Musterlösungen Blatt 4

1a) $I = \int \sqrt{8-2x}\, dx \quad , \quad z = 8-2x \quad , \quad \frac{dz}{dx} = -2$

$$= \int z^{1/2} \, -\tfrac{1}{2}\, dz$$

$$= -\frac{1}{2}\,\frac{2}{3}\, z^{3/2} + C \quad = \underline{\underline{-\frac{1}{3}(8-2x)^{3/2} + C}}$$

b) $I = \int \frac{\ln(x)}{x^2}\, dx \quad = \int \underbrace{\ln(x)}_{u} \cdot \underbrace{x^{-2}}_{v'}\, dx \qquad u' = \frac{1}{x}$

$v = -x^{-1}$

$$= \underbrace{-\frac{1}{x}\ln(x)}_{u \cdot v} - \int \frac{1}{x}\left(-x^{-1}\right) dx$$

$$= -\frac{\ln(x)}{x} + \int x^{-2} dx \quad = \underline{\underline{-\frac{\ln(x)}{x} - x^{-1} + C}}$$

c) $I = \int x\sqrt{1+x^2}\, dx \quad , \quad z = 1+x^2 \quad , \quad \frac{dz}{dx} = 2x$

$$= \int x \cdot \sqrt{z}\;\frac{dz}{2x} \quad = \frac{1}{2}\int z^{1/2}\, dz \quad = \frac{1}{2}\cdot\frac{2}{3} z^{3/2} + C$$

$$\underline{\underline{= \frac{1}{3}\left(1+x^2\right)^{3/2} + C}}$$

d) $I = \int \frac{x^2}{x(x-1)^2}\, dx \quad = \quad \int \frac{x}{(x-1)^2}\, dx$

$$= \int \left\{ \frac{x-1}{(x-1)^2} + \frac{1}{(x-1)^2} \right\} dx$$

$$= \int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} \quad = \underline{\underline{\ln|x-1| + (-1)(x-1)^{-1} + C}}$$

e) $I = \int \frac{1}{\sqrt{x}+1}\, dx \quad , \quad z = \sqrt{x} \quad , \quad \frac{dz}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2z}$

$$= \int \frac{1}{z+1} \cdot 2z\, dz \quad = 2\int \frac{z}{z+1}\, dz = 2\left\{ \int \frac{z+1}{z+1}\, dz - \int \frac{1}{z+1}\, dz \right\}$$

$$= 2\left\{ z - \ln|z+1| \right\} + C = \underline{\underline{2\left\{ \sqrt{x} - \ln|\sqrt{x}+1| \right\} + C}}$$

2a) $I = \int_0^1 \frac{1}{e^x + e^{-x}}\, dx \quad , \; z = e^x \; , \; \frac{dz}{dx} = e^x = z$

$$= \int_{e^0}^{e^1} \frac{1}{z + \frac{1}{z}} \; \frac{dz}{z} = \int_1^e \frac{1}{z^2+1}\, dz$$

$$= \arctan(z)\Big|_1^e = \underline{\underline{\arctan(e) - \arctan(1)}}$$

b) $I = \int_1^e \underbrace{\sqrt{x}}_{v'}\, \underbrace{\ln(x)}_{u}\, dx \quad , \quad u' = \frac{1}{x} \; , \; v = \frac{2}{3} x^{3/2}$

$$= \left[\frac{2}{3} x^{3/2} \ln(x)\right]_1^e - \int_1^e \frac{1}{x} \cdot \frac{2}{3} x^{3/2}\, dx$$

$$= \frac{2}{3} e^{3/2} - \frac{2}{3} \int_1^e x^{1/2}\, dx = \frac{2}{3} e^{3/2} - \frac{2}{3} \cdot \frac{2}{3} x^{3/2}\Big|_1^e$$

$$= \frac{2}{3} e^{3/2} - \frac{4}{9} e^{3/2} + \frac{4}{9} = \underline{\underline{\frac{2}{9} e^{3/2} + \frac{4}{9}}}$$

c) $I = \int_0^\pi \frac{\sin(x)}{4 - \cos^2(x)}\, dx \quad , \; z = \cos(x) \; , \; \frac{dz}{dx} = -\sin(x)$

$$= \int_{\cos(0)}^{\cos(\pi)} \frac{\cancel{\sin(x)}}{4 - z^2} \; \frac{dz}{-\cancel{\sin(x)}} = \int_1^{-1} \frac{dz}{z^2 - 4}$$

$$= \int_1^{-1} \frac{1}{(z+2)(z-2)}\, dz \quad , \quad \text{Ansatz } \frac{1}{(z+2)(z-2)} = \frac{A}{z+2} + \frac{B}{z-2}$$

$$= \frac{1}{4} \int_1^{-1} \left\{ \frac{1}{z-2} - \frac{1}{z+2} \right\} dz \qquad 1 = A(z-2) + B(z+2)$$

$$= \frac{1}{4} \Big[\ln|z-2| - \ln|z+2| \Big]_1^{-1} \qquad \Rightarrow A = -\frac{1}{4} \; , \; B = \frac{1}{4}$$

$$= \frac{1}{4} \left\{ \left(\ln(3) - \underbrace{\ln(1)}_{0} \right) - \left(\underbrace{\ln(1)}_{0} - \ln(3) \right) \right\} = \underline{\underline{\frac{1}{2} \ln(3)}}$$

3.) $a>0$, $f:[-a,a]\to\mathbb{R}$ stetig.

3a) zu zeigen: $f(x)=f(-x)\ \forall x\in[-a,a]$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \qquad (1)$$

Bew.: $$\int_{-a}^{0} f(x)\,dx = \int_{z=a}^{z=0} \underbrace{f(-z)}_{=f(z)}(-dz) \quad \text{mit } z=-x,\ dz=-dx$$

$$= -\int_a^0 f(z)\,dz = \int_0^a f(z)\,dz$$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$$= \int_0^a f(z)\,dz + \int_0^a f(x)\,dx = \underline{\underline{2\int_0^a f(x)\,dx}}$$

$\Rightarrow (1)$ ∎

3b) zu zeigen: $f(x)=-f(-x)\ \forall x\in[-a,a]$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = 0 \qquad (2)$$

Beweis: $$\int_{-a}^{0} f(x)\,dx = \int_{z=a}^{z=0} \underbrace{f(-z)}_{-f(z)}(-1)\,dz \quad \text{mit } z=-x,\ dz=-dx$$

$$= \int_a^0 f(z)\,dz = -\int_0^a f(z)\,dz$$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$$= -\int_0^a f(z)\,dz + \int_0^a f(z)\,dz = \underline{\underline{0}} \Rightarrow (2)$$ ∎