Approximation of Eigenvalues and of Eigenfunctions for the Stokes Operator on an open Square

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Abstract
The study of the final states at two-dimensional decaying turbulence in wall-bounded domains is connected immediately with investigations to the Stokes eigenvalue problem. ([1]) We regard the Stokes operator **A** on an open bounded square Q, where no-slip (homogeneous Dirichlet) boundary conditions are required. The square Q is a bounded and convex domain with a Lipschitz-continuous boundary, what from follows that one knows the weak regularity of the eigenfunctions w (respectively of the corresponding stream function φ) to exhibit generalized derivations of second (resp. third) order with vanishing traces in the Sobolev-sense. ([2], [3] and [4]) Our path to tackle the Stokes eigenvalue problem is the following: We formulate the problem in the framework of equivalent sequence-spaces in the Fourier coefficients of the Fourier expansion of the stream (eigen-)function φ (of w) in half-periodic sinusoidal functions. The Sobolev-spaces with generalized derivations are explained equivalently in the form of weighted sequence-spaces in the Fourier coefficients and Lagrange multipliers involve the boundary conditions as an infinite sequence of constraints as boundary functionals. We get as the Euler-Lagrange formulation of the eigenvalue problem a linear system of equations in indefinite Lagrange multipliers, where the searched eigenvalue λ is a root of a transcendental equation expressed by infinite-dimensional determinants. Two different finite-dimensional approximations result in inclusions of every eigenvalue by upper and lower bounds, which converge for increasing dimension of approximations to λ. Finally, we approximate the corresponding eigenfunctions w by successive approximation. The described method improves former results [5] especially in the calculation of the eigenvalues.

Keywords: *Stokes Operator, Stokes Eigenfunctions, Stream Function, Inclusion Methods*

1. Introduction

The system of Stokes eigenfunctions $\{\underline{w}_{k}\}_{k=1}^{\infty}$ span the spatial Sobolev spaces for the mathematical treatment of the incompressible Navier-Stokes equations as an initial-boundary-value problem, where the existence of a complete system of Stokes eigenfunctions is ensured by the applicability from the elliptic partial differential equations theory. However, there are only rare cases of spatial domains with known systems of Stokes eigenfunctions as [6] and [7]. The explicit shape of every eigenfunction and the eigenvalues have to be identified separately even for such a simple domain as a square Q. It is worth to note, that the eigenvalues of the Stokes operator **A** are of particular interest too in addition to the eigenfunctions, especially for estimates and studies to the clamped plate equation. The Stokes eigenfunctions to the smallest eigenvalues are the significant modes in the observation of the final states at two-dimensional decaying turbulence in wall-bounded squares.

We use the open bounded square Q of the side length 2 as the domain, where $Q = \{\underline{x} \in \mathbb{R}^2 : 0 < x_1, x_2 < 2\}$ with the boundary ∂Q. The open square Q is to categorize as a convex Lipschitzian domain, what means that the boundary ∂Q is Lipschitz-continuous boundary - written as $Q \in C^{0,1}$.

The Stokes eigenvalue problem in the unknown $\underline{u}$(velocity), λ(eigenvalue) (and p(pressure)) is explained by

$$-\triangle \underline{u} + \underline{\nabla} p = \lambda \underline{u} \ , \ 0 = \underline{\nabla}^T \underline{u} = div\, \underline{u} \ \text{(in } Q), \ \underline{u}_{|\partial Q} = \underline{0}. \qquad (1)$$

One may observe the problem in an equivalent eigenvalue formulation for the stream function φ with the same eigenvalue λ also - but without a correct examination of regularity – as:

$$-\Delta^2 \varphi = \lambda \Delta \varphi \,, \; \underline{u} = curl\, \varphi \;\; (\text{in } Q), \varphi_{|\partial Q} = \frac{\partial \varphi}{\partial \underline{v}} = 0, \tag{2}$$

here the equation (2) is closely spaced to the clamped plate equation from mechanics.

Our way to tackle the problem is adumbrated as follows. We use Fourier series to describe the unknown eigenfunction $\underline{u}$ in the space of zero-divergence fields, where the fields are taken as normed curl of products of half-periodic sinusoidal functions, respectively:

$$\underline{u} = \underline{u}\,(\underline{x}) = curl \left(\sum_{m,n=1}^{\infty} \hat{z}_{m,n}^{\underline{u}} \frac{2}{\pi^2 \sqrt[2]{m^2+n^2}} \sin\frac{m\pi x_1}{2} \sin\frac{n\pi x_2}{2} \right) = \; curl\, \varphi(x_1, x_2) \tag{3}$$

We involve the no-slip boundary conditions for $\underline{u}$ by the requirement of vanishing functionals, where we use the distinction of cases in odd and even wave numbers. We set:

$$\langle c_m^1, \underline{u} \rangle := \sum_{l=0}^{\infty} \hat{z}_{m,2l+1}^{\underline{u}} \frac{2l+1}{\sqrt[2]{m^2+(2l+1)^2}} \; ; \; \langle c_m^2, \underline{u} \rangle := \sum_{l=1}^{\infty} \hat{z}_{m,2l}^{\underline{u}} \frac{2l}{\sqrt[2]{m^2+(2l)^2}} \quad \forall m \in \mathbb{N} \tag{4}$$

$$\langle d_n^1, \underline{u} \rangle := \sum_{q=0}^{\infty} \hat{z}_{2q+1,n}^{\underline{u}} \frac{2q+1}{\sqrt[2]{(2q+1)^2+n^2}} \; ; \; \langle d_n^2, \underline{u} \rangle := \sum_{q=1}^{\infty} \hat{z}_{2q,n}^{\underline{u}} \frac{2q}{\sqrt[2]{(2q)^2+n^2}} \quad \forall n \in \mathbb{N} \tag{5}$$

Finally, the partial differential equations of (1) and (2) are rewritten in the weak form in series of the Fourier coefficients $\{\hat{z}_{m,n}^{\underline{u}}\}_{m,n\in\mathbb{N}}$ and $\{\hat{z}_{m,n}^{\underline{v}}\}_{m,n\in\mathbb{N}}$, where we substitute $\lambda = \frac{\pi^2}{4}\lambda^*$.

$$\lambda^* \sum_{m,n=1}^{\infty} \hat{z}_{m,n}^{\underline{u}} \hat{z}_{m,n}^{\underline{v}} = \sum_{m,n=1}^{\infty} (m^2+n^2) \hat{z}_{m,n}^{\underline{u}} \hat{z}_{m,n}^{\underline{v}} \tag{6}$$

Here we have denoted by $\underline{v}$ a zero-divergence field with the same properties as $\underline{u}$, what means that $\underline{v}$ satisfies the same regularity and boundary conditions as $\underline{u}$. The relations from (3) to (6) provide the basis for the application of the variational calculus to the Stokes eigenvalue problem. We are going to describe the further approach of the eigenvalue problem in the subsequent sections. The essential function spaces and correspondent spaces of sequences are introduces in section 2, where the properties of the spaces are sketched in the required accuracy only. We explain the Stokes operator and the Stokes eigenpairs (eigenvalues and eigenfunctions) there. Afterwards we establish the variational formulation of the eigenvalue problem there as a constrained variational problem. Standard procedures of variation are applied there to reach the Euler-Lagrange equations as the necessary conditions for a minimum. The section is concluded with the description of the Fourier coefficients in Lagrange multipliers, which are obtained by the examination of the Euler-Lagrange equations. Section 3 is devoted to the systems of linear equations deduced from the conditions (6). There we explain infinite matrices, their approximations as finite matrices and determinants of finite and infinite matrices. Finally, the numerical methods and results are presented as a condensed account in section 4.There are presented images of decaying turbulence in comparison to our results as well.

2. Stokes eigenvalue problem and variational formulation

The Stokes operator **A** is explained as a mapping from $\underline{\mathbb{S}}^2$ on $\underline{\mathbb{S}}^0 = \underline{\mathbb{S}}$. There are introduced the function spaces $\underline{\mathbb{S}}^k$ (for $k = 0{,}1$) by the closures of the set $\underline{\mathcal{V}} = \{\underline{v} \in \underline{C}_0^\infty(Q): div\, \underline{v} = 0\}$ in the norms of $\underline{\mathbb{L}}_2 = \underline{\mathbb{W}}_2^0(Q)$ and $\underline{\mathbb{W}}_2^1(Q)$. The space $\underline{\mathbb{S}}^2$ is defined by $\underline{\mathbb{S}}^2 = \underline{\mathbb{S}}^1 \cap \underline{\mathbb{W}}_2^2(Q)$, where $\underline{\mathbb{L}}_2$ denotes the Lebesgue space and $\underline{\mathbb{W}}_2^k(Q)$ are Sobolev spaces, both on vector functions. The mathematical setting of the Stokes operator **A** is given by: $\mathrm{A} := -\mathcal{Y} \cdot \Delta$, where the projection $\mathcal{Y}$ ($\mathcal{Y}: \underline{\mathbb{L}}_2 \mapsto \underline{\mathbb{S}}$) is called the Leray-Helmholtz projection. We call a function $\underline{u} \in \underline{\mathbb{S}}^2$ an eigenfunction to the eigenvalue λ of A iff $\underline{u} \neq \underline{0}$ satisfies the equation $\mathrm{A}\underline{u} = \lambda \underline{u}$. The sequence $\{\lambda_k, \underline{w_k}\}_{k=1}^{\infty}$ is denoted as the system of Stokes eigenpairs (eigenvalues and normalized eigenfunctions), where the succession of pairs is given by the ordered eigenvalues (counted in multiplicity). We use the representation (3) for the treatment in the context of sequence spaces. The function spaces $\hat{\underline{\mathbb{S}}}^k$ are explained as the sets of all functions $\underline{u}$ according to (3) with

$$\|\underline{u}\|_k = \sqrt[2]{\sum_{m,n=1}^{\infty} \left(1 + \frac{\pi^2}{4}(m^2+n^2)\right)^k \left|\hat{z}_{m,n}^{\underline{u}}\right|^2} < \infty \text{ for all } k \in \mathbb{N}_0. \tag{7}$$

We are going to use the spaces $\underline{\widehat{\mathbb{S}}}^1$ (with (4) and (5)) and $\underline{\mathbb{S}}^1$ especially with regards to the so-called weak formulation of the eigenvalue problem. The kernels (null-spaces) of the linear functionals in (4) and (5) are closed subspaces of $\underline{\widehat{\mathbb{S}}}^1$ due to their continuity for $k = 1$. To get an equivalent description of in the $\underline{\mathbb{S}}^1$ in the setting of (7) we define the space $\underline{\widehat{\mathbb{S}}}_0^1$ by $\underline{\widehat{\mathbb{S}}}_0^1 = \{\underline{u} \in \underline{\widehat{\mathbb{S}}}^1 | \langle c_m^i, \underline{u}\rangle = \langle d_n^i, \underline{u}\rangle = 0 \; \forall m,n \in \mathbb{N}, i = 1,2\}$.

We introduce the simple sequence spaces $\mathbb{l}_2$ and $\mathbb{l}_2 \times \mathbb{l}_2$ for the implementation of the constraints (4) and (5). We denote by $\boldsymbol{\alpha}^i = \{\alpha_m^i\}_{m=1}^{\infty}$ and by $\boldsymbol{\beta}^i = \{\beta_n^i\}_{n=1}^{\infty}$ sequences of Lagrange multipliers respectively for $i = 1,2$. We explain the space $\mathbb{l}_2$ by the linear space of all $\boldsymbol{\alpha}$ which fulfill the condition $\sum_{m=1}^{\infty}|\alpha_m|^2 < \infty$, the square of the $\mathbb{l}_2$-norm. The space $\mathbb{l}_2 \times \mathbb{l}_2$ is defined as the Cartesian product of two $\mathbb{l}_2$-factors each one endowed with the norm of the product space. We use the spaces $\underline{\widehat{\mathbb{S}}}_0^1$ and $\underline{\mathbb{S}}^1$ for the variational formulation of problem $A\underline{u} = \lambda \underline{u}$ in a weak sense as a constrained variational problem. The following variational principle is to understand as an adaption of Courants minimax method. The Euler-Lagrange function $\mathcal{F}$ is defined by: $\mathcal{F} = \kappa\left(\sum_{m,n=1}^{\infty}(m^2+n^2-\lambda^*)\left(\hat{z}_{m,n}^{\underline{u}}\right)^2\right) + \eta\left(1 - \sum_{m,n=1}^{\infty}\left(\hat{z}_{m,n}^{\underline{u}}\right)^2\right) + 2\sum_{i=1}^{2}\left(\sum_{m=1}^{\infty}\alpha_m^i\langle c_m^i, \underline{u}\rangle + \sum_{n=1}^{\infty}\beta_n^i\langle d_n^i, \underline{u}\rangle\right)$ The necessary conditions for stationary points of $\mathcal{F}$ are determined in a standard way by partial derivatives with respect to all Fourier coefficients $\{\hat{z}_{m,n}^{\underline{u}}\}_{m,n=1}^{\infty}$, where we set $\kappa = 1$ and $\eta = 0$ firstly. We get the key relations from the necessary conditions by simple calculations. So we have received the representation of the Fourier coefficients in Lagrange multipliers:

$$\hat{z}_{m,n}^{\underline{u}} = \frac{n\alpha_m^i + m\beta_n^j}{(m^2+n^2-\lambda^*)\sqrt{n^2+m^2}} \; \forall l,q \in \mathbb{N} \text{ , with } j = \begin{cases} 1 \; \forall m = 2q-1 \\ 2 \quad \forall m = 2q \end{cases} \text{ and } i = \begin{cases} 1 \; \forall n = 2l-1 \\ 2 \quad \forall n = 2l \end{cases} \tag{8}$$

3. Matrix equations and finite approximation matrices

The spaces $\underline{\widehat{\mathbb{S}}}^1$ and $\underline{\widehat{\mathbb{S}}}_0^1$ admit orthogonal decomposition in subspaces of odd and even wavenumbers m and n, expressed by the indicators $i,j = 1,2$. So we get by way of example for the space $\underline{\widehat{\mathbb{S}}}^1$: $\underline{\widehat{\mathbb{S}}}^1 = \underline{\widehat{\mathbb{S}}}^1{}_{1,1} \oplus \underline{\widehat{\mathbb{S}}}^1{}_{1,2} \oplus \underline{\widehat{\mathbb{S}}}^1{}_{2,1} \oplus \underline{\widehat{\mathbb{S}}}^1{}_{2,2}$, where the correspondent pairs of subsequences of Lagrange multipliers are $(\boldsymbol{\alpha}_1^1, \boldsymbol{\beta}_1^1), (\boldsymbol{\alpha}_2^1, \boldsymbol{\beta}_1^2), (\boldsymbol{\alpha}_1^2, \boldsymbol{\beta}_2^1), (\boldsymbol{\alpha}_2^2, \boldsymbol{\beta}_2^2)$. We get the following systems of linear equations with infinite block matrices of coefficients by putting the representations (8) of the Fourier coefficients in Lagrange multipliers into the constraints (4) and (5) as zero equations, at which we denote by $\underline{\mathbf{0}}$ and $\underline{\boldsymbol{\alpha}}_j^i$, $\underline{\boldsymbol{\beta}}_j^i$ a sequence filled with zeros the correspondent and subsequences written as infinite columns.

$$\begin{bmatrix} \underline{\underline{\mathcal{D}}}{}_j^i & \underline{\underline{\mathcal{A}}}_i^j \\ \underline{\underline{\mathcal{A}}}_j^i & \underline{\underline{\mathcal{D}}}{}_i^j \end{bmatrix} \begin{bmatrix} \underline{\boldsymbol{\alpha}}_j^i \\ \underline{\boldsymbol{\beta}}_i^j \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{0}} \\ \underline{\mathbf{0}} \end{bmatrix} \tag{9}$$

There are written the infinite matrices for elements of block matrices as $\underline{\underline{\mathcal{A}}}$ or $\underline{\underline{\mathcal{D}}}$, afterwards the square finite approximation matrices are referred as $\underline{\underline{A}}$ and $\underline{\underline{D}}$ respectively. For example we give the infinite matrix $\underline{\underline{\mathcal{A}}}_1^1 = \left[\frac{(2q-1)(2l-1)}{((2q-1)^2+(2l-1)^2-\lambda^*)\sqrt{(2q-1)^2+(2l-1)^2}}\right]_{q=1,l=1}^{\infty,\infty}$ and the infinite diagonal matrix $\underline{\underline{\mathcal{D}}}{}_1^1 = diag\left[\sum_{l=1}^{\infty}\frac{(2l-1)^2}{((2q-1)^2+(2l-1)^2-\lambda^*)\sqrt{(2q-1)^2+(2l-1)^2}}\right]_{q=1}^{\infty}$. The transposed matrix of $\underline{\underline{\mathcal{A}}}_i^j$ is the infinite matrix $\underline{\underline{\mathcal{A}}}_j^i$ there. We note, that there is a strong transcendental dependence of all elements of the infinite block matrices of coefficients in (9) from the value λ^*. Since the value λ^* has to be a multiple of the sought eigenvalue, it is a simple conclusion that there have to exist nontrivial solutions of (9) too. We use the standard for the existence of nontrivial solutions of (9) as our criterion for eigenvalues: We look for values of λ^* with vanishing determinant of the block matrices. Finite dimensional approximations provide lower and upper bounds for every λ^*. Two types of definitions explain the correspondent block matrices. We set according to the choice of example $i, j = 1$ above:

$\underline{\underline{A}}_{1,M}^{1} = \left[\frac{(2q-1)(2l-1)}{((2q-1)^2+(2l-1)^2-\lambda^*)\sqrt{(2q-1)^2+(2l-1)^2}}\right]_{q=1,l=1}^{M,M}$,

$\underline{\underline{\breve{D}}}_{1,M}^{1} = diag\left[\sum_{l=1}^{\infty}\frac{(2l-1)^2}{((2q-1)^2+(2l-1)^2-\lambda^*)\sqrt{(2q-1)^2+(2l-1)^2}}\right]_{q=1}^{M}$ for the lower bounds $\breve{\lambda}^*$ and $\underline{\underline{\hat{D}}}_{1,M}^{1} = diag\left[\sum_{l=1}^{M}\frac{(2l-1)^2}{((2q-1)^2+(2l-1)^2-\lambda^*)\sqrt{(2q-1)^2+(2l-1)^2}}\right]_{q=1}^{M}$ for the upper bounds $\hat{\lambda}^*$. The versions of the received block matrices are just like the matrix (9). The approximations of the Lagrange multipliers as unknown are written as $\underline{\breve{\alpha}}_{1,M}^{1} = \left[\breve{\alpha}_{2q-1}^{1,M}\right]_{q=1}^{M}$ and $\underline{\breve{\beta}}_{1,M}^{1} = \left[\breve{\beta}_{2q-1}^{1,M}\right]_{q=1}^{M}$ et cetera. Finally we state the determinant for infinite block matrices by:

$$\det\begin{bmatrix}\underline{\underline{\mathcal{D}}}_{j}^{i} & \underline{\underline{\mathcal{A}}}_{i}^{j}\\ \underline{\underline{\mathcal{A}}}_{j}^{i} & \underline{\underline{\mathcal{D}}}_{i}^{j}\end{bmatrix} = \lim_{M\to\infty}\det\begin{bmatrix}\underline{\underline{\hat{D}}}_{j,M}^{i} & \underline{\underline{A}}_{i,M}^{j}\\ \underline{\underline{A}}_{j,M}^{i} & \underline{\underline{\hat{D}}}_{i,M}^{j}\end{bmatrix} = \lim_{M\to\infty}\begin{vmatrix}\underline{\underline{\hat{D}}}_{j,M}^{i} & \underline{\underline{A}}_{i,M}^{j}\\ \underline{\underline{A}}_{j,M}^{i} & \underline{\underline{\hat{D}}}_{i,M}^{j}\end{vmatrix} = \lim_{M\to\infty}\left|\underline{\underline{\hat{B}}}_{2M}^{i,j}\right|.$$

4. Numerical methods and results

For the numerical study we use a preprocessing steps, where the determinants of $\underline{\underline{\hat{B}}}_{2M}^{i,j}$ or $\underline{\underline{\breve{B}}}_{2M}^{i,j}$ are expressed by determinants of the matrices from format M. We apply there standard tools for determinants of 2×2 block matrices. To secure that the determinants are not identical vanishing functions of λ^* for regular values (not poles) there is diminished the format to $M-1$ if necessary. The zeros λ^* of the determinants are calculated numerically with MAPLE combined with the bisection method and the secant method, where the region of values is restricted by $\breve{\lambda}^* < \lambda^* < \hat{\lambda}^*$. Finally, we approach the corresponding Lagrange multipliers $\underline{\breve{\alpha}}_{j,M}^{i}, \underline{\breve{\beta}}_{i,M}^{j}, \underline{\hat{\alpha}}_{j,M}^{i}, \underline{\hat{\beta}}_{i,M}^{j}$ by successive approximation for a reconstruction of the functions with (8). Our main result is the inclusion method for all eigenvalues $\lambda_{k} = \frac{\pi^2}{4}\lambda_{k}^*$ at $k = k(i,j)$: $\lambda_{k}^* = \lim_{M\to\infty}\breve{\lambda}_{k,M}^* = \lim_{M\to\infty}\hat{\lambda}_{k,M}^*$ with limits of monotonically sequences. Here, we abridge some findings in a condensed form. The first eigenvalue is $\lambda_1^* \approx 5.30362606$. Figure 1 illustrates the behavior of the vorticity in decaying turbulence in comparison with stream functions of eigenfunctions both represented in contour colors. We point to the lecture for a lot of further findings and comparisons with [8] and [9].

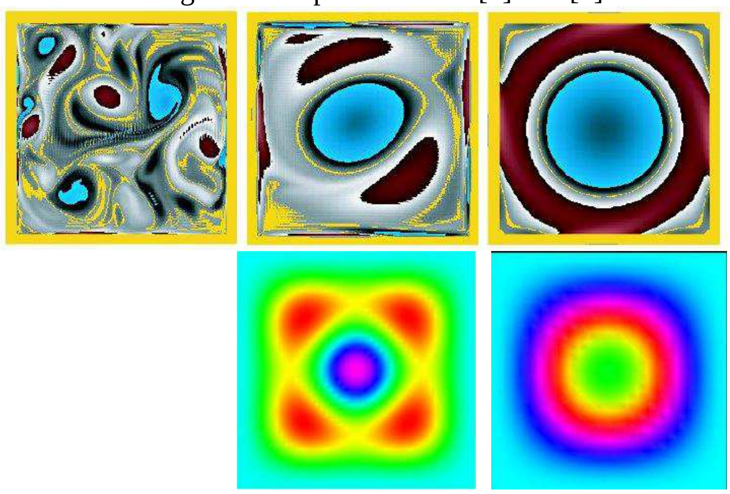

Fig. 1 Decaying turbulence [1] in comparison to Stokes eigenfunctions as stream functions

5. References

1. K. Schneider, M. Farge. Physica D 237, 2228-2233, 2008.
2. P. Grisward. Elliptic Problems in Nonsmooth Domains. Boston, Pitman Publ., 1985
3. R. Teman. Navier-Stokes Equations. Amsterdam, North Holland, 1984
4. R.B. Kellogg, J.E. Osborn. J. Functional Anal.21, 397-431, 1976
5. F. Goerisch, H. Haunhorst. ZAMM 65(2), 129-135, 1985
6. D. -S. Lee, B. Rummler. ZAMM 82 (6), 399-407, 2002
7. B. Rummler. ZAMM 77(8), 619-627, 1997
8. E. Leriche, G. Labrosse. J. Comput. Phys., 200, 489-511, 2004
9. G. Labrosse, E. Leriche, P. Lallemand. Theor. & Comp.Fluid Dyn. 28(3), 335-356, 2014