot A) = \frac{1}{d}\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E_2$$

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1b) <u>sei $d = 0$:</u>

• rang-erhaltende Zeilenumformung liefert :

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \frac{a_{21}}{a_{11}} * \text{Zeile } 1 \qquad \underline{\text{im Fall}} \ \boxed{a_{11} \neq 0}$$

$$\rightarrow A^{(1)} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & \frac{d}{a_{11}} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} \Rightarrow rg(A^{(1)}) = rg(A) \leq 1$$

wäre A regulär, so müßte gelten $rg(A) = 2$

<u>Wdspr</u> $\lightning$

$\Rightarrow$ A ist <u>singulär</u>

• <u>der Fall $a_{11} = 0$</u> : $\Rightarrow d = -a_{12} \cdot a_{21} = 0$

$\Rightarrow a_{12} = 0$ <u>oder</u> $a_{21} = 0$

$\Rightarrow A = \begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix}$ <u>oder</u> $A = \begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}$

im beiden Fällen folgt : $rg(A) \leq 1$

$\Rightarrow$ (wie im Fall $a_{11} \neq 0$) A ist <u>singulär</u> //

2) $\mathcal{A} = \{v_1, v_2\}$, $\mathcal{B} = \{w_1, w_2\}$ mit

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

<u>Schritt 1</u>: bestimme Matrix $S = (s_{ij}) \in M(2 \times 2; \mathbb{R})$ mit

$$w_j = \sum_{i=1}^{2} s_{ij} v_i$$

- $$w_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = s_{11} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{v_1} + s_{21} \underbrace{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}_{v_2}$$

$$\Rightarrow \left.\begin{aligned} 3 &= s_{11} - s_{21} \\ 2 &= s_{11} + 2 s_{21} \end{aligned}\right\} \Rightarrow \begin{aligned} &2 * \text{Zei } 1 + \text{Zei } 2 : \\ &8 = 3 s_{11} \Rightarrow \boxed{s_{11} = \frac{8}{3}} \end{aligned}$$

$$\Rightarrow s_{21} = s_{11} - 3 = \frac{8}{3} - \frac{9}{3} \Rightarrow \boxed{s_{21} = -\frac{1}{3}}$$

- $$w_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = s_{12} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + s_{22} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} 0 &= s_{12} - s_{22} \\ 2 &= s_{12} + 2 s_{22} \end{aligned}\right\} \Rightarrow 2 = 3 s_{12} \Rightarrow \boxed{s_{12} = \frac{2}{3}}$$

$$\Rightarrow s_{22} = s_{12} \Rightarrow \boxed{s_{22} = \frac{2}{3}}$$

$$\Rightarrow \underline{\underline{S = \begin{pmatrix} 8/3 & 2/3 \\ -1/3 & 2/3 \end{pmatrix}}}$$

Schritt 2: $T = S^{-1} \overset{1c)}{=} \frac{1}{s_{11}s_{22} - s_{12}s_{21}} \begin{pmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{pmatrix}$

$$\Rightarrow T = \frac{1}{\frac{16}{9} + \frac{2}{9}} \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{8}{3} \end{pmatrix} = \underline{\underline{\frac{1}{2} \cdot \frac{1}{3} \begin{pmatrix} 2 & -2 \\ 1 & 8 \end{pmatrix}}}$$

- Koordinaten von $x = v_1 + v_2$ bzgl. $\mathcal{A}$-, $\mathcal{B}$-Basis

$x = \alpha_1 v_1 + \alpha_2 v_2$ mit $\alpha = (\alpha_1, \alpha_2) = \underline{\underline{(1,1)}}$

$x = \beta_1 w_1 + \beta_2 w_2$, wobei $\beta = T\alpha$

also: $\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 2 & -2 \\ 1 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 0 \\ 9 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 \\ 3/2 \end{pmatrix}}}$

3a) $A = \begin{pmatrix} a & a & b \\ a & a & c \end{pmatrix}$, $a, b, c \in \mathbb{R}$

1. Fall $a = 0$ $\Rightarrow A = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}$

$$\Rightarrow rg(A) = \begin{cases} 0, & \text{falls } b = c = 0 \\ 1, & \text{sonst (d.h. } b \neq 0 \text{ oder } c \neq 0) \end{cases}$$

2. Fall $a \neq 0$

$$A = \begin{pmatrix} a & a & b \\ a & a & c \end{pmatrix} \begin{matrix} \\ -\text{Zei } 1 \end{matrix} \longrightarrow A^{(1)} = \begin{pmatrix} a & a & b \\ 0 & 0 & c-b \end{pmatrix}$$

$$\Rightarrow rg(A) = rg(A^{(1)}) = \begin{cases} 2, & \text{falls } c \neq b \\ 1, & \text{falls } c = b \end{cases}$$

3b) rang-erhaltende Umformungen liefern

$$A = \begin{pmatrix} 2 & -3 & -5 \\ -6 & 9 & 13 \\ 14 & -23 & -34 \end{pmatrix} \begin{array}{l} \\ +3 Z_1 \\ -7 Z_1 \end{array}$$

$$\rightarrow A^{(1)} = \begin{pmatrix} 2 & -3 & -5 \\ 0 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix} \begin{array}{l} \\ \updownarrow \text{Zei-Tausch} \\ \end{array}$$

$$\rightarrow A^{(2)} = \begin{pmatrix} 2 & -3 & -5 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\Rightarrow \quad rg(A^{(2)}) = rg(A) = \underline{\underline{3}}$$