# Musterlösungen Blatt 8

1.) $f: \mathbb{R}^2 \to \mathbb{R}$, $\quad f(x,y) := \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

<u>1. Fall:</u> $a = (0,0)$

- wir betrachten die Folge $(x_n, y_n) = (\frac{1}{n}, 0)$
- es gilt offenbar $(x_n, y_n) \longrightarrow a = (0,0)$ für $n \to \infty$

und

$$f(x_n, y_n) = \frac{(\frac{1}{n})^2 - 0}{(\frac{1}{n})^2 + 0} = 1 \quad \forall n$$

$$\Rightarrow \lim_{n\to\infty} f(x_n, y_n) = 1$$

- $1 \neq f(0,0) = 0 \implies$ <u>$f$ ist nicht stetig in $a = (0,0)$</u>

<u>2. Fall: $a \neq (0,0)$</u>

- $\Rightarrow \exists \varepsilon > 0 : \boxed{(x,y) \neq (0,0) \quad \forall \; (x,y) \in B_\varepsilon(a)} \quad (1)$
- sei $(x_n, y_n) \in \mathbb{R}^2$ beliebige Folge mit

$$(x_n, y_n) \xrightarrow{n\to\infty} a = (a_1, a_2)$$

$$\Rightarrow \exists n_0 : \boxed{(x_n, y_n) \in B_\varepsilon(a) \quad \forall n > n_0} \quad (2)$$

$$\xRightarrow{(1),(2)} (x_n, y_n) \neq (0,0) \quad \forall n > n_0$$

$$\Rightarrow f(x_n, y_n) = \frac{x_n^2 - y_n^2}{x_n^2 + y_n^2} \quad \forall n > n_0$$

nach Grenzwertsatz folgt $\lim_{n\to\infty} f(x_n, y_n) = \dfrac{a_1^2 - a_2^2}{a_1^2 + a_2^2} = f(a_1, a_2)$

$\Rightarrow$ <u>$f$ ist stetig in $a$</u>

2.) $f(x,y,z) = x^{y+z} = e^{(y+z)\ln(x)}$

- $f_x = (y+z)\, x^{y+z-1}$

$f_y = \frac{\partial}{\partial y}\left( e^{(y+z)\ln(x)} \right) = e^{(y+z)\ln(x)} \cdot \ln(x)$

$f_z = \frac{\partial}{\partial z}\left( e^{(y+z)\ln(x)} \right) = e^{(y+z)\ln(x)} \cdot \ln(x)$

$\Rightarrow \operatorname{grad} f(x,y,z) = \left[ f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z) \right]$

$= \underline{\underline{\left[ (y+z)\, x^{y+z-1},\ e^{(y+z)\ln(x)} \cdot \ln(x),\ e^{(y+z)\ln(x)} \cdot \ln(x) \right]}}$

- $f_{xx} = (y+z)(y+z-1)\ x^{y+z-2}$

$f_{yy} = e^{(y+z)\ln(x)} \cdot [\ln(x)]^2 = x^{y+z} \cdot [\ln(x)]^2$

$f_{zz} = e^{(y+z)\ln(x)} \cdot [\ln(x)]^2 = $ — " —

$f_{xy} = 1 \cdot x^{y+z-1} + (y+z) \cdot \frac{\partial}{\partial y}\left( e^{(y+z-1)\cdot\ln(x)} \right)$

$= x^{y+z-1} + (y+z)\ e^{(y+z-1)\cdot\ln(x)} \cdot \ln(x)$

$= \underline{\underline{x^{y+z-1} + (y+z)\cdot\ln(x)\ x^{y+z-1}}}$

analog $f_{xz} = \underline{\underline{x^{y+z-1} + (y+z)\cdot\ln(x)\ x^{y+z-1}}}$

$$f_{yz} = \frac{\partial}{\partial z}\left( e^{(y+z)\cdot \ln(x)} \cdot \ln(x) \right)$$

$$= \ln(x) \cdot e^{(y+z)\cdot \ln(x)} \cdot \ln(x)$$

$$\underline{\underline{= [\ln(x)]^2 \, x^{y+z}}}$$

- nach dem Satz von Schwarz gilt:

$$f_{yx} = f_{xy}, \quad f_{zx} = f_{xz}, \quad f_{zy} = f_{yz}$$

$\Rightarrow$ Hesse-Matrix $H_f(x,y,z) = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix}$

3.) $g: \mathbb{R}^2 \to \mathbb{R}^2$, $g(u,v) = \begin{pmatrix} \cos(u-2v) \\ (u+v)\sin(3u) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 y - x y^2$

- nach Def. ist:

$$(f \circ g)(u,v) = f\big(\underbrace{g_1(u,v)}_{=x}, \underbrace{g_2(u,v)}_{=y}\big)$$

$\Rightarrow$ mit Kettenregel

$$\frac{\partial (f\circ g)}{\partial u}(u,v) = \frac{\partial f}{\partial x}\cdot\frac{\partial g_1}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial g_2}{\partial u}$$

$$= (2xy - y^2)\cdot(-\sin(u-2v)\cdot 1)$$
$$\underline{\underline{+ (x^2 - 2xy)\{1\cdot\sin(3u) + (u+v)\cos(3u)\cdot 3\}}}$$

wobei in obiger Formel gilt:

$$x = g_1(u,v) = \cos(u-2v)$$

$$y = g_2(u,v) = (u+v)\sin(3u)$$

- analog erhält man

$$\frac{\partial (f\circ g)}{\partial v}(u,v) = \frac{\partial f}{\partial x}\frac{\partial g_1}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial g_2}{\partial v}$$

$$= (2xy - y^2)\cdot\{-\sin(u-2v)\cdot(-2)\}$$

$$\underline{\underline{+ (x^2 - 2xy)\cdot\{\sin(3u)\}}}$$