Constrained H^1-interpolation on quadrilateral and hexahedral meshes with hanging nodes

Vincent Heuveline * Friedhelm Schieweck †

Abstract

We propose a Scott-Zhang type finite element interpolation operator of first order for the approximation of H^1-functions by means of continuous piecewise mapped bilinear or trilinear polynomials. The novelty of the proposed interpolation operator is that it is defined for general non-affine equivalent quadrilateral and hexahedral elements and so-called 1-irregular meshes with hanging nodes. We prove optimal local approximation properties of this interpolation operator for functions in H^1. As necessary ingredients we provide an abstract definition of a hanging node and a rigorous analysis of the issue of constrained approximation which cover both the two- and three-dimensional case in a unified fashion.

Keywords: finite-element interpolation, hanging nodes, constrained approximation, non-smooth functions, quadrilateral and hexahedral elements

Mathematics Subject Classification (2000): 65N15, 65N30, 65N50

1 Introduction

Unstructured quadrilateral and hexahedral meshes are widely used in the context of numerical simulations based on the finite element method. An attractive feature of quadrilateral and hexahedral finite elements is that they allow to construct higher order basis functions and suitable integration rules based on the tensor-product of the one-dimensional case.

Today, a well established technique to solve a partial differential equation efficiently by means of finite elements is based on adaptivity in the sense that the method is able to refine the mesh locally. Having refined some elements selected by an error estimator, this leads in a first step to an irregular mesh due to the incompatibility to the neighbored unrefined elements. However, the usual second step to do a conforming closure of this irregular mesh by means of special refinement rules for the neighboring elements is unsuitable for the

*Rechenzentrum und Institut für Angewandte Mathematik, Universität Karlsruhe, Zirkel 2, 76128 Karlsruhe, Germany.

†Institut für Analysis und Numerik, Otto-von-Guericke-Universität Magdeburg, Postfach 4120, D-39016 Magdeburg, Germany. URL: `http://www-ian.math.uni-magdeburg.de/~schiewec/`

case of quadrilateral or hexahedral elements. Therefore, a popular alternative in practice is to allow a certain irregularity of the mesh and to work with so-called *hanging nodes*. An advantage of this approach is that one gets well-shaped elements which converge to parallelograms or parallelepipeds if the mesh size tends to zero. However, hanging nodes need a special treatment in theory since the degrees of freedom are reduced and the problem arises whether this fact leads to a reduction in the order of approximation.

Our main goal in this paper is to study the question how H^1-functions can be approximated by means of general non-affine equivalent quadrilateral and hexahedral finite elements of first order on a so-called *1-irregular mesh* with hanging nodes. To this end, we construct and analyze an interpolation operator that is defined for H^1-functions and has optimal local and global approximation properties. Such an interpolation operator is required, for example, in the analysis of a posteriori estimators (see e.g. [18, 19, 1] and references therein) as well as for the proof of the inf-sup condition in the context of mixed finite elements [9, 4]. This last application in the case of hp-FEM on general meshes with hanging nodes (see [12]) has been the starting point of this work. Despite its high relevance in practice, little attention has been paid in the literature to the derivation and analysis of such interpolation operators in the above mentioned general setup. Our aim in this paper is to fill this gap.

Publications, where an interpolation operator for H^1-functions is constructed and analyzed also for the case of quadrilateral or hexahedral elements, are for example [7, 19, 16, 2]. However, the treated configurations are either restricted to the special case of affine equivalent elements or to the 2D-case solely, or the case of hanging nodes is excluded. For quadrilateral or hexahedral meshes, the assumption of affine equivalent elements means that only such elements are admitted which are parallelograms or parallelepipeds, respectively. This assumption is clearly too restrictive for many applications.

In this paper, we generalize the H^1-interpolation operator of Scott and Zhang [17] to the case of a general non-affine equivalent family of quadrilateral and hexahedral meshes with hanging nodes. Here the nodal values are defined locally by averaging over $(d-1)$-dimensional faces of the d-dimensional elements. This approach has the advantage that it preserves piecewise polynomial boundary conditions of the approximated function in a natural way. The specificity in our context is that special nodal functionals have to be defined for hanging nodes and the fact that, for non-affine equivalent hexahedral elements, the two-dimensional faces can be curved in general. This fact causes extra difficulties in the estimation of the corresponding nodal functionals.

A large part of this paper is devoted to some ingredients that are necessary for the construction and analysis of our interpolation operator. As a first ingredient, we provide an abstract definition of a hanging node. This definition is independent of the dimension d of the considered domain Ω and might be of interest elsewhere. A second ingredient is the exact characterization under which conditions an unconstrained finite element function belongs globally to the space H^1 if it is defined elementwise by its values in the regular and hanging nodal points. By means of this characterization we can introduce suitable nodal functionals for the hanging nodes. With these nodal functionals and Scott-Zhang-type nodal functionals for the regular nodes we define the interpolant separately on each

element. The choice of the nodal functionals for the hanging nodes guarantees that the interpolant belongs globally to the space H^1.

The outline of this paper is as follows: In §2, we state the assumptions to be made on the meshes, we introduce the concept of *1-irregular* multilevel adaptive meshes and give an abstract definition of a hanging node. In §3, we first treat the question of a constrained approximation which ensures a globally continuous finite element function on a 1-irregular mesh with hanging nodes. Then, we present the construction of the interpolation operator and the proof of optimal local estimates in H^1 and L^2. The specific properties of general quadrilateral and hexahedral meshes with hanging nodes which are needed for the analysis of the proposed interpolation operator are derived in the appendix.

2 Meshes with hanging nodes

2.1 General notations

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain where $d \in \{2, 3\}$. For a measurable set $G \subset \Omega$, let $(\cdot, \cdot)_G$ and $\|\cdot\|_{0,G}$ denote the inner product and the norm in $L^2(G)$, respectively. Furthermore, let $|\cdot|_{m,G}$ and $\|\cdot\|_{m,G}$ denote the seminorm and norm in the Sobolev space $H^m(G)$, respectively. We denote by $\mathbb{Q}_m(G)$ the space of all polynomials on the domain $G \subset \mathbb{R}^d$ where the maximum power in each coordinate is less or equal to m.
By $\mathrm{card}(J)$ we denote the number of elements of a finite set J. The Euclidean norm of a vector x will be denoted by $\|x\|$. For a set $G \subset \mathbb{R}^d$, we denote by $\mathrm{int}(G)$ and $\overline{G}$ the interior and closure of G, respectively. If $G \subset \Omega$ is a d-dimensional set then we will write $|G|$ for the d-dimensional measure of G and if E is a $(d-1)$-dimensional manifold we denote by $|E|$ the $(d-1)$-dimensional measure of E. The meaning will be clear from the context. Throughout this paper, C, C', $\widetilde{C}$ will denote generic constants which may have different values at different places. All these constants occurring in any estimates will be independent of the local and global mesh parameter h_K and h defined below.

2.2 Multilevel adaptive meshes of quadrilateral or hexahedral elements

Let the bounded domain $\Omega \subset \mathbb{R}^d$ be decomposed by a *mesh* $\mathcal{T}$ of elements $K \in \mathcal{T}$ which are assumed to be open quadrilaterals in the 2D-case and open hexahedrons in the 3D-case such that $\overline{\Omega} = \cup_{K \in \mathcal{T}} \overline{K}$. We denote by h_K the diameter of the element $K \in \mathcal{T}$ and by ρ_K the diameter of the largest ball that can be inscribed into K. The *mesh width* h of $\mathcal{T}$ is given by $h := \max_{K \in \mathcal{T}} h_K$. We assume that the mesh is *shape regular* in the sense of [13]. Note that for general non-affine equivalent families of quadrilateral or hexahedral meshes the usual shape regularity assumption $\frac{h_K}{\rho_K} \leq C$ for all $K \in \mathcal{T}$ is not sufficient. The *shape regularity assumption* of [13] imposes that the distortion of the quadrilateral or hexahedral elements from a parallelogram or parallelepiped, respectively, is uniformly bounded. This guarantees that the mapping $F_K : \overline{\widehat{K}} \to \overline{K}$ between the closure of the

reference element $\widehat{K} = (-1,1)^d$ and the closure of the original element $K \in \mathcal{T}$ is bijective. Note that the mapping F_K is multi-linear, i.e., $F_K \in (\mathbb{Q}_1(\overline{\widehat{K}}))^d$, which implies that in the three dimensional case $d = 3$ the faces of a hexahedral element are in general curved.

In the following, we describe the shape regularity assumption of [13] in more detail. By a Taylor expansion of $F_K(\hat{x})$ we get

$$F_K(\hat{x}) = b_K + B_K\ \hat{x} + G_K(\hat{x}), \tag{1}$$

with $b_K := F_K(0)$, $B_K := DF_K(0)$ and $G_K(\hat{x}) := F_K(\hat{x}) - F_K(0) - B_K\hat{x}$. We denote by $\widehat{S} \subset \widehat{K}$ the d-simplex with the vertices $(0,\dots,0)$, $(1,0,\dots,0)$, $\dots$, $(0,\dots,0,1)$ and by S_K the image of $\widehat{S}$ under the affine mapping $\hat{x} \mapsto B_K\ \hat{x} + b_K$. For the simplices S_K, $K \in \mathcal{T}$, we assume the usual shape regularity assumption

$$\frac{h_{S_K}}{\rho_{S_K}} \le C \qquad \forall K \in \mathcal{T}, \tag{2}$$

where $h_{S_K} := \mathrm{diam}(S_K)$ and ρ_{S_K} is the diameter of the largest ball inscribed into S_K. Note that by standard arguments for simplicial elements [6] we get

$$\|B_K\| \le C h_{S_K}, \qquad \|B_K^{-1}\| \le C h_{S_K}^{-1} \qquad \forall K \in \mathcal{T}, \tag{3}$$

where $\|B_K\|$ denotes the matrix norm induced by the Euclidean vector norm in $\mathbb{R}^d$. For each element $K \in \mathcal{T}$, we define the constant γ_K by

$$\gamma_K := \sup_{\hat{x} \in \widehat{K}} \|B_K^{-1}\ DF_K(\hat{x}) - I\|, \tag{4}$$

which is a measure of the deviation of K from a parallelogram or a parallelepiped, respectively. Note that $\gamma_K = 0$ if and only if the mapping F_K is affine.

Definition 1 *A mesh $\mathcal{T}$ of quadrilateral or hexahedral elements is called* **shape regular** *if the condition (2) and*

$$\gamma_0 := \max_{K \in \mathcal{T}} \gamma_K < 1\,, \tag{5}$$

are satisfied.

From the literature, different conditions are known that describe the concept of a shape regular quadrilateral mesh. An overview and a proof of the equivalence of several conditions for the two-dimensional case is given in [5, 14]. Sufficient conditions for (5) in the three-dimensional case, which can be easily checked in practical computations, can be found in [13].

In this paper, the main point of interest in deriving an H^1-stable interpolation operator is the treatment of meshes with hanging nodes. That means that the usual assumption of a *regular* mesh has to be weakened. In the following, we will describe the type of meshes $\mathcal{T}$ treated in this paper. We assume $\mathcal{T}$ to be a *multilevel adaptive mesh* generated by a refinement process in the following way. We start with a partition $\mathcal{T}^0$ of the domain Ω into

elements $K \in \mathcal{T}^0$ of mesh-level 0, i.e., $\Omega = \mathrm{int}\big(\bigcup_{K\in\mathcal{T}^0} \overline{K}\big)$. The mesh $\mathcal{T}^0$ is assumed to be *regular* in the usual sense, i.e., for any two different elements $K_1, K_2 \in \mathcal{T}^0$ the intersection $\overline{K_1} \cap \overline{K_2}$ is either empty or a common $(d-m)$-dimensional face of K_1 and K_2 where $m \in \{1, \ldots, d\}$. Now, starting with the elements $K \in \mathcal{T}^0$, an existing element K can be refined, i.e., it can be splitted into 2^d many new elements called *son-elements* of K and denoted by $\sigma_i(K)$, $i = 1, \ldots, 2^d$ (see Figure 1). These son-elements are constructed by

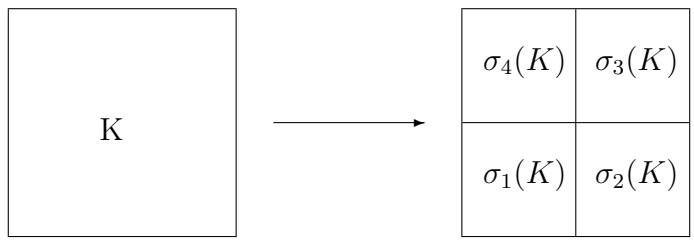

Figure 1: *refinement of element K into son-elements $\sigma_i(K)$, $i = 1, \ldots, 2^d$, $d = 2$*

connecting the barycenters of opposite $(d-1)$-dimensional faces of K and additionally, in the 3D-case, by connecting the midpoints of opposite element edges in the two-dimensional faces of K. For a new element $K' = \sigma_i(K)$, we will say that K is the *father-element* of K' and we will write $K = \mathcal{F}(K')$. If an element K is refined then, in the partition of the domain Ω, it is replaced by the set of its son-elements $\sigma_i(K)$, $i = 1, \ldots, 2^d$. The new elements can be refined again and again and so the final partition $\mathcal{T}$ of Ω is created. Examples of such meshes used in practical computations can be found in [10, 11].

Definition 2 *For an element $K \in \mathcal{T}$, generated from the initial mesh $\mathcal{T}^0$ by the refinement process described above we define the* **refinement level** *$\ell(K)$ as $\ell(K) := 0$ if $K \in \mathcal{T}^0$ and $\ell(K) := m \geq 1$ if there exists a chain of m father-elements K_i, $i = 1, \ldots, m$, starting from $K_0 := K$ and defined by $K_i := \mathcal{F}(K_{i-1})$ for $i = 1, \ldots, m$, such that $K_m \in \mathcal{T}^0$.*

The above defined *refinement level* $\ell(K)$ is equal to the number of refinement steps that are needed to generate element K from an element of the coarsest mesh $\mathcal{T}^0$.

Definition 3 *A mesh $\mathcal{T}$, which has been generated by the above defined refinement process from a regular initial mesh $\mathcal{T}^0$, is called* **1-irregular** *if the property*

$$|\ell(K) - \ell(K')| \leq 1\,, \tag{6}$$

holds for any pair of different neighbored elements $K, K' \in \mathcal{T}$ where $\partial K \cap \partial K'$ is a one- or two-dimensional manifold.

Note that the property (6) need not to be satisfied for pairs of elements K, K' having only one vertex in common. For the 2D-case, the term *1-irregular* can already be found in [8].

In this paper, we only consider meshes $\mathcal{T}$ which are *1-irregular*. In principle our results can be generalized to m-irregular meshes where in condition (6) the right hand side is

replaced by an integer $m \geq 1$. However, for practical computations, 1-irregular meshes are sufficient to describe even complicated structures in a reasonable way (see e.g. [10, 11]). Moreover, the constants in the estimates would blow up if the parameter m would be increased.

2.3 Abstract definition of hanging nodes

The definition and analysis of our interpolation operator require additional notation on geometrical details of the mesh. We denote by $\mathcal{E}(K)$ the set of all $(d-1)$-dimensional faces of an element K. Let $\mathcal{E} := \bigcup_{K\in\mathcal{T}} \mathcal{E}(K)$ be the set of all element faces of the mesh. We split $\mathcal{E}$ as follows

$$\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_\Gamma \,,$$

where $\mathcal{E}_\Gamma$ describes the set of all faces located at the boundary Γ of Ω and $\mathcal{E}_0$ denotes the set of the inner faces of $\mathcal{E}$. For any face $E \in \mathcal{E}$, we define the set

$$\mathcal{T}(E) := \{K \in \mathcal{T} : E \in \mathcal{E}(K)\} \,,$$

as the set of all elements having E as one of their faces. Let $\mathcal{E}_r$ denote the set of the *regular inner faces* defined as

$$\mathcal{E}_r := \{E \in \mathcal{E}_0 : \mathrm{card}(\mathcal{T}(E)) = 2\} \,.$$

For each regular face $E \in \mathcal{E}_r$, there exist exactly two different elements denoted by $K(E)$ and $K'(E)$ such that E is one of their faces, i.e.,

$$\mathcal{T}(E) = \{K(E), K'(E)\} \qquad \forall E \in \mathcal{E}_r.$$

For all other faces $E \in \mathcal{E} \setminus \mathcal{E}_r$, there is only one element denoted by $K(E)$ which has E as one of its faces, i.e.,

$$\mathcal{T}(E) = \{K(E)\} \qquad \forall E \in \mathcal{E} \setminus \mathcal{E}_r.$$

A face $\widetilde{E} \in \mathcal{E}$ is called a *son-face* of a face $E \in \mathcal{E}$ if $\widetilde{E} \subset E$ and $|\widetilde{E}| < |E|$ (see Figure 2). We denote by $\sigma(E)$ the set of all son-faces of E. Note that for each regular face $E \in \mathcal{E}_r$, the set $\sigma(E)$ is empty. We define the set $\mathcal{E}_i$ of all *irregular inner faces* as follows

$$\mathcal{E}_i := \{E \in \mathcal{E}_0 : \sigma(E) \neq \emptyset\} \,.$$

Let $\widetilde{E} \in \sigma(E)$ be a *son-face* of $E \in \mathcal{E}_i$, then the face E is called the *father-face* of $\widetilde{E}$ and we will write $E = \mathcal{F}(\widetilde{E})$. We define the set of all *son-faces* by

$$\mathcal{E}_\sigma := \bigcup_{E\in\mathcal{E}_i} \sigma(E).$$

Using these definitions, the set $\mathcal{E}_0$ of all inner faces can be decomposed as

$$\mathcal{E}_0 = \mathcal{E}_r \cup \mathcal{E}_i \cup \mathcal{E}_\sigma.$$

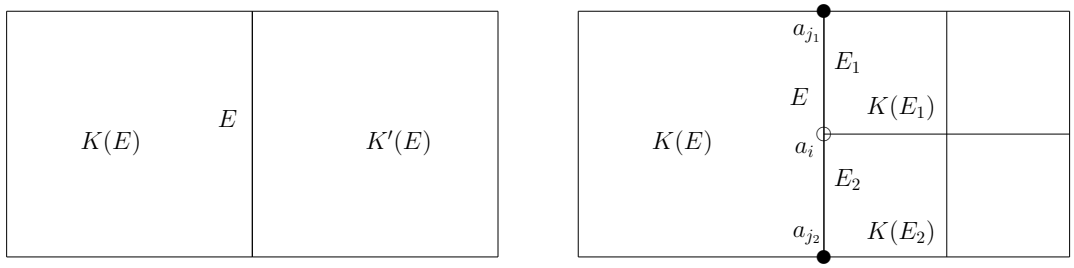

Figure 2: *(left) regular inner face $E \in \mathcal{E}_r$ with the two associated elements $K(E), K'(E)$; (right) irregular inner face $E \in \mathcal{E}_i$ with the son-faces $E_1, E_2 \in \sigma(E)$ where to each face only one element is associated and denoted by $K(E), K(E_1), K(E_2)$; nodal point a_i of a hanging node i and nodal points a_j, $j \in \Lambda_i := \{j_1, j_2\}$, of associated regular nodes $j \in \Lambda_i$.*

Let $\hat{a}_m$, for $m \in \{1, \ldots, 2^d\}$, denote the vertices of the reference element $\widehat{K} := (-1, +1)^d$. We introduce the global index set J of all vertices $\{a_j\}_{j \in J}$ of all elements $K \in \mathcal{T}$. An index $j \in J$ will be called a *node* and the associated vertex a_j the *nodal point* of j. Obviously, for each nodal point a_j, there exists at least one element $K \in \mathcal{T}$ such that $a_j = F_K(\hat{a}_m)$ for some $m \in \{1, \ldots, 2^d\}$. Let us define by

$$J(K) := \left\{ j \in J : \quad a_j \in \{F_K(\hat{a}_m) : \ 1 \le m \le 2^d\} \right\},$$

the set of all vertex nodes of the element $K \in \mathcal{T}$.

Definition 4 *A node $i \in J$ is called a* **hanging node** *if there exists a son-face $E \in \mathcal{E}_\sigma$ such that*

$$a_i \in \overline{E} \qquad \text{and} \qquad i \in J(K(E)) \setminus J(K(\mathcal{F}(E))),$$

where $K(E)$ and $K(\mathcal{F}(E))$ denote the uniquely determined elements associated with the face E and its father-face $\mathcal{F}(E)$, respectively. A node $j \in J$ is called a **regular node** *if it is not a hanging node. The subset of J associated with the hanging nodes is denoted by J_h and $J_r := J \setminus J_h$ denotes the set of all regular nodes. The corresponding local sets of regular and hanging nodes of an element $K \in \mathcal{T}$ are defined as $J_h(K) := J_h \cap J(K)$ and $J_r(K) := J_r \cap J(K)$.*

A typical 2D-configuration with a hanging node i and associated regular nodes $j \in \Lambda_i$ is depicted in Figure 2. A node $i \in J$ is a hanging node if there exists an element $K \in \mathcal{T}$ such that $a_i \in \partial K$ but a_i is not a vertex of K. This type of characterization of a hanging node can be found, e.g., in [8]. In the following lemma, we provide a further characterization of a hanging node which will be used below for a rigorous analysis of the question of constrained approximation.

Lemma 5 *Let $\mathcal{T}$ be a 1-irregular multilevel adaptive mesh consisting of quadrilateral or hexahedral elements. Then, the nodal point a_i associated to a hanging node $i \in J_h$ can be represented uniquely as a linear combination of nodal points of regular nodes in the*

following way

$$a_i = \sum_{j \in \Lambda_i} c_{i,j} a_j \qquad \textit{and} \qquad \Lambda_i \subset J_r, \tag{7}$$

where either

$$\Lambda_i := \{j_1, j_2\}, \qquad c_{i,j} := \frac{1}{2} \quad \forall\, j \in \Lambda_i, \tag{8}$$

and a_i is the midpoint of an element edge e with the vertices a_{j_1} and a_{j_2} in the 2D- or 3D-case, or

$$\Lambda_i := \{j_1, j_2, j_3, j_4\}, \qquad c_{i,j} := \frac{1}{4} \quad \forall\, j \in \Lambda_i, \tag{9}$$

and a_i is the barycenter of a two-dimensional face $E \in \mathcal{E}_i$ with the vertices $a_{j_1}, \ldots, a_{j_4}$ in the 3D-case.

Proof. Let $i \in J_h$ be a hanging node. Due to Definition 4, there exists a son-face $\widetilde{E} \in \mathcal{E}_\sigma$ with the father-face $E := \mathcal{F}(\widetilde{E}) \in \mathcal{E}_i$ such that the following holds: a_i is a vertex of the element $\widetilde{K} := K(\widetilde{E})$ and a_i is located at the boundary of the element $K := K(E)$ (since $a_i \in \overline{E} \subset \partial K$), but a_i is not a vertex of element K. Therefore, the vertex a_i of element $\widetilde{K}$, which is also a vertex of the face $\widetilde{E}$ but not a vertex of its father-face E, can only be a midpoint of an element edge e of K or the barycenter of E. This proves the representation of a_i by means of equation (7) with the definitions (8) or (9).

In order to show $\Lambda_i \subset J_r$, let us suppose that one of the nodes $j \in \Lambda_i$, say j_1, is a hanging node. Since a_{j_1} is a vertex of the father-face E of $\widetilde{E}$, there exists a son-element K_1 of the father-element $\mathcal{F}(\widetilde{K})$ of $\widetilde{K}$ such that a_{j_1} is a vertex of K_1. Furthermore, since j_1 is a hanging node, there exists a neighboring element K_2 of K_1 such that $a_{j_1} \in \partial K_2$ but a_{j_1} is not a vertex of K_2. For a 1-irregular mesh $\mathcal{T}$, the common boundary of two different elements can be either empty, a common vertex or a one- or two-dimensional manifold. Since $a_{j_1} \in \partial K_1 \cap \partial K_2$ is not a vertex of K_2, the common boundary of K_1 and K_2 must be a one- or two-dimensional manifold. Moreover, since $a_{j_1} \in \partial K \cap \partial K_2$ is not a vertex of K_2, the common boundary of K and K_2 must be a one- or two-dimensional manifold. Thus, from Definition 3 we conclude that

$$|\ell(K_1) - \ell(K_2)| \le 1 \qquad \text{and} \qquad |\ell(K) - \ell(K_2)| \le 1. \tag{10}$$

Since the vertex a_{j_1} of K is not a vertex of K_2 but located at ∂K_2, it follows that $\ell(K) > \ell(K_2)$ which implies with (10) that $\ell(K) = \ell(K_2) + 1$. Thus, we obtain $\ell(K_1) = \ell(\widetilde{K}) = \ell(K) + 1 = \ell(K_2) + 2$. However, this is a contradiction to (10), i.e., j_1 cannot be a hanging node.

Finally we show that the representation (7) with (8) or (9) is unique. For a 1-irregular mesh $\mathcal{T}$, all the midpoints of element edges and barycenters of two-dimensional element faces are different points. Therefore, due to the above proved properties of a hanging node $i \in J_h$, the associated nodal point a_i uniquely determines its element edge e or its two-dimensional element face E together with the set of the corresponding vertices a_j, $j \in \Lambda_i$. $\square$

3 Interpolation operator

3.1 Constrained approximation

We consider the following finite element spaces

$$D_h := \{\phi \in L^2(\Omega) : \ \phi\big|_K \circ F_K \in \mathbb{Q}_1(\widehat{K}) \quad \forall K \in \mathcal{T}\}, \tag{11}$$

$$S_h := D_h \cap H^1(\Omega)\,. \tag{12}$$

The space D_h contains functions that are in general discontinuous across the element boundaries. In the following, we will treat the issue of *constrained H^1-approximation*, i.e., we will characterize the subspace S_h of those functions $v_h \in D_h$ that are globally in $H^1(\Omega)$. To this end, we define the operator

$$\Phi_h : \mathbb{R}^n \to D_h \qquad \text{with} \qquad n := \operatorname{card}(J)\,,$$

that assigns to a nodal vector $\underline{v} = (v_j)_{j\in J} \in \mathbb{R}^n$ a finite element function $v_h = \Phi_h \underline{v} \in D_h$ which is elementwise defined by

$$\Phi_h \underline{v}\big|_K(x) := \sum_{j\in J(K)} v_j \psi_j^K(x) \qquad \forall\, x \in K,\ \ \forall\, K \in \mathcal{T}, \tag{13}$$

where $\psi_j^K : \overline{K} \to \mathbb{R}$ denotes the usual local mapped basis function with respect to the node $j \in J(K)$ determined by $\psi_j^K \circ F_K \in \mathbb{Q}_1(\overline{\widehat{K}})$ and $\psi_j^K(a_i) = \delta_{ij}$ for all $i \in J(K)$. Note that the operator Φ_h is not surjective since the functions $\Phi_h \underline{v}$ by construction are already continuous in the regular nodal points a_j, $j \in J_r$. However $\Phi_h \underline{v}$ can be discontinuous at nodal points associated with hanging nodes.

To be more precisely, we have to explain what the continuity of a function $v_h := \Phi_h \underline{v}$ at a point x on the element boundaries $\bigcup_{K\in\mathcal{T}} \partial K$ means since the values of v_h at such points are not defined by (13) due to the fact that the elements $K \in \mathcal{T}$ are open sets. At first, for a given function $v_h \in D_h$ and a given point x at the boundary of an element K, we define the value $v_h\big|_K(x)$ by continuous extension of the smooth restriction $v_h\big|_K$ given by (11). Then, the function v_h is called continuous at a point $x \in \bigcup_{K\in\mathcal{T}} \partial K$ if the values $v_h\big|_K(x)$ are identical for all elements $K \in \mathcal{T}$ with $x \in \partial K$.

Now, we will derive necessary and sufficient conditions (constraints) for the nodal vector $\underline{v} \in \mathbb{R}^n$ that guarantee that the associated finite element function $v_h := \Phi_h \underline{v}$ belongs to the space S_h, i.e., that v_h is globally continuous. By definition the restriction of $\Phi_h \underline{v}$ to an element $K \in \mathcal{T}$ is arbitrarily smooth up to the element boundary. Therefore, the global continuity of v_h is equivalent to the continuity of v_h in the points at the element boundaries. In the following, we will describe this continuity in the sense that the *jump* $[v_h]_E$ of v_h over a $(d-1)$-dimensional face E is zero for all faces $E \in \mathcal{E}_r \cup \mathcal{E}_\sigma$.

The jump $[v_h]_E$ is defined as follows. For a regular face $E \in \mathcal{E}_r$, let $K(E)$ and $K'(E)$ denote the two elements associated with E as defined in Section 2.3. For a son-face $E \in \mathcal{E}_\sigma$, let $K(E)$ be the element associated with E and $K'(E) := K(\mathcal{F}(E))$ the element associated

with the father-face $\mathcal{F}(E)$ of E (see Section 2.3). Then, for a point $x \in \overline{E}$ and a function $v_h \in D_h$, we define the jump of v_h at x across E as

$$[v_h]_E(x) := v_h\big|_{K(E)}(x) - v_h\big|_{K'(E)}(x). \tag{14}$$

Now, by means of elementary arguments we can prove the following statement.

Lemma 6 *A function $v_h \in D_h$ is globally continuous if and only if*

$$[v_h]_E(x) = 0 \qquad \forall\, x \in \overline{E},\ \ \forall\, E \in \mathcal{E}_r \cup \mathcal{E}_\sigma. \tag{15}$$

For a given face $E \in \mathcal{E}$, let $J(E)$ denote the set of those nodes $j \in J$ such that a_j is a vertex of E, i.e.,

$$J(E) := \left\{ j \in J(K(E)) : \quad a_j \in \overline{E} \right\}. \tag{16}$$

Exploiting the analytical structure of the local basis functions ψ_j^K, $j \in J(K)$, we can show in an elementary way the next lemma.

Lemma 7 *For a given function $v_h \in D_h$ and a face $E \in \mathcal{E}_r \cup \mathcal{E}_\sigma$, it holds*

$$[v_h]_E(x) = 0 \qquad \forall\, x \in \overline{E}$$

if and only if

$$[v_h]_E(a_j) = 0 \qquad \forall\, j \in J(E). \tag{17}$$

Using Lemma 5, 6 and 7, we can prove our final result on the issue of constrained approximation.

Lemma 8 *Let $\underline{v} = (v_j)_{j\in J} \in \mathbb{R}^n$ be a given nodal vector and $v_h = \Phi_h \underline{v} \in D_h$ the associated finite element function defined elementwise by* (13)*. Then, the function v_h is globally continuous, i.e., $v_h \in S_h$, if and only if the constraints*

$$v_i = \sum_{j\in\Lambda_i} c_{i,j} v_j \qquad \forall\, i \in J_h, \tag{18}$$

are satisfied where the index set Λ_i and the coefficients $c_{i,j}$ are defined in Lemma 5.

Proof. Let us assume that the constraints (18) for the components of the nodal vector $\underline{v}$ are satisfied. We will show that then, for the function $v_h = \Phi_h \underline{v}$, the jump conditions (17) are satisfied for all faces $E \in \mathcal{E}_r \cup \mathcal{E}_\sigma$. This implies by Lemma 6 and Lemma 7 that the function v_h is globally continuous.

Let $E \in \mathcal{E}_r \cup \mathcal{E}_\sigma$ be a given face and $i \in J(E)$ an arbitrary vertex node of E, i.e., $i \in J(K(E))$. Due to the condition $\psi_j^K(a_i) = \delta_{i,j}$ for all $i, j \in J(K)$, we have

$$v_h\big|_K(a_i) = \sum_{j\in J(K)} v_j \psi_j^K(a_i) = v_i \qquad \forall\, i \in J(K) \quad \forall\, K \in \mathcal{T}. \tag{19}$$

Thus, we conclude $v_h\big|_{K(E)}(a_i) = v_i$. In the case $E \in \mathcal{E}_r$, it holds that E is a face of the element $K'(E)$ which implies that the vertex a_i of E is also a vertex of $K'(E)$, i.e., $i \in J(K'(E))$ and $v_h\big|_{K'(E)}(a_i) = v_i$ due to (19). Therefore, we get $[v_h]_E(a_i) = 0$, i.e., condition (17) is satisfied for all regular faces $E \in \mathcal{E}_r$ without any constraint.

Now we consider the case of a son-face $E \in \mathcal{E}_\sigma$. Here, the element $K'(E) := K(\mathcal{F}(E))$ is the unique element associated with the father-face $\mathcal{F}(E)$ of the face E. In the case where a_i is a vertex of $K'(E)$, i.e., $i \in J(K'(E))$, it follows by (19) that $v_h\big|_{K'(E)}(a_i) = v_i$, i.e., the jump condition (17) is satisfied without any constraint for the components of the nodal vector $\underline{v}$. Let us now consider the case where $i \notin J(K'(E))$. Since $a_i \in \overline{E}$ and $i \in J(K(E))$ we get by Definition 4 that i is a hanging node. Because a_i is a vertex of E but not a vertex of its father-face $\mathcal{F}(E)$, it follows that a_i is either the midpoint of an edge e of element $K(\mathcal{F}(E)) = K'(E)$ or the barycenter of the face $\mathcal{F}(E)$ in the 3D-case. Therefore, a_i can be represented as a linear combination of some vertices a_j of element $K'(E)$. Using Lemma 5 and the fact that such a representation is unique, we get that

$$a_i = \sum_{j \in \Lambda_i} c_{i,j} a_j \qquad \text{where} \qquad \Lambda_i \subset J(K'(E)).$$

If a_i is the midpoint of an edge e of element $K'(E)$ with the vertices a_j, $j \in \Lambda_i$, we get from the analytical structure of the basis functions ψ_j^K that $\psi_j^{K'(E)}(a_i) = \frac{1}{2}$ if $j \in \Lambda_i$ and $\psi_j^{K'(E)}(a_i) = 0$ if $j \in J(K'(E)) \setminus \Lambda_i$. If, in the 3D-case, a_i is the barycenter of the two-dimensional face $\mathcal{F}(E)$ of $K'(E)$ with the vertices a_j, $j \in \Lambda_i$, we get $\psi_j^{K'(E)}(a_i) = \frac{1}{4}$ if $j \in \Lambda_i$ and $\psi_j^{K'(E)}(a_i) = 0$ if $j \in J(K'(E)) \setminus \Lambda_i$. This implies

$$v_h\big|_{K'(E)}(a_i) = \sum_{j \in J(K'(E))} v_j \psi_j^{K'(E)}(a_i) = \sum_{j \in \Lambda_i} v_j c_{i,j}, \tag{20}$$

where the $c_{i,j}$ are the coefficients defined in Lemma 5. From (20) and the constraint (18) we conclude $v_h\big|_{K'(E)}(a_i) = v_i$, i.e., that the jump condition (17) is satisfied.

Now, we prove that the constraints (18) are necessary for the global continuity of v_h. Let the function $v_h = \Phi_h \underline{v}$ be continuous. This implies by Lemma 6 and Lemma 7 that the jump condition (17) is satisfied for all faces $E \in \mathcal{E}_r \cup \mathcal{E}_\sigma$. Let $i \in J_h$ be an arbitrary hanging node. Then, by Definition 4 there exists a son-face $E \in \mathcal{E}_\sigma$ such that $a_i \in \overline{E}$ and $i \in J(K(E)) \setminus J(K(\mathcal{F}(E)))$. From (19) we get $v_h\big|_{K(E)}(a_i) = v_i$. In the same way as above, we obtain on the element $K'(E) := K(\mathcal{F}(E))$ the formula (20) for the value of v_h at the point a_i. Finally, the jump condition (17) implies that

$$v_h\big|_{K(E)}(a_i) = v_i = v_h\big|_{K'(E)}(a_i) = \sum_{j \in \Lambda_i} c_{i,j} v_j,$$

i.e., the constraint (18) is satisfied for the node i. $\square$

3.2 Construction of the operator R_h

Similar to Scott-Zhang [17], we assign to each node $j \in J$ with the nodal point a_j, a face $E_j \in \mathcal{E}_r \cup \mathcal{E}_i \cup \mathcal{E}_\Gamma$, such that the following conditions are satisfied:

$$\begin{aligned} a_j &\in \overline{E_j}, && (21)\\ a_j \in \partial\Omega &\Rightarrow E_j \subset \partial\Omega, && (22)\\ j \in J_h &\Rightarrow E_j \in \mathcal{E}_i \quad \text{and} \quad j \notin J(K(E_j)). && (23) \end{aligned}$$

Note that these conditions do not lead to a unique mapping $j \mapsto E_j$, i.e., we have several possible choices for E_j. The definition of a face E_j for a hanging node $j \in J_h$ is needed only in the analysis of the interpolation operator.

Now, for the definition and the subsequent analysis of the interpolation operator R_h, we use the general concept of nodal functionals [6, 3, 15]. For a *regular node* $j \in J_r$, we define the *nodal functional* $N_j : H^1(\Omega) \to \mathbb{R}$ by

$$N_j(v) := |E_j|^{-1} \int_{E_j} v \, ds \qquad \forall\, v \in H^1(\Omega), \quad j \in J_r. \tag{24}$$

Based on this definition we define, for a *hanging node* $i \in J_h$, the nodal functional $\widetilde{N}_i : H^1(\Omega) \to \mathbb{R}$ as the following linear combination of nodal functionals for regular nodes

$$\widetilde{N}_i(v) := \sum_{j \in \Lambda_i} c_{i,j} \, N_j(v) \qquad \forall\, v \in H^1(\Omega), \quad i \in J_h, \tag{25}$$

where the index set Λ_i and the coefficients $c_{i,j}$ are defined in Lemma 5. Then, for a given function $v \in H^1(\Omega)$, we define the interpolate $R_h v \in D_h$ locally on each element $K \in \mathcal{T}$ as

$$R_h v|_K := \sum_{j \in J_r(K)} N_j(v) \, \psi_j^K + \sum_{i \in J_h(K)} \widetilde{N}_i(v) \, \psi_i^K. \tag{26}$$

3.3 Properties of the operator R_h

One can easily show that the elementwise definition (26) yields a globally continuous function, i.e., for $v \in H^1(\Omega)$ we have $R_h v \in S_h$. In fact, the interpolate can be written as $R_h v = \Phi_h \underline{v}$ where the components of the nodal vector $\underline{v} = (v_j)_{j \in J} \in \mathbb{R}^n$ are defined by $v_j := N_j(v)$ for regular nodes $j \in J_r$ and $v_i := \widetilde{N}_i(v)$ for hanging nodes $i \in J_h$. Then, due to the choice of the nodal functional $\widetilde{N}_i(\cdot)$, it holds the condition $v_i = \sum_{j \in \Lambda_i} c_{i,j} v_j$ for each hanging node $i \in J_h$. Thus, from Lemma 8 we conclude that $R_h v = \Phi_h \underline{v} \in H^1(\Omega)$.

Furthermore, for a function $v \in H_0^1(\Omega)$, we have due to (22) that $N_j(v) = 0$ for each boundary node $j \in J$ with $a_j \in \partial\Omega$. Therefore, it holds

$$R_h v \in S_h \cap H_0^1(\Omega) \qquad \forall v \in H_0^1(\Omega), \tag{27}$$

i.e., the operator R_h preserves homogeneous boundary conditions in a natural way. Finally, one can easily show that R_h reproduces the constant functions, i.e.,

$$R_h v = v \qquad \text{if} \qquad v(x) = c \in \mathbb{R} \quad \forall x \in \Omega. \tag{28}$$

Note that this property already allows to prove that the interpolate $R_h v$ is of first order accurate in the $L^2(\Omega)$-norm which is an optimal result if the function v has not more regularity than $v \in H^1(\Omega)$. However, our interpolate would not be of optimal second order accurate in case of the higher regularity $v \in H^2(\Omega)$. In order to achieve this optimality, it would be necessary to choose more complicated nodal functionals $N_j(\cdot)$ for the regular nodes $j \in J_r$ (see [17]). Since we are only interested in the approximation of H^1-functions we have chosen the simpler nodal functionals in (24).

We are now able to state the main result of this paper.

Theorem 9 *For $K \in \mathcal{T}$, the operator R_h defined in (26) satisfies*

$$|R_h v|_{1,K} \le C|v|_{1,\omega(K)} \qquad \forall v \in H^1(\Omega), \tag{29}$$

$$\|v - R_h v\|_{0,K} \le C h_K |v|_{1,\omega(K)} \qquad \forall v \in H^1(\Omega), \tag{30}$$

where

$$\omega(K) := \bigcup_{\widetilde{K} \in \mathcal{N}(K)} \delta(\widetilde{K}) \qquad \textit{with} \qquad \delta(\widetilde{K}) := \bigcup_{K' \in \mathcal{N}(\widetilde{K})} \overline{K'}$$

and

$$\mathcal{N}(K) := \left\{ \widetilde{K} \in \mathcal{T} : \overline{K} \cap \overline{\widetilde{K}} \neq \emptyset \right\} \tag{31}$$

denoting the set of all neighboring elements of the element $K \in \mathcal{T}$.

Proof. For a regular node $j \in J_r(K)$, we have $a_j \in \overline{K}$ and $a_j \in \overline{E_j} \subset \overline{K(E_j)}$ due to (21). This implies for the element $\widetilde{K} := K(E_j)$ that $\overline{K} \cap \overline{\widetilde{K}} \neq \emptyset$, i.e., $\widetilde{K} \subset \delta(K)$. Thus, using the definition (24) of $N_j(\cdot)$, we obtain for $v \in H^1(\Omega)$

$$\begin{aligned} |N_j(v)| &\le |E_j|^{-1/2} \|v\|_{L^2(E_j)}, \\ &\le C h_{\widetilde{K}}^{1-d/2} \left\{ h_{\widetilde{K}}^{-1} \|v\|_{0,\widetilde{K}} + |v|_{1,\widetilde{K}} \right\}, \\ &\le C h_K^{1-d/2} \left\{ h_K^{-1} \|v\|_{0,\delta(K)} + |v|_{1,\delta(K)} \right\}, \end{aligned} \tag{32}$$

where the last two estimates follow from (53) and (52).

Now, let $i \in J_h(K)$ be a hanging node. Based on the definition (25) of $\widetilde{N}_i(v)$ we obtain

$$|\widetilde{N}_i(v)| \le \sum_{j \in \Lambda_i} |c_{i,j}| \, |N_j(v)| \le \sum_{j \in \Lambda_i} \frac{1}{2} |N_j(v)|, \tag{33}$$

where the last inequality directly follows from the definition of the $c_{i,j}$ in Lemma 5.

From condition (21) for the choice of E_i we get $a_i \in \overline{E_i} \subset \overline{K(E_i)}$ which implies together with $a_i \in \overline{K}$ that $K(E_i) \subset \delta(K)$. Condition (23) for E_i says that the nodal point a_i at

the boundary of $K(E_i)$ is not a vertex of $K(E_i)$. Therefore, a_i must be the midpoint of an edge of $K(E_i)$ or the barycenter of a face of $K(E_i)$. We conclude that all vertices a_j with $j \in \Lambda_i \subset J_r$ have to be among the vertices of element $K(E_i)$, i.e., it holds $\Lambda_i \subset J_r(K(E_i))$. If we now apply, for each node $j \in \Lambda_i$, the estimate (32) to $N_j(v)$ with the element $K(E_i)$ instead of K and use the relation $\delta(K(E_i)) \subset \omega(K)$ and the estimate $C' h_K \le h_{K(E_i)} \le C h_K$ due to (52), then we obtain from (33) the estimate

$$|\widetilde{N}_i(v)| \le C h_K^{1-d/2} \left\{ h_K^{-1} \|v\|_{0,\omega(K)} + |v|_{1,\omega(K)} \right\}. \tag{34}$$

From (26), (32) and (34) we can conclude

$$\begin{aligned} |R_h v|_{1,K} &\le \sum_{j \in J_r(K)} |N_j(v)| \, |\psi_j^K|_{1,K} + \sum_{i \in J_h(K)} |\widetilde{N}_i(v)| \, |\psi_i^K|_{1,K} \\ &\le C \left\{ h_K^{-1} \|v\|_{0,\omega(K)} + |v|_{1,\omega(K)} \right\}, \end{aligned} \tag{35}$$

since

$$|\psi_j^K|_{1,K} \le C h_K^{d/2-1} \qquad \forall K \in \mathcal{T}, \qquad \forall j \in J(K).$$

For any constant $c \in \mathbb{R}$ we obtain from (28) the equality

$$|R_h v|_{1,K} = |R_h v - c|_{1,K} = |R_h(v-c)|_{1,K},$$

which combined with (35) leads to

$$|R_h v|_{1,K} \le C \left\{ h_K^{-1} \|v - c\|_{0,\omega(K)} + |v|_{1,\omega(K)} \right\}.$$

Now let $c \in \mathbb{R}$ be the constant L_2-projection of v on the patch $\omega(K)$, then we get the estimate (29) by means of standard interpolation results.

The interpolation property (30) is proven as follows. From (26), (32) and (34) we obtain

$$\begin{aligned} \|R_h v\|_{0,K} &\le \sum_{j \in J_r(K)} |N_j(v)| \, \|\psi_j^K\|_{0,K} + \sum_{i \in J_h(K)} |\widetilde{N}_i(v)| \, \|\psi_i^K\|_{0,K} \\ &\le C \left\{ \|v\|_{0,\omega(K)} + h_K |v|_{1,\omega(K)} \right\}, \end{aligned} \tag{36}$$

where the last inequality follows from

$$\|\psi_j^K\|_{0,K} \le C h_K^{d/2} \qquad \forall K \in \mathcal{T}, \qquad \forall j \in J(K).$$

Let $c \in \mathbb{R}$ again be the constant L_2-projection of v on $\omega(K)$. Then we get

$$\|R_h(v-c)\|_{0,K} \le C \left\{ \|v - c\|_{0,\omega(K)} + h_K |v|_{1,\omega(K)} \right\} \le C h_K |v|_{1,\omega(K)}. \tag{37}$$

This estimate combined with (28) directly leads to

$$\|v - R_h v\|_{0,K} \le \|v - c\|_{0,K} + C h_K |v|_{1,\omega(K)} = C h_K |v|_{1,\omega(K)} \qquad \forall v \in H^1(\Omega),$$

which concludes the proof. □

Remark 10 *Using the above defined scalar interpolation operator $R_h : H^1(\Omega) \to S_h$, we can define an interpolation operator $\boldsymbol{R}_h : (H^1(\Omega))^m \to (S_h)^m$ for vector-valued functions in the following way. Let $v \in (H^1(\Omega))^m$ be a vector-valued function with the components $v = (v_1, \dots, v_m)^T$. Then, the k-th component $(\boldsymbol{R}_h v)_k \in S_h$ of the interpolate $\boldsymbol{R}_h v \in (S_h)^m$ is defined by*

$$(\boldsymbol{R}_h v)_k := R_h v_k, \qquad k = 1, \dots, m.$$

If we apply the estimates of Theorem 9 to $R_h v_k$, we get the analogous estimates also for the vector-valued interpolate $\boldsymbol{R}_h v$.

4 Appendix

In this section, we prove some properties for the considered non-affine equivalent quadrilateral and hexahedral meshes with hanging nodes which are needed for the proof of Theorem 9. The subsequent results, which are in their quintessence very similar to the well known case of simplicial elements, could not be found in the literature by the authors and are therefore quoted here for the sake of completeness. Note that in the three-dimensional case of non-affine equivalent hexahedral elements, the two dimensional faces of the elements can be curved. This fact causes some extra difficulties in the proof of some estimates which are easy to get in the case of affine equivalent elements.

Lemma 11 *Let $\mathcal{T}$ be a 1-irregular mesh which is shape regular in the sense of Definition 1. Furthermore, let h_{S_K} denote the diameter of simplex S_K associated to the element $K \in \mathcal{T}$ as described in Section 2.2. Then, for each element $K \in \mathcal{T}$, the following estimates are satisfied*

$$Ch_K \le h_{S_K} \le C' h_K, \tag{38}$$

$$\|DF_K(\hat{x})\| \le Ch_K \qquad \forall \hat{x} \in \widehat{K}, \tag{39}$$

$$Ch_K^d \le |\det(DF_K(\hat{x}))| \le C' h_K^d \qquad \forall \hat{x} \in \widehat{K}. \tag{40}$$

Proof. In order to prove (38), we consider the affine mapping

$$F_K^{\text{aff}}(\hat{x}) := F_K(0) + DF_K(0)\hat{x} = b_K + B_K \hat{x},$$

which maps $\widehat{S}$ to S_K. Then, we get

$$h_{S_K} = \sup_{\hat{x},\hat{y} \in \widehat{S}} \|F_K^{\text{aff}}(\hat{x}) - F_K^{\text{aff}}(\hat{y})\| \le 2\sqrt{d}\, \|B_K\|. \tag{41}$$

A direct computation shows that the column vectors of the matrix B_K can be represented as a linear combination of differences of the vectors $a_{K,m} = F_K(\hat{a}_m)$, $m = 1, \dots, 2^d$, where $\hat{a}_m$ are the vertices of $\widehat{K} = (-1, 1)^d$. The vectors $a_{K,m}$ correspond to the vertices of the original element K. Therefore, we obtain the estimate $\|B_K\| \le Ch_K$ which implies by

means of (41) the estimate $h_{S_K} \le C' h_K$. In [13], it has been proven that for a shape regular mesh in the sense of Definition 1, the following estimates hold

$$d!\,(1-\gamma_K)^d |S_K| \le |\det(DF_K(\hat{x}))| \le d!\,(1+\gamma_K)^d |S_K| \qquad \forall \hat{x} \in \widehat{K}, \tag{42}$$

$$\|DF_K(\hat{x})\| \le (1+\gamma_K)\|B_K\| \qquad \forall \hat{x} \in \widehat{K}. \tag{43}$$

Using the estimates (43), (5) and (3) we get

$$\begin{aligned} h_K &= \sup_{\hat{x},\hat{y}\in\widehat{K}} \|F_K(\hat{x}) - F_K(\hat{y})\| \le 2\sqrt{d} \sup_{\hat{\xi}\in\widehat{K}} \|DF_K(\hat{\xi})\| \\ &\le C(1+\gamma_K)\|B_K\| \le C(1+\gamma_0) h_{S_K}, \end{aligned}$$

which proves the estimate $Ch_K \le h_{S_K}$.

Now, the estimate (39) is a simple consequence of (43), (5), (3) and the estimate $h_{S_K} \le C' h_K$. Using the estimates (42), (5), (2) and (38) we get

$$|\det(DF_K(\hat{x}))| \ge d!(1-\gamma_0)^d |S_K| \ge C\rho_{S_K}^d \ge C h_{S_K}^d \ge C h_K^d \qquad \forall \hat{x} \in \widehat{K},$$

which proves the lower bound in (40). The upper bound in (40) is a simple consequence of (39). □

Lemma 12 *Let $\mathcal{T}$ be a 1-irregular mesh which is shape regular in the sense of Definition 1. Then for any element $K \in \mathcal{T}$ and for any face $E \in \mathcal{E}(K)$ the following estimates hold*

$$C h_K^{d-1} \le |E| \le C' h_K^{d-1}. \tag{44}$$

Proof. First we consider the two dimensional case $d = 2$. Let $\widehat{E} = F_K^{-1}(E)$ denote the edge of $\hat{K}$ which is associated with the original edge E. We assume that $\widehat{E}$ corresponds to the edge with $\hat{x}_1 = -1$. This means that $\widehat{E}$ can be parameterized as

$$\hat{x} = \widehat{\gamma}(t) := (-1, t)^T, \qquad \text{where } t \in (-1, 1),$$

and E as

$$x = \gamma(t) := F_K(\widehat{\gamma}(t)) \qquad \forall\, t \in (-1, 1).$$

Then we have

$$|E| = \int_{-1}^{+1} \|\dot{\gamma}(t)\|\, dt = \int_{-1}^{+1} \left\| \frac{\partial F_K}{\partial \hat{x}_2}(\hat{\gamma}(t)) \right\| dt. \tag{45}$$

From (39) we get the upper estimate in (44). Let $d_m := \dfrac{\partial F_K}{\partial \hat{x}_m}(\hat{\gamma}(t)) \in \mathbb{R}^2$, $m = 1, 2$, such that $DF_K(\hat{\gamma}(t)) = [d_1, d_2]$. It is well known that $|\det([d_1, d_2])|$ corresponds to the area of the parallelogram spanned by the vectors d_1 and d_2, i.e.,

$$|\det(DF_K(\hat{\gamma}(t)))| = \|d_1\|\, \|d_2\|\, \sin(\alpha),$$

where α is the angle between d_1 and d_2. Then, based on the estimate (40) of Lemma 11 and the estimate $\|d_1\| \leq Ch_K$, which is a consequence of (39), we obtain

$$\|d_2\| \geq \frac{Ch_K^2}{\|d_1\| \sin(\alpha)} \geq Ch_K.$$

Using (45), this implies the lower estimate in (44). The proof for the remaining cases where $\widehat{E}$ corresponds to $\hat{x}_i = \pm 1$ is completely analogous.

For the three dimensional case $d = 3$, we assume E to be defined such that $\widehat{E} = F_K^{-1}(E)$ corresponds to the face $\hat{x}_1 = -1$. Then, we can represent each point $\hat{x} \in \widehat{E}$ as

$$\hat{x} = \widehat{\gamma}(t_1, t_2) := (-1, t_1, t_2)^T \qquad \text{where } (t_1, t_2) \in G := (-1, 1)^2, \tag{46}$$

and each point $x \in E$ as

$$x = \gamma(t_1, t_2) := F_K(\widehat{\gamma}(t_1, t_2)), \qquad \forall (t_1, t_2) \in G. \tag{47}$$

Further, we have

$$|E| = \int_G \|N_E(t_1, t_2)\| \, dt_1 \, dt_2, \tag{48}$$

with

$$N_E(t_1, t_2) := \frac{\partial \gamma}{\partial t_1} \times \frac{\partial \gamma}{\partial t_2}\Big|_{(t_1,t_2)} = \frac{\partial F_K}{\partial \hat{x}_2} \times \frac{\partial F_K}{\partial \hat{x}_3}\Big|_{\hat{x}=\hat{\gamma}(t_1,t_2)}.$$

By definition we have

$$\begin{aligned} \det(DF_K(\hat{\gamma}(t_1, t_2))) &= \frac{\partial F_K}{\partial \hat{x}_1} \cdot \left(\frac{\partial F_K}{\partial \hat{x}_2} \times \frac{\partial F_K}{\partial \hat{x}_3} \right)\Big|_{\hat{x}=\hat{\gamma}(t_1,t_2)} \\ &= \frac{\partial F_K}{\partial \hat{x}_1} \cdot N_E(t_1, t_2) = \|\frac{\partial F_K}{\partial \hat{x}_1}\| \cdot \|N_E(t_1, t_2)\| \cdot \cos(\alpha), \end{aligned} \tag{49}$$

where α is the angle between $\frac{\partial F_K}{\partial \hat{x}_1}$ and $N_E(t_1, t_2)$. From (49) we deduce

$$\|N_E(t_1, t_2)\| = \frac{\det(DF_K(\hat{\gamma}(t_1, t_2)))}{\|\frac{\partial F_K}{\partial \hat{x}_1}\| \cdot \cos(\alpha)} \tag{50}$$

Then, from (40) and (39) we get

$$\|N_E(t_1, t_2)\| \geq Ch_K^2. \tag{51}$$

Together with (48) we get $|E| \geq Ch_K^2$, which is the lower estimate in (44). The upper estimate in (44) follows directly from (48) and (39). Obviously the proof where $\widehat{E}$ corresponds to $\hat{x}_i = \pm 1$ is completely analogous. $\square$

Lemma 13 *Let $\mathcal{T}$ be a 1-irregular and shape regular mesh. Then for all $K \in \mathcal{T}$, it holds*

$$Ch_{\widetilde{K}} \le h_K \le C' h_{\widetilde{K}} \qquad \forall \widetilde{K} \in \mathcal{N}(K), \tag{52}$$

where $\mathcal{N}(K)$ is the set of all neighboring elements of K defined in (31).

Proof. It is sufficient to show the estimate $h_K \le C' h_{\widetilde{K}}$ since the estimate $Ch_{\widetilde{K}} \le h_K$ follows by exchanging the role of K and $\widetilde{K}$. For each $\widetilde{K} \in \mathcal{N}(K)$, there exists a chain of face neighbored elements K_i, $i = 0, \ldots, s$, with $K_0 = K$ and $K_s = \widetilde{K}$ such that for the common boundary $E_i := \partial K_{i-1} \cap \partial K_i$ one the following three conditions is satisfied for $i = 1, \ldots, s$:

$$\begin{aligned}
&\text{Case 1:} \;\; E_i \in \mathcal{E}(K_{i-1}) \;\text{ and }\; E_i \in \mathcal{E}(K_i),\\
&\text{Case 2:} \;\; E_i \in \mathcal{E}(K_{i-1}) \cap \mathcal{E}_\sigma \;\text{ and }\; \mathcal{F}(E_i) \in \mathcal{E}(K_i),\\
&\text{Case 3:} \;\; \mathcal{F}(E_i) \in \mathcal{E}(K_{i-1}) \;\text{ and }\; E_i \in \mathcal{E}(K_i),
\end{aligned}$$

where $\mathcal{F}(E_i)$ denotes the father-face of E_i. Using (44), we get for each $i \in \{1, \ldots, s\}$ the estimates

$$\begin{aligned}
&\text{Case 1:} \;\; h_{K_{i-1}} \le C|E_i|^{1/(d-1)} \le \widetilde{C} h_{K_i},\\
&\text{Case 2:} \;\; h_{K_{i-1}} \le C|E_i|^{1/(d-1)} \le C|\mathcal{F}(E_i)|^{1/(d-1)} \le \widetilde{C} h_{K_i},\\
&\text{Case 3:} \;\; h_{K_{i-1}} \le C|\mathcal{F}(E_i)|^{1/(d-1)} \le C|E_i|^{1/(d-1)} \le \widetilde{C} h_{K_i}.
\end{aligned}$$

Thus, we obtain $h_K \le \widetilde{C}^s h_{\widetilde{K}}$. Due to the shape regularity of the mesh $\mathcal{T}$, the number s of the chain connecting K with $\widetilde{K} \in \mathcal{N}(K)$ is uniformly bounded which implies the estimate $h_K \le C' h_{\widetilde{K}}$. $\square$

Lemma 14 *Let $E \in \mathcal{E}_r \cup \mathcal{E}_i$ and $K \in \mathcal{T}$ such that $E \in \mathcal{E}(K)$. Then, the following estimate holds*

$$\|v\|_{L^2(E)} \le C\, h_K^{1/2} \left\{ h_K^{-1} \|v\|_{0,K} + |v|_{1,K} \right\} \qquad \forall v \in H^1(K). \tag{53}$$

Proof. At first, we will prove the estimate

$$\|v\|_{L^2(E)} \le C\, h_K^{(d-1)/2} \|\hat{v}\|_{L^2(\widehat{E})}, \tag{54}$$

where the function $\hat{v} \in H^1(\widehat{K})$ is defined by $\hat{v}(\hat{x}) := v(F_K(\hat{x}))$ for all $\hat{x} \in \widehat{K}$. We present the proof of (54) only for the three dimensional case $d = 3$. The case $d = 2$ follows easily by simple transformation of the integrals corresponding to both sides of (54).

We consider only the special case where $\widehat{E} = F_K^{-1}(E)$ corresponds to $\hat{x}_1 = -1$. For the face E, we get

$$\|v\|^2_{L^2(E)} = \int_G v(\gamma(t_1, t_2))^2 \, \|N_E(t_1, t_2)\| \, dt_1 dt_2,$$

with

$$N_E(t_1,t_2) = (\frac{\partial\gamma}{\partial t_1} \times \frac{\partial\gamma}{\partial t_2})\big|_{(t_1,t_2)}.$$

The definition of G (resp. γ) is given by means of the expression (46) (resp. (47)). Now, considering the estimate (39),

$$\|N_E(t_1,t_2)\| \leq \left\|\frac{\partial F_K}{\partial \hat{x}_2}(\widehat{\gamma}(t_1,t_2))\right\| \cdot \left\|\frac{\partial F_K}{\partial \hat{x}_3}(\widehat{\gamma}(t_1,t_2))\right\| \leq Ch_K^2,$$

we obtain

$$\|v\|^2_{L^2(E)} \leq Ch_K^2 \int_G v(\gamma(t_1,t_2))^2 \, dt_1 dt_2. \tag{55}$$

The parameterization of $\widehat{E}$ is given by $\hat{x} = \widehat{\gamma}(t_1,t_2) = (-1,t_1,t_2)^T$ for $(t_1,t_2) \in G$ and the corresponding normal vector is

$$\widehat{N}_{\widehat{E}}(t_1,t_2) := (\frac{\partial\widehat{\gamma}}{\partial t_1} \times \frac{\partial\widehat{\gamma}}{\partial t_2})\big|_{(t_1,t_2)} = (1,0,0)^T.$$

Therefore, we get

$$\|\hat{v}\|^2_{L^2(\widehat{E})} = \int_G \hat{v}(\widehat{\gamma}(t_1,t_2))^2 \, \|\widehat{N}_{\widehat{E}}(t_1,t_2)\| dt_1 dt_2 = \int_G v(\gamma(t_1,t_2))^2 dt_1 dt_2.$$

Together with (55), this proves (54) for the case $d = 3$. Now, we apply the trace theorem on the reference element $\widehat{K}$ and well-known estimates between the norms of $\hat{v}$ and $\widehat{K}$ and the norms of v on K (see e.g. [6]) and get

$$\|\hat{v}\|_{L^2(\widehat{E})} \leq C\|\hat{v}\|_{0,\widehat{K}} + C|\hat{v}|_{1,\widehat{K}} \leq Ch_K^{-d/2}\|v\|_{0,K} + Ch_K^{1-d/2}|v|_{1,K}.$$

The proof where $\widehat{E}$ corresponds to $\hat{x}_i = \pm 1$ is completely analogous. Together with (54), this proves the estimate (53). $\square$

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