Musterlösungen Blatt 5

1a) $I = \int_1^\infty \frac{\ln(t)}{t^2}\,dt$, $u = \ln(t)$, $v' = t^{-2}$
$u' = t^{-1}$, $v = -t^{-1}$

$$= \lim_{R\to\infty} \left\{ \int_1^R \underbrace{\ln(t)}_{u} \cdot t^{-2}\,dt \right\}$$

$$= \lim_{R\to\infty} \left\{ -t^{-1}\ln(t)\Big|_1^R - \int_1^R t^{-1}\cdot(-t^{-1})\,dt \right\}$$

$$= \lim_{R\to\infty} \left\{ -\underbrace{\frac{\ln(R)}{R}}_{\text{Typ } \frac{\infty}{\infty}\text{, l'Hospital}} + \left[-t^{-1}\right]_1^R \right\}$$

$$= \lim_{R\to\infty} -\underbrace{\frac{1/R}{1}}_{\to 0} - \lim_{R\to\infty} \left(\underbrace{\frac{1}{R}}_{\to 0} - 1\right) = \underline{\underline{1}}$$

$\Rightarrow$ Integral ist konvergent, Wert $I = 1$

1b) $I = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$

$$= \lim_{\varepsilon\to 0} \int_{-1+\varepsilon}^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\varepsilon\to 0} \underbrace{\arcsin(x)}_{\text{ist stetig}}\Big|_{-1+\varepsilon}^{1-\varepsilon}$$

$$= \arcsin(1) - \arcsin(-1)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \underline{\underline{\pi}}$$

$\Rightarrow$ Integral ist konvergent, Wert $I = \pi$

1c) $I = \int_0^{\pi/2} \tan(x)\,dx$

$$= \lim_{\varepsilon\to 0} \int_0^{\frac{\pi}{2}-\varepsilon} \frac{\sin(x)}{\cos(x)}\,dx \;, \qquad z = \cos(x), \quad dz = -\sin(x)\,dx$$

$$= \lim_{\varepsilon\to 0} \underbrace{\left(\int -\frac{dz}{z}\right)}_{\text{Stammfkt}} = \lim_{\varepsilon\to 0} -\ln\big(\underbrace{\cos(x)}_{>0}\big)\Big|_0^{\pi/2-\varepsilon}$$

$$\Rightarrow I = \lim_{\varepsilon\to 0} \Big\{ \underbrace{-\ln\big(\underbrace{\cos(\tfrac{\pi}{2}-\varepsilon)}_{\to 0}\big)}_{\to +\infty} + \underbrace{\ln(1)}_{=0} \Big\}$$

$$= \underline{\underline{+\infty}}$$

$\Rightarrow$ Integral ist <u>divergent</u>.

(2) $$I_p = \int_1^\infty \frac{\sqrt{x}}{\sqrt{1+x^p}}\,dx$$

<u>Fall $p>3$:</u>

- $$|f(x)| = \frac{\sqrt{x}}{\sqrt{1+x^p}} \le \frac{x^{1/2}}{x^{p/2}} = x^{\frac{1}{2}-\frac{p}{2}} =: g(x)$$
- $$\int_1^\infty g(x)\,dx = \lim_{R\to\infty}\int_1^R x^{\frac{1}{2}-\frac{p}{2}}\,dx = \lim_{R\to\infty} \frac{1}{\frac{3}{2}-\frac{p}{2}}\, x^{\frac{3}{2}-\frac{p}{2}}\Big|_1^R$$

$$= \lim_{R\to\infty} \frac{2}{3-p}\Big\{ \underbrace{\boxed{\frac{1}{R^{\frac{p-3}{2}}}}}_{\to 0,\ \text{da } p>3} - 1 \Big\} = \frac{-2}{3-p}$$

- nach dem Vergleichskriterium ist damit I_p konvergent für $p>3$ ∎

<u>Fall $p \le 3$:</u>

- $$f(x) = \frac{x^{1/2}}{(1+x^p)^{1/2}} \underbrace{\overset{x\ge 1}{\ge} \frac{x^{1/2}}{(x^p+x^p)^{1/2}}}_{\text{falls } p\ge 0} = \underbrace{\frac{1}{\sqrt{2}}\,\frac{1}{x^{\frac{p-1}{2}}} \ge \frac{1}{\sqrt{2}}\cdot\frac{1}{x}}_{\text{da } p\le 3,\ x\ge 1}$$
- $$\int_1^\infty \frac{1}{\sqrt{2}}\frac{1}{x}\,dx = \lim_{R\to\infty} \frac{1}{\sqrt{2}}\ln(x)\Big|_1^R = \lim_{R\to\infty}\frac{1}{\sqrt{2}}\Big\{\underbrace{\ln(R)}_{\to\infty} - \underbrace{\ln(1)}_{=0}\Big\}$$

$$= +\infty \quad \text{divergent}$$

- nach Vergleichskriterium ist damit auch I_p <u>divergent</u>

Fall $p<0$:

- $f(x) = \dfrac{x^{1/2}}{1+\frac{1}{x^{-p}}}$, wegen $x \geq 1$, gilt $x^{-p} \geq 1$

$\Rightarrow \dfrac{1}{x^{-p}} \leq 1$

$$\geq \frac{x^{1/2}}{1+1} = \frac{1}{2} x^{1/2}$$

- $\displaystyle\int_1^\infty \frac{1}{2} x^{1/2} \, dx = \lim_{R\to\infty} \frac{1}{2} \cdot \frac{2}{3} x^{3/2} \Big|_1^R$

$$= \frac{1}{3} \lim_{R\to\infty} \Big\{ \underbrace{R^{3/2}}_{\to\infty} - 1 \Big\} = +\infty$$

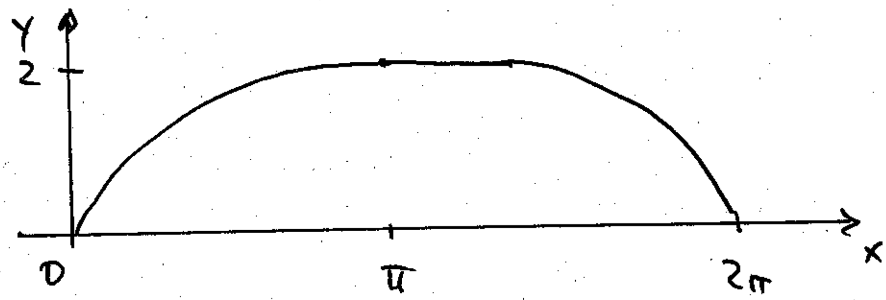
- nach Vergleichskriterium ist dann auch I_p divergent ■

3a) Skizzieren Sie die Kurve

$$\varphi(t) = (t - \sin(t),\ 1-\cos(t)), \quad t \in [0, 2\pi]$$

u. berechnen Sie deren Länge.

Skizze:



$$\varphi'(t) = (1-\cos(t),\ \sin(t))$$

$$\|\varphi'(t)\| = \sqrt{(1-\cos(t))^2 + \sin^2(t)}$$

$$= \sqrt{1 + \underbrace{\cos^2 t + \sin^2 t}_{=1} - 2\cos t}$$

$$= \sqrt{2(1-\cos(t))} = \sqrt{4\sin^2(\tfrac{t}{2})}$$

$$= 2\left|\sin(\tfrac{t}{2})\right| , \quad \tfrac{t}{2} \in [0,\pi] \Rightarrow \sin(\tfrac{t}{2}) > 0$$

$$= 2\sin(\tfrac{t}{2})$$

$$\Rightarrow \text{Länge } L = \int_0^{2\pi} \|\varphi'(t)\|\, dt = \int_0^{2\pi} 2\sin\left(\tfrac{t}{2}\right) dt$$

$$= -2\cdot 2\cos\left(\tfrac{t}{2}\right)\Big|_{t=0}^{2\pi} = -4\,\Big(\underbrace{\cos(\pi)}_{-1} - 1\Big) = \underline{\underline{8}}$$

3b) Berechnen Sie die Länge des Graphen

$$f : [0, \ln(2)] \to \mathbb{R}\ , \quad f(t) = 2e^{t} + \tfrac{1}{8}e^{-t}$$

$$\Rightarrow \text{Kurve } \quad \varphi(t) = \left(t\, ,\, 2e^{t} + \tfrac{1}{8}e^{-t}\right),\ t \in [0, \ln(2)]$$

$$\varphi'(t) = \left(1\, ,\, 2e^{t} - \tfrac{1}{8}e^{-t}\right)$$

$$\|\varphi'(t)\|^2 = 1 + \left(2e^{t} - \tfrac{1}{8}e^{-t}\right)^2$$

$$= 1 + 4e^{2t} + \tfrac{1}{64}e^{-2t} + 2\cdot 2\left(-\tfrac{1}{8}\right)\underbrace{e^{t}e^{-t}}_{=1}$$

$$= \tfrac{1}{2} + 4e^{2t} + \tfrac{1}{64}e^{-2t}$$

$$= 4e^{2t} + \tfrac{1}{64}e^{-2t} + 2\cdot 2\cdot \tfrac{1}{8}\underbrace{e^{t}e^{-t}}_{=1}$$

$$= \Big(\underbrace{2e^{t}}_{>0} + \underbrace{\tfrac{1}{8}e^{-t}}_{>0}\Big)^2$$

$$\Rightarrow \|\varphi'(t)\| = 2e^{t} + \tfrac{1}{8}e^{-t}$$

$$\Rightarrow \text{Länge } L = \int_0^{\ln(2)} \left(2e^{t} + \tfrac{1}{8}e^{-t}\right) dt$$

$$= 2\cdot e^{t}\Big|_0^{\ln(2)} - \tfrac{1}{8}\cdot e^{-t}\Big|_0^{\ln(2)}$$

$$= 2\,(2-1) - \tfrac{1}{8}\left(\tfrac{1}{e^{\ln(2)}} - 1\right)$$

$$= 2 - \tfrac{1}{8}\left(-\tfrac{1}{2}\right) = \underline{\underline{2 + \tfrac{1}{16}}}$$