Some Strange Numerical Solutions of the Non-stationary Navier-Stokes Equations in Pipes

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Abstract

A general class of boundary-pressure-driven flows of incompressible Newtonian fluids in three-dimensional pipes with known steady laminar realizations is investigated. Considering the laminar velocity as a 3D-vector-function of the cross-section-circle arguments, we fix the scale for the velocity by the L_2-norm of the laminar velocity. The usual new variables are introduced to get dimension-free Navier-Stokes equations. The characteristic physical and geometrical quantities are subsumed in the energetic Reynolds number Re and a parameter ψ, which involves the energetic ratio and the directions of the boundary-driven part and the pressure-driven part of the laminar flow. The solution of non-stationary dimension-free Navier-Stokes equations is sought in the form $\underline{\mathbf{u}} = \mathbf{u}_L + \mathbf{u}$, where $\mathbf{u}_L$ is the scaled laminar velocity and periodical conditions in center-line-direction are prescribed for $\mathbf{u}$. An autonomous system (S) of ordinary differential equations for the time-dependent coefficients of the spatial Stokes eigenfunction is got by application of the Galerkin-method to the dimension-free Navier-Stokes equations for $\mathbf{u}$. The finite-dimensional approximations $\mathbf{u}_{N(\lambda)}$ of $\mathbf{u}$ are defined in the usual way. A class of timely periodical solutions near to the laminar velocities but different from them was found by parameter studies for the numerical solution of finite-dimensional subsystems of (S). This class of timely periodical strange numerical solutions seems to stay for one of the first links in the bifurcation chain to turbulence.

Keywords: Navier-Stokes equations, Stokes eigenfunctions, Galerkin methods, transition to turbulence

2000 Mathematics subject classifications: 34A34, 35Q30, 65M60, 76F06, 76F65

1 Introduction

The so-called pipe flow of incompressible Newtonian fluids is well known as an old-established object of preference in theoretical and applied fluid-dynamic research. Owing to the simple geometry of the domain - the pipe flow is ideal qualified for studies to the transition to turbulence and for investigations to extraction of further deterministic features from a random, fine-grained turbulent flow.

It is the purpose of our investigations to check and to clarify the possibilities of using low-dimensional Galerkin spaces defined by Stokes eigenfunctions for fact finding of the mechanism of the transition to turbulence in the non-stationary 3D-Navier-Stokes equations. Especially, our studies are targeted on the behaviour of such approximations in the vicinities of critical Reynolds numbers.

We explore a general class of scaled pipe flows of incompressible Newtonian fluids in unbounded pipes in $\mathbf{R}^3$, which can be described by the sum of laminar boundary-driven Couette (angular momentum) flow $\mathbf{u}_{L,C}$, of laminar pressure-driven Poiseuille flow $\mathbf{u}_{L,P}$ and of a time-dependent part $\mathbf{u}$. The first and second addenda are used to define the energetic Reynolds number and a weighting parameter ψ for the energetic ratio and the direction of action of the boundary- and pressure-driven parts from $\mathbf{u}_L$. Low-dimensional approximation spaces of Stokes eigenfunctions with periodic conditions in the center-line-direction are applied for the direct numerical study of systems of Galerkin equations.

The essential notations and governing equations supplemented with initial and boundary conditions are given in section 2. After convenient scaling we decompose the velocity fields $\underline{\mathbf{u}}(t, \mathbf{x})$ into the laminar flow $\mathbf{u}_L(\mathbf{x})$ and the remaining velocity $\mathbf{u}(t, \mathbf{x})$ (fulfilling homogeneous Dirichlet conditions on the boundary of the pipe: $\underline{\mathbf{u}} = \mathbf{u}_L + \mathbf{u}$. Additionally, periodic conditions in the center-line-direction are required for $\mathbf{u}$. The pressure is decomposed similarly. The energetic Reynolds numbers and a weighting parameter ψ for the energetic ratio and the direction of action of the boundary- and pressure-driven parts of $\mathbf{u}_L$ are also defined in this section.

The weak formulation of the Navier-Stokes initial-boundary value problem for the remaining velocities $\mathbf{u}$ is introduced in section 3. We give the existence-theorem for weak solutions of the Navier-Stokes initial-boundary value problem and tools for its proof in this section. Especially, there are explained the Stokes operator and the Galerkin approximation $\mathbf{u}_N := \sum_{j=1}^{N} g_j \mathbf{w}_j$ of the weak solutions of the Navier-Stokes equations for the remaining velocities $\mathbf{u}$, where the $\{\mathbf{w}_j\}_{j=1}^{\infty}$ are the Stokes eigenfunctions. The Galerkin equations (as an autonomous system of ordinary differential equations for the coefficients $g_j(t)$ of the eigenfunctions $\{\mathbf{w}_j(\mathbf{x})\}_{j=1}^{N}$) are there stated explicitly.

Section 4 is devoted to the numerical method and the results. At first a fixed period $2 \cdot l = 2 \cdot 2,69$ in the center-line-direction was chosen for historical reasons (cf. [14]). With this choice is the dimension of the Galerkin-space $N(\lambda) = N(\lambda_{max}) = 114$ determined by a bound $\lambda_{max} := 50$ for the eigenvalues λ. λ_{max} is taken in such a way, that the Galerkin-space includes two significant modes for the modification of the mean velocity as well as for pure Couette-flow and pure Poiseuille-flow. The calculation of the coefficients in forming the system of ordinary differential equations is very expensive. For it universalized tools of combined C- and MAPLE-routines are used together with implemented rules of general addition theorems in form of allocation-lists. MAPLE-tools are also utilized for the handling of Stokes eigenfunctions and for symbolic computations.

The corresponding systems of ordinary differential equations were solved numerically for several values of the parameters Re, ψ and a set of initial values $\{g_j(0)\}, j = 1, ..., N$ (small $\mathbf{u}_{N,(0)} := \sum_{j=1}^{N} g_j(0)\mathbf{w}_j$), where the kinetic energy $E(t) := \sum_{j=1}^{N} g_j^2(t)$ of the Galerkin approximations was used $\mathbf{u}_N$ as a measure of turbulence.

The evaluation of our numerical investigations for general pipe flows shows matches with our perceptions for general channel flows. There are very good agreements with measurements for the transition from laminar to turbulent flows in the vicinity of the critical Reynolds numbers and satisfactory results for the mean velocities of the turbulent flow (even for small dimension of our approximation space). However, the agreement with other experimental data for the Reynolds stresses and the root-mean-square values of the fluctuating velocities is less satisfactory, but due to the small dimension of our approximation space most probably.

But the evaluation of our numerical investigations for general pipe flows yields a new significant speciality. Much to our surprise we found by our calculations time periodic non-laminar solutions of a constant kinetic energy $E(t, Re)$ at fixed ψ-values out of a small interval of such arguments about a range of Reynolds numbers Re. This set of timely periodical strange numerical solutions can be interpreted as the numerical result of stabilization by the physical property of angular momentum in the flows. From the mathematical point of view it seems to be one of the first links in the bifurcation chain to turbulence.

2 The Basic Notations and Equations

The non-stationary Navier-Stokes equations describe the time evolution of an incompressible Newtonian fluid. We are interested on the pipe flow, what means, that fluid is filling an open unbounded cylindrical domain $\mathbf{\Omega}'$ in $\mathbf{R}^3$:

$$\mathbf{\Omega}' \quad := \quad \{\mathbf{y} = (y_1, y_2, y_3)^T \in \mathbf{R}^3 : \sqrt{y_2^2 + y_3^2} < R\},$$

where R in [m] is the radius of the pipe.

The unknown are the the velocity field $\mathbf{v}$ in [m/s],

$$\mathbf{v}(\tau,\mathbf{y}) \quad = \quad (v^1(\tau,\mathbf{y}), v^2(\tau,\mathbf{y}), v^3(\tau,\mathbf{y}))^T \quad , \quad \mathbf{y} \in \mathbf{\Omega}', \quad \tau \in \quad [0,\Theta], (\Theta > 0)$$

and the kinematic pressure $p'(\tau,\mathbf{y})$ in $[m^2/s^2]$, both depending on the actual time in τ [s]. The initial-boundary value problem is given by:

Problem 2.1 *We seek for* $\mathbf{v}$ *and* p' *fulfilling:*

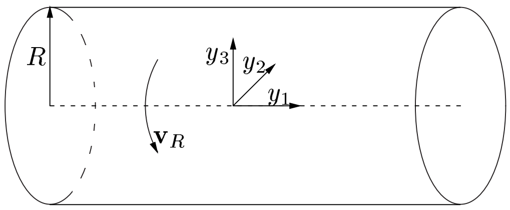
$$\frac{\partial \mathbf{v}}{\partial \tau} + \sum_{j=1}^{3} v^j D_j^{\cdot} \mathbf{v} - \nu \triangle_{\mathbf{y}} \mathbf{v} + \nabla_{\mathbf{y}} p' \quad = \quad \mathbf{0} \quad , \; in \quad (0,\Theta) \times \mathbf{\Omega}' \tag{1}$$

$$\nabla_{\mathbf{y}} \cdot \mathbf{v} \quad = \quad 0 \tag{2}$$

$$\mathbf{v}(0,\mathbf{y}) \quad = \quad \mathbf{v}_{(0)}(\mathbf{y}) \tag{3}$$

$$\mathbf{v}(\tau,.)_{|\partial \mathbf{\Omega}_P} \quad = \quad \mathbf{v}_R \quad := \quad c_c(0,-y_3,y_2)^T_{|y_2^2+y_3^2=R^2}, \tag{4}$$

where we have used the kinematic viscosity ν *in* $[m^2/s]$ *and the acronym* $D_j^{\cdot} := \frac{\partial .}{\partial y_j}$, $\quad j = 1,2,3.$



It's a matter of common knowledge, that there exist stationary solutions of the Navier-Stokes equations (2.1), (2) for the boundary conditions (4). These solutions are given by the laminar velocity fields and the corresponding laminar pressures:

$$\begin{aligned} \mathbf{v}_L(\mathbf{y}) &:= c_c(0,-y_3,y_2)^T + c_p \chi R(1 - \frac{y_2^2+y_3^2}{R^2}, 0, 0)^T = \mathbf{v}_{L,c} + \mathbf{v}_{L,p}, \\ p'_L(\mathbf{y}) &:= \frac{c_c^2}{2}((y_2^2+y_3^2) - \frac{R^2}{2}) - \nu c_p \chi \frac{4}{R} y_1 \quad , \end{aligned} \tag{5}$$

where c_c, c_p are velocity parameters in [1/s] and $\chi = \sqrt{\frac{3}{2}}$ is chosen in such a way that for $c_c = c_p$ the velocity fields $\mathbf{v}_{L,.,c}$ and $\mathbf{v}_{L,.,p}$ result the equal kinetic energy in a control volume. The characteristic (laminar) velocity v_{char} is introduced as the scale of the velocities by:

$$v_{char} \quad := \quad R\sqrt{c_c^2 + c_p^2}. \tag{6}$$

Additionally the parameter $\psi \in [0, 2\pi)$ is defined as an indicator for the energetic ratio and the direction of action of the parts of the laminar velocity:

$$\cos\psi := \frac{c_c}{\sqrt{c_c^2 + c_p^2}} \quad , \quad \sin\psi := \frac{c_p}{\sqrt{c_c^2 + c_p^2}} \tag{7}$$

The Problem 2.1 is formulated in dimension-free terms by the use of R as the length scale, $\frac{R^2}{\nu}$ as time scale, v_{char} as the velocity scale and $\nu\sqrt{c_c^2 + c_p^2}$ as the scale for the kinematic pressure.

In this way we get as the image of $\mathbf{\Omega}'$: $\mathbf{\Omega} := \{\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbf{R}^3 : \sqrt{x_2^2 + x_3^2} < 1\}$ and

$$\mathbf{x} = \frac{1}{R}\mathbf{y}\,, \quad t = \frac{\nu\tau}{R^2}\,, \quad \underline{\mathbf{u}}(.,.) = \frac{1}{v_{char}}\mathbf{v}(.,.)\,, \quad \underline{p}(.,.) = \frac{1}{\nu\sqrt{c_c^2 + c_p^2}}p'(.,.)\,. \tag{8}$$

for the variables. The velocity and pressure fields are handled by the use of splitting up formulas:

$$\underline{\mathbf{u}}(.,.) = \mathbf{u}_{L,M}(.) + \mathbf{u}(.,.), \quad \underline{p}(.,.) = p_{L,M}(.) + p(.,.), \tag{9}$$

where $\mathbf{u}_L$ and p_L are the scaled laminar fields (cf.(5)):

$$\begin{aligned} \mathbf{u}_L(\mathbf{x}) &:= \cos\psi(0, -y_3, y_2)^T + \sin\psi(\chi_P(1 - x_2^2 - x_3^2), 0, 0)^T = \\ &= \cos\psi\,\mathbf{u}_{L,c} + \sin\psi\,\mathbf{u}_{L,p}, \\ p_L(\mathbf{x}) &:= \cos^2\psi((x_2^2 + x_3^2) - \frac{1}{2}) - 4\sin\psi\,\chi_P x_1 \end{aligned}$$

Finally the *energetic* Reynolds number is denoted by Re:

$$Re = Re(c_c, c_p) := \frac{R^2}{\nu}\sqrt{c_c^2 + c_p^2} \tag{10}$$

and periodic conditions for $\mathbf{u}$ and p according to (9) are required instead of conditions at infinity:

$$\textbf{(P):}\ \mathbf{u}(t, x_1, x_2, x_3) = \mathbf{u}(t, x_1 + 2l, x_2, x_3) \quad , \quad p(t, x_1, x_2, x_3) = p(t, x_1 + 2l, x_2, x_3) \tag{11}$$

Then is the initial-boundary value problem for the unknowns $\mathbf{u}(t, \mathbf{x})$ and $p(t, \mathbf{x})$ given by:

Problem 2.2 *: We seek solutions $\mathbf{u}(t, \mathbf{x})$ and $p(t, \mathbf{x})$ fulfilling the following equations:*

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \triangle_{\mathbf{x}}\mathbf{u} + Re\Big(&\sum_{j=1}^{3} u^j D_j\mathbf{u} + \cos\psi\sum_{j=1}^{3} u^j_{L,c} D_j\mathbf{u} + \sin\psi\sum_{j=1}^{3} u^j_{L,p} D_j\mathbf{u} + \\ &+ \cos\psi\sum_{j=1}^{3} u^j D_j\mathbf{u}_{L,c} + \sin\psi\sum_{j=1}^{3} u^j D_j\mathbf{u}_{L,p}\Big) + \nabla_{\mathbf{x}} p = \mathbf{0} \\ \nabla_{\mathbf{x}}\cdot\mathbf{u} &= 0 \quad \mathit{in} \quad (0, T) \times \mathbf{\Omega} \\ \mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) &= \underline{\mathbf{u}}(0, .) - \mathbf{u}_L(.) \\ \mathbf{u}(t, x_1, x_2, x_3)_{|(x_2)^2 + (x_3)^2 = 1} &= \mathbf{0}, \quad \textit{and}\ \textbf{(P):}\ \textit{cf.(11)} \end{aligned}$$

with $D_j := \frac{\partial .}{\partial x_j}, \quad j = 1, 2, 3\,.$

3 Explanation of the Galerkin-Approximations

Firstly, the domain $\mathbf{\Omega}$ is restricted to the open bounded sub-domain T_l under the terms of the periodic conditions (11):

$$T_l \;:=\; \{\mathbf{x} \in \mathbf{\Omega} : \mathbf{x} = (x_1, x_2, x_3)^T, |x_1| < l\}, \tag{12}$$

in which we suppose $l \geq 1$. Additionally the notation σ for the slice plane will be used in our considerations:

$$\sigma \quad := \quad \{\{(x_2, x_3)^T \;:\; x_2^2 + x_3^2 \;<\; 1\} \tag{13}$$

Definition 3.1 *: Let* $\mathbf{S}(= \mathbf{S}^0)$ *be the closure of the set*

$$\mathcal{V}_l \quad := \quad \{\mathbf{s} \in (C^\infty(\mathbf{T}_l))^3 \,:\, \nabla_{\mathbf{x}} \cdot \mathbf{s} \,=\, 0,\, \mathbf{s}(\mathbf{x}) := \sum_{\kappa \in \mathbf{Z}} \hat{\mathbf{s}}_\kappa(x_2, x_3) \exp(\frac{i\kappa\pi x_1}{l}),\, \hat{\mathbf{s}}_\kappa(x_2, x_3) \;:=$$

$$:= \frac{1}{2l} \int_{-l}^{l} \mathbf{s}(\mathbf{x}) \exp(\frac{-i\pi\kappa x_1}{l}) dx_1,\, \hat{\mathbf{s}}_\kappa(.) = \overline{\hat{\mathbf{s}}_{-\kappa}(.)},\, \forall\, \kappa \in \mathbf{Z},\, \hat{\mathbf{s}}_\kappa(.) \,\in (C_0^\infty(\sigma))^3\}$$

in the norm of $\mathbf{H}(= \mathbf{H}^0) = (L_2(T_l))^3$*. The space* $\mathbf{S}^1$ *arises from the closure of* $\mathcal{V}_{M,l}$ *in the norm of* $\mathbf{H}^1 = (W_2^1(T_{M,l}))^3$ *and the spaces* $\mathbf{S}^k$ *,* $k = 2, 3, 4, \ldots$*, are explained by the relation:* $\mathbf{S}^k = \mathbf{S}^1 \cap \mathbf{H}^k$*, where* $\mathbf{H}^k = (W_2^k(T_{M,l}))^3$*.*

The norms of the spaces $\mathbf{S}^k$ *agree for* $k = 0, 1, 2, \ldots$ *on* $\mathbf{S}^k$ *with the norms of* $\mathbf{H}^k$ *denoted by* $\|.\|_k$ *and the scalar product on* $\mathbf{H}$ *is termed by* $(.,.)$*. Finally let* $\mathbf{S}^{-1}$ *be the dual of* $\mathbf{S}^1$*.*

Remark 3.2 *: One can also endow the space* $\mathbf{S}^1$ *with the scalar product* $(.,.)_{D,2}$

$$(\mathbf{u}, \mathbf{w})_{D,2} \quad := \quad (\sum_{j=1}^{3} \sum_{k=1}^{3} (D_k u^j, D_k w^j)_{L_2(T_{M,l})}) \quad ,$$

since the Poincaré inequality holds by the boundedness of the domains $\mathbf{\Omega}_M$ *in at least one direction and vanishing traces of the elements of* $\mathbf{S}^1$ *on* $\partial\mathbf{\Omega} \cap \partial T_l$*.*

Definition 3.3 *: Let* π_l *be the Leray-Helmholtz projection of* $\mathbf{H}$ *onto* $\mathbf{S}$*:* $\pi_l : \mathbf{H} \to \mathbf{S}$*.*
The Stokes operator is termed by $\mathbf{A}$*,* $\mathbf{A} := -\pi_l \triangle_{\mathbf{x}} : D(\mathbf{A}) \,=\, \mathbf{S}^2 \to \mathbf{S}$*,*
where the defining equation is to understand in the sense of Friedrichs extention.

The Stokes operator $\mathbf{A}$ plays a central role in the treatment of incompressible Navier-Stokes equations. The most important properties of $\mathbf{A}$ will be specified in the following statements.

Theorem 3.4 *: The Stokes operator* $\mathbf{A}$ *is positive and self-adjoint. The inverse of the Stokes operator* $\mathbf{A}^{-1}$ *is injective, self-adjoint and compact.*

The proof of Theorem 3.4 is a simple modification of the Theorems 4.3 and 4.4 in [3]. The essential tools for the proofs of these theorems are the theorems of Rellich and Lax-Milgram. In detail Theorem 3.4 is proven in [10].

By the use of a well known theorem of Hilbert and regularity results like Proposition I.2.2 of [15] one shows additionally:

Corollary 3.5 : *The Stokes operator is an operator with a pure point spectrum. All the eigenvalues* λ_j *of* $\mathbf{A}$ *are real and of finite multiplicity. The associated eigenfunctions* $\{\mathbf{w}_j(\mathbf{x})\}_{j=1}^{\infty}$ *of the Stokes operator* $\mathbf{A}$ *(counted in multiplicity) are an orthogonal basis of* $\mathbf{S}$ *and* $\mathbf{S}^1$. *We obtain that*

(i)	$\mathbf{A}\mathbf{w}_j$	$:= \lambda_j \mathbf{w}_j\,, \quad \mathbf{w}_j \in D(\mathbf{A}), \quad \forall\, j = 1, 2, \ldots$
(ii)	0	$< \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots$
(iii)	$\lim_{j\to\infty} \lambda_j$	$= \infty$
(iv)	$\{\mathbf{w}_j(\mathbf{x})\}_{j=1}^{\infty}$,	$\|\|\mathbf{w}_j\|\|_0\, \forall\, j = 1, 2, \ldots,$ *form a complete orthonormal system in* $\mathbf{S}$

The eigenfunctions $\mathbf{w}_j(\mathbf{x})\ \forall\, j = 1, 2, \ldots$ *are elements of* $\mathbf{C}^{\infty}(\overline{\mathbf{T}_l})$.

Some explicit calculated real eigenfunctions $\{\mathbf{w}_j(\mathbf{x})\}_{j=1}^{\infty}$ are used in our investigations. These eigenfunctions (as complex functions) were deduced in [11]. The real eigenfunctions are inscribed in [9], where also the following theorem is proven:

Theorem 3.6 : *The explicit calculated real eigenfunctions* $\{\mathbf{w}_j(\mathbf{x})\}_{j=1}^{\infty}$ *form a a complete orthonormal system in* $\mathbf{S}$ *and agree in this sense with the set of eigenfunctions of Corollary 3.5.*

The Galerkin spaces are determined by the use of bound λ for the permitted eigenvalues. Let $N = N(\lambda)$ denote the number of eigenvalues of $\mathbf{A}$ with $\lambda_j \le \lambda$ (counted in multiplicity):

$$N(\lambda) \ := \sum_{\lambda_j \le \lambda} 1 \tag{14}$$

The spaces $\mathbf{S}^1$, $\mathbf{S}$, $\mathbf{S}^{-1}$ will be used as evolution triple for the weak formulations of the Problem 2.2. Let us denote by $L_p(0, T, \mathbf{X})$, $p \in [1, \infty]$, with an arbitrary Banach space $\mathbf{X}$, the usual evolutionary Lebesgue spaces, by $C^k(0, T, \mathbf{X})$, $k = 0, 1, 2$ the Banach spaces of k-times continuously differentiable functions with values in $\mathbf{X}$ and by $C_w(0, T, \mathbf{X})$ the spaces of weakly continuous functions.

The weak formulation of Problem 2.2 is what follows:

Problem 3.7 *: Let* $\mathbf{u}_0 \in \mathbf{S}$*. We seek a function* $\mathbf{u} \in L_2(0,T,\mathbf{S}^1) \cap C_w(0,T,\mathbf{S})$ *such that:*

$$\begin{aligned}\frac{d}{dt}(\mathbf{u},\mathbf{s}) + (\mathbf{u},\mathbf{s})_{D,2} \quad + \quad & Re\{(\sum_{j=1}^{3} u^j D_j \mathbf{u},\mathbf{s}) + \cos\psi[(\sum_{j=1}^{3} u^j_{L,c} D_j \mathbf{u},\mathbf{s}) + (\sum_{j=1}^{3} u^j D_j \mathbf{u}_{L,c},\mathbf{s})] \\ & \sin\psi[(\sum_{j=1}^{3} u^j_{L,p} D_j \mathbf{u}\,,\mathbf{s}) + (\sum_{j=1}^{3} u^j D_j \mathbf{u}_{L,p}\mathbf{s})]\} = 0 \ \ \textit{for all} \mathbf{s} \in \mathbf{S}^1 \\ \mathbf{u}(0) \quad = \quad \mathbf{u}_0 & \qquad \textit{in the sense of } C_w(0,T,\mathbf{S})\end{aligned}$$

Remark 3.8 *: We note that the boundary conditions on* $\mathbf{u}$ *and the vanishing divergence of* $\mathbf{u}$ *are included in the definition of the spaces* $\mathbf{S}$ *and* $\mathbf{S}^1$*. Finally the gradient fields like* $\nabla_{\mathbf{x}} p$ *are included in the orthogonal complement of* $\mathbf{S}$ *in* $\mathbf{H}$*. The decomposition and trace theorems for* $\mathbf{H}$ *and* $\mathbf{H}^1$ *respectively* $\mathbf{S}$ *and* $\mathbf{S}^1$ *are given in [8], [9] and [10].*

In [10] was proven the following result to weak solutions of Problem 3.7 as a straight forward extrapolation of Leray's statement (cf. [10] Theorem 5.8 and Remark 5.9):

Theorem 3.9 *: There exists at least one weak solution of Problem 3.7 for every* $\mathbf{u}_0 \in \mathbf{S}$*.*

Note 3.10 *: Further results to the existence of weak and strong solutions, to the regularity of these solutions and energetic estimates are also given and proven in [10]. The proofs of the existence of statistical solutions of the Navier-Stokes equations as probability measures on weighted Sobolev spaces on* $(0,T) \times \mathbf{\Omega}$ *were achieved in [9]. Invariance properties of these statistical solutions were established by the availability of these invariance properties for the Galerkin approximations of weak solutions on* $(0,T) \times T_l$ *for all* $l > 1$*. Particularly the invariance via translations in the argument belonging to the unbounded direction of* $\mathbf{\Omega}$ *has excelled in the evaluation of scalings and numerical methods.*

The Galerkin method is an essential tool for the proof of Theorem 3.9. Some notations for the explanation of the Galerkin equations will be declared in what follows.

Definition 3.11 *: Let N (cf. 14) be the dimension of the Galerkin space. The span of* $\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_N\}$ *as a subspace of* $\mathbf{S}$ *is denoted by* $\mathbf{M}_N$*:*

$$\mathbf{M}_N = \mathbf{M}_N(\mathbf{T}_{M,l}) \quad := \quad \overline{\{\mathbf{w}_j(\mathbf{x})\}_{j=1}^{N}}^{\mathbf{H}(\mathbf{T}_{M,l})} \subset \mathbf{S}$$

The orthogonal projector from $\mathbf{S}$ *to* $\mathbf{M}_N$ *is termed by* P_N*, where* P_N *will be used* P_N *also in the sense:* $P_N : \mathbf{S}^1 \to \mathbf{M}_N$*, since* $dim(\mathbf{M}_N) < \infty$*.*

For the actual formulation of the Galerkin approximations of Problem 3.7. one uses $b(.,.,.)$:

$$b(\mathbf{u},\mathbf{q},\mathbf{s}) := (\sum_{j=1}^{3} u^j D_j \mathbf{q}, \mathbf{s}) \ \forall (\mathbf{u},\mathbf{q},\mathbf{s}) \in L_2(0,T,\mathbf{S}^1) \times L_2(0,T,\mathbf{H}^1) \times L_2(0,T,\mathbf{S}^1)(*)\,, \tag{15}$$

as the notation of the trilinear-form, where $(*)$ means that permutations of the factors in the product space are allowed. The trilinear-form $b(.,.,.)$ is antisymmetric in relation to permutation of $\mathbf{q}$ and $\mathbf{s}$.

Problem 3.12 *: Let $\mathbf{u}_0 \in \mathbf{S}$. We seek a function $\mathbf{u}_N \in C^1(0,T,\mathbf{M}_N)$ such that:*

$$\begin{aligned}
\frac{d}{dt}(\mathbf{u}_N,\mathbf{w}_j) + (\mathbf{u}_N,\mathbf{w}_j)_{D,2} + Re\{b(\mathbf{u}_N,\mathbf{u}_N,\mathbf{w}_j) \ &+ \ \cos\psi\, b(\mathbf{u}_{L,c},\mathbf{u}_N,\mathbf{w}_j) \ + \\
+ \cos\psi\, b(\mathbf{u}_N,\mathbf{u}_{L,c},\mathbf{w}_j) + \sin\psi[b(\mathbf{u}_{L,p},\mathbf{u}_N\,\mathbf{w}_j) \ &+ \ b(\mathbf{u}_N,\mathbf{u}_{L,p},\mathbf{w}_j)]\} = \ 0\,, \\
&\textit{for all } j \in \{1,\ldots,N\} \\
\mathbf{u}_N(0) \ &= \ P_N\mathbf{u}_0
\end{aligned}$$

A solution $\mathbf{u}_N$ of this problem is called Galerkin approximation of the weak solution $\mathbf{u}$ to Theorem 3.9.

Using energetic apriori estimates and the theorem of Picard-Lindelöf one shows in the proof of Theorem 3.9 that there exists a unique Galerkin approximation in the sense of Problem 3.12. Since $\mathbf{u}_N \in C^1(0,T,\mathbf{M}_N)$, one can receive the equivalent representation of $\mathbf{u}_N$ by:

$$\mathbf{u}_N(t,\mathbf{x}) := P_N u(t,\mathbf{x}) = \sum_{j=1}^{N} g_j(t)\mathbf{w}_j(\mathbf{x}), \tag{16}$$

with $g_j(t) \quad := \quad (\mathbf{u}(t,\mathbf{x}),\mathbf{w}_j)\,, \quad j = 1,...,N, \quad$ (i.e. $\underline{\mathbf{u}}_N = \mathbf{u}_L + \mathbf{u}_N$)

Corollary 3.13 *: The solution of Problem 3.12 is equivalent to solution of the initial value problem of the autonomous system of ordinary differential equations:*

$$\begin{aligned}
\dot{g}_k = -\lambda_k g_k \quad -Re \quad (&\sum_{i,s=1}^{N} b_{k,i,s}g_ig_s + \cos\psi\sum_{i=1}^{N} q^I_{k,i}g_i + \sin\psi\sum_{i=1}^{N} q^{II}_{k,i}g_i + \\
&+ \cos\psi\sum_{i=1}^{N} r^I_{k,i}g_i + \sin\psi\sum_{i=1}^{N} r^{II}_{k,i}g_i), \qquad k = 1,2,...,N \\
g_k(0) = g_{k,0} \quad := \quad &(\mathbf{u}_{(0)}(\mathbf{x}),\mathbf{w}_k), \ \ k = 1,2,...,N,
\end{aligned}$$

with

$$b_{k,i,s} \quad := \quad b(w_i,\mathbf{w}_s,\mathbf{w}_k) \quad (= -b_{s,i,k}), \quad i,k,s = 1,2,...,N\,,$$

$$\begin{aligned}
q^I_{k,i} &:= b(\mathbf{w}_i,\mathbf{u}_{L,c},\mathbf{w}_k)\,, & r^I_{k,i} &:= b(\mathbf{u}_{L,c},\mathbf{w}_i,\mathbf{w}_k)\,(= -r^I_{i,k}), \\
q^{II}_{k,i} &:= b(\mathbf{w}_i,\mathbf{u}_{L,p},\mathbf{w}_k)\,, & r^{II}_{k,i} &:= b(\mathbf{u}_{L,p},\mathbf{w}_i,\mathbf{w}_k)\,(= -r^{II}_{i,k}), \\
& & & i,k = 1,2,...,N,\ .
\end{aligned}$$

Finally the kinetic energy of our Galerkin approximation of the remaining velocity and its rate of change in time is termed by:

$$E(t) \quad := \quad \sum_{k=1}^{N} g_k^2(t) \quad \text{and} \tag{17}$$

$$\dot{E}(t) \quad = \quad -2(\sum_{k=1}^{N} \lambda_k g_k^2(t) + Re(\cos\psi \sum_{i,k=1}^{N} q_{k,i}^{I} g_k g_i \; + \sin\psi \sum_{i,k=1}^{N} q_{k,i}^{II} g_k g_i)) \tag{18}$$

4 Numerical Experiments and Results

The first step in the numerical treatment of the autonomous systems of ordinary differential equations to Corollary 3.13 are some run-up procedures. One has to fix a period l and after that an upper bound λ for the eigenvalues of the considered Stokes eigenfunctions - respectively N as the dimension of the Galerkin space. The energetic Reynolds number Re and the control parameter ψ (cf. (7)) stay furthermore free available to adjust the type of flow. The variables l and λ are the necessary inputs for the first preparation procedure. There are generated the Stokes eigenfunctions $\{\mathbf{w}_j(\mathbf{x})\}_{j=1}^{N}$ and the required partial derivatives of these functions. In a second preparation procedure are calculated the coefficients of the ordinary differential equations to Corollary 3.13. Here are used universalized tools of combined C- and MAPLE-routines together with implemented rules of general addition theorems in form of allocation-lists.

The main step in the numerical treatment of the Galerkin approximations of the Navier-Stokes equations are studies of the numerical solutions of the systems to Corollary 3.13 in the dependence of the the energetic Reynolds number Re, the control parameter ψ (cf. (7)) and the initial values $g_k(0) := g_{k,0} = 0, \forall k = 1, 2, ..., N,$ as free parameters. There are implemented and used Dormand-Price methods (DOPRI5) and Runge-Kutta-Fehlberg (RKF45) methods (cf. [4]) with step size control for the numerical solution of the systems of ordinary differential equations.

The initial values $g_k(0) := g_{k,0} = 0, \forall k = 1, 2, ..., N,$ stand for the laminar velocities $\mathbf{u}_{L,M}$ as initial values of the Navier-Stokes systems. In this sense, small initial values $g_{k,0}$ are to understood as small perturbations of the laminar flow at the time $t = 0$, or as small disturbance of the laminar velocity field at time $t = 0$ from a physical point of view. (This agrees with experimental results where the laminar velocity field is also a realisation of the real flow in the domain of over-critical Reynolds numbers (e.g. if the walls are very smooth).) In other words, the initial conditions for the g_k have been chosen as small random values out of a ball of radius σ around the origin in $\mathbf{R}^N$. If the initial conditions $g_{k,0}$ have a distance smaller than $\rho < \sigma$ from the origin in $\mathbf{R}^N$, where ρ depends on the

Reynolds number Re, our numerical solutions of the autonomous systems of ordinary differential equations of Corollary 3.13 tend to the zero of $\mathbf{R}^N$. The reason is the asymptotic stability of the laminar flow in the sense of Ljapunov. The radius ρ of this ball of asymptotic stability depends nearly exponentially on Re.

The kinetic energy $E(t)$ (cf.(17)) of the approximated remaining velocity was used for the evaluation of the solutions. The behavior in time of $E(t)$ shows whether the solution of the systems (Corollary 3.13) tends to the origin of $\mathbf{R}^N$ (asymptotic stability) or not ?

The numerical experiments were the following: The numerical solutions of the Galerkin equations were calculated for several intervals of time a set of initial values for varying parameters Re and ψ. The behavior of these solutions was studied with respect to behavior of the kinetic energy $E(t)$ (our indicator-measure of turbulence or bifurcations).

In a second turn of studies, the numerical solutions were calculated for a set of parameters Re and ψ with fixed initial values chosen as random values out of spherical shells around the origin of $\mathbf{R}^N$ and for a fixed intervals of time.

For a better understanding, let us present some of the used eigenfunctions and their corresponding eigenvalues. The period l in our present investigations was chosen to be $2l = 2 \times 2,69$ for historical reasons. In the following are used cylindrical coordinates x_1, r, φ for the arguments and a Cartesian representation for the vector fields.

$$\mathbf{w}_{i^*} = \mathbf{w}^{I,R}_{0,j,0}(\mathbf{x}) := \frac{J_0(\sqrt{\lambda}r)}{\sqrt{2l\pi}J_1(\sqrt{\lambda})}\begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad j \in \mathbf{N}\,,\ J_0(\sqrt{\lambda}) = 0\,, \quad \mathbf{w}_{i^*} = w_{i^*}(r)\mathbf{e}_{x_1}$$

and

$$\mathbf{w}_{i^*} = \mathbf{w}^{I,r}_{0,j,0}(\mathbf{x}) := \frac{J_1(\sqrt{\lambda}r)}{\sqrt{2l\pi}J_2(\sqrt{\lambda})}\begin{pmatrix}0\\-\sin\varphi\\\cos\varphi\end{pmatrix}, \quad j \in \mathbf{N}\,,\ J_1(\sqrt{\lambda}) = 0\,, \quad \mathbf{w}_{i^*} = w_{i^*}(r)\mathbf{e}_{\varphi}$$

$$\mathbf{w}^{(S),I,g,r,R}_{\kappa,j,n}(\mathbf{x}) := \frac{1}{\tilde{c}}\left[\begin{pmatrix} 2(I_n(\frac{\pi}{l}\kappa r) - \alpha_B J_n(\sqrt{\lambda_\kappa}r))\cos(\frac{\pi}{l}\kappa x_1)\sin n\varphi \\ (I_{n-1}(\frac{\pi}{l}\kappa r) - \alpha_{B-}J_{n-1}(\sqrt{\lambda_\kappa}r))\sin(\frac{\pi}{l}\kappa x_1)\sin(n-1)\varphi \\ (I_{n-1}(\frac{\pi}{l}\kappa r) - \alpha_{B-}J_{n-1}(\sqrt{\lambda_\kappa}r))\sin(\frac{\pi}{l}\kappa x_1)\cos(n-1)\varphi \end{pmatrix} + \right.$$

$$\left. + \begin{pmatrix} 0 \\ (I_{n+1}(\frac{\pi}{l}\kappa r) - \alpha_{B+}J_{n+1}(\sqrt{\lambda_\kappa}r))\sin(\frac{\pi}{l}\kappa x_1)\sin(n+1)\varphi \\ -(I_{n+1}(\frac{\pi}{l}\kappa r) - \alpha_{B+}J_{n+1}(\sqrt{\lambda_\kappa}r))\sin(\frac{\pi}{l}\kappa x_1)\cos(n+1)\varphi \end{pmatrix}\right],$$

$$\text{with} \quad \alpha_B := I_n(\frac{\pi}{l}\kappa)(J_n(\sqrt{\lambda_\kappa}))^{-1} \quad , \quad \alpha_{B-} := I_{n-1}(\frac{\pi}{l}\kappa)(J_{n-1}(\sqrt{\lambda_\kappa}))^{-1}$$
$$\text{and} \quad \alpha_{B+} := I_{n+1}(\frac{\pi}{l}\kappa)(J_{n+1}(\sqrt{\lambda_\kappa}))^{-1}\,.$$

The first 114 eigenvalues are written down in the following table, where M denotes the multiplicity of the eigenvalue λ_j.

Table of eigenvalues: cylinder $l = 2.69$

j	λ_j	$\sqrt{\lambda_{\kappa,j}}$	n	κ	M	type
$1*$	5.78318596	2.40482555	0	0	1	R
$2-5$	14.00353173	3.55522038	1	1	4	g, r, R
$6*$	14.68197064	3.83170597	0	0	1	r
$7-8$			1	0	2	R
$9-12$	15.44219494	3.16013226	1	2	4	g, r, R
$13-14$	16.04591039	3.83170597	0	1	2	r
$15-16$	20.13772963	3.83170597	0	2	2	r
$17-20$	20.75026890	2.91115289	1	3	4	g, r, R
$21-22$	26.37461643	5.13562230	1	0	2	g, r
$23-24$			2	0	2	R
$25-28*$	26.53943566	5.01751890	2	1	4	g, r, R
$29-30$	26.95742836	3.83170597	0	3	2	r
$31-32$	27.10106525	5.07317706	0	1	2	g, R
$33-36$	28.19952031	5.18030699	1	1	4	g, r, R
$37-40$	28.28023245	4.77749656	2	2	4	g, r, R
$41-44$	29.49065461	2.76904652	1	4	4	g, r, R
$45-46$	29.69742364	4.92358250	0	2	2	g, R
$47*$	30.47126234	5.52007811	0	0	1	R
$48-51$	32.77224938	5.22651799	1	2	4	g, r, R
$52-55$	33.03561689	4.55633176	2	3	4	g, r, R
$56-57$	34.87484339	4.75388111	0	3	2	g, R
$58-59$	36.50500661	3.83170597	0	4	2	r
$60-63$	39.80837362	5.24718170	1	3	4	g, r, R
$64-65$	40.70646582	6.38016189	2	0	2	g, r
$66-67$			3	0	2	R
$68-71$	41.11877055	4.39269104	2	4	4	g, r, R
$72-75$	41.22711826	6.31372936	3	1	4	g, r, R
$76-79$	41.30088185	2.68372654	1	5	4	g, r, R
$80-83*$	42.45059017	6.40988692	2	1	4	g, r, R
$84-85$	43.01834131	4.60383593	0	4	2	g, R
$86-89$	43.37712401	6.15803256	3	2	4	g, r, R
$90-93$	47.18741703	6.46000449	2	2	4	g, r, R
$94-97$	48.09226325	5.98471432	3	3	4	g, r, R
$98-101$	48.52539446	6.86741980	1	1	4	g, r, R
$102-103$	48.78046433	3.83170597	0	5	2	r
$104*$	49.21845632	7.01558667	0	0	1	r
$105-106$			1	0	2	R
$107-110$	49.43922306	5.25511057	1	4	4	g, r, R
$111-114$	49.65210407	6.64803317	1	2	4	g, r, R

The behavior of the kinetic energies $E(t)$ of the solutions of the systems for three values of the parameter ψ, fixed Reynolds number $Re = 1450$ and fixed initial values is illustrated in Figure 1. $\psi = 0$ stands for the pure boundary-driven Couette (angular momentum) flow and $\psi = pi/2$ stands for the pure pressure-driven Poiseuille flow. Our numerical investigations provide good agreement with experimental results for critical Reynolds numbers, for initial values in the order of magnitude of physical perturbations. If the initial values $g_{k,0}$ are chosen not too small, the corresponding functions $E(t)$ tend to zero for $Re \leq Re_{crit.}$ and remain at about the same level for $Re \geq Re_{crit.}$. For small values of $\psi \approx 0, \pi, 2\pi$ is the laminar boundary-driven Couette flow stable. $(E(t) \to 0)$ The same is true for the longtime-behavior of the solutions.

The time mean values of the coefficients of the remaining velocity field are defined by

$$\bar{g_j} := \frac{1}{T - t_0}\left(\int\limits_{t_0}^{T} g_j(t)\,dt\right) \approx \frac{1}{n_1}\sum_{i=1}^{n_1} g_j(t_i)\,,\ t_i \in [t_0, T]\,,\ j = 1, \ldots, N\,, \tag{19}$$

where n_1 is the number of Dormand-Price - resp. Runge-Kutta-Fehlberg time steps, t_i are the calculation points in time and t_0 is a time for which the kinetic energy starts to show an aperiodic oscillating behavior between an upper and lower level. The mean velocities are given by

$$\begin{aligned}\underline{\bar{\mathbf{u}}}_N(r) &= \mathbf{u}_{L,P}(r) + \frac{1}{4l\pi(T - t_0)}\int\limits_{t_0}^{T}\int\limits_{0}^{2\pi}\int\limits_{-l}^{l} \mathbf{u}_N(t,\mathbf{x})\,dx_1\,d\varphi\,dt \\ &\approx \mathbf{u}_{L,P}(r) + \sum_{i^\star \in I^\star} \bar{g_{i^\star}}\,\mathbf{w}_{i^\star}(\mathbf{x}),\end{aligned}$$

where the significant coefficients are given by $i^\star = 6,\ 104$ for the Couette flow and by $i^\star = 1,\ 47$ for the Poiseuille flow. Let call us the corresponding eigenfunctions Stokes modes. It is worth noting that the time averages $\bar{g_{i^\star}}, \bar{g_{j^\star}}$ according (20) assume much larger values than the time averages of all the other coefficients $\bar{g_j}$ (19).

The critical Reynolds numbers for our calculations are

$$Re^{COUETTE}_{crit.} \approx \infty, \quad Re^{POIS.}_{crit.} \in (1400, 20000)\,.$$

Much to our surprise we found by our calculations time periodic non-laminar solutions of a constant kinetic energy $E(t, Re)$ at fixed ψ-values out of a small ψ-interval ($\psi \approx \pi/3$) about a range of Reynolds numbers Re. These solutions are termed strange numerical solution. An example of such strange numerical solutions is set in what follows. The Reynolds number was taken to be $Re = 5000$. We found beginning with $t_0 = 1.03607$ constant values of the coefficients to the Stokes modes (1, 6, 47 and 104) and only eight other non-vanishing coefficients with:

$g_{25}(t) \approx g_{28}(t)$, $-g_{26}(t) \approx g_{27}(t)$,

$g_{80}(t) \approx g_{83}(t)$ and $-g_{81}(t) \approx g_{82}(t)\ \forall\, t \in [t_0, T]$

(The modes with non-vanishing coefficients was signed by $*$ in the table.)

For a simple linear combination of the corresponding eigenfunctions

$$\mathbf{w}_A(\mathbf{x}) := \frac{\mathbf{w}_{27}(\mathbf{x}) - \mathbf{w}_{26}(\mathbf{x})}{\sqrt{2}} \quad \text{and} \quad \mathbf{w}_B(\mathbf{x}) := \frac{\mathbf{w}_{25}(\mathbf{x}) + \mathbf{w}_{28}(\mathbf{x})}{\sqrt{2}}$$

as well as

$$\mathbf{w}_C(\mathbf{x}) := \frac{\mathbf{w}_{82}(\mathbf{x}) - \mathbf{w}_{81}(\mathbf{x})}{\sqrt{2}} \quad \text{and} \quad \mathbf{w}_D(\mathbf{x}) := \frac{\mathbf{w}_{80}(\mathbf{x}) + \mathbf{w}_{83}(\mathbf{x})}{\sqrt{2}},$$

are the received coefficients:

$$g_A(t) := \sqrt{2}g_{27}(t) \quad \text{and} \quad g_B(t) := \sqrt{2}g_{25}(t)$$

as well as

$$g_C(t) := \sqrt{2}g_{82}(t) \quad \text{and} \quad g_D(t) := \sqrt{2}g_{80}(t)$$

shown in Figure 2.

The mean velocity of a calculated time periodic non-laminar solution of constant kinetic energy $E(t)$ at $Re = 5000$ is finally compared with the corresponding laminar profile in Figure 3. The rotation of the pipe acts stabilizing to the pressure-driven part of the velocity, like a bullet is stabilized by the spiral fluted barrel of a gun. We put the mean calculated velocity with the same driving pressure but without any rotation of the pipe also in this Figure for a better illustration of this effect.

The illustrated set of timely periodical strange numerical solutions will be an object of our further research in the next years.

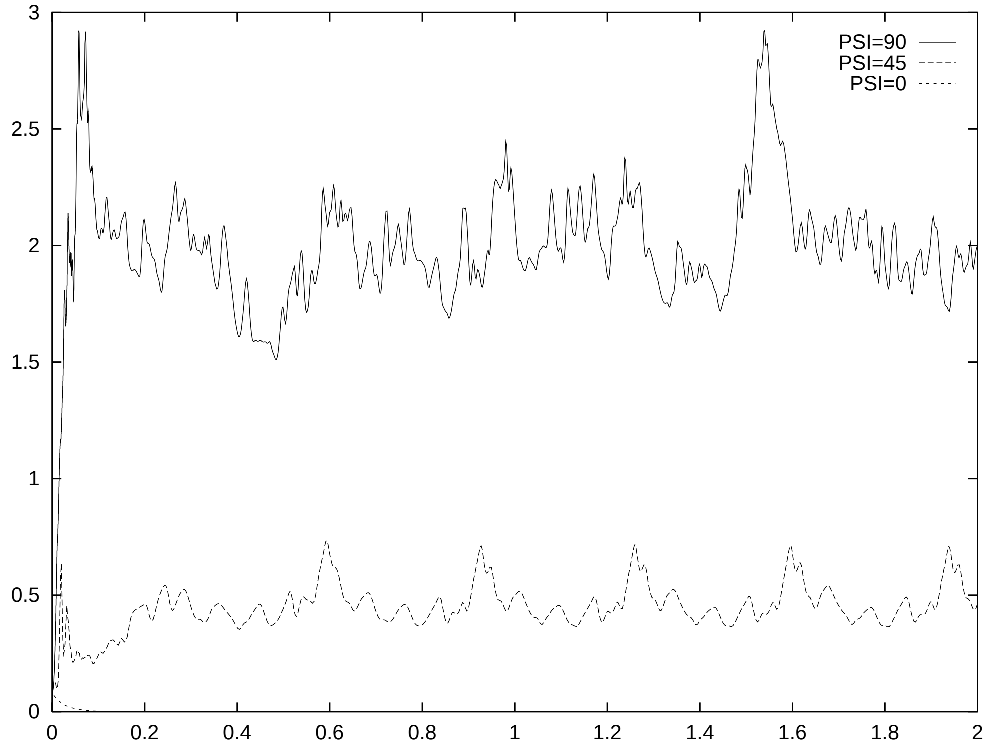


Figure 1: Behavior of $E(t)$ over t at Re=1450 for $\psi = 0\,(E(t) \equiv 0)\,, \pi/4, \pi/2$

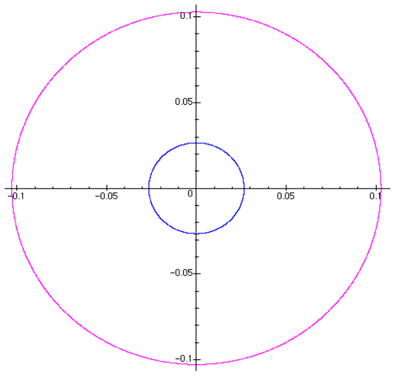


Figure 2: Trajectories in the planes of modes

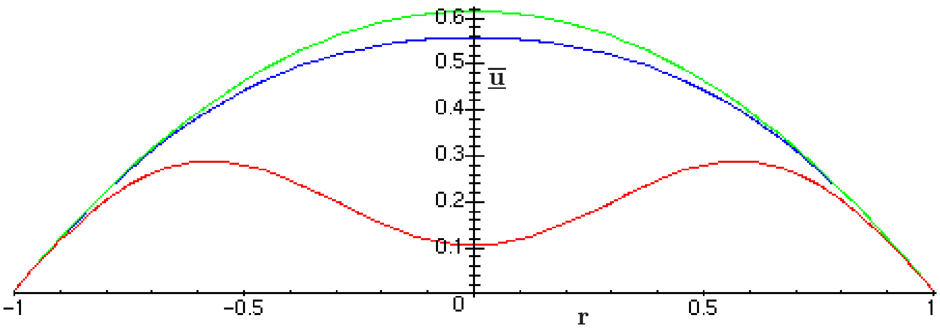


Figure 3: Flow in a pipe ($\psi = \pi/6$) Re=5000, laminar profile, calculations of the quasistationary solution in comparison with calculations($\psi = \pi/2$) Re=2500

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