Blatt 2 Musterlösungen

① Vor: $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha+\beta+\gamma = 0$, $\alpha^2+\beta^2+\gamma^2 > 0$

$x = (\alpha, \beta, \gamma)$, $y = (\gamma, \alpha, \beta)$

Beh: $\varphi = \measuredangle(x,y) = \frac{2\pi}{3}$

Bew: $$\cos(\varphi) = \frac{\langle x,y\rangle}{\|x\| \cdot \|y\|} = \frac{\alpha\gamma + \alpha\beta + \beta\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2} \cdot \sqrt{\gamma^2+\alpha^2+\beta^2}}$$

$$\overset{\gamma = -\alpha-\beta}{=} \frac{\alpha\beta + (\alpha+\beta)(-\alpha-\beta)}{\alpha^2+\beta^2+(\alpha+\beta)^2}$$

$$= \frac{-\alpha^2-\beta^2-\alpha\beta}{2\alpha^2+2\beta^2+2\alpha\beta} = -\frac{1}{2}$$

wegen $\varphi \in [0,\pi]$ folgt daraus $\varphi = \frac{2\pi}{3}$ ▯

② z.z.: $$\boxed{(u \times v) \times w = \langle u,w\rangle v - \langle v,w\rangle u} \quad (1)$$

$\forall u,v,w \in \mathbb{R}^3$

1. Komponente von (1):

$$((u\times v)\times w)_1 = (u\times v)_2 w_3 - (u\times v)_3 w_2$$

$$= (u_3v_1 - u_1v_3)w_3 - (u_1v_2 - u_2v_1)w_2$$

$$= (u_3w_3 + u_2w_2)v_1 - (v_3w_3 + v_2w_2)u_1$$

$$= (u_3w_3 + u_2w_2 + \underbrace{u_1w_1)v_1} - (v_3w_3 + v_2w_2 + \underbrace{v_1w_1)u_1}$$

hebt sich auf

$$= \langle u,w\rangle v_1 - \langle v,w\rangle u_1$$

$$= \underline{\underline{(\langle u,w\rangle v - \langle v,w\rangle u)_1}}$$

2. Komponente von (1):

$$((u \times v) \times w)_2 = (u \times v)_3 w_1 - (u \times v)_1 w_3$$
$$= (u_1 v_2 - u_2 v_1) w_1 - (u_2 v_3 - u_3 v_2) w_3$$
$$= (u_1 w_1 + u_3 w_3) v_2 - (v_1 w_1 + v_3 w_3) u_2$$
$$= (u_1 w_1 + u_2 w_2 + u_3 w_3) v_2 - (v_1 w_1 + v_2 w_2 + v_3 w_3) u_2$$
$$= \langle u, w \rangle v_2 - \langle v, w \rangle u_2$$
$$= (\langle u, w \rangle v - \langle v, w \rangle u)_2$$

3. Komponente von (1):

$$((u \times v) \times w)_3 = (u \times v)_1 w_2 - (u \times v)_2 w_1$$
$$= (u_2 v_3 - u_3 v_2) w_2 - (u_3 v_1 - u_1 v_3) w_1$$
$$= (u_2 w_2 + u_1 w_1) v_3 - (v_2 w_2 + v_1 w_1) u_3$$
$$= (u_2 w_2 + u_1 w_1 + u_3 w_3) v_3 - (v_2 w_2 + v_1 w_1 + v_3 w_3) u_3$$
$$= \langle u, w \rangle v_3 - \langle v, w \rangle u_3$$
$$= (\langle u, w \rangle v - \langle v, w \rangle u)_3$$

(3) Ebene E durch: $p = (3,5,1) \in E$, $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ in E

a) berechne $\varphi = \sphericalangle(v, w)$:

$\|v\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ $\qquad \langle v, w \rangle = 2 + 1 + 3 = 6$

$\|w\| = \sqrt{4 + 1 + 9} = \sqrt{14}$

$$\Rightarrow \cos(\varphi) = \frac{6}{\sqrt{3}\sqrt{14}} \Rightarrow \varphi = \arccos\left(\frac{6}{\sqrt{42}}\right) \approx 0.3876$$
$$\approx 22.2077^\circ$$

b) Hessesche Normalform von E:

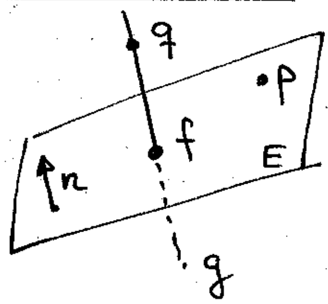
$$N = \begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}1\cdot 3 - 1\cdot 1\\ -1\cdot 3 + 2\cdot 1\\ 1\cdot 1 - 1\cdot 2\end{pmatrix} = \begin{pmatrix}2\\-1\\-1\end{pmatrix}$$

$$\|N\| = \sqrt{4+1+1} = \sqrt{6} \quad \Rightarrow \quad n = \frac{1}{\|N\|} N = \frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\-1\end{pmatrix}$$

$$\Rightarrow E = \{x \in \mathbb{R}^3 \mid \langle x - p, n\rangle = 0\}$$

$$= \left\{x \in \mathbb{R}^3 \,\middle|\, \left\langle \begin{pmatrix}x_1 - 3\\ x_2 - 5\\ x_3 - 1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\-1\end{pmatrix}\right\rangle = 0 \right\}$$

Fußpunkt $f \in E$ des Lotes durch $q = \begin{pmatrix}0\\1\\0\end{pmatrix} \perp E$



- sei $g \perp E$ Lotgerade durch q

  dann ist $n = \frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\-1\end{pmatrix}$ Richtungsvektor von g

$$\left.\begin{array}{l} \bullet\ f \in g \Rightarrow \exists\, t \in \mathbb{R}: \quad f = q + t\cdot n \\ \bullet\ f \in E \Rightarrow \langle f - p, n\rangle = 0 \end{array}\right\} \Rightarrow \langle q + tn - p, n\rangle = 0$$

$$\Rightarrow \langle q - p, n\rangle + t\underbrace{\langle n, n\rangle}_{=1} = 0$$

$$t = -\langle q - p, n\rangle = -\left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}3\\5\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\-1\end{pmatrix}\right\rangle$$

$$= \frac{-1}{\sqrt{6}}\left\langle \begin{pmatrix}-3\\-4\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\\-1\end{pmatrix}\right\rangle = -\frac{1}{\sqrt{6}}(-6+4+1) = \frac{1}{\sqrt{6}}$$

$$\Rightarrow f = q + t\cdot n = \begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{1}{\sqrt{6}}\cdot\frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\-1\end{pmatrix} = \begin{pmatrix}1/3\\ 5/6\\ -1/6\end{pmatrix}$$

Abstand d von q zu E

$$d = \| f - q \| = \| (q + t \cdot n) - q \| = \| t n \| = |t| \cdot \underbrace{\|n\|}_{=1}$$

$$= |t| = \left| \frac{1}{\sqrt{6}} \right| = \underline{\underline{\frac{1}{\sqrt{6}}}}$$

c) $g := \{ r + tu \mid t \in \mathbb{R} \}$ Gerade mit $r = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$, $u = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$

berechne Schnittpunkt $s = g \cap E$

$$\left.\begin{array}{l} s \in g \Rightarrow \exists t \in \mathbb{R}: \quad s = r + tu \\ s \in E \Rightarrow \langle s - p, n \rangle = 0 \end{array}\right\} \Rightarrow \langle r + tu - p, n \rangle = 0$$

$$\Rightarrow \langle r - p, n \rangle + t \langle u, n \rangle = 0 \quad \Rightarrow t = \frac{\langle p - r, n \rangle}{\langle u, n \rangle}$$

$$\langle u, n \rangle = \left\langle \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right\rangle = \frac{1}{\sqrt{6}} (-2 - 2 - 4) = \underline{\underline{\frac{-8}{\sqrt{6}}}}$$

$$\langle p - r, n \rangle = \frac{1}{\sqrt{6}} \left\langle \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right\rangle = \frac{1}{\sqrt{6}} (0 - 3 - 1) = \underline{\underline{\frac{-4}{\sqrt{6}}}}$$

$$\Rightarrow t = \frac{-4}{\sqrt{6}} \frac{\sqrt{6}}{-8} = \underline{\underline{\frac{1}{2}}}$$

$$\Rightarrow s = r + tu = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2{,}5 \\ 3 \\ 2 \end{pmatrix}}}$$