Musterlösungen Blatt 4

Blatt 4
-1-

1a) <u>ges:</u> alle $z \in \mathbb{C}$ mit $\boxed{|z| < 1 - \mathrm{Re}(z)}$ (1)

- sei $z = x + iy$, dann ist

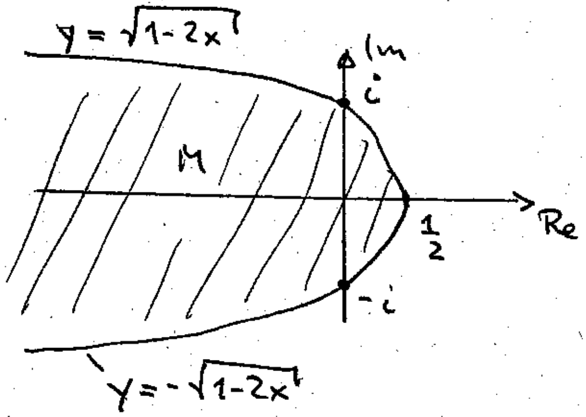
$(1) \iff \sqrt{x^2 + y^2} < 1 - x$ (notwendig ist $\boxed{x < 1}$)

$(1) \iff x^2 + y^2 < 1 + x^2 - 2x$

$(1) \iff y^2 < 1 - 2x$ (notwendig ist $\boxed{x < \frac{1}{2}}$,

sonst $\exists$ keine Lösung)

$(1) \iff -\sqrt{1-2x} < y < \sqrt{1-2x}$ mit $x < \frac{1}{2}$



$M = \{z \in \mathbb{C} \mid z \text{ erfüllt } (1)\}$

1b) $M = \{z \in \mathbb{C} : \underbrace{\left|\frac{z-1}{z+i}\right| = 2}_{(2)}\}$

- sei $z = x + iy$, dann ist

$$(2) \iff \sqrt{\frac{(x-1)^2 + y^2}{x^2 + (y+1)^2}} = 2$$

$$\iff (x-1)^2 + y^2 = 4x^2 + 4(y+1)^2$$

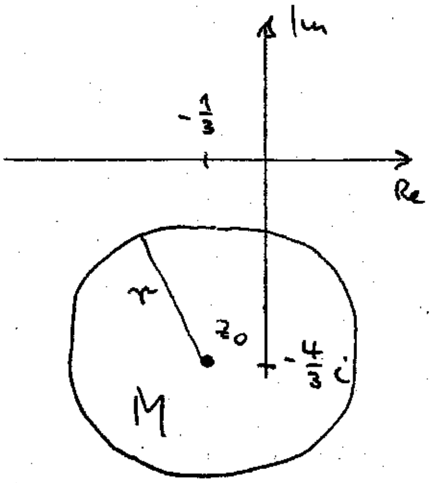
$$\iff x^2 - 2x + 1 + y^2 = 4x^2 + 4y^2 + 8y + 4$$

$$\iff (3x^2 + 2x) + (3y^2 + 8y) + 3 = 0 \quad |:3$$

$$\iff \left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) + \left(y^2 + \frac{8}{3}y + \frac{16}{9}\right) = -1 + \frac{1}{9} + \frac{16}{9}$$

$$\Leftrightarrow \left(x+\frac{1}{3}\right)^2 + \left(y+\frac{4}{3}\right)^2 = \frac{8}{9} = \left(\frac{2\sqrt{2}}{3}\right)^2$$

Kreisgleichung



M = Kreis mit Mittelpunkt

$z_0 = -\frac{1}{3} - \frac{4}{3}i$

u. Radius $r = \frac{2\sqrt{2}}{3}$

2.) geg.: $p, q \in \mathbb{C}$, $D := \frac{p^2}{4} - q$, $w \in \mathbb{C}$ mit $\boxed{w^2 = D}$ (2)

a) z.z.: $z_{1/2} = -\frac{p}{2} \pm w$ ist Lösung von

$$z^2 + p \cdot z + q = 0 \qquad (1)$$

Bew: $z_{1/2}^2 = \left(-\frac{p}{2} \pm w\right)^2 = \frac{p^2}{4} \mp pw + \underbrace{w^2}_{=D}$

$$\overset{(2)}{=} \frac{p^2}{4} \mp pw + \left(\frac{p^2}{4} - q\right) = \frac{p^2}{2} \mp pw - q$$

$$p z_{1/2} = -\frac{p^2}{2} \pm pw$$

$$\Rightarrow z_{1/2}^2 + p z_{1/2} + q = \left(\cancel{\frac{p^2}{2}} \mp pw - \cancel{q}\right) + \left(-\cancel{\frac{p^2}{2}} \pm pw\right) + \cancel{q}$$

$$= \mp pw \pm pw = 0 \quad //$$

b) $p = -4 - i$, $q = 5 + 5i$. Gesucht sind D, $\mathrm{Re}(D)$, $\mathrm{Im}(D)$

$$D = \frac{1}{4}p^2 - q = \frac{1}{4}(16 + 8i - 1) - 5 - 5i$$

$$= \frac{15}{4} + 2i - \frac{20}{4} - 5i = \underline{\underline{-\frac{5}{4} - 3i}}$$

$$\Rightarrow \mathrm{Re}(D) = -\frac{5}{4}, \quad \mathrm{Im}(D) = -3$$

c) bestimme zu $D = -\frac{5}{4} - 3i$ ein $w \in \mathbb{C}$ mit $w^2 = D$

Ansatz: $\boxed{w = a + bi}$ $\Rightarrow w^2 = (a^2 - b^2) + 2ab\, i = D$

$$\Rightarrow \begin{array}{lll} a^2 - b^2 = -\frac{5}{4} & (\mathrm{I}) & \\ 2ab = -3 & (\mathrm{II}) & \xrightarrow{a \neq 0} \boxed{b = -\frac{3}{2a}} \end{array}$$

$$\overset{(\mathrm{I})}{\Rightarrow} a^2 - \frac{9}{4a^2} = -\frac{5}{4} \quad \Big| \cdot 4a^2$$

$$\Rightarrow 4a^4 - 9 = -5a^2, \quad \text{Substitution } \boxed{x = a^2}, \ a \in \mathbb{R}$$

$$\Rightarrow \quad 4x^2 + 5x - 9 = 0$$

$$\Rightarrow \quad x^2 + \frac{5}{4}x - \frac{9}{4} = 0$$

$$\Rightarrow \quad x_{1/2} = -\frac{5}{8} \pm \sqrt{\frac{25+144}{64}} = \frac{-5 \pm 13}{8}$$

$$x_1 = \frac{-5+13}{8} = 1 = a_1^2 \quad \Longrightarrow \quad a_1 = \pm 1$$

$x_2 = -\frac{18}{8} = a_2^2$ ist nicht möglich, da $a \in \mathbb{R}$

$\Rightarrow$ Lösungen für a, b: $\quad a = \pm 1 \; , \; b = -\frac{3}{2a} = \mp \frac{3}{2}$

$$\Rightarrow \quad \boxed{w = \pm\left(1 - \frac{3}{2}i\right)}$$

$\Rightarrow$ Lösungen $z_{1/2}$ von (1)

$$z_{1/2} = -\frac{p}{2} \pm w = \frac{1}{2}(4+i) \pm \left(1 - \frac{3}{2}i\right)$$

$$\Rightarrow \boxed{z_1 = 3 - i} \quad , \quad \boxed{z_2 = 1 + 2i}$$

Bl.4

3 sei $a_n := \frac{n^2-1}{3n^2+1}$

a) $n = 10^6 \Rightarrow a_n = 0.333\ 333\ 333\ 332\ 889$

$\Rightarrow$ Vermutung $g = \lim_{n\to\infty} a_n = \underline{\underline{\frac{1}{3}}}$

b) Beweis der Vermutung mittels ε-N-Def :

$$|a_n - g| = \left| \frac{n^2-1}{3n^2+1} - \frac{\frac{1}{3}(3n^2+1)}{3n^2+1} \right|$$

$$= \left| \frac{-1-\frac{1}{3}}{3n^2+1} \right| = \frac{\frac{4}{3}}{3n^2+1} < \underbrace{\frac{4/3}{3n^2} \overset{!}{<} \varepsilon}_{(u)}$$

$$(u) \Leftrightarrow \frac{4}{3} < 3n^2\varepsilon$$

$$\overset{\varepsilon>0}{\Leftrightarrow} 0 < \frac{4}{9\varepsilon} < n^2 \quad \Big| \sqrt{\ldots}$$

$$\Leftrightarrow n > \sqrt{\frac{4}{9\varepsilon}} =: n_0(\varepsilon)$$

$$\Rightarrow \forall \varepsilon > 0\ \exists N > n_0(\varepsilon) : |a_n - g| < \varepsilon \quad \forall n > N \quad \blacksquare$$