A Multigrid Stokes Solver Using a Discretely Divergence Free Basis

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Abstract

We consider the discrete Stokes problem generated by a finite element method with continuous Q_r elements for the velocity components and discontinuous P_{r-1} elements for the pressure based on a quadrilateral mesh. Our aim is to construct an efficient solver for the arising indefinite system of equations. A key ingredient of our method is the construction of a local and practically useful basis of the discretely divergence free subspace of the velocity space. Thus, the computation of the velocity is reduced to the solution of symmetric positive definite problem in this subspace which can be done efficiently by a multigrid method. The pressure can be obtained by a simple post-processing procedure. The ideas for the construction of a discretely divergence free basis can be extended easily to the general case of an adaptive mixed mesh with triangular and quadrilateral elements and hanging nodes. The approach can be applied also to the Navier-Stokes equations.

1 Introduction

In this paper we consider the numerical solution of the Stokes equations. For simplicity we restrict our presentation to the stationary two-dimensional case with homogeneous

Dirichlet boundary conditions, i.e. we want to find a velocity $u : \Omega \to \mathbb{R}^2$ and a pressure $p : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\triangle u + \nabla p &= f \quad \text{in } \Omega \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain and f a given body force. We use a conforming finite element method with continuous polynomial approximation of order $r \geq 2$ for the velocity and discontinuous approximation of order $r-1$ for the pressure. Our aim is to create an efficient multigrid solver for the corresponding indefinite system of equations.

The main idea of our solver is to use a local discretely divergence free basis for the velocity space. The advantage of this approach is that it leads to a symmetric positive definite problem for the velocity which is completely decoupled from the pressure. Thus the difficulty to find a good solver for the indefinite coupled problem is circumvented. Once the velocity is solved the pressure can be calculated easily by a post-processing procedure.

The idea to use a discretely divergence free basis for the velocity space is not new. We refer to [13], [14] for the case of continuous pressure approximation and to [5], [6], [10], [7], [2], [11], [12] for the case of a discontinuous pressure ansatz. An advantage of a discontinuous pressure space is that the divergence free constraint can be characterized element-wise which simplifies the construction of a discretely divergence free basis. Among the mentioned papers, a multigrid method is considered in [2], [11], [12]. To the authors knowledge the new contribution of this paper is the general construction process of a local, practically useful basis of the discretely divergence free subspace of the velocity in the case of higher order elements. This process can be applied to a general *hp*-finite element discretization of the Stokes problem where a discontinuous piecewise polynomial approximation for the pressure is used. Let us emphasize that a discontinuous pressure approximation is not a disadvantage, it can be easily smoothed by a simple post-processing procedure which maintains the optimal order of approximation (see [8]).

The ideas of this paper can be applied also in the case of the stationary or non-stationary Navier - Stokes equations.

2 Preliminaries and Notation

The weak formulation of problem (1) reads:

$$\begin{array}{rcll} a(u,v) + b(v,p) & = & (f,v)_\Omega & \forall\, v \in X \\ b(u,q) & = & 0 & \forall\, q \in M \end{array} \tag{2}$$

where $(\cdot,\cdot)_G$ denotes the inner product in $L^2(G)$ or $(L^2(G))^2$, respectively, $X := (H_0^1(\Omega))^2$, $M := L_0^2(\Omega)$ with $L_0^2(G) := \{q \in L^2(G) : (q,1)_G = 0\}$,

$$a(u,v) := (\nabla u, \nabla v)_\Omega \qquad \text{and} \qquad b(v,p) := -(\nabla \cdot v, p)_\Omega\,. \tag{3}$$

Let the domain Ω be decomposed into quadrilateral elements $K \in \mathcal{T}^0$ forming the coarsest grid level $\ell = 0$, i.e. $\overline{\Omega} = \cup_{K\in\mathcal{T}^0}\overline{K}$. The next finer grid $\mathcal{T}^{\ell+1}$, $\ell = 0,\ldots,L-1$, is defined from the previous grid $\mathcal{T}^\ell$ by a uniform refinement of each cell $K \in \mathcal{T}^\ell$ into four finer cells. The diameter of an element K is denoted by h_K and the global mesh-size of the grid $\mathcal{T}^\ell$ is defined as $h_\ell := \max_{K\in\mathcal{T}^\ell} h_K$. By $\mathcal{E}^\ell$ we denote the set of all edges of the grid $\mathcal{T}^\ell$, by $\mathcal{E}^\ell(\partial\Omega)$ the set of all boundary edges and by $\mathcal{E}_0^\ell := \mathcal{E}^\ell \setminus \mathcal{E}^\ell(\partial\Omega)$ the set of all inner edges. For some curve $\Gamma \subset \overline{\Omega}$, the inner product in $L^2(\Gamma)$ will be denoted by $\langle\,\cdot,\cdot\,\rangle_\Gamma$.

For an element K, let $F_K : \widehat{K} \to K$ be the one-to-one mapping between the reference element $\widehat{K} := [-1,+1]^2$ and K. We denote by $\mathbb{P}_m(G)$ the space of all polynomials on the domain $G \subset \mathbb{R}^2$ with total degree less or equal to m and by $\mathbb{Q}_m(G)$ the space of those polynomials where the maximum power in each coordinate is less or equal to m. Let $r \geq 2$ be some given integer. Then, the finite element space of the velocity $X^\ell \subset X$ associated with grid level ℓ is defined as $X^\ell := S^\ell \times S^\ell$ with

$$S^\ell := \{\phi \in H_0^1(\Omega) :\ \phi\big|_K \circ F_K \in \mathbb{Q}_r(\widehat{K}) \quad \forall K \in \mathcal{T}^\ell\} \tag{4}$$

and the finite element space $M^\ell \subset M$ for the approximation of the pressure as

$$M^\ell := \{q \in L_0^2(\Omega) :\ q\big|_K \in \mathbb{P}_{r-1}(K) \quad \forall K \in \mathcal{T}^\ell\}\,. \tag{5}$$

Now the discrete Stokes problem on grid level ℓ reads: *Find* $(u^\ell, p^\ell) \in (X^\ell, M^\ell)$ *such that*

$$\begin{array}{rcll} a(u^\ell, v^\ell) + b(v^\ell, p^\ell) & = & (f, v^\ell)_\Omega & \forall\, v^\ell \in X^\ell \\ b(u^\ell, q^\ell) & = & 0 & \forall\, q^\ell \in M^\ell\,. \end{array} \tag{6}$$

The pair (X^ℓ, M^ℓ) fulfills the inf-sup-condition uniformly with respect to the mesh-size h_ℓ (see [3]) which implies the existence of a unique and stable solution (u^ℓ, p^ℓ) of problem (6).

Finally, let us introduce some general notation. We denote by $\mathrm{card}(J)$ the number of elements of a finite set J. Let X' be the dual space of a given space X. For a set $G \subset \mathbb{R}^2$, we denote by $\mathrm{int}(G)$ and $\overline{G}$ the interior and closure of G, respectively.

3 Construction of a Discretely Divergence Free Basis

Let $V^\ell \subset X^\ell$ be the linear subspace of all *discretely divergence free velocity functions*, i.e.

$$V^\ell := \{v^\ell \in X^\ell : \; b(v^\ell, q^\ell) = 0 \quad \forall q^\ell \in M^\ell\} \,. \tag{7}$$

Then the solution $u^\ell \in X^\ell$ of the discrete Stokes problem (6) is uniquely determined as the solution of the subspace problem

$$a(u^\ell, v^\ell) = (f, v^\ell)_\Omega \qquad \forall\, v^\ell \in V^\ell \,, \tag{8}$$

which is symmetric positive definite and completely decoupled from the pressure. If we could create a locally defined basis of V^ℓ we would only need a multigrid solver for the elliptic problem (8). Once the velocity is solved, the pressure $p^\ell \in M^\ell$ due to the inf-sup-condition is uniquely determined by

$$b(v^\ell, p^\ell) = (f, v^\ell)_\Omega - a(u^\ell, v^\ell) \qquad \forall\, v^\ell \in X^\ell \setminus V^\ell \,. \tag{9}$$

As it will be shown in Section 5, this problem can be solved by a simple post-processing procedure, where the fact is exploited that the pressure is defined element-wise without any continuity constraints at the element interfaces.
Another advantage of the discontinuous pressure space M^ℓ is that the divergence free constraint can be characterized element-wise, i.e. a velocity function $v^\ell \in X^\ell$ belongs to the space V^ℓ if and only if for all elements $K \in \mathcal{T}^\ell$ it holds

$$(\nabla \cdot v^\ell, q^\ell)_K = 0 \qquad \forall\, q^\ell \in \mathbb{P}_{r-1}(K) \,. \tag{10}$$

Now we want to construct the basis functions of V^ℓ. Let us denote the usual Lagrange basis functions of X^ℓ by φ_j where the indices $j \in J$ are called the "nodes" of X^ℓ. Here and in the following we will omit for all basis functions and sets of nodes the index ℓ indicating the level of the corresponding finite element space X^ℓ or V^ℓ. Our aim is to find the basis functions $\widetilde{\varphi}_j$ of V^ℓ where $j \in \widetilde{J}$ with

$$\mathrm{card}(\widetilde{J}) = \mathrm{card}(J) - \dim(M^\ell) \,. \tag{11}$$

At first, for a large subset of nodes $j \in J$, we want to modify the basis function $\varphi_j \in X^\ell$ into a discretely divergence free basis function $\widetilde{\varphi}_j \in V^\ell$. Suppose that each node $j \in J$ of the vector-valued basis $\{\varphi_j\}$ of X^ℓ is associated with a scalar basis function $\phi_j \in S^\ell$ such that

$$\varphi_j(x) = \phi_j(x)\, y^{(j)}\,, \tag{12}$$

where $y^{(j)} = (1,0)^T$ if the index j is an odd integer and $y^{(j)} = (0,1)^T$ if j is even. That means that the index set J consists of pairs of nodes having the same scalar basis function. Let us subdivide the set of nodes J into disjoint subsets

$$J = J_v \cup J_e \cup J_c \cup J_b\,.$$

where J_v is the set of nodes associated with a vertex of $\mathcal{T}^\ell$, J_e the set of nodes associated with the interior of an edge $E \in \mathcal{E}_0^\ell$, J_c the set of *corrector bubble nodes* and J_b the set of *rest bubble nodes* both satisfying $\mathrm{supp}(\varphi_j) \subset \overline{K}$ for some element $K \in \mathcal{T}^\ell$. We define the *space of corrector bubble functions* $X_c^\ell \subset X^\ell$ as

$$X_c^\ell := \mathrm{span}\{\varphi_j : j \in J_c\}\,.$$

For a given element $K \in \mathcal{T}^\ell$, we denote by $J(K)$ the set of nodes related to K, i.e.

$$J(K) := \{j \in J : \mathrm{supp}(\varphi_j) \cap \mathrm{int}(K) \neq \emptyset\}\,. \tag{13}$$

and by $J_s(K) := J(K) \cap J_s$ the corresponding subset for each index $s \in \{v, e, c, b\}$. Furthermore we introduce the local pressure space orthogonal to the constant functions $M_0^\ell(K) := \mathbb{P}_{r-1}(K) \cap L_0^2(K)$ and its basis functions χ_m^K, $m = 1, \ldots, n_0$. For the choice of X_c^ℓ we assume that

$$\mathrm{card}(J_c(K)) = \dim(M_0^\ell(K)) = n_0 \qquad \forall\, K \in \mathcal{T}^\ell \tag{14}$$

and that the following matrices $D^K \in \mathbb{R}^{n_0 \times n_0}$ are regular, i.e.

$$\det(D^K) \neq 0 \quad \forall\, K \in \mathcal{T}^\ell \quad \text{where} \quad D^K = (D^K_{mj}) := \big((\nabla \cdot \varphi_j, \chi_m^K)_K\big)\,. \tag{15}$$

Note that condition (15) is equivalent to a local Babuška-Brezzi condition for the spaces $X_c^\ell(K) := \mathrm{span}\{\varphi_j : j \in J_c(K)\}$ and $M_0^\ell(K)$. It is easy to prove that there exists a space X_c^ℓ satisfying (14) and (15).

Lemma 1 *Let $v \in X_n(\mathcal{T}^\ell)$ with*

$$X_n(\mathcal{T}^\ell) := \{v \in (H^1(\Omega))^2 : \left\langle v \cdot n^K, 1 \right\rangle_{\partial K} = 0 \quad \forall K \in \mathcal{T}^\ell\}, \tag{16}$$

where n^K denotes the unit normal vector on ∂K directed outward with respect to K. Assume that the space X_c^ℓ satisfies the conditions (14) and (15). Then there exists a projection $P_c v \in X_c^\ell$ such that for all $K \in \mathcal{T}^\ell$ it holds

$$(\nabla \cdot P_c v, q^\ell)_K = (\nabla \cdot v, q^\ell)_K \quad \forall\, q^\ell \in \mathbb{P}_{r-1}(K)\,. \tag{17}$$

Proof. Since $(P_c v)\big|_K \in (H_0^1(K))^2$ and $v \in X_n(\mathcal{T}^\ell)$ we get that (17) is satisfied for constant functions q^ℓ. For the function $P_c v \in X_c^\ell$ we use the ansatz $P_c v = \sum_{K \in \mathcal{T}^\ell} \sum_{j \in J_c(K)} \underline{v}_j \varphi_j$. Then (17) for a given element K and all $q^\ell \in M_0^\ell(K)$ is equivalent to

$$\sum_{j \in J_c(K)} D_{mj}^K \, \underline{v}_j = (\nabla \cdot v, \chi_m^K)_K \qquad \forall\, m = 1, \ldots, n_0\,, \tag{18}$$

which is a linear system of dimension n_0 with the system matrix D^K. Because of (15) there exists, for each $K \in \mathcal{T}^\ell$, a unique vector $(\underline{v}_j)_{j \in J_c(K)}$ satisfying (18), which completes the proof of (17) for the ansatz $P_c v$. ■

A consequence of Lemma 1 is that a discrete function $v^\ell \in X^\ell \cap X_n(\mathcal{T}^\ell)$ can be modified to a function $\tilde{v}^\ell := v - P_c v \in V^\ell$. That means we can modify a given basis function $\varphi_j \in X^\ell$ into a discretely divergence free function $\widetilde{\varphi}_j \in V^\ell$ by first changing φ_j into a function $\overline{\varphi}_j \in X^\ell \cap X_n(\mathcal{T}^\ell)$ and then taking $\widetilde{\varphi}_j := \overline{\varphi}_j - P_c\, \overline{\varphi}_j$. Note that the computation of the projection $P_c\, \overline{\varphi}_j$ can be done element-wise with low computational costs by solving local linear systems of type (18).

In the following, we want to construct for each basis function $\varphi_j \in X^\ell$ the corresponding function $\overline{\varphi}_j \in X^\ell \cap X_n(\mathcal{T}^\ell)$. The vector-valued functions $\overline{\varphi}_j$ will be defined by means of modified scalar basis functions $\overline{\phi}_j \in S^\ell$ defined below. These functions $\overline{\phi}_j$, except two per edge, are chosen such that $\left\langle \overline{\phi}_j, 1 \right\rangle_E = 0$ for all edges $E \in \mathcal{E}_0^\ell$ and therefore the vector-valued function

$$\overline{\varphi}_j(x) := \overline{\phi}_j(x)\, \overline{y}^{(j)} \tag{19}$$

with some unit vector $\overline{y}^{(j)} \in \mathbb{R}^2$ satisfies $\overline{\varphi}_j \in X^\ell \cap X_n(\mathcal{T}^\ell)$.

Now, for the different types of nodes j, we will define the associated functions $\overline{\phi}_j$ and the vectors $\overline{y}^{(j)}$.

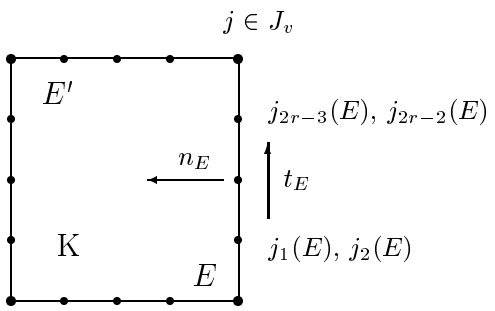

Figure 1: vertex node $j \in J_v$ and inner edge nodes $j_m(E)$, $m = 1, \ldots, 2r-2$

Type 1: $j \in J_e$. For each edge $E \in \mathcal{E}_0^\ell$ we define a unit vector $n_E = (n_{E,1}, n_{E,2})^T$ normal to E which is directed outward with respect to one of the two adjacent elements $K, K' \in \mathcal{T}^\ell$ with $E = \partial K \cap \partial K'$. As a tangential vector associated with E we take $t_E := (n_{E,2}, -n_{E,1})^T$. We introduce, for each edge E, the scalar quadratic edge bubble function $b_E \in S^\ell$ defined by $b_E\big|_K \circ F_K \in \mathbb{Q}_2(\widehat{K})$ for all $K \in \mathcal{T}^\ell$ and the conditions

$$b_E\big|_{E'} = 0 \quad \forall\, E' \in \mathcal{E}^\ell \setminus \{E\} \qquad \text{and} \qquad \langle\, b_E, 1 \,\rangle_E = |E|\,. \tag{20}$$

Furthermore, let $J_e(E) := \{j_1(E), j_2(E), \ldots j_{2r-2}(E)\}$ be the set of all nodes associated with the interior of the edge E and enumerated in the direction of the tangential vector t_E (see Figure 1). Then we define for all $j \in J_e(E)$

$$\overline{\phi}_j(x) := \begin{cases} b_E(x) & \text{if } j \in \{j_1(E), j_2(E)\} \\ \phi_j(x) \;-\; |E|^{-1} \langle\, \phi_j, 1 \,\rangle_E \; b_E(x) & \text{otherwise} \end{cases} \tag{21}$$

and $\overline{y}^{(j)} := n_E$ for $j = j_1(E)$, $\overline{y}^{(j)} := t_E$ for $j = j_2(E)$ and $\overline{y}^{(j)} := y^{(j)}$ otherwise where $y^{(j)}$ is the unit vector corresponding to (12).

Type 2: $j \in J_v$. Let $K \in \mathcal{T}^\ell$ be an arbitrary element such that $j \in J(K)$ and E, E' the two edges of K associated with j (see Figure 1). Then we define the function $\overline{\phi}_j(x)$ for all $x \in \overline{K}$ by

$$\overline{\phi}_j(x) := \phi_j(x) \;-\; |E|^{-1} \langle\, \phi_j, 1 \,\rangle_E \; b_E(x) \;-\; |E'|^{-1} \langle\, \phi_j, 1 \,\rangle_{E'} \; b_{E'}(x) \tag{22}$$

and $\overline{y}^{(j)} := y^{(j)}$. On elements $K \in \mathcal{T}^\ell$ with $j \notin J(K)$ we define $\overline{\phi}_j := \phi_j = 0$. Note that $\overline{\phi}_j : \Omega \to \mathbb{R}$ is a continuous function that is contained in the space S^ℓ.

Type 3: $j \in J_b$. Here we have $\phi_j(x) = 0$ for all $x \in \partial K$, $K \in \mathcal{T}^\ell$. That means it is

sufficient to choose $\overline{\phi}_j(x) := \phi_j(x)$ and $\overline{y}^{(j)} := y^{(j)}$ in order to ensure $\overline{\varphi}_j \in X_n(\mathcal{T}^\ell)$ for $\overline{\varphi}_j$ defined by (19).

Now, let us define the index set

$$J_e^* := \bigcup_{E \in \mathcal{E}_0^\ell} \big(J_e(E) \setminus \{ j_1(E) \} \big) . \tag{23}$$

Then, for all nodes $j \in J_v \cup J_e^* \cup J_b$, we have by construction that $\overline{\varphi}_j \in X_n(\mathcal{T}^\ell) \cap X^\ell$. Therefore, using Lemma 1, we get the discretely divergence free modification of φ_j as

$$\widetilde{\varphi}_j := \overline{\varphi}_j - P_c \overline{\varphi}_j \in V^\ell \qquad \forall\, j \in J^* := J_v \cup J_e^* \cup J_b \,. \tag{24}$$

Now, a simple calculation of the dimension of V^ℓ yields

$$\dim V^\ell = \mathrm{card}(J^*) \,+\, \{ \mathrm{NE}_0 - \mathrm{NEL} + 1 \} \,,$$

where NEL denotes the number of elements and NE_0 the number of inner edges of $\mathcal{T}^\ell$. Let us assume that the boundary of Ω consists of the closed components Γ_i, $i = 0, \ldots, n_b$, where Γ_0 denotes the exterior boundary and Γ_i, $i = 1, \ldots, n_b$, the boundary of the i-th hole of Ω. Then, using a well-known theorem on the relation between the numbers of the d-dimensional objects, $d = 0, 1, 2$, of the mesh $\mathcal{T}^\ell$, we get

$$\dim V^\ell = \mathrm{card}(J^*) \,+\, \mathrm{NV}_0 \,+\, n_b \,, \tag{25}$$

where NV_0 denotes the number of the inner vertices of $\mathcal{T}^\ell$. This shows that we have to construct $\mathrm{NV}_0 + n_b$ further basis functions of V^ℓ in addition to the functions $\widetilde{\varphi}_j$ with $j \in J^*$.

For the construction of the first NV_0 functions, we can use the same idea as in [5], [2] where to each inner vertex of the mesh a vector-valued basis function is assigned for the approximation of the **curl** of a scalar stream function. In the following, we want to describe the construction of these *vortex basis functions* ψ_j for $j \in J_\psi$. Each node $j \in J_\psi$ is uniquely associated with an inner vertex a_j of the mesh $\mathcal{T}^\ell$. Let us denote by $\mathcal{E}^\ell(a_j)$ the set of all edges $E \in \mathcal{E}^\ell$ corresponding to the vertex a_j in the sense that $a_j \in \overline{E}$. Furthermore, to a given vertex a_j, we assign the unit vectors $n_{j,E}$ which are normal to the edges $E \in \mathcal{E}^\ell(a_j)$ and directed counterclockwise around a_j (see Figure 2). Then we define the function $\psi_j \in X^\ell$ as

$$\psi_j(x) := \sum_{E \in \mathcal{E}^\ell(a_j)} |E|^{-1} b_E(x)\, n_{j,E} \,. \tag{26}$$

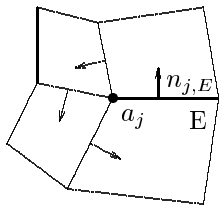

Figure 2: vertex node a_j and associated normal vectors $n_{j,E}$

One can easily verify that $\psi_j \in X_n(\mathcal{T}^\ell)$. Thus, for each node $j \in J_\psi$ we get the discretely divergence free basis function

$$\widetilde{\varphi}_j := s_j(\psi_j - P_c\psi_j) \qquad \text{with} \qquad s_j := \{\text{card}(\mathcal{E}^\ell(a_j))\}^{-1} \sum_{E\in\mathcal{E}^\ell(a_j)} |E|\,. \tag{27}$$

In contrast to [5], [2], we have introduced the scaling factors $s_j = O(h_\ell)$ in order to ensure that the L^2-norm of these functions is of the same order (with respect to h_ℓ) as for the former constructed functions $\widetilde{\varphi}_j$ with $j \in J^*$. An advantage of this scaling is that the condition numbers of the corresponding stiffness matrix and mass matrix have a better order than without scaling [9].

To complete our basis of V^ℓ we have to construct n_b many functions, say $\widetilde{\varphi}_j$ with $j \in J_\Gamma$. For each node $j \in J_\Gamma$, we uniquely assign a boundary component $\Gamma_{k(j)}$ with $1 \le k(j) \le n_b$. Let us denote by $\mathcal{E}^\ell(\Gamma_{k(j)})$ the set of all inner edges $E \in \mathcal{E}_0^\ell$ touching the boundary component $\Gamma_{k(j)}$, i.e. $\overline{E} \cap \Gamma_{k(j)} \neq \emptyset$. Furthermore, to $\Gamma_{k(j)}$ we assign the set of unit vectors $n_{j,E}$ which are normal to the edges $E \in \mathcal{E}^\ell(\Gamma_{k(j)})$ and directed counterclockwise around the corresponding vertex $a_{j,E} \in \overline{E}$ that is located at $\Gamma_{k(j)}$. Then we define a function $\theta_j \in X^\ell$ as

$$\theta_j(x) := \sum_{E\in\mathcal{E}^\ell(\Gamma_{k(j)})} |E|^{-1} b_E(x)\, n_{j,E}\,, \qquad j \in J_\Gamma\,. \tag{28}$$

For this function, it holds $\theta_j \in X_n(\mathcal{T}^\ell)$ such that for each node $j \in J_\Gamma$ we get a discretely divergence free basis function

$$\widetilde{\varphi}_j := s_j(\theta_j - P_c\theta_j) \qquad \text{with} \qquad s_j := \{\text{card}(\mathcal{E}^\ell(\Gamma_{k(j)}))\}^{-1} \sum_{E\in\mathcal{E}^\ell(\Gamma_{k(j)})} |E|\,. \tag{29}$$

We can prove the following result about the set of the constructed functions $\widetilde{\varphi}_j$ [9].

Theorem 2 *The functions $\widetilde{\varphi}_j \in X^\ell$ defined by (24), (27) and (29) form a basis of the subspace of discretely divergence free functions $V^\ell \subset X^\ell$, i.e.*

$$V^\ell = span\{\, \widetilde{\varphi}_j \,:\, j \in \widetilde{J} := J_v \cup J_e^* \cup J_b \cup J_\psi \cup J_\Gamma \}\,. \tag{30}$$

The functions $\widetilde{\varphi}_j$, $j \in J_\Gamma$, in contrast to the other functions $\widetilde{\varphi}_j$, $j \in \widetilde{J} \setminus J_\Gamma$, have a non-local support which is a strip of elements located adjacent to the corresponding boundary component $\Gamma_{k(j)}$. This would be a disadvantage concerning the sparsity pattern of the stiffness matrix associated with the basis functions $\widetilde{\varphi}_j$. However, this disadvantage can be avoided in the same way as in [11]. There it has been proposed to use, instead of the functions $\widetilde{\varphi}_j$, $j \in J_\Gamma$, the local functions $\widetilde{\varphi}_j$ of the same type as defined in (27) and (26) where each node j is associated with a vertex of $\mathcal{T}^\ell$ located at one of the boundary components $\Gamma_1, \ldots, \Gamma_{n_b}$. Then, all these local basis functions generate a larger space, say $\widetilde{V}^\ell$, that does not satisfy the boundary conditions of X^ℓ. However, a function $v = \sum_j \underline{v}_j \widetilde{\varphi}_j \in \widetilde{V}^\ell$ is contained in the right space V^ℓ if the coefficients $\underline{v}_j$ of the new local "vortex basis functions" are identical for all vertices a_j that are located at the same boundary component Γ_i, $1 \le i \le n_b$. This can be managed by a projection technique [11].

The presented technique for the construction of a discretely divergence free basis can be generalized in a straightforward way to the case where the polynomial degree of the finite element functions can vary from one element to another. The ideas of our construction can also be applied to the case of triangular elements. Moreover, adaptive mesh refinement based on the use of hanging nodes as well as more general boundary conditions can be treated (see [9]).

4 Multigrid Algorithm

Most of the work in this paper was needed to construct basis functions of the discretely divergence free subspace V^ℓ of the original velocity space X^ℓ. Now, the design of a multigrid method for solving the subspace problem (8) is relatively standard. For each level $\ell = 0, 1, \ldots, L$, we generate the stiffness matrix $A^\ell = (A_{ij}^\ell) \in \mathbb{R}^{\tilde{n}_\ell \times \tilde{n}_\ell}$ defined by

$$A_{ij}^\ell := a(\widetilde{\varphi}_j^\ell, \widetilde{\varphi}_i^\ell) \qquad \forall\, i, j \in \widetilde{J}^\ell\,, \tag{31}$$

where the $\widetilde{\varphi}_j^\ell$, $j \in \widetilde{J}^\ell$, denote the discretely divergence free basis functions of the subspace V^ℓ at level ℓ and $\tilde{n}_\ell := \dim(V^\ell)$. Then, we use all these matrices A^ℓ to apply several

multigrid cycles to the finest grid equation

$$A^L \underline{u}^L = \underline{F}^L \,, \tag{32}$$

where $\underline{u}^L \in \mathbb{R}^{\tilde{n}_L}$ is the nodal vector for our discrete solution $u^L \in V^L$ in the basis representation

$$u^L = \sum_{j \in \tilde{J}^L} \underline{u}_j^L \widetilde{\varphi}_j^L$$

and $\underline{F}^L = (\underline{F}_i^L) := ((f, \widetilde{\varphi}_i^L)_\Omega) \in \mathbb{R}^{\tilde{n}_L}$. For the definition of the multigrid cycle we refer to [2, §4] and [11, §4]. Whereas in [2] and [11] a lot of work has been spent for the construction of intergrid transfer operators $I_{\ell-1}^{\ell} : V^{\ell-1} \to V^{\ell}$, we can simply use the general transfer operator $R_h^{(g)} : L^1(\Omega) \to V_h^{(2)}$, defined in [8, §3], where the "target space" $V_h^{(2)}$ in our context is the space V^ℓ. The local nodal functionals $N_{j,K}^{(g)}(\cdot)$ are defined from the basis functions $\widetilde{\varphi}_j^\ell \in V^\ell$ by local L^2-projection on the reference element (see [8, eq. (19) and (20)]). This transfer operator is defined locally and fulfills the conditions of the analysis in [2]. We think it will be possible to show an analogous result on the convergence of the multigrid algorithm as in [2, 11]. For the proof and numerical tests, we refer to a forthcoming paper.

5 Computation of the Pressure

Once the velocity $u^L \in V^L$ has been computed by the multigrid algorithm, the pressure $p^L \in M^L$ can be calculated by the following post-processing procedure without solving any global system of equations. For all elements, the local pressure is decomposed uniquely into a constant $p_{0,K}$ and its orthogonal complement $\widetilde{p}_K \in L_0^2(K) \cap \mathbb{P}_{r-1}(K)$:

$$p^L(x) := p_{0,K} + \widetilde{p}_K(x) \qquad \forall\, x \in K \,,\; K \in \mathcal{T}^L \,. \tag{33}$$

In a first step, for each element $K \in \mathcal{T}^L$, the local pressure $\widetilde{p}_K$ can be determined from (9) by solving the local system of dimension n_0 (see (14))

$$b(\varphi_i, \widetilde{p}_K) = (f, \varphi_i)_\Omega - a(u^L, \varphi_i) \qquad \forall\, i \in J_c(K) \,.$$

Because of the assumption (15) the matrix of this system is regular.

Then, in a second step, all pressure constants $p_{0,K} \in \mathbb{R}$ for $K \in \mathcal{T}^L$ are determined. We start by setting $p_{0,K_0} = 0$ for an arbitrary element $K_0 \in \mathcal{T}^L$ located at the exterior

boundary of Ω. Now, we pass through all other elements by a marching strategy where the next element K is chosen adjacent to an element K' that has been passed already. Let $E \in \mathcal{E}_0^L$ the common edge of K and K'. We choose in equation (9) as a test function $v^L(x) = b_E(x) n'_E$ where n'_E is the unit vector normal to E and directed outward with respect to K', This gives

$$\{p_{0,K} - p_{0,K'}\} \langle b_E, 1 \rangle_E = (f, b_E n'_E)_\Omega - a(u^L, b_E n'_E) + (\nabla \cdot (b_E n'_E), \widetilde{p}_K + \widetilde{p}_{K'})_{K \cup K'} ,$$

where the local pressures $\widetilde{p}_K$ and $\widetilde{p}_{K'}$ on the right hand side are extended by zero on K' and K, respectively. From this equation the constant $p_{0,K}$ can be computed. At the end, a global constant c_Ω can be calculated such that the discrete pressure solution of (6) is $p^L + c_\Omega \in L_0^2(\Omega)$ where p^L denotes the pressure computed by the above described procedure.

6 Concluding Remarks

We can apply our technique also in the case of a continuous pressure approximation. Let us consider, for example, the Taylor-Hood element Q_2-Q_1 for quadrilateral elements or P_2-P_1 for triangles [3]. Here the condition (14) for the choice of the space X_c^ℓ of the corrector bubble functions requires to have more element bubble functions (with the support of just one element) than in the original velocity space Q_2 or P_2, respectively. However, such elements are known as for instance the Q_2^+-Q_1 element or the P_2^+-P_1 element in [1]. The additional bubble functions are not a disadvantage since they can be eliminated during the assembling of the stiffness matrix.

Many of the presented ideas for the construction of a discretely divergence free basis can be applied also in the 3D case. In particular, we can construct a generating system for the space V^ℓ such that $V^\ell = \text{span}\{\widetilde{\varphi}_j : j \in \widetilde{J}\}$ where some of the functions $\widetilde{\varphi}_j$ are linearly dependent. It should be possible to create a multigrid method also for the case of a generating system. For an example of such methods from the literature we refer to [4].

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