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The Eigenfunctions of the Stokes Operator in the open Unit Ball and in the open spherical Annulus

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Abstract

We consider the eigenvalue problem of the Stokes operator $\mathbf{A}_\sigma$ on the open unit ball Ω_o ($\sigma = 0$) - respectively on an open spherical annulus Ω_σ ($0 < \sigma < 1$) - both in $\mathbb{R}^3$, where one constructs the domain $D(\mathbf{A}_\sigma)$ of $\mathbf{A}_\sigma$ in such a way, that the requirement of homogeneous Dirichlet boundary conditions on the frontier $\partial\Omega_\sigma$ is fulfilled in a general sense. Supplementary, one protects by theoretical standard methods of partial differential equations (PDE), that the Stokes eigenfunctions are smooth, such that the Stokes eigenfunctions have to fulfill homogeneous Dirichlet boundary conditions in the classical sense too.
The existence of a complete system of eigenfunctions of the Stokes operator $\mathbf{A}_\sigma$ is known as a theoretical result of (PDE), but a system of explicitly calculated eigenfunctions of $\mathbf{A}_o$ was given for the first time by the author (cf. [12]) just ten years ago. It is to note, that the proof of the completeness for the system of the explicitly calculated eigenfunctions to $\mathbf{A}_o$ was not given in [12]. In what follows, we prove the completeness for the system and give a new method for the representation of the calculated Stokes eigenfunctions. The proof is based on a decomposion theorem of solenoidal vector fields on Ω_* into toroidal and poloidal parts, where these toroidal and poloidal vector fields are defined by toroidal and poloidal scalar functions respectively. We derive from the original eigenvalue problem of the Stokes operator $\mathbf{A}_\sigma$ in the unknowns λ and $\underline{u}$ (the eigenvalue and the vectorial eigenfunction) scalar eigenvalue problems in the unknowns λ and η (the eigenvalue and a scalar eigenfunction), where we use the decomposion theorem and surface spherical harmonics. Especially we apply the nice properties of the Laplace-Beltrami operator $\mathbf{B}$ on the spherical surfaces and the surface spherical harmonics as eigenfunctions of $\mathbf{B}$. Our investigations result in a new explicite derivation of the Stokes eigenfunctions, which are written in spherical coordinates here, where the eigenvalues of $\mathbf{A}_o$ are exactly the eigenvalues given in [12]. Additionally we give hints for the application of our methods to the Stokes operator on the open spherical annulus Ω_σ between two concentrical spherical surfaces. We note, that the Stokes eigenfunctions are an essential tool in the study of flows, where we cover Newtonian flows in the kernel of the earth and around the earth by the systems of Stokes eigenfunctions.

Keywords: Stokes eigenfunctions, spherical harmonics, Bessel functions, poloidal fields, toroidal fields

1. Introduction

In an open series on *Eigenfunctions of the Stokes Operator in Special Domains* (cf. [8], [13] and [14]), we have given some complete systems of calculated (complex-valued) Stokes eigenfunctions, whereby we have applied periodic conditions in the former unbounded directions of the 3D-domains.
The proofs of completeness of this systems were pointed out in [12], where the reader will find the real-valued eigenfunctions and the possibilities for the use of low-dimensional Galerkin spaces defined by Stokes eigenfunctions for fact finding of the mechanism of the transition to turbulence in the non-stationary 3D-Navier-Stokes equations (NSE) for the flow of incompressible Newtonian fluids in 3D-channels and in 3D-pipes too (cf. also [11]).
The eigenvalue problems for the Stokes operator $\mathbf{A}_\sigma$ on 3D-balls or on 3D-annuli are boundary value problems in bounded 3D-domains with rigid frontiers, what means, that we do not need additional periodic conditions in any unbounded directions of the 3D-domains.
In the recent paper we establish a general method to decompose solenoidal vector fields in a ball (or in a spherical annulus (a spherical layer)), what finally results in the proof of completeness for the system of calculated real-valued eigenfunction written in Cartesian coordinates given in [12]. It is worth to note, that we have already used the decomposition idea of solenoidal fields into toroidal and poloidal vector fields

implicitly to reach our results in [12], where we have also profit by the pure scalar action of the vector Laplacian on every Cartesian coordinate of vector fields written as functions of polar coordinates as a big advantage.
The Stokes eigenvalue problem written as a boundary value problem in the unknowns $\underline{u}$, λ and p is given by:
<u>PROBLEM 1:</u> We seek solutions $\underline{u}$, λ and p (for $\sigma : 0 \leq \sigma < 1$) fulfilling:

$-\triangle \underline{u} + \underline{\nabla}\, p = \lambda \underline{u} \quad$ and $\quad div\, \underline{u} \,=\, \underline{\nabla}^T \cdot \underline{u} \,=\, 0 \quad$ in Ω_σ , with $\underline{u} \,=\, \underline{0}$ on $\partial\Omega_\sigma$

One can write the PROBLEM 1 in an operator formulation by the use of appropriate function spaces of solenoidal vector fields (cf. [6], [12] and [17]) as:
<u>PROBLEM 2:</u> We seek solutions $\underline{u}$ and λ fulfilling: $\mathbf{A}_\sigma \underline{u} \,=\, \lambda \underline{u}$, where $\mathbf{A}_\sigma$ denotes the Stokes operator. So we can use the full power of functional analysis to get, that $\mathbf{A}_\sigma$ is an operator with a pure real point spectrum, what means that all the eigenvalues $\lambda_j \,>\, 0$ are of finite multiplicity. The associated eigenfunctions $\{\underline{w}_j(\underline{x})\}_{j=1}^{\infty}$ of $\mathbf{A}_\sigma$ (counted in the multiplicity of λ_j) are regular. (cf. [1], [4], [17] and [18])
The recent paper is designed as follows:
The essential notations, the essential function spaces and the governing equations supplemented with boundary conditions are outlined in section 2. The Stokes operator $\mathbf{A}_\sigma$, the Leray-Helmholtz projector Υ and the Laplace-Beltrami operator $\mathbf{B}$ are introduced there.
The section 3 is devoted to the characterization of solenoidal fields on Ω_σ. In a first step we give a general characterization lemma for solenoidal fields there. The central point of this section is the decomposion theorem of solenoidal vector fields on Ω_* into toroidal and poloidal parts, where these toroidal and poloidal vector fields are defined by toroidal and poloidal scalar functions respectively.
The eigenvalue problem for the Stokes operator $\mathbf{A}_o$ is treated in section 4. We restrict ourselves there to the operator $\mathbf{A}_o$ in order to show the solution of the principal task. For the general results for $\mathbf{A}_\sigma$ we refer to [19] and [15]. We give in this section a summary on our results, where we sketch an outlook on applications also.
Finally some helpful relations of the vector analysis and the definitions of surface spherical harmonics are collected in the Appendix, which closes the paper.

2. Notations, Function Spaces and Operators

We denote by $\underline{x} \,=\, (x_1, x_2, x_3)^T$ the spatial coordinates of points in $\mathbb{R}^3$, by $p \,=\, p(\underline{x})$ scalar fields and by $\underline{u} \,=\, (u_1, u_2, u_3)^T \,=\, \underline{u}(\underline{x})$ 3D-vector fields, where we use all these quantities in convenient scaling and $(.,.,.)^T$ as the notation for transposed columns.
We regard these vector fields p and $\underline{u}$ as functions on the domain $\overline{\Omega}_o \,:=\, \Omega_o \cup \partial\Omega_o$ respectively on the domains $\overline{\Omega}_\sigma := \Omega_\sigma \cup \partial\Omega_o$, where $\Omega := \Omega_o$ stands for the open unit ball:
$\Omega_o \,:=\, \{\underline{x} \in \mathbb{R}^3 :\, ||\underline{x}|| := \sqrt{x_1^2 + x_2^2 + x_3^2} < 1\}$, with the frontier $\omega := \partial\Omega_o \,:=\, \{\underline{x} \in \mathbb{R}^3 :\, ||\underline{x}|| = 1\}$,
respectively where we denote by Ω_σ for fixed $\sigma : 0 < \sigma < 1$ the open annulus:
$\Omega_\sigma \,:=\, \{\underline{x} \in \mathbb{R}^3 :\, \sigma < ||\underline{x}|| < 1\}$, with the frontier $\partial\Omega_\sigma \,:=\, \{\underline{x} \in \mathbb{R}^3 :\, ||\underline{x}|| = \sigma \,\text{or}\, ||\underline{x}|| = 1\}$.

We will use $\underline{\mathfrak{e}}_j \,:=\, (\delta_{j,1}, \delta_{j,2}, \delta_{j,3})^T$, $\forall\, j \,=\, 1, 2, 3$, with Kronecker's delta $\delta_{j,k}$, for the unit vectors in the Cartesian coordinate system and the notations $\{\underline{\mathfrak{e}}_r, \underline{\mathfrak{e}}_\vartheta, \underline{\mathfrak{e}}_\varphi\}$ for the system of unit vectors in polar coordinates. Any vector field $\underline{u}$ is customary representable in both systems as:
$\underline{u} \,=\, \sum_{j=1}^3 u_j \underline{\mathfrak{e}}_j \,=\, \sum_{j=1}^3 u_{j,\mathsf{c}} \underline{\mathfrak{e}}_j \,=\, u_r \underline{\mathfrak{e}}_r + u_\vartheta \underline{\mathfrak{e}}_\vartheta + u_\varphi \underline{\mathfrak{e}}_\varphi \,=\, u_{r,\mathsf{p}} \underline{\mathfrak{e}}_r + u_{\vartheta,\mathsf{p}} \underline{\mathfrak{e}}_\vartheta + u_{\varphi,\mathsf{p}} \underline{\mathfrak{e}}_\varphi$
The transformation from one coordinate system to the other one can be written as $\underline{u}_\mathsf{c} \,=\, \underline{T}_{\mathsf{c},\mathsf{p}} \underline{u}_\mathsf{p}$ respectively $\underline{u}_\mathsf{p} \,=\, \underline{T}_{\mathsf{c},\mathsf{p}}^{-1} \underline{u}_\mathsf{c} \,=\, \underline{T}_{\mathsf{p},\mathsf{c}} \underline{u}_\mathsf{c}$, where we have used conception pure columns of coordinates and the transformation matrices:

$$\underline{T}_{\mathsf{c},\mathsf{p}} \,:=\, \begin{bmatrix} \sin\vartheta\cos\varphi & \sin\vartheta\cos\varphi & -\sin\varphi \\ \sin\vartheta\sin\varphi & \sin\vartheta\sin\varphi & \cos\varphi \\ \cos\vartheta & -\sin\vartheta & 0 \end{bmatrix} \qquad \underline{T}_{\mathsf{p},\mathsf{c}} \,:=\, \underline{T}_{\mathsf{c},\mathsf{p}}^{-1} \,=\, \underline{T}_{\mathsf{c},\mathsf{p}}^{T}$$

We denote by $\underline{\nabla}$ the Nabla operator, written in Cartesian coordinates as: $\quad \underline{\nabla} := \sum_{j=1}^3 \frac{\partial}{\partial x_j} \underline{\mathfrak{e}}_j$.
The vector product of $\underline{u}$ and $\underline{v} \,\in\, \mathbb{R}^3$ is stated by: $\underline{u} \wedge \underline{v}$.
We utilize the Nabla operator $\underline{\nabla}$ to represent the fundamental differential expressions of the vector analysis in $\mathbb{R}^3$ like: $\quad div\, \underline{u}(\underline{x}) \,:=\, \underline{\nabla}^T \underline{u}(\underline{x}), \quad curl\, \underline{u}(\underline{x}) \,:=\, \underline{\nabla} \wedge \underline{u}(\underline{x}), \quad$ and $\quad grad\, \eta(\underline{x}) \,:=\, \underline{\nabla}\eta(\underline{x})$,

for vector fields $\underline{u}$ and scalar functions η, where we understand the derivations at least in the generalized sense.
It is worth to note, that the vector product and the differential expressions (div, $curl$ and $grad$) are independent of the special choice of the orthonormal right-handed coordinate system in $\mathbb{R}^3$ by their definitions. We are going to use this property for the derivation of differential relations for $curl$'s of vector fields $\underline{u} := \eta(\underline{x}) \cdot \underline{x}$ in the Appendix.
Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_o$ be the set of non-negative integers and Ω any 'nice' (e.g of class $\mathcal{C}^2$) open bounded set (domain) in $\mathbb{R}^n$ with $n \in \mathbb{N}$ and the boundary $\partial\Omega$.
We regard the points $\underline{x}$ of $\overline{\Omega}$ like the usual multi-indeces $\underline{\kappa}$ further on as columns. Following Schwartz' notation for generalized derivations, we use $\underline{\kappa} \in \mathbb{N}_o^n$ as a multi-index, with $|\underline{\kappa}| := \sum_{j=1}^n \kappa_j$.

We write the partial derivations as $$D^{\underline{\kappa}} := \frac{\partial^{|\underline{\kappa}|}}{\partial x_1^{\kappa_1} \ldots \partial x_n^{\kappa_n}} \,, \; \forall \underline{\kappa}, \text{ where } |\underline{\kappa}| \geq 1 \,.$$

For $\underline{\kappa} = \underline{0}$ and any real-valued (or complex-valued) function $f : D(f) \subset \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{C}$) we set $D^{\underline{0}} f(\underline{x}) := f(\underline{x}) \; \forall \underline{x} \in D(f)$.
For $m \in \mathbb{N}$ we denote by $\mathbb{C}^m(\overline{\Omega})$ the Banach space of real-valued (or complex-valued) functions f that are continuous on $\overline{\Omega}$ and all of whose derivatives $D^{\underline{\kappa}} f$ up to and including the order $m = |\underline{\kappa}|$ are continuous on Ω and can be extended by continuity to $\overline{\Omega}$. We write $\mathbb{C}(\overline{\Omega})$ for $m = 0$. Finally we denote by $\mathcal{C}^\infty(\tilde{\Omega})$ the linear vector space of all real-(or complex-)valued functions f which possess in all points $\underline{\text{x}}$ of any simple connected set $\tilde{\Omega}$ in $\mathbb{R}^n$ continuous derivations of arbitrary order $|\underline{\kappa}| \in \mathbb{N}_o$ and by $\mathcal{C}_o^\infty(\tilde{\Omega})$ the subspace of $\mathcal{C}^\infty(\tilde{\Omega})$ of real- (or complex-)valued functions with compact supports in and different from $\tilde{\Omega}$.
We apply the Lebesgue spaces $\mathbb{L}_2(\Omega)$ (and $\mathbb{L}_2(\partial\Omega)$) the *(equivalence classes)* of real (complex) absolutely quadratic integrable functions defined on Ω (resp. on $\partial\Omega$) for the Lebesgue measure. The Sobolev spaces $\mathbb{W}_2^m(\Omega)$ (resp. $\mathbb{W}_2^m(\Omega)$ are the spaces of functions in $\mathbb{L}_2(\Omega)$ (resp. in $\mathbb{L}_2(\partial\Omega)$) with derivatives up to and including the order $m = |\underline{\kappa}|$ in $\mathbb{L}_2(\Omega)$ (resp. in $\mathbb{L}_2(\partial\Omega)$). The spaces $\mathbb{L}_2(.)$ and $\mathbb{W}_2^m(.)$ are Hilbert spaces with the norm $||f||_{\mathbb{W}_2^m(.)} := (\sum_{m \leq |\underline{\kappa}|} (||D^{\underline{\kappa}} f||_{\mathbb{L}_2(.)})^2)^{(1/2)}$. For $m \in \mathbb{N}$ we will apply the Hilbert spaces of functions with vanishing boundary values in the generalized sense by $\mathbb{W}_{2,o}^m(\Omega) := \overline{\mathcal{C}_o^\infty(\Omega)}^{\mathbb{W}_2^m(.)}$, where the right hand side of this equation stands for the closure of $\mathcal{C}_o^\infty(\Omega)$ in $\mathbb{W}_2^m(\Omega)$.
For any linear space $\mathcal{X}$ or any Banach space $\mathbb{X}$ we will denote by $\underline{\mathcal{X}}$ resp. $\underline{\mathbb{X}}$ the spaces of vector-valued functions with all components in $\mathcal{X}$ resp. $\mathbb{X}$.
Let us define by $\underline{\mathcal{V}}_\sigma := \{\underline{v} \in \underline{\mathcal{C}}_o^\infty(\Omega_\sigma) : div\, \underline{v} = 0\}$ for $\sigma : 0 \leq \sigma < 1$. The closures of $\underline{\mathcal{V}}_\sigma$ in the sense of $\underline{\mathbb{L}}_2(\Omega_\sigma)$ and $\underline{\mathbb{W}}_{2,o}^1(\Omega_\sigma)$ will be called by $\underline{\mathbb{S}}(\Omega_\sigma)$ respectively by $\underline{\mathbb{S}}^1(\Omega_\sigma)$. We abbreviate the space $\underline{\mathbb{W}}_2^2(\Omega_\sigma) \cap \underline{\mathbb{S}}^1(\Omega_\sigma)$ with $\underline{\mathbb{S}}^2(\Omega_\sigma) := \underline{\mathbb{W}}_2^2(\Omega_\sigma) \cap \underline{\mathbb{S}}^1(\Omega_\sigma)$. The advantage of these spaces is, that their elements are solenoidal fields and fulfill the boundary conditions in a general sense.

<u>DEFINITION 1:</u> We denote by $\triangle = div\, grad : \underline{\mathbb{W}}_2^2(\Omega_\sigma) \to \underline{\mathbb{L}}_2(\Omega_\sigma)$ the Laplace operator in the sense of Friedrichs extension. The Leray-Helmholtz projector Υ is the projector on solenoidal fields explained by: $\Upsilon : \underline{\mathbb{L}}_2(\Omega_\sigma) \to \underline{\mathbb{S}}(\Omega_\sigma)$. The Stokes operator is stated as the product of Υ and $-\triangle$ and defined by: $-\Upsilon\triangle : \underline{\mathbb{S}}^2(\Omega_\sigma) \to \underline{\mathbb{S}}(\Omega_\sigma)$.

<u>DEFINITION 2:</u> The Laplace-Beltrami operator is explained by

$$\mathbf{B^o}\, Y := -\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\frac{\partial Y}{\partial\vartheta}) - \frac{1}{\sin^2\vartheta}\frac{\partial^2 Y}{\partial\varphi^2}\,, \qquad \forall Y \in D(\mathbf{B^o}) = C^\infty(\omega) \subset \mathbb{L}_2(\omega)$$

We denote the Friedrichs extension of $\mathbf{B^o}$ by $\mathbf{B} := \overline{\mathbf{B^o}}$. The operator $\mathbf{B}$ is also called Laplace-Beltrami operator for abbreviation.

It is usual to use the eigenvalues and eigenfunctions of the Laplace-Beltrami operator for the definion of function spaces on ω cf. e.g. [6], p. 33). We denote by
$D(\mathbf{B}^m) = \{Y \in \mathbb{L}_2(\omega_3) : \sum_{j=0}^\infty (1 + \nu_j^{2m})|(Y, Y_j)_{\mathbb{L}_2(\omega)}|^2 < \infty\}$ for $m \in \mathbb{N}_o$ in account with THEOREMS B and C of the Appendix.

3. Decomposition of solenodal fields into toroidal and poloidal fields

We show in this section, that every solenoidal field on Ω_σ (for $\sigma : 0 \leq \sigma < 1$) is the sum of a toroidal field $\underline{\mathfrak{t}} := curl(\psi(\underline{x}) \cdot \underline{x})$ and a poloidal field $\underline{\mathfrak{p}} := curl(curl(\chi(\underline{x}) \cdot \underline{x}))$, where we use similar argumentations like in [2], [3], [5], [7], [9] (and also like in [16] but with spherical harmonics). We are going to apply the notations $\psi(.) = \eta_{\underline{\mathfrak{t}}}(.)$ and $\chi(.) = \eta_{\underline{\mathfrak{p}}}(.)$ for the use of the relations given in the Appendix too.
Additionally, we formulate a representation of the gradients $\underline{\nabla}\, p$ of harmonic pressures p as poloidal fields $\underline{\mathfrak{p}}$ here.
The following lemma gives an essential characterization of solenoidal fields on Ω_σ, $\sigma : 0 \leq \sigma < 1$:
<u>LEMMA 1:</u> Let Ω_σ be the open unit ball Ω_o in $\mathbb{R}^3$, $\underline{x} \in \mathbb{R}^3$ and $r := \sqrt{\sum_{j=1}^{3} x_j^2}$ the radius-coordinate in a polar coordinate system. Then there exists no solenoidal vector function $\underline{F} \in (C^1(\Omega_o))^3$, which is only depending of the variable r, with $\underline{F}(r) \neq \underline{c}\, \forall\, r \in (0,1)$, where $\underline{c}$ denotes any constant vector.
PROOF: Let the vector field $\underline{F}$ be written in Cartesian coordinates:

$$\underline{F}^T(r) = (F_1(r), F_2(r), F_3(r))$$

Since $\underline{F}(r)$ is in $(C^1(.))^3$, such that $\underline{F}(r) \neq \underline{c}$, there has to be at least one $F_{j^*}(r)$, $j^* \in \{1,2,3\}$ with

$$\frac{dF_{j^*}(r)}{dr} \neq 0 \quad . \tag{1}$$

In addition, the gradient of $r = r(\underline{x})$ is for $r \neq 0$ a point on the unit sphere: $\quad \underline{\nabla}_{\underline{x}}\, r = \frac{1}{r}\underline{x}$
So the $\frac{x_j}{r}$ are simply products of sinusoidal functions of the angles ϑ, φ. Because of (1) we get

$$\frac{x_{j^*}}{r} \cdot \frac{dF_{j^*}(r)}{dr} \not\equiv 0 \quad \text{and} \quad div\,\underline{F} := \underline{\nabla}_{\underline{x}}^T \cdot \underline{F}(r) = \sum_{j=1}^{3} \frac{x_j}{r} \cdot \frac{dF_j(r)}{dr} \not\equiv 0 \ ,$$

what contradicts our assumption $div\,\underline{F} = 0$. □

We formulate the conditions for the decomposition of a solenoidal vector field $\underline{u}$ on Ω_σ into a toroidal field $\underline{\mathfrak{t}}$ and a poloidal field $\underline{\mathfrak{p}}$, $\underline{u}(\underline{x}) = \underline{\mathfrak{t}}(\underline{x}) + \underline{\mathfrak{p}}(\underline{x})$ in what follows. Let the fields $\underline{\mathfrak{t}}$ and $\underline{\mathfrak{p}}$ be explained by:

$$\underline{\mathfrak{t}} := curl(\psi\,\underline{x}) = grad\,\psi \wedge \underline{x} \quad \text{and} \quad \underline{\mathfrak{p}} := curl(curl(\chi\,\underline{x})) \ , \tag{2}$$

where we restrict $\underline{\mathfrak{t}}$ and $\underline{\mathfrak{p}}$ by the conditions:

$$div\,\underline{\mathfrak{t}} = \underline{x}^T\underline{\mathfrak{t}} = 0 \quad \text{and} \quad div\,\underline{\mathfrak{p}} = \underline{x}^T curl(\underline{\mathfrak{p}}) = 0 \ . \tag{3}$$

We denote by ω_r for $r \in [\sigma, 1]$, $0 < \sigma$, the spherical surface: $\omega_r := \{\underline{x} \in \mathbb{R}^3 : \ ||\underline{x}|| = r\}$. The conditions for the unique determination of $\underline{\mathfrak{t}}$ and $\underline{\mathfrak{p}}$ are stated by vanishing mean values:

$$\frac{1}{|\omega_r|}\int\limits_{\omega_r} \underline{x}^T \underline{u}\, d\omega_r = 0 \quad \text{and}\ \forall\, f \in \mathbb{C}[\sigma,1] : \quad 0 = \frac{1}{|\omega_r|}\int\limits_{\omega_r} f \cdot \psi(\underline{x}) d\omega_r = \frac{1}{|\omega_r|}\int\limits_{\omega_r} f \cdot \chi(\underline{x}) d\omega_r \ , \tag{4}$$

where one can understand the equations of (4) as conclusions of the LEMMA 1 given above.
<u>THEOREM 1:</u> Let $\underline{u} \in \underline{\mathbb{W}}_2^2(\Omega_\sigma)$ be a solenoidal vector field defined on Ω_σ written in polar coordinates, which fulfills the first equations of (4). Then there exist vector fields $\underline{\mathfrak{t}}$ and $\underline{\mathfrak{p}}$ what are unique determined by (2),(3) and (4), with $\underline{u} = \underline{\mathfrak{t}} + \underline{\mathfrak{p}}$ (at least in the sense of $\underline{\mathbb{L}}_2(\Omega_\sigma)$), where the regularity of $\underline{\mathfrak{t}}$ and $\underline{\mathfrak{p}}$ is inferred by that of ψ and χ.
<u>REMARK:</u> The regularity of ψ and χ is inferred by the regularity of $\underline{x}^T curl(\underline{u})$ and $\underline{x}^T\underline{u}$. There one has to study subspaces of $\underline{\mathbb{L}}_2(\Omega_\sigma)$ as weighted Sobolev spaces $\underline{\mathbb{W}}_{2,r}^m(\sigma, 1; D(\mathbf{B}^k)/\mathbb{R})$ with $m,\ k \in \mathbb{N}$.
Now we give a shortened proof of THEOREM 1.
PROOF: (i) In the first step we write down the explicit shape of of $\underline{\mathfrak{t}}$ and $\underline{\mathfrak{p}}$ in polar coordinates:

$$\underline{\mathfrak{t}}_{\mathfrak{s}} = (0, \frac{1}{\sin\vartheta}\frac{\partial\psi}{\partial\varphi}, -\frac{\partial\psi}{\partial\vartheta})_{\mathfrak{s}}^T \tag{5}$$

$$\underline{\mathfrak{p}}_{\mathfrak{s}} = \frac{1}{r}(\mathbf{B}\chi, \frac{\partial}{\partial\vartheta}\frac{\partial(r\chi)}{\partial r}, \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\frac{\partial(r\chi)}{\partial r})^T_{\mathfrak{s}} = (-r\cdot\triangle\chi, 0, 0)^T_{\mathfrak{s}} + (grad(\frac{\partial(r\chi)}{\partial r}))_{\mathfrak{s}} , \quad (6)$$

where one gets these relations by simple calculations.
For more regular $\underline{\mathfrak{t}}, \underline{\mathfrak{p}} \in \underline{\mathbb{C}}^2(\Omega_\sigma)$ the $curls$ of toroidal fields are poloidal fields and the $curls$ of poloidal fields are toroidal fields:

$$(curl\,\underline{\mathfrak{t}})_{\mathfrak{s}} = \tilde{\underline{\mathfrak{p}}}_{\mathfrak{s}} \quad \text{and} \quad (curl\,\underline{\mathfrak{p}})_{\mathfrak{s}} = \tilde{\underline{\mathfrak{t}}}_{\mathfrak{s}} . \quad (7)$$

Finally one gets the following conservation of the character for toroidal fields and poloidal fields, if one applies the $curl$ once more:

$$(curl\,curl\,\underline{\mathfrak{t}})_{\mathfrak{s}} = \breve{\underline{\mathfrak{t}}}_{\mathfrak{s}} \quad \text{and} \quad (curl\,curl\,\underline{\mathfrak{p}})_{\mathfrak{s}} = \breve{\underline{\mathfrak{p}}}_{\mathfrak{s}} . \quad (8)$$

We obtain these equations for $\underline{\mathfrak{t}}, \underline{\mathfrak{p}} \in \underline{\mathbb{W}}^2_2(\Omega_\sigma)$ by density arguments too.
(ii) In the second step we note orthogonality properties of toroidal and poloidal fields on spherical surfaces. Let us apply the surface spherical harmonics $\{Y_j\}_{j=0}^{\infty}$ in the notation of THEOREM B in the Appendix. Making use of expansions of ψ and χ in $\{Y_j\}_{j=0}^{\infty}$ one gets by the restrictions (4), that

$$\psi(\underline{x}) = \sum_{j=1}^{\infty}\hat{\psi}_j(r)Y_j(\vartheta,\varphi) \quad \text{and} \quad \chi(\underline{x}) = \sum_{j=1}^{\infty}\hat{\chi}_j(r)Y_j(\vartheta,\varphi) . \quad (9)$$

The we define by

$$\underline{\mathfrak{t}}^{(j)}_{\mathfrak{s}} := (curl(\hat{\psi}_j(r)Y_j(\vartheta,\varphi)\underline{x}))_{\mathfrak{s}} \quad \text{and} \quad \underline{\mathfrak{p}}^{(j)}_{\mathfrak{s}} := (curl(curl(\hat{\chi}_j(r)Y_j(\vartheta,\varphi)\underline{x})))_{\mathfrak{s}}\ \forall\, j \in \mathbb{N} . \quad (10)$$

Let us denote the systems of toroidal fields by $\{\underline{\mathfrak{t}}^{(j)}_{\mathfrak{s}}\}_{j=1}^{\infty}$ and the systems of poloidal fields by $\{\underline{\mathfrak{p}}^{(j)}_{\mathfrak{s}}\}_{j=1}^{\infty}$. One shows the following orthogonality properties in the sense of $\underline{\mathbb{L}}_2(\omega)$ similar to [5] p.623-623:

$$\forall\, j, k \in \mathbb{N}\,\text{with}\ j \neq k\ :$$
$$r^2(\int_\omega(\underline{\mathfrak{t}}^{(j)}_{\mathfrak{s}})^T\underline{\mathfrak{t}}^{(k)}_{\mathfrak{s}}d\omega = \int_\omega(\underline{\mathfrak{p}}^{(j)}_{\mathfrak{s}})^T\underline{\mathfrak{p}}^{(k)}_{\mathfrak{s}}d\omega = \int_\omega(\underline{\mathfrak{t}}^{(j)}_{\mathfrak{s}})^T\underline{\mathfrak{p}}^{(k)}_{\mathfrak{s}}d\omega = \int_\omega(\underline{\mathfrak{t}}^{(j)}_{\mathfrak{s}})^T\underline{\mathfrak{p}}^{(j)}_{\mathfrak{s}}d\omega) = 0 \quad (11)$$

where one uses gradients, curls and the divergence on surfaces.
(iii) The proof is finished by the unique solution of the equations for the determination of ψ and χ:

$$\mathbf{B}\psi = \underline{x}^T(curl\,\underline{u}) \quad \text{and} \quad \mathbf{B}\chi = \underline{x}^T\underline{u} \qquad \Box. \quad (12)$$

Now we introduce some details for the consideration of gradients of harmonic functions.
Following [9](cf. sect. 2.4.1) and [18](cf. sect. 6.4.5) we use, that the set of all harmonic functions on Ω_o respectively on Ω_σ is spanned by the systems of functions
$H_o := \{r^l\cdot C_l^k(\vartheta,\varphi)\}_{k=0,\,l\in\mathbb{N}_0}^l \cup \{r^l\cdot S_l^k(\vartheta,\varphi)\}_{k=1,\,l\in\mathbb{N}}^l$ and by $H_\sigma :=$
$\{\{r^l\cdot C_l^k(\vartheta,\varphi)\}_{k=0}^l \cup \{r^{-(l+1)}\cdot C_l^k(\vartheta,\varphi)\}_{k=0}^l\}_{l\in\mathbb{N}_0} \cup \{\{r^l\cdot S_l^k(\vartheta,\varphi)\}_{k=1}^l \cup \{r^{-(l+1)}\cdot S_l^k(\vartheta,\varphi)\}_{k=1}^l\}_{l\in\mathbb{N}}$,
where the C_l^k and S_l^k are surface spherical harmonics on the unit sphere ω (cf. Appendix, DEFINITION A).
<u>LEMMA 2:</u> Let $p \in H_\sigma$ be a harmonic function on Ω_σ, $\sigma : 0 \leq \sigma < 1$. Then are the gradients $\underline{\nabla}\,p$ of $p \in H_\sigma$ poloidal fields $\underline{\mathfrak{p}}$.
PROOF: The statement of the Lemma is a simple conclusion of the application of (6) on $\chi(\underline{x}) := p(\underline{x})$, where one uses: $0 = \triangle\chi = \triangle p\ \forall\, p \in H_\sigma$. $\Box$

We are going to apply our results toroidal and poloidal fields in the next section on the eigenvalue problem for the Stokes operator in the unit ball.

4. The eigenvalue problem for the Stokes operator in the unit ball

We regard the eigenvalue problem of the Stokes Operator $\mathbf{A}_o$ formulated in section 1 as the PROBLEMS 1 for $[\underline{u}, p] \in \underline{\mathbb{C}}^2(\Omega_o) \times \mathbb{C}^1(\Omega_o)$ respectively as the PROBLEMS 2 for $\underline{u} \in \underline{\mathbb{S}}^2(\Omega_o)$.
Let the Bessel functions of first kind be denoted by

$$J_{l+\frac{1}{2}}(t) \ , \ J_{-(l+\frac{1}{2})}(t) \qquad \forall\, l \in \mathbb{N}_o \, . \tag{13}$$

The Bessel function $J_{l+\frac{1}{2}}(t)$ has infinitely many positive zeros $\{\mu^j_{l+\frac{1}{2}}\}_{j=1}^{\infty}$, with $0 < \mu^1_{l+\frac{1}{2}} < \mu^2_{l+\frac{1}{2}}, < \ldots$ and $\mu^j_{l+\frac{1}{2}} \to \infty$ for $j \to \infty$.
In what follows we will use the normed surface spherical harmonics (cf. Appendix THEOREM B:)) to define the following toroidal fields for $k = 0, \ldots, l$ and $k = 0, \ldots, l$:

$$\underline{\mathfrak{t}}_{c,\mathfrak{s}}^{(j),l,k} := \,(curl(\frac{J_{l+\frac{1}{2}}(\mu^j_{l+\frac{1}{2}} r)}{\sqrt{r}} \tilde{C}^k_l\, \underline{x}))_{\mathfrak{s}} \quad \text{and} \quad \underline{\mathfrak{t}}_{s,\mathfrak{s}}^{(j),l,k} := \,(curl(\frac{J_{l+\frac{1}{2}}(\mu^j_{l+\frac{1}{2}} r)}{\sqrt{r}} \tilde{S}^k_l\, \underline{x}))_{\mathfrak{s}} \,\forall\, j,\, l \in \mathbb{N} \tag{14}$$

For the definition of the poloidal fields we will use the following scalar functions χ: $\forall\, j,\, l \in \mathbb{N}$

$$\underline{\mathfrak{p}}_{c,\mathfrak{s}}^{(j),l,k} := \,(curl(curl(\chi^{(j),l,k}_c \underline{x})))_{\mathfrak{s}} \ , \ \chi^{(j),l,k}_c := \,(\frac{J_{l+\frac{1}{2}}(\mu^j_{l+\frac{3}{2}} r)}{\sqrt{r}} - \frac{\mu^j_{l+\frac{3}{2}} J_{l-\frac{1}{2}}(\mu^j_{l+\frac{3}{2}}) r^l}{2l+1}) \tilde{C}^k_l \quad , \quad k = 0, \ldots, l$$

$$\underline{\mathfrak{p}}_{s,\mathfrak{s}}^{(j),l,k} := \,(curl(curl(\chi^{(j),l,k}_s \underline{x})))_{\mathfrak{s}} \ , \ \chi^{(j),l,k}_s := \,(\frac{J_{l+\frac{1}{2}}(\mu^j_{l+\frac{3}{2}} r)}{\sqrt{r}} - \frac{\mu^j_{l+\frac{3}{2}} J_{l-\frac{1}{2}}(\mu^j_{l+\frac{3}{2}}) r^l}{2l+1}) \tilde{S}^k_l \quad , \quad k = 1, \ldots, l \tag{15}$$

We collect the toroidal fields and the poloidal fields in the following system of solenodal functions:

$$\{\{\underline{\mathfrak{p}}_{s,\mathfrak{s}}^{(j),l,l}, \ldots, \underline{\mathfrak{p}}_{c,\mathfrak{s}}^{(j),l,0}, \ldots, \underline{\mathfrak{p}}_{c,\mathfrak{s}}^{(j),l,l}\} \cup \{\underline{\mathfrak{t}}_{s,\mathfrak{s}}^{(j),l,l}, \ldots, \underline{\mathfrak{t}}_{c,\mathfrak{s}}^{(j),l,0}, \ldots, \underline{\mathfrak{t}}_{c,\mathfrak{s}}^{(j),l,l}\}\}_{j,l \in \mathbb{N}} \tag{16}$$

We are now able to formulate our main result, where we will use the Bessel functions (13) and the definitions (14) and (15):

<u>THEOREM 2:</u> The solenoidal fields (16) form a complete system of eigenfunctions for the Stokes operator $\mathbf{A}_o$ in the unit ball. The eigenvalues are the $\lambda_{(j),l,k} = (\mu^j_{l+\frac{1}{2}})^2$ with the multiplicity $(2l+1)$ in the class of toroidal fields and the values $\lambda_{(j),l,k} = (\mu^j_{l+\frac{3}{2}})^2$ with the multiplicity $(2l+1)$ in the class of poloidal fields.

PROOF: The fields of the system (16) are solenoidal by their definition. Additionally we get the orthogonality of the fields (16) through the application of THEOREM 1. From the definition of the $\underline{\mathfrak{t}}_{\mathfrak{s}}$ (5) and the $\underline{\mathfrak{p}}_{\mathfrak{s}}$ (6), it is apparent that one gets from the Stokes eigenvalue problem boundary value problems of ordinary differential equations for the functions $\hat{\psi}(r)$ and $\hat{\chi}(r)$ Fourier coefficients of ψ and χ developed in the surface spherical harmonics. Here are substantially applied the tools of the REMARK A of the Appendix.
The proof is finished by the solution and a discussion of the boundary value problems of ordinary differential equations for the $\hat{\psi}(r)$ and $\hat{\chi}(r)$. □

<u>REMARK:</u> The Stokes eigenfunctions for $\mathbf{A}_\sigma$ and the equations for the determination of the eigenvalues are given by the methods explained above in [19].

As a short resume it is to note that our tools and investigations provide complete systems of Stokes eigenfunctions for $\mathbf{A}_\sigma$. One can use these functions to construct Galerkin schemes for the numerical investigations of (turbulent) in balls and spherical annuli.

Appendix

The notation of the surface spherical harmonics will be explained in what follows.
We introduce the associated Legendre functions for $l \in \mathbb{N}_0$, and $k \in \{0, 1, \ldots, l\}$ by:

$$P_l^k(t) := \frac{(-1)^{l+k}}{2^l l!}(1-t^2)^{\frac{k}{2}} \frac{d^{l+k}}{dt^{l+k}}(1-t^2)^l,\ t \in [-1,1]$$

<u>DEFINITION A:</u> We will denote by:

$$\begin{aligned} C_l^k(\vartheta,\varphi) &:= \cos(k\varphi)P_l^k(\cos\vartheta),\ l \in \mathbb{N}_0\,, k \in \{0,1,\ldots,l\} \qquad \text{and} \\ S_l^k(\vartheta,\varphi) &:= \sin(k\varphi)P_l^k(\cos\vartheta),\ l \in \mathbb{N}\,, k \in \{1,2,\ldots,l\}\ , \end{aligned}$$

the surface spherical harmonics of degree $l \in \mathbb{N}_0$ on the unit sphere ω (, for the arguments $\vartheta \in [0,\pi]$ and $\vartheta \in [0, 2\pi)$).
<u>DEFINITION B:</u> The normed systems of surface spherical harmonic is defined by:

$$\tilde{C}_l^k(.,.) := (\sqrt{2})^{-\delta_{k,0}}\sqrt{\frac{(l-k)!(2l+1)}{(l+k)!2\pi}}C_l^k(.,.) \qquad \text{and} \qquad \tilde{S}_l^k(.,.) := \sqrt{\frac{(l-k)!(2l+1)}{(l+k)!2\pi}}S_l^k(.,.)$$

Using the notations of section 2 DEFINITION 1 for the Laplace-Beltrami operator $\mathbf{B^o}$ and its Friedrichs extension $\mathbf{B} := \overline{\mathbf{B^o}}$ (also called as Laplace-Beltrami operator) we cite the following theorems:
<u>THEOREM A:</u> The Laplace-Beltrami operator $\mathbf{B^o}$ is essentially self-adjoint in $\mathbb{L}_2(\omega)$.
The associated bilinear form $\beta(Y,Z) := (\mathbf{B^o}Y,Z)_{\mathbb{L}_2(\omega)}\ \forall Y,Z \in D(\mathbf{B^o})$ is nonnegative, i.e. $\beta(Y,Y) \geq 0\,\forall Y \in D(\mathbf{B^o})$ and β is $\mathbb{W}_2^1(\omega)$-coercive.
The Friedrichs extension $\mathbf{B}$ of $\mathbf{B^o}$ is an operator with pure point spectrum.
<u>THEOREM B:</u> The eigenvalues of $\mathbf{B}$ are the $\nu_{(l)} := l(l+1)$ and the multiplicity of every $\nu_{(l)}$ is $N(\nu_{(l)}) := 2l+1\,\forall\, l \in \mathbb{N}_0$. The system of eigenfunctions of $\mathbf{B}$:
$\{\tilde{Y}_l^l, \ldots, \tilde{Y}_l^1, Y_l^0, Y_l^1, \ldots, Y_l^l\}_{l=0}^{\infty} := \{\tilde{S}_l^l, \ldots, \tilde{S}_l^1, \tilde{C}_l^0, \tilde{C}_l^1, \ldots, \tilde{C}_l^l\}_{l=0}^{\infty}$
is a complete orthonormal system in $\mathbb{L}_2(\omega)$.
<u>THEOREM C:</u> The eigenvalues of $\mathbf{B}$ can be ordered by their (absolute) values, taking into account their multiplicities. If the $\{\nu_j\}_{j=0}^{\infty}$ are the ordered eigenvalues and the $\{Y_j\}_{j=0}^{\infty}$ the correspondent orthonormal eigenfunctions, then the system $\{Y_j\}_{j=0}^{\infty}$ is complete in $\mathbb{L}_2(\omega)$ and $D(\mathbf{B}) = \{Y \in \mathbb{L}_2(\omega_3) : \sum_{j=0}^{\infty}(1+\nu_j^2)|(Y,Y_j)_{\mathbb{L}_2(\omega)}|^2 < \infty\}$ with $\mathbf{B}Y = \sum_{j=1}^{\infty} \nu_j(Y,Y_j)_{\mathbb{L}_2(\omega)}Y_j$, since $\nu_0 = 0$.
<u>REMARK A:</u> We will complete the Appendix with a collection of tools for the proof of the THEOREMS 1 and 2. One shows by a simple calculation that the Laplace-Beltrami operator $\mathbf{B}$ and the scalar Laplacian commute for all functions $f \in \mathbb{C}^4(\overline{\Omega}_\sigma)$ at least in all inner points $\underline{x}$ of $\overline{\Omega}_\sigma$ (in $\mathcal{C}^4(\Omega_\sigma)$) for $\sigma : 0 \leq \sigma < 1$.
We regard now an arbitrary vector field of the form: $\underline{\mathfrak{y}} = \eta(\underline{x})\underline{x}$. We will derive simple relations for the application of the 3D-Laplacian on toroidal and poloidal vector fields.
We will omit the index for the coordinate system in what follows, since the definitions of the differential operators $grad\,, div$ and $curl$ are independent of the chosen orthonormal right-handed coordinate system.
Let us start with: $curl\,\underline{\mathfrak{y}}$. Here we got by the use of standard relations of the $\underline{\nabla}-$calculus:

$$\begin{aligned} curl\,(\eta(\underline{x})\underline{x}) &= (grad\,\eta(\underline{x})) \wedge \underline{x} \quad \text{and} \\ curl\,(curl\,(\eta(\underline{x})\underline{x})) &= grad\,(div\,(\eta(\underline{x})\underline{x})) - \triangle\,(\eta(\underline{x})\underline{x}) \end{aligned}$$

A simple calculation gives for the second term on the right hand side:

$$-\triangle\,(\eta(\underline{x})\underline{x}) = (-\triangle\,\eta)\underline{x} - 2grad\,\eta$$

The repeated application of the $curl$ operator yields the following equations:

$$\triangle\,(curl\,(\eta\underline{x})) = curl\,([\triangle\,\eta]\underline{x}))$$

and

$$\triangle\,(curl\,curl\,(\eta\underline{x})) = curl\,curl\,([\triangle\,\eta]\underline{x}))$$

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