On the formulas for $\pi(x)$ and $\psi(x)$ of Riemann and von-Mangoldt

Matthias Kunik *
Institute for Analysis and Numerics, Otto-von-Guericke-Universität
Postfach 4120 . D-39106 Magdeburg, Germany

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*matthias.kunik@mathematik.uni-magdeburg.de

Abstract

Using the Mellin transform and the complex exponential integral we derive various representation formulas for the factors of the entire functions in Hadamards product theorem. The application of these results on Riemann's zeta function leads to a derivation of Riemann's prime number formula for $\pi(x)$. We also derive explicit formulas with the nontrivial zeros of the ζ function for regularizations of von Mangoldt's function $\psi(x)$. The regularizations are based on cardinal B-splines and Gaussian integration kernels, which are related by the Central Limit Theorem. These results will then be generalized to a windowed Mellin or Fourier transform with a Gaussian window function.

1 Introduction

The purpose of this article is threefold.

First we give a detailed introduction to the theory of the complex exponential integral from the analytical as well as numerical point of view and derive some related Mellin and Fourier transforms which can be used for the study of the entire functions in Hadamards product theorem. Parts of these results are widely spread in literature or can only be found without proofs in mathematical tables of higher functions, see for example the handbook of Abramowitz and Stegun [1]. One of the various representations of the exponential integral is

$$\mathrm{Ei}(z) := \gamma + \log z + \mathrm{Ei}_0(z) \tag{1.1}$$

with Euler's constant $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right)$ and the entire function

$$\mathrm{Ei}_0(z) := \sum_{k=1}^{\infty}\frac{z^k}{k\cdot k!} = \int_0^z \frac{e^t-1}{t}\,dt = \int_0^1 \frac{e^{uz}-1}{u}\,du\,. \tag{1.2}$$

In contrast to the entire function Ei_0, the functions Ei and log are only defined on the cut plane $D_{log} := \{\, z\in\mathbb{C} \mid z \notin (-\infty,0]\,\} = \mathbb{C}\setminus(-\infty,0]\,.$

The second purpose is the application of the theory developed here on the Riemann ζ function and the primes. The exponential integral rather than the logarithmic integral plays a key role for the correct formulation of Riemann's famous formulas obtained in [12] for the number $\pi(x)$ of primes less than a given number $x > 0$. It was C.F. Gauss who has observed from extensive calculations of prime tables that the logarithmic integral $\mathrm{Li}(x) := \mathrm{Ei}(\log x)$ is a "very good" approximation for the number of primes less than x.

And indeed, it was shown in 1896 by Hadamard and independently by de la Vallée-Poussin that for appropriate constant $c > 0$

$$\mathrm{Li}(x) = \pi(x) + O(xe^{-c\sqrt{\log x}}) \qquad \text{for } x \to \infty\,, \tag{1.3}$$

a result which is much stronger than the prime number theorem $\pi(x) \sim \dfrac{x}{\log x}$ for $x \to \infty$.

Riemann did not give a proof of the prime number theorem, but from the study of the complex ζ function and their zeros he obtained 1859 in [12] the following explicit formula for $\pi_*(x) := \sum\limits_{k=1}^{\infty} \frac{1}{k}\pi(\sqrt[k]{x})$, which is valid at each point $x > 1$ of continuity

$$\pi_*(x) = \mathrm{Li}(x) - \lim_{T\to\infty} \sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1 \\ |Im\,\rho|\le T}} \mathrm{Ei}(\rho\log x) + \int\limits_x^{\infty} \frac{dt}{t(t^2-1)\log t} - \log 2\,. \tag{1.4}$$

Due to an incorrect use of the analytic continuation principle this formula is usually formulated in a wrong way, where the expressions $\mathrm{Ei}(\rho\log x)$ are replaced by $\mathrm{Li}(x^\rho)$, while numerical calculations of $\pi_*(x)$ are performed due to the correct expression in (1.4) or due to correct approximations of (1.4).

From the formula for $\pi_*(x)$, which is denoted by $f(x)$ in Riemann's article [12], one can also obtain explicit formulas for $\pi(x)$ by using the Möbius inversion formula

$$\pi(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k}\pi_*(\sqrt[k]{x})\,, \tag{1.5}$$

see Zagier's article [15] for a short and interesting overview.

There are two major ingredients for the derivation of explicit prime number formulas involving the zeros of the zeta function. The first one is the relation of the ζ function to number theory via Euler's product decomposition valid for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p\,prime}^{\infty} \frac{1}{1-p^{-s}} = \exp(\,s\int\limits_1^{\infty} \frac{\pi_*(x)}{x^{s+1}}\,dx\,)\,. \tag{1.6}$$

The second one is the following analytical product decomposition of the ζ function, which was proved and generalized by Hadamard,

$$\zeta(s) = \frac{1}{s-1}\,\frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}\cdot\frac{1}{2}\lim_{T\to\infty}\prod_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|\le T}}\left(1-\frac{s}{\rho}\right). \tag{1.7}$$

According to Riemann's approach in [12], the desired explicit representation formulas for the primes result from a comparison of both product decompositions by using appropriate Mellin- or Fourier inversion formulas.

Riemann's formulas for $\pi_*(x)$ or $\pi(x)$ are often neglected nowadays, because they turned out to be equivalent to the simplier formula of von Mangoldt (for its derivation see the textbooks of Edwards [6] and Ingham [7]), also valid at each point $x > 1$ of continuity, namely

$$\psi(x) := \sum_{p^k\le x}\log p = x - \lim_{T\to\infty}\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|\le T}}\frac{x^\rho}{\rho} + \frac{1}{2}\log\frac{x^2}{x^2-1} - \log(2\pi)\,, \tag{1.8}$$

where the sum is performed with respect to all prime powers $p^k \ge 2$ with an exponent $k \ge 1$. Von Mangoldt's derivation of Riemann's formula for $\pi_*(x)$ can be found in the textbook of Edwards [6], its starting point is the following modification of (1.8) which is valid for $r > 0$, $r \ne 1$ and for almost all $x > 1$,

$$\int_0^x x^{-r}\,d\psi(x) = \frac{x^{1-r}}{1-r} - \lim_{T\to\infty}\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|\le T}}\frac{x^{\rho-r}}{\rho-r} - \sum_{n=1}^{\infty}\frac{x^{-2n-r}}{-2n-r} - \frac{\zeta'(r)}{\zeta(r)}\,. \tag{1.9}$$

However, this formula is also difficult to prove. We will present an interesting alternative derivation of (1.4) by reducing it to von Mangoldt's original formula (1.8). For this purpose we prove a representation which is generally useful for the study of Hadamards product decomposition of certain entire functions with exponential growth like $e^s(s-1)\,\zeta(s)$,

$$(1-\frac{s}{\rho})\,\exp(\frac{s}{\rho}) = \exp\left(-s^2\int_1^\infty\frac{\varphi_\rho(x)\cdot\log x - \frac{x^\rho}{\rho}}{x^{s+1}}\,dx\right) \tag{1.10}$$

with the following abbreviation for $x > 1$

$$\varphi_\rho(x) := \gamma + \log(-\rho) + \log(\log x) + \mathrm{Ei}_0(\rho\log x)\,. \tag{1.11}$$

Formula (1.10) is valid for $\mathrm{Re}(\rho) < \mathrm{Re}(s)$ and $\rho \in \mathbb{C} \setminus [0, \infty)$, where the integral exists in the Lebesgue sense.

Moreover, beside (1.7) and (1.10) we use the following representation, which is valid for $\mathrm{Re}(s) > 1$

$$e^{s}\,(s-1)\,\zeta(s) = \exp\left(s^2 \int\limits_1^{\infty} \frac{(\pi_*(x) - \mathrm{Li}(x)) \cdot \log x - (\psi(x) - x)}{x^{s+1}}\, dx \right) . \tag{1.12}$$

We will also investigate explicit formulas for regularizations of von Mangoldt's function $\psi(x)$ with cardinal B-splines and Gaussian kernels in terms of the nontrivial zeros of the zeta function which are certain counterparts of the von Mangoldt formula (1.8). Cardinal B-splines and Gaussian kernels are important tools in the wavelet-Fourier theory, see the textbook of Chui [4]. More precisely, we consider regularizations of the Lebesgue integrable function $\eta : (0, \infty) \to \mathbb{R}$ with

$$\eta(x) := \psi(x) - \chi_{(1,\infty)}(x) \cdot \left(x + \frac{1}{2} \log \frac{x^2}{x^2 - 1} - \log(2\pi) \right) , \tag{1.13}$$

where $\chi_{(1,\infty)}$ is the characteristic function of the interval $(1, \infty)$. We show that the following Gaussian mollifier of the function η

$$(G_\delta\, \eta)(x) := \frac{1}{\sqrt{2\pi\delta}} \int\limits_{-\infty}^{+\infty} \exp\left(-\frac{u^2}{2\,\delta} \right) \eta\,(x e^{u})\; du \tag{1.14}$$

results in a very natural way from the cardinal B-spline mollifiers and the Central Limit Theorem. Then we derive the following explicit formula for the Gaussian mollifier

$$(G_\delta\, \eta)(x) = - \sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1}} \frac{x^{\rho}}{\rho} \exp\left(\frac{\delta}{2}\, \rho^2 \right) \Phi\left(\frac{\rho\,\delta + \log x}{\sqrt{\delta}} \right) , \tag{1.15}$$

where Φ is the complex Gaussian error function

$$\Phi(z) := \int\limits_0^{\infty} \frac{e^{-\frac{(z-u)^2}{2}}}{\sqrt{2\pi}}\, du\,,\ \ z \in \mathbb{C}\,. \tag{1.16}$$

A further important tool which is used for (1.15) is a formulation of the Mellin or Fourier inversion formula based on the Lebesgue integration theory. We will derive a very efficient asymptotic relation for (1.15). The representation (1.15) will also be generalized to some kind of windowed Mellin or Fourier transform with the Gaussian window function in (1.14).

2 Representation formulas for the zeta function and their logarithmic derivative via the Euler product formula

Riemann's zeta function is given for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \,. \tag{2.1}$$

Here $n^s = e^{s \ln n}$, is already defined by the real logarithm ln. The representation (2.1) converges absolutely for $\mathrm{Re}(s) > 1$ due to $|n^s| = n^{Re(s)}$ and

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{Re(s)}} < 1 + \int_1^{\infty} \frac{1}{\xi^{Re(s)}} \, d\xi.$$

Theorem (2.1)

There holds the following Euler product formula for $\mathrm{Re}(s) > 1$

$$\zeta(s) = \prod_{p\,prime}^{\infty} \frac{1}{1 - p^{-s}} \,. \tag{2.2}$$

Proof: We use that every integer number $m \geq 2$ has a unique product decomposition into prime factors,

$$m = \prod_{k=1}^{\infty} p_k^{\alpha_k(m)}$$

where p_k is the k-th prime number, $p_1 = 2$, $p_2 = 3$, ... and so on, and $\alpha_k(m) \geq 0$ the integer exponent of m with respect to p_k. Then

$$\begin{aligned} \Pi_n(s) &:= \prod_{k=1}^{n} \frac{1}{1 - p_k^{-s}} \\ &= \prod_{k=1}^{n} \sum_{j=0}^{\infty} \frac{1}{p_k^{js}} \\ &= \sum_{m=1}^{n} \frac{1}{m^s} + \sum_{m \in S_n} \frac{1}{m^s} \end{aligned}$$

with the set

$S_n := \{m \in \mathbb{N} \,|\, m > n \;\&\; \text{m has only prime factors } p_k \text{ with index } k \leq n\}\,.$

Due to

$$\sum_{m\in S_n} \left|\frac{1}{m^s}\right| \leq \sum_{m>n} \frac{1}{m^{Re(s)}} \longrightarrow 0 \;\text{ for } n \to \infty$$

the product $\Pi_n(s)$ converges to $\zeta(s)$ for $n \to \infty$ and $\mathrm{Re}(s) > 1$. ∎

Theorem (2.2)

For any integer number $n \geq 1$ we define the *von Mangoldt function*

$$\Lambda(n) := \begin{cases} \ln p & , \quad \text{for } n = p^m,\; m \geq 1,\; p \text{ prime} \\ 0 & , \quad \text{otherwise}\,, \end{cases}$$

and for $x \geq 1$ the functions

$$\psi(x) := \sum_{n\leq x} \Lambda(n)\,, \quad \pi(x) := \sum_{p\leq x,\; p\; prime} 1\,, \quad \pi_*(x) := \sum_{1<n\leq x} \frac{\Lambda(n)}{\ln n} = \sum_{n=1}^{\infty} \frac{\pi(\sqrt[n]{x})}{n}\,,$$

where $\pi(x)$ is the number of primes $\leq x$. Then we obtain for $\mathrm{Re}(s) > 1$

$$\zeta(s) = \exp\Big(\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-s}\Big) = \exp\Big(s \int_1^{\infty} \frac{\pi_*(x)}{x^{s+1}}\, dx \Big) \tag{2.3}$$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^{\infty} \Lambda(n)\, n^{-s} = s \int_1^{\infty} \frac{\psi(x)}{x^{s+1}}\, dx\,. \tag{2.4}$$

Proof: The integrals in (2.3) and (2.4) are absolutly convergent for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. For $\mathrm{Re}(s) \geq 1 + \varepsilon > 1$ they converge uniformly due to

$$\pi_*(x) := \sum_{n\leq \frac{\ln x}{\ln 2}} \frac{\pi(\sqrt[n]{x})}{n} \leq x \frac{\ln x}{\ln 2}, \qquad \psi(x) := \sum_{n\leq x} \Lambda(n) \leq x \ln x.$$

We take the logarithm of the Euler product for $\zeta(s)$, restrict to real $s > 1$

and obtain

$$\begin{aligned}
\ln \zeta(s) &= -\sum_{k=1}^{\infty} \ln\left(1 - p_k^{-s}\right) \\
&= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{1}{p_k}\right)^{ms} = \sum_{k,m=1}^{\infty} \frac{1}{m}\left(\frac{1}{p_k^m}\right)^{s} \\
&= \sum_{k,m=1}^{\infty} \frac{\Lambda(p_k^m)}{\ln(p_k^m)}\left(\frac{1}{p_k^m}\right)^{s} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-s}
\end{aligned}$$

due to the unique decomposition of the integer numbers into prime factors. The analytical continuation principle thus gives the proof of the first part in (2.3) for $\mathrm{Re}(s) > 1$. The second part follows immediately from

$$\begin{aligned}
s\int_1^{\infty} \frac{\pi_*(x)}{x^{s+1}}\,dx &= s\sum_{k=1}^{\infty}\int_k^{k+1} \frac{\pi_*(k)}{x^{s+1}}\,dx = s\sum_{k=1}^{\infty} \pi_*(k)\left[\frac{x^{-s}}{(-s)}\right]_{x=k}^{x=k+1} \\
&= \sum_{k=1}^{\infty} \pi_*(k)\left(\frac{1}{k^s} - \frac{1}{(k+1)^s}\right) \\
&= \sum_{k=2}^{\infty} \frac{\pi_*(k)}{k^s} - \sum_{k=1}^{\infty} \frac{\pi_*(k)}{(k+1)^s} \quad (\text{since } \pi_*(1) = 0) \\
&= \sum_{k=1}^{\infty} \frac{\pi_*(k+1) - \pi_*(k)}{(k+1)^s} = \sum_{k=1}^{\infty} \frac{\Lambda(k+1)/\ln(k+1)}{(k+1)^s} \\
&= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-s}\,.
\end{aligned}$$

However, in the sequel we will also use Stieltjes integrals. Then the calculation above reduces to

$$\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\ln n} n^{-s} = \int_1^{\infty} \frac{d\pi_*(x)}{x^s} = s\int_1^{\infty} \frac{\pi_*(x)}{x^{s+1}}\,dx\,.$$

Now we obtain (2.4) by taking the logarithmic derivative of the first part in (2.3) and then by performing partial summation. ∎

For our study of the Riemann ζ function we need some properties of the complex Γ function, which are proven in an elegant way in the textbook of

Andrews, Askey and Roy [2]:

Theorem (2.3) For all complex numbers $s \neq 0, -1, -2, ...$, the Γ function is defined by

$$\Gamma(s) = \lim_{n\to\infty} \frac{n!\, n^s}{s(s+1)(s+2)...(s+n)} \,.$$

The Γ function has the following properties:

(a) $\Gamma(n+1) = n!$ and $\Gamma(s+1) = s\Gamma(s)$ for $n \in \mathbb{N}_0$, $s \neq 0, -1, -2, \dots$.

(b) With Euler's constant $\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) = 0.5772156...$ we have

$$\frac{1}{s\,\Gamma(s)} = \frac{1}{\Gamma(s+1)} = e^{\gamma s} \prod_{k=1}^{\infty} \left(1 + \frac{s}{k}\right) e^{-s/k} = \prod_{k=1}^{\infty} \frac{1+\frac{s}{k}}{\left(1+\frac{1}{k}\right)^s} \,.$$

(c) If $s \neq 0, -1, -2, ...$, then $-\frac{\Gamma'(s)}{\Gamma(s)} = \gamma + \frac{1}{s} + \sum_{k=1}^{\infty} \left(\frac{1}{s+k} - \frac{1}{k} \right)$, and especially $\Gamma'(1) = -\gamma$.

(d) There holds Euler's reflection formula

$$\frac{1}{\Gamma(1+s)\Gamma(1-s)} = \frac{\sin(\pi s)}{\pi s} \,.$$

(e) For $\mathrm{Re}(s) > 0$

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t}\, dt \,.$$

(f) There holds Legendre's duplication formula

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\!\left(\frac{s+1}{2}\right) \Gamma\!\left(\frac{s}{2}\right) .$$

The following theorem shows that the Riemann ζ function is everywhere defined in the complex plane, except at $s = 1$, where $\zeta(s)$ has a simple pole. We make essentially use of Theorem (2.3) in order to sketch its proof.

Theorem (2.4) Analytic continuation of $\zeta(s)$, functional equation

(a) *For* $\mathrm{Re}(s) > -1$ *the* ζ *function is analytically continued by the expression*

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s \lim_{N\to\infty} \int_1^N \frac{x - \lfloor x \rfloor - \frac{1}{2}}{x^{s+1}}\, dx\,.$$

Thus the ζ *function has a simple pole at* $s = 1$ *with residue* 1.

(b) *For* $-1 < \mathrm{Re}(s) < 0$ *the* ζ *function is given by*

$$\zeta(s) = s \lim_{N\to\infty} \int_0^N \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{s+1}}\, dx\,.$$

(c) *For* $\mathrm{Re}(s) < 0$ *the continuation of the* ζ *function is given by*

$$\zeta(s) = 2(2\pi)^{s-1} \sin\frac{\pi s}{2}\, \Gamma(1-s)\, \zeta(1-s)\,.$$

Since the Γ*-function has no zeros and only simple poles at* $s = 0, -1, -2, \ldots$*, we conclude that the zeta function has the so called* **trivial zeros** *at* $s = -2, -4, -6, ..$

$$\zeta(-2n) = 0 \qquad \forall n \in \mathbb{N}\,.$$

(d) *If we define the analytic function* ξ *on the entire complex plane by*

$$\xi(s) := \frac{s(s-1)}{2}\, \pi^{-\frac{s}{2}}\, \Gamma(\frac{s}{2})\, \zeta(s)\,,$$

then the functional equation for the Riemann zeta function can be written for all $s \in \mathbb{C}$ *in the more symmetric form* $\xi(s) = \xi(1-s)$.

Sketch of a proof: Part (a) results from Euler's summation formula, part (b) from (a). If we use the Fourier expansion

$$\lfloor x \rfloor - x + \frac{1}{2} = \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{\pi k}$$

and substitute this in (b), then termwise integration can be justified, which leads to (c) by analytic continuation, see Titchmarch [14]. The part (d) results from (c), as for the derivation of (c) from (b), by using Theorem (2.3) for the Γ-function. ∎

The following theorem can be proven thanks to a famous trick of Hadamard which shows that $\zeta(s)$ has no zero on the axis $\mathrm{Re}(s) = 1$.

Theorem (2.5) Zeros of $\zeta(s)$ and $\xi(s)$ and special values at $t = 0$

(a) Beside the trivial zeros at $s = -2, -4, -6, \dots$, $\zeta(s)$ *has only zeros in the critical strip* $0 < \mathrm{Re}(s) < 1$ *. There holds* $\zeta(0) = -\frac{1}{2}$*. Here* $\zeta(s)$ *is defined for* $s \in \mathbb{C} \setminus \{1\}$*.*

(b) The function $\xi(s)$ *defined in Theorem (2.3) has only zeros in the critical strip* $0 < \mathrm{Re}(s) < 1$ *. It holds* $\xi(0) = \frac{1}{2}$*, and* $\xi(s)$ *is defined for all* $s \in \mathbb{C}$*.*

Proof: For $\mathrm{Re}(s) > 1$ we have $\zeta(s) \neq 0$ due to Theorem (2.2), formula (2.3). Recall that the Γ-function has no zeros and only simple poles at $s = 0, -1, -2, \dots$. Thus we conclude from Theorem (2.4)(c) that $\zeta(s)$ has no other zeros for $\mathrm{Re}(s) < 0$ beside the trivial zeros at $s = -2, -4, -6, \dots$, which result from the $\sin \frac{\pi s}{2}$-term. From Theorem (2.4)(a) we obtain that

$$\lim_{s\to 0} \sin \frac{\pi s}{2} \zeta(1-s) = -\frac{\pi}{2}\,, \quad \lim_{s \to 1} (1-s)\zeta(s) = 1\,.$$

On the other hand, $\Gamma(1) = 1$ and $\frac{s}{2}\Gamma(\frac{s}{2}) = \Gamma(1 + \frac{s}{2})$ by Theorem (2.3)(a). Thus we obtain from (c) that $\zeta(0) = -\frac{1}{2}$ and $\xi(0) = \frac{1}{2}$.

If we can show that $\zeta(s) \neq 1$ for $\mathrm{Re}(s) = 1$, then Theorem (2.5) follows from Theorem (2.3) and Theorem (2.4)(c).

Here is Hadamard's famous trick in order to establish this result:

We assume that $\zeta(1 + it) = 0$ for some fixed $t \in \mathbb{R} \setminus \{0\}$. Then we define the auxiliary function $h : (1, \infty) \to \mathbb{R}$ by $h(x) := |\zeta(x)^3\, \zeta(x + it)^4\, \zeta(x + 2it)|$. We use formula $(2.3)_1$ in Theorem (2.2) and obtain for any $x > 1$

$$\begin{aligned}
h(x) &= \left|\zeta(x)^3 \zeta(x + it)^4 \zeta(x + 2it)\right| \\
&= \left| \exp\Big(\sum_{n=2}^{\infty} \underbrace{\frac{\Lambda(n)}{\ln n}}_{=c_n \geq 0} \{3n^{-x} + 4n^{-x-it} + n^{-x-2it}\}\Big)\right| \\
&= \exp\left(\sum_{n\geq 2} \frac{c_n}{n^x} \left(3 + 4\cos(t \ln n) + \cos(2t \ln n)\right)\right) \\
&= \exp\left(\sum_{n \geq 2} \frac{2c_n}{n^x} (1 + \cos(t \ln n))^2 \right) \geq 1\,.
\end{aligned}$$

But due to our assumption the limit $\lim_{x\to 1} h(x)$ should be zero according to the combination

$$\text{(pole of order 3)}\cdot\text{(at least a zero of order 4)}\cdot\text{(reg. function)}\,,$$

which contradicts the estimate above.

Thus we have shown that $\zeta(1+it) \neq 0$ for all $t \in \mathbb{R}\setminus\{0\}$. ■

3 A special version of Hadamard's product representation

In the previous section we have given number theoretic representations of the zeta function and their logarithmic derivative based on Euler's product representation over the prime numbers.

In this section we will give analytic representations using Hadamard's product theorem and the zeros of the zeta function. Roughly speaking, the comparison of both representations is needed for the prime number formulas of Riemann and von-Mangoldt.

We formulate only a special version of Hadamard's product representation for certain entire functions, which is sufficient for the application on the prime number formulas of Riemann and von-Mangoldt.

Theorem (3.1) Hadamard's product representation, special version

Let be $f:\mathbb{C}\to\mathbb{C}$ be analytic in the entire complex plane with $f(0)\neq 0$ and zeros $\rho_1,\rho_2,\rho_3,\ldots$, such that each zero of order m is included m times in this sequence.

There may be a constant $0<\alpha<2$ such that the growing rate of f is bounded by

$$(*) \qquad |f(s)| \le \exp(|s|^\alpha) \qquad \textit{for } |s| >> 1\,.$$

Then there are constants $a,b\in\mathbb{C}$ with $f(s) = e^{as+b}\prod_{\rho_k}\left(1-\frac{s}{\rho_k}\right)e^{s/\rho_k}$

such that $\sum_{\rho_k}\frac{1}{|\rho_k|^2} < \infty$. ■

The following three examples will be used for our application on the prime number formulas.

Example 1: With the analytic continuation $\lim\limits_{\substack{s\to 0\\ s\neq 0}} \frac{\sin(\pi s)}{\pi s} = 1$ there holds the following product representation for all $s \in \mathbb{C}$

$$f(s) := \frac{\sin(\pi s)}{\pi s} = \prod_{k\in\mathbb{Z}\setminus\{0\}} \left(1 - \frac{s}{k}\right) e^{s/k} = \prod_{k=1}^{\infty} \left(1 - \frac{s^2}{k^2}\right) .$$

Here inequality $(*)$ is valid for any $\alpha > 1$.

Example 2: We define $1/\infty := 0$ and $1/0 := \infty$ to formulate for all $s \in \mathbb{C}$ the product representation of the Γ function in Theorem(2.3)(a) as

$$f(s) := \frac{1}{s\Gamma(s)} = \frac{1}{\Gamma(s+1)} = e^{\gamma s} \prod_{k=1}^{\infty} \left(1 + \frac{s}{k}\right) e^{-s/k} .$$

In this case $(*)$ is satisfied for all $\alpha > 1$.

Example 3: For all $s \in \mathbb{C}$ we have

$$\xi(s) := \frac{s(s-1)}{2}\, \pi^{-\frac{s}{2}}\, \Gamma(\frac{s}{2})\, \zeta(s) = \frac{1}{2} \lim_{T\to\infty} \prod_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|\le T}} \left(1 - \frac{s}{\rho}\right) ,$$

in short notation written as $\xi(s) = \frac{1}{2} \prod_{\rho}' \left(1 - \frac{s}{\rho}\right)$, where $(*)$ is again satisfied for all $\alpha > 1$. Recall that the value $\xi(0) = \frac{1}{2}$ was already evaluated in Theorem (2.5)(b).

A more general formulation and proof of Hadamard's product representation to higher order polynomials in the exponential function can be found in Lorenz [9]. It can be used for other applications like the study of elliptic functions. For a special derivation of the product representation of $\xi(s)$ see Edwards [6]. It was already known to Riemann [12].

4 The exponential integral and representations for $1/(s-1)$ and $(1 - \frac{s}{\rho})$

In Hadamard's product formula Theorem (3.1) for the entire function f under consideration there occur the factors $(1 - \frac{s}{\rho})$ with the roots ρ of f. Recall the application of this theorem on the ξ-function in Example 3 of the last section and the formula (2.3) for $\zeta(s)$ in Theorem (2.2), namely

$$\zeta(s) = \exp(s \int_1^\infty \frac{\pi_*(x)}{x^{s+1}}\, dx) , \qquad \mathrm{Re}(s) > 1 . \tag{4.1}$$

In order to obtain Riemann's explicit formula for $\pi_*(x)$ and $\pi(x)$, one compares the product formula for $\xi(s)$ with the formula (4.1) for $\zeta(s)$. Therefore it is quite natural to look first for representations of the form

$$1 - \frac{s}{\rho} = \exp(- s \int_1^\infty \frac{\varphi_\rho(x)}{x^{s+1}}\, dx) , \qquad \mathrm{Re}(s) > 1 , \tag{4.2}$$

with appropriate complex, but constant values ρ. This will be done next, i.e. for some given values of ρ a function $\varphi_\rho : (1, \infty) \to \mathbb{C}$ will be calculated such that (4.2) is satisfied. By the way, we will also obtain a useful representation of $1/(s-1)$ in order to handle the pole of the zeta function.

We first define the <u>exponential integral</u>

$$\mathrm{Ei}(z) := \gamma + \log z + \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!} . \tag{4.3}$$

We only use the logarithmic main branch $\log : D_{log} \to \mathbb{C}$ on the domain

$$D_{log} := \{ z \in \mathbb{C} \,|\, z \notin (-\infty, 0] \} = \mathbb{C} \setminus (-\infty, 0] \tag{4.4}$$

given for $z := a + ib \in D_{log}$ and $a, b \in \mathbb{R}$ by $\log z := \ln |z| + i\, \mathrm{arc}(z)$ with

$$\mathrm{arc}(z) := \begin{cases} \arctan \frac{b}{a} , & a > 0 \\ \frac{\pi}{2} \,\mathrm{sign}(b) - \arctan \frac{a}{b} , & a \le 0 . \end{cases} \tag{4.5}$$

Thus we conclude that $\mathrm{Ei}(z)$ is also defined for $z \in D_{log}$.

There always hold $\mathrm{Im}(\log z) = \mathrm{arc}(z) \in (-\pi, \pi)$ and

$$\arctan x + \arctan \frac{1}{x} = \begin{cases} \frac{\pi}{2} , & x > 0 \\ -\frac{\pi}{2} , & x < 0 , \end{cases} \tag{4.6}$$

and log is analytic in D_{log} with $e^{\log z} = z \; \forall z \in D_{log}$ and

$\log e^z = z - 2\pi i \lfloor \dfrac{\mathrm{Im}(z) + \pi}{2\pi} \rfloor$ if $z \in \mathbb{C}$ and $\mathrm{Im}(z) \neq \pi(2k+1) \, \forall k \in \mathbb{Z}$.

We also need the regular part of the exponential integral, the entire function

$$\mathrm{Ei}_0(z) := \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!} = \int_0^z \frac{e^t - 1}{t}\, dt = \int_0^1 \frac{e^{uz} - 1}{u}\, du\,. \tag{4.7}$$

These representations of $\mathrm{Ei}_0(z)$ are defined for all $z \in \mathbb{C}$.

In order to calculate the exponential integral we generalize Ei_0 and define for $n \in \mathbb{N}_0$ the entire functions $\mathrm{Ei}_n : \mathbb{C} \to \mathbb{C}$ by

$$\mathrm{Ei}_n(z) := \int_0^1 (1-u)^n \cdot \frac{e^{uz} - 1}{u}\, du \tag{4.8}$$

as well as the entire functions $Q_n : \mathbb{C} \to \mathbb{C}$ with

$$Q_n(z) := \int_0^1 (1-u)^n \cdot e^{uz}\, du = e^z \int_0^1 u^n \cdot e^{-uz}\, du\,. \tag{4.9}$$

We obtain by induction for all $n \in \mathbb{N}_0$ and $z \in \mathbb{C} \setminus \{0\}$ the important recurrence equations

$$Q_0(z) = \frac{e^z - 1}{z}\,, \quad Q_{n+1}(z) = \frac{n+1}{z}\, Q_n(z) - \frac{1}{z}\,. \tag{4.10}$$

This implies

$$Q_n(z) = \frac{n!}{z^{n+1}} \left(e^z - \sum_{k=0}^{n} \frac{z^k}{k!} \right)\,. \tag{4.11}$$

The next theorem states that the functions Q_n and Ei_n enable an efficient calculation of $\mathrm{Ei}_0(z)$ for any $z \in \mathbb{C}$ by using a combination of the recurrence equations (4.10) with Taylor expansions of Ei_n.

Theorem (4.1)

(a) For any $z \in \mathbb{C}$ and any $n \in \mathbb{N}_0$ we have

$$\mathrm{Ei}_0(z) = \sum_{m=0}^{n-1} \left(Q_m(z) - \frac{1}{m+1} \right) + Ei_n(z)\,.$$

(b) For any $z \in \mathbb{C}$ and any $n \in \mathbb{N}_0$ there holds the Taylor expansion

$$\mathrm{Ei}_n(z) = n! \sum_{k=1}^{\infty} \frac{z^k}{k \cdot (n+k)!} = \sum_{k=1}^{\infty} \frac{1}{k} \prod_{\nu=1}^{k} \frac{z}{n+\nu} \,.$$

Remark: The recurrence equations (4.10) can be used in combination with the Taylor expansion of Ei_n by choosing $n := \lfloor |z| \rfloor$ for given $z \in \mathbb{C}$ in Theorem (4.1) to calculate $\mathrm{Ei}_0(z)$.

Proof: (a) From (4.9) we immediately obtain for any $z \in \mathbb{C}$ and any $m \in \mathbb{N}_0$ that

$$\int_0^1 (1-u)^m \cdot (e^{uz} - 1) \; du = Q_m(z) - \frac{1}{m+1} \,. \tag{4.12}$$

For $0 < u < 1$ we evaluate the finite geometric series

$$\sum_{m=0}^{n-1} (1-u)^m = \frac{1-(1-u)^n}{u} \tag{4.13}$$

and conclude

$$\sum_{m=0}^{n-1} \int_0^1 (1-u)^m \, (e^{uz} - 1) \; du = \int_0^1 \frac{1-(1-u)^n}{u} \, (e^{uz} - 1) \; du \tag{4.14}$$

in order to establish Theorem (4.1)(a) with (4.12).

(b) From Ei_n in (4.8) we take the k-th derivative for $k \geq 1$,

$$\mathrm{Ei}_n^{(k)}(z) = \int_0^1 u^{k-1} (1-u)^n \cdot e^{uz} \, du \,, \tag{4.15}$$

and obtain at $z = 0$ that $\mathrm{Ei}_n^{(k)}(0) = \int_0^1 u^{k-1}(1-u)^n \, du = \dfrac{n! \cdot k!}{k \cdot (n+k)!} \,.$

Regarding also $\mathrm{Ei}_n(0) = 0$ we also obtain part (b) of Theorem (4.1). ∎

Now we are also able to calculate the exponential integral $\mathrm{Ei}(z)$ with arbitrary accuracy, even for large $|z|$. However, this method has to be complemented by asymptotic formulas for the exponential integral, which will also be derived in this section.

Lemma (4.2) *Let s be any complex number with* $\mathrm{Re}(s) > 0$.

(a) For all $n \in \mathbb{N}_0$ there holds

$$\frac{1}{s^{n+1}} = \int_1^\infty \frac{(\log x)^n}{n!} \frac{dx}{x^{s+1}} .$$

(b) Assume that $\rho \in \mathbb{C}$ satisfies $\mathrm{Re}(\rho) < \mathrm{Re}(s)$. Then $\mathrm{Ei}_0(\rho \log x)/x^{s+1}$ is Lebesgue integrable on the interval $(1, \infty)$, and there holds the relation

$$\frac{1}{1 - \frac{\rho}{s}} = \exp\left(s \int_1^\infty \frac{\mathrm{Ei}_0(\rho \log x)}{x^{s+1}} \, dx \right) .$$

(c) The expression $(\gamma + \log(\log x))/x^{s+1}$ is Lebesgue integrable on the interval $(1, \infty)$, and there holds the relation

$$\frac{1}{s} = \exp\left(s \int_1^\infty \frac{\gamma + \log(\log x)}{x^{s+1}} \, dx \right) .$$

Proof: (a) The integral on the right hand side is obviously well defined. The statement is true for $n = 0$. Assume the statement is already established for some $n \in \mathbb{N}_0$. Then we put $u(x) := (\log x)^{n+1}/(n+1)!$, $v(x) := 1/x^s$, and obtain by partial integration with zero boundary terms

$$\frac{1}{s^n} = \int_1^\infty u'(x) v(x) \, dx = - \int_1^\infty u(x) v'(x) \, dx = s \int_1^\infty \frac{(\log x)^{n+1}}{(n+1)!} \frac{dx}{x^{s+1}} .$$

This is the proof of part (a).

(b) For $|z| \le 1$ we have $|\mathrm{Ei}_0(z)| \le \mathrm{Ei}_0(1)$ from the Taylor expansion (4.7), and for $|z| > 1$ we obtain

$$\begin{aligned} |\mathrm{Ei}_0(z)| &\le \left| \int_0^{\frac{z}{|z|}} \frac{e^t - 1}{t} \, dt \right| + \left| \int_{\frac{z}{|z|}}^{z} \frac{e^t - 1}{t} \, dt \right| \\ &\le \mathrm{Ei}_0(1) + |z| \max(e + 1, e^{Re(z)} + 1) . \end{aligned}$$

We put $z := \rho \log x$ in the last inequality in order to conclude from $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(\rho) < \mathrm{Re}(s)$ that $\frac{1}{1-\frac{\rho}{s}}$ and $\int_1^\infty \frac{\mathrm{Ei}_0(\rho \log x)}{x^{s+1}}\, dx$ are well defined.

Next we show that the formula in Lemma (4.2)(b) is true in the special case $\mathrm{Re}(s) > |\rho|$. Then the formula is also true for $\mathrm{Re}(\rho) < \mathrm{Re}(s)$ by the analytic continuation principle.

In order to evaluate the integral in Lemma (4.2)(b) we perform partial integration, define $u(x) := -\mathrm{Ei}_0(\rho \log x)$, $v(x) := x^{-s}$ with $v'(x) = -sx^{-(s+1)}$ and $u'(x) = -(x^\rho - 1)/(x \log x)$, and obtain with vanishing boundary terms

$$s \int_1^\infty \frac{\mathrm{Ei}_0(\rho \log x)}{x^{s+1}}\, dx = \int_1^\infty \frac{x^\rho - 1}{\log x} \cdot \frac{dx}{x^{s+1}}\,, \tag{4.16}$$

where x^ρ is an abbreviation for $\exp(\rho \log x)$. Using the Taylor expansion of the exponential function we obtain immediately

$$\frac{x^\rho - 1}{\log x} = \sum_{n=1}^\infty \frac{\rho^n\, (\log x)^{n-1}}{n \cdot (n-1)!}\,. \tag{4.17}$$

For our restriction $\mathrm{Re}(s) > |\rho|$ we have $|\rho/s| < 1$ and conclude from part (a)

$$\sum_{n=1}^\infty \int_1^\infty \frac{\rho^n\, (\log x)^{n-1}}{n \cdot (n-1)!} \cdot \frac{dx}{x^{s+1}} = \sum_{n=1}^\infty \frac{1}{n} \left(\frac{\rho}{s}\right)^n = \log \frac{1}{1-\frac{\rho}{s}}\,. \tag{4.18}$$

Next we show that summation and integration in (4.18) can be interchanged. Then we have shown part (b) of the Lemma due to (4.16), (4.17), (4.18).

We define for $N \in N$ and $x > 1$ the L_1 functions f_N by

$$f_N(x) := \sum_{n=1}^N \frac{\rho^n\, (\log x)^{n-1}}{n \cdot (n-1)!} \cdot \frac{1}{x^{s+1}}\,.$$

Then we have already shown the pointwise convergence to

$$f(x) := \lim_{N \to \infty} f_N(x) = \frac{x^\rho - 1}{\log x} \cdot \frac{1}{x^{s+1}} \qquad \forall x > 1\,.$$

Moreover, we have for all $N \in \mathbb{N}$ and $x > 1$

$$|f_N(x)| \leq \sum_{n=1}^\infty \frac{|\rho|^n\, (\log x)^{n-1}}{n \cdot (n-1)!} \cdot \frac{1}{|x^{s+1}|} = F(x) := \frac{x^{|\rho|} - 1}{\log x} \cdot \frac{1}{x^{Re(s)+1}}$$

with $F \in L_1(1,\infty)$ due to our restriction $\mathrm{Re}(s) > |\rho|$. We finally conclude by the Lebesgue dominated convergence theorem that

$$\lim_{N\to\infty} \int_1^\infty f_N(x)\,dx = \int_1^\infty f(x)\,dx\,.$$

This means the interchange of summation and integration in (4.18), and we have shown part (b).

In order to prove part (c) we employ the substitution $u = \log x$ in the integral and obtain

$$\int_1^\infty \frac{\gamma + \log(\log x)}{x^{s+1}}\,dx = \int_0^\infty (\gamma + \log u)\,e^{-su}\,du\,. \tag{4.19}$$

It follows first from $\mathrm{Re}(s) > 0$ that these integrals are well defined.

However, in contrast to the integral in part (b), we have a singular behaviour of $\log(\log x)$ at $x = 1$ and of $\log u$ at $u = 0$, but the integrals in (4.19) still converge in the absolute sense.

Next we define the auxiliary function

$$g(s) := s\int_0^\infty e^{-su}\,\log u\,du \tag{4.20}$$

and form its derivative by applying partial integration and regarding $\frac{d}{du}(u\log u - u) = \log u$,

$$\begin{aligned} g'(s) &= \int_0^\infty e^{-su}\,\log u\,du - s\int_0^\infty e^{-su}\,u\log u\,du \\ &= \int_0^\infty e^{-su}\,\log u\,du + \int_0^\infty (u\log u - u)\cdot(-s)e^{-su}\,du \\ &+ \int_0^\infty u\cdot(-s)e^{-su}\,du = -\frac{1}{s}\,. \end{aligned}$$

We conclude that there is a constant $C \in \mathbb{C}$ such that

$$g(s) = s\int_0^\infty e^{-su}\,\log u\,du = -C + \log\frac{1}{s}\,, \qquad \mathrm{Re}(s) > 0\,. \tag{4.21}$$

In order to determine C we put $s := 1$ and obtain first that

$$C = -\int_0^\infty e^{-u} \log u \, du \,. \tag{4.22}$$

Due to Theorem (2.3)(e) and (c) we have

$$\Gamma'(1) = \int_0^\infty e^{-u} \log u \, du = -\gamma \,,$$

this means $C = \gamma$, and from (4.21) and (4.22) we finally conclude the part (c) of Lemma (4.2). ■

Theorem (4.3) *Let s be any complex number with $\mathrm{Re}(s) > 0$.*

(a) Then $\mathrm{Ei}(\log x)/x^{s+1}$ is Lebesgue integrable on the interval $(1, \infty)$, and there holds the relation

$$\frac{1}{s-1} = \exp\left(s \int_1^\infty \frac{\mathrm{Ei}(\log x)}{x^{s+1}} \, dx \right) .$$

(b) Assume that $\rho \in \mathbb{C} \setminus [0, \infty)$ and $\mathrm{Re}(\rho) < \mathrm{Re}(s)$. Then

$$1 - \frac{s}{\rho} = \exp\left(-s \int_1^\infty \frac{\gamma + \log(-\rho) + \log(\log x) + \mathrm{Ei}_0(\rho \log x)}{x^{s+1}} \, dx \right) .$$

<u>Proof:</u> Part (a) follows from Lemma (4.2)(b) with $\rho := 1$ and from Lemma (4.2)(c) by multiplication.

Part (b) also follows from Lemma (4.2)(b),(c) by multiplication

$$1 - \frac{s}{\rho} = \frac{s}{(-\rho)} \cdot (1 - \frac{\rho}{s}) = \exp\left(-s \int_1^\infty \frac{\gamma + \log(-\rho) + \log(\log x)}{x^{s+1}} \, dx \right)$$

$$\cdot \exp\left(-s \int_1^\infty \frac{\mathrm{Ei}_0(\rho \log x)}{x^{s+1}} \, dx \right) .$$

■

Theorem (4.4)

Let s, α be any complex numbers with $\mathrm{Re}(s) > 0$, $\mathrm{Re}(\alpha) > 0$. *Then*

$$\text{(a)} \ \exp\left(s \int_1^\infty \frac{\int_x^\infty \frac{t^{-(\alpha+1)}\,dt}{\log t}}{x^{s+1}}\,dx \right) = 1 + \frac{s}{\alpha}\,.$$

(b) For any real $x > 1$ there holds

$$-\int_x^\infty \frac{t^{-(\alpha+1)}\,dt}{\log t} = \gamma + \log\,\alpha + \log(\log x) + \mathrm{Ei}_0(-\alpha\,\log x)\,.$$

Proof: (a) The left hand side in (b) has only a logarithmic singularity at $x = 1$, and thus the left hand side in (a) defines for any fixed s with $\mathrm{R}e(s) > 0$ an analytic function in the half plane $\mathrm{R}e(\alpha) > 0$. We conclude that

$$\frac{\partial}{\partial\alpha}\left\{ s \int_1^\infty \frac{\int_x^\infty \frac{t^{-(\alpha+1)}\,dt}{\log t}}{x^{s+1}}\,dx \right\} = -s \int_1^\infty \frac{\int_x^\infty t^{-(\alpha+1)}\,dt}{x^{s+1}}\,dx$$

$$= -s \int_1^\infty \frac{\frac{x^{-\alpha}}{\alpha}}{x^{s+1}}\,dx = \frac{1}{s+\alpha} - \frac{1}{\alpha}\,,$$

and therefore

$$s \int_1^\infty \frac{\int_x^\infty \frac{t^{-(\alpha+1)}\,dt}{\log t}}{x^{s+1}}\,dx = f_1(s) + \log\left(1 + \frac{s}{\alpha}\right)$$

with an integration constant $f_1(s)$ depending only on s. But $f_1(s)$ must be zero for all s, because both sides of the last equality must be zero in the limit $\alpha \to \infty$ along the positive real axis.

(b) Differentiation with respect to $x > 1$ on both sides in (b) gives the same result $\dfrac{x^{-(\alpha+1)}}{\log x}$. This means that both sides in (b) can only differ in an integration constant $f_2(\alpha)$. We put $\rho = -\alpha$ in Theorem (4.3)(b) in order to conclude with (a) that $f_2(\alpha) = 0$ for all real $\alpha > 0$. The general result for

$\mathrm{Re}(\alpha) > 0$ follows by analytic continuation. ∎

In the next theorem we derive some integral representations for the exponential integral. A resulting asymptotic expansion for $\mathrm{Ei}(z)$ is very efficient for $|\mathrm{Im}(z)| \gg |\mathrm{Re}(z)|$.

Theorem (4.5) Integral representations and asymptotic expansion for $\mathrm{Ei}(z)$

(a) Let α be any complex number with $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Im}(\alpha) \neq 0$. Then we obtain for all $y > 0$

$$\mathrm{Ei}(-\alpha y) = -i\pi\,\mathrm{sign}(\mathrm{Im}(\alpha)) - \int_y^\infty \frac{e^{-\alpha u}}{u}\,du\,.$$

(b) Let be $\sigma, t \in \mathbb{R}$ and $t \neq 0$. Then we put $z := \sigma + it$ and obtain

$$\mathrm{Ei}(z) = i\pi\,\mathrm{sign}(\mathrm{Im}(z)) + \int_0^\infty \frac{e^{z-u}}{z-u}\,du\,.$$

Moreover, for all $n \in \mathbb{N}_0$ we have the asymptotic expansion

$$\int_0^\infty \frac{e^{z-u}}{z-u}\,du = \frac{e^z}{z}\sum_{k=0}^{n}\frac{k!}{z^k} + (n+1)!\int_0^\infty \frac{e^{z-u}}{(z-u)^{n+2}}\,du\,,$$

and for all $k \in \mathbb{N}_0$ the estimate

$$\left|\int_0^\infty \frac{e^{z-u}}{(z-u)^k}\,du\right| \leq \frac{e^\sigma}{|t|^k}\,.$$

(c) For $\mathrm{Im}(z) > 0$ we have

$$\mathrm{Ei}(z) = i\pi\,\mathrm{sign}(\mathrm{Im}(z)) - \lim_{T\to\infty}\int_{\mathrm{Im}(z)}^{T} \frac{e^{\sigma+i\vartheta}}{\sigma+i\vartheta}\,i\,d\vartheta\,.$$

<u>Proof:</u> (a) The left and right hand side of the equation in (a) define analytic functions in the quarter plane $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Im}(\alpha) > 0$. By performing the derivative with respect to α on both sides we obtain that

$$\frac{\partial}{\partial\alpha}\mathrm{Ei}(-\alpha y) = \frac{\partial}{\partial\alpha}\left(\mp i\pi - \int_y^\infty \frac{e^{-\alpha u}}{u}\,du\right) = \frac{e^{-\alpha y}}{\alpha}\,.$$

It remains to determine the constants $\mp i\pi$ on the right hand side. We apply the substitution $u = \log t$ on the integral in Theorem (4.4)(b) and obtain for the number $x > 1$ given there that

$$-\int\limits_{\log x}^{\infty} \frac{e^{-\alpha u}}{u}\, du = \gamma + \log\, \alpha + \log(\log x) + \mathrm{Ei}_0(-\alpha\, \log x)\,.$$

In this equation we put $y := \log x > 0$ and conclude for $\mathrm{Im}(\alpha) > 0$

$$-\int\limits_{y}^{\infty} \frac{e^{-\alpha u}}{u}\, du = \log\, \alpha + \log y - \log(-\alpha y) + \mathrm{Ei}(-\alpha y) = i\pi + \mathrm{Ei}(-\alpha y)\,.$$

We obtain Theorem (4.5)(a) by regarding that $\overline{\mathrm{Ei}(-\alpha y)} = \mathrm{Ei}(-\overline{\alpha} y)\,.$

Note that the condition $\mathrm{Im}(\alpha) > 0$ guarantees that the argument $-\alpha y$ does not pass the cut of the logarithm function in the exponential integral. This condition is not necessary in Theorem (4.4)(b), which makes use of the entire function Ei_0.

It is not difficult to check that the representations for $\mathrm{Ei}(z)$ in (b) and (c) define analytic functions in the half planes $t = \mathrm{Im}(z) > 0$ and $t = \mathrm{Im}(z) < 0$. Differentiation with respect to $z = \sigma + it$ reduces the proof of the representation formulas to the determination of the integration constants, which just follow from part (a) in the limit $\sigma \to -\infty$. From the representation for $\mathrm{Ei}(z)$ in (b) we also obtain the asymptotic expansion by partial integration. The estimate for the asymptotic expansion of $\mathrm{Ei}(z)$ in (b) results from

$$\left| \int\limits_{0}^{\infty} \frac{e^{z-u}}{(z-u)^k}\, du \right| \le \int\limits_{0}^{\infty} \frac{|e^{(\sigma-u)+it}|}{|(\sigma-u)+it|^k}\, du \le \int\limits_{0}^{\infty} \frac{e^{\sigma-u}}{|t|^k}\, du = \frac{e^{\sigma}}{|t|^k}\,.$$

∎

The functions Ei and Ei_0 are closely related to the cosine and sine integral functions, for which we obtain the following

Corollary (4.6)

The cosine and sine integral functions $\mathrm{Ci} : D_{log} \to \mathbb{C}$ and $\mathrm{Si} : \mathbb{C} \to \mathbb{C}$ are given by

$$\mathrm{Ci}(z) := \gamma + \log z + \int_0^z \frac{\cos(t) - 1}{t} \, dt \,,$$

$$\mathrm{Si}(z) := \int_0^z \frac{\sin t}{t} \, dt = \frac{\mathrm{Ei}_0(iz) - \mathrm{Ei}_0(-iz)}{2i} \,.$$

They satisfy the asymptotic relations

$$\lim_{x\to\infty} \mathrm{Ci}(x) = 0 \,, \quad \lim_{x\to\infty} \mathrm{Si}(x) = \frac{\pi}{2} \,.$$

■

Theorem (4.7)

Let s, ρ *be any complex numbers with* $\mathrm{Re}(s) > 0$ *and* $\mathrm{Re}(\rho) < \mathrm{Re}(s)$*. By* $\frac{d\log}{ds} f(s) := \frac{f'(s)}{f(s)}$ *we denote the logarithmic derivative of an analytic expression* $f(s)$*. Then*

$$(a) \;\; s \int_1^\infty \frac{(x^\rho - 1)/\rho}{x^{s+1}} \, dx = \frac{d\log}{ds} (1 - \frac{s}{\rho}) = \frac{s}{\rho(s-\rho)} - \frac{1}{\rho} = \frac{1}{s-\rho} \,,$$

$$(b) \;\; s \int_1^\infty \frac{\gamma + \log(x-1)}{x^{s+1}} \, dx = -\frac{d\log}{ds} \Gamma(s) = -\frac{\Gamma'(s)}{\Gamma(s)} \,,$$

$$(c) \;\; s \int_1^\infty \frac{\frac{1}{2} \log \frac{x^2}{x^2-1}}{x^{s+1}} \, dx = \sum_{k=1}^\infty \frac{s}{2k(s+2k)} = \frac{\gamma}{2} + \frac{d\log}{ds} \Gamma(\frac{s}{2} + 1) \,.$$

Proof: (a) is trivial, (b) follows with Lemma (4.2)(a) and Theorem (2.3)(c) from $\log(x-1) = \log x - \log \dfrac{1}{1 - 1/x} = \log x - \sum_{k=1}^\infty \dfrac{x^{-k}}{k} \,, \quad x > 1$, and the proof of (c) is analogous to (b). ■

5 Regularizations for von Mangoldt's $\psi(x)$

In the sequel we assume that

$$\sigma_0, \sigma_1 \in \mathbb{R} \cup \{-\infty, +\infty\} \quad \text{and} \quad \sigma_0 < \sigma_1 \,. \tag{5.1}$$

Depending on σ_0, σ_1 we define the strip $\mathcal{S}(\sigma_0, \sigma_1)$ in the complex plane by

$$\mathcal{S}(\sigma_0, \sigma_1) := \{ s \in \mathbb{C} \,|\, \sigma_0 < \mathrm{Re}(s) < \sigma_1 \} \,. \tag{5.2}$$

In the preceeding section there often occurs the Mellin transform of a given function $f : (0, \infty) \to \mathbb{C}$,

$$(\mathcal{M}f)(s) := \int_0^\infty \frac{f(x)}{x^{s+1}} \, dx \,, \tag{5.3}$$

defined for all s in an appropriate strip $\mathcal{S}(\sigma_0, \sigma_1)$, provided that $f(x)/x^{s+1}$ is Lebesgue integrable on $(0, \infty)$ for all fixed $s \in \mathcal{S}(\sigma_0, \sigma_1)$. There are slightly different but equivalent definitions of a Mellin transform, however the definition above is better suited for our study of the ζ function and the primes.

Mellin transforms include the Fourier-Laplace transforms, which can be seen by applying the integral substitution $x = e^u$ in (5.3), namely

$$(\mathcal{M}f)(s) = \int_{-\infty}^\infty f(e^u) \, e^{-su} \, du \,. \tag{5.4}$$

Note that $f(x) = 0$ for $0 < x < 1$ leads to the lower integration limit 0 instead of $-\infty$ in (5.4), such that (5.4) contains the usual Laplace transformation for $\mathrm{Im}(s) = 0$ and $\mathrm{Re}(s) > 0$ as a special case.

The relation between the Mellin transform and the Fourier-Laplace transform is important, because we can use the Fourier-theory in order to obtain the following uniqueness result and inversion formula:

Theorem (5.1) *The function $f : (0, \infty) \to \mathbb{C}$ may be given in such a way that $|f(x)|/x^{\sigma+1}$ is Lebesgue integrable on $(0, \infty)$ for any fixed $\sigma \in (\sigma_0, \sigma_1)$. Then there hold*

(a) If $(\mathcal{M}f)(\sigma + it) = 0$ for all $t \in \mathbb{R}$ and fixed $\sigma \in (\sigma_0, \sigma_1)$, then $f(x) \equiv 0$ for almost all $x > 0$.

(b) *Mellin inversion formula*

Assume that the Mellin transform $(\mathcal{M}f)(\sigma + it)$ *is Lebesgue integrable on* $\mathbb{R}$ *with respect to* $t \in \mathbb{R}$ *for fixed* $\sigma \in (\sigma_0, \sigma_1)$*. Then the function* $g : (0, \infty) \to \mathbb{C}$ *with*

$$g(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathcal{M}f)(\sigma + it)\, x^{\sigma+it}\, dt = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{(\mathcal{M}f)(s)\, x^s}{2\pi i}\, ds$$

is well defined and continuous, and there holds $g(x) = f(x)$ *for almost all* $x > 0$*.*

Remark: The uniqueness result in part (a) of this theorem is a special case of part (b). A proof of part (b) for piecewise smooth f can be found in the textbook of Courant and Hilbert [5]. Here we have just used the Fourier inversion formula for Lebesgue integrable Fourier transforms. ■

Recall the definition of von Mangoldt's function ψ in Theorem (2.2). Due to Theorem (2.3)(a) and Theorem (2.4)(d) we can write

$$\zeta(s) = \frac{\xi(s)}{s-1} \cdot \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}\,. \tag{5.5}$$

Due to Hadamard's product representations for $\xi(s)$ and $\Gamma(\frac{s}{2}+1)$ the Mellin transform of ψ is given for $\mathrm{Re}(s) > 1$ by the absolutely convergent series

$$-\frac{1}{s}\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} - \sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1}} \frac{1}{\rho(s-\rho)} + \sum_{k=1}^{\infty} \frac{1}{2k(s+2k)} - \frac{\zeta'(0)}{s\,\zeta(0)}\,. \tag{5.6}$$

Having Theorem (4.7) in mind, it is natural to conjecture von Mangoldt's formula at each point $x > 1$ of continuity, namely

$$\psi(x) = x - \lim_{T\to\infty} \sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1 \\ |Im\,\rho|\le T}} \frac{x^\rho}{\rho} + \frac{1}{2}\log\frac{x^2}{x^2-1} - \frac{\zeta'(0)}{\zeta(0)}\,. \tag{5.7}$$

A proof of (5.7) is technical difficult and was first presented by von Mangoldt, see the textbook of Edwards [6]. The numerical constant in (5.7) turns out to be

$$\frac{\zeta'(0)}{\zeta(0)} = \log(2\pi)\,. \tag{5.8}$$

Here we will only derive some similar formulas for appropriate regularizations of $\psi(x)$ which can be obtained easier.

The following theorem is used for the derivation of von Mangoldt's formula (5.7), but we will also use the resulting Lemma (5.3) for Riemann's prime number formula in the next section. It provides some information about the vertical distribution of the zeros of the zeta function in the critical strip. The following result was used by Riemann in [12] and gives the asymptotic density of the nontrivial roots. It was first shown by von Mangoldt [11] and then simplified by Backlund [3].

Theorem (5.2) *For $T \geq 2$ let $N(T)$ be the number of zeros $\rho = \sigma + it$ of the ζ-function in the critical strip $0 < \sigma < 1$ with $0 < t \leq T$, regarding the multiplicity of the roots. Then*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)\,. \tag{5.9}$$

■

Theorem (5.2) can be applied very easily for the proof of the following Lemma by using the Stieltjes calculus.

Lemma (5.3) *There hold the following asymptotic relations for $T \to \infty$*

(a) $$\sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1 \\ |Im\,\rho|\leq T}} \frac{1}{|\rho|} = \frac{1}{2\pi} (\log T)^2 + O(\log T)\,,$$

(b) $$\sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1 \\ |Im\,\rho|>T}} \frac{1}{|Im\,\rho|^2} = \frac{1}{\pi} \frac{\log T}{T} + O(1/T)\,.$$

<u>Proof:</u> We use Stieltjes integral representations as follows in order to apply Theorem (5.2)

$$\sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1 \\ |Im\,\rho|\leq T}} \frac{1}{|\mathrm{Im}\rho|} = 2\int_0^T \frac{dN(t)}{t} = 2\int_0^T \frac{N(t)}{t^2}\,dt = \frac{1}{2\pi} (\log T)^2 + O(\log T)\,,$$

where the boundary term of partial integration is zero because $\zeta(s)$ has no zero on the line segment $0 < s < 1$. The first part of the Lemma is now a consequence of

$$\frac{1}{|Im\,\rho|} - \frac{1}{|Im\,\rho|^2} < \frac{1}{|Im\,\rho|+1} < \frac{1}{|\rho|} < \frac{1}{|Im\,\rho|-1} < \frac{1}{|Im\,\rho|} + \frac{1}{|Im\,\rho|^2}\,,$$

since the imaginary part of ρ is larger than 1, while the real part is between 0 and 1. For the second part we obtain

$$\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|>T}} \frac{1}{|Im\,\rho|^2} = 2\int_T^\infty \frac{dN(t)}{t^2} = 4\int_T^\infty \frac{N(t)}{t^3}\,dt - 2\frac{N(T)}{T^2} = \frac{1}{\pi}\,\frac{\log T}{T} + O(1/T)\,.$$

■

Assume that $f(x)/x^{s+1}$ is Lebesgue integrable on $(0,\infty)$ with respect to x on some strip $\sigma_0 < \mathrm{Re}(s) < \sigma_1$ and that $f(x) = 0$ for $0 < x < \delta$ and some constant $\delta > 0$. We define for a positive parameter ε and all $x > 0$ the regularization of f by the integral mean value

$$(R_\varepsilon f)(x) := \int_{-\frac{1}{2}}^{\frac{1}{2}} f\left(x\,\exp(\varepsilon\,u)\right)\,du\,. \tag{5.10}$$

Then the Mellin transform of $(R_\varepsilon f)$ is

$$\int_0^\infty \frac{(R_\varepsilon f)(x)}{x^{s+1}}\,dx = \frac{\sinh(\frac{s\,\varepsilon}{2})}{(\frac{s\,\varepsilon}{2})}\int_0^\infty \frac{f(y)}{y^{s+1}}\,dy\,. \tag{5.11}$$

Note that $f(x) = 0$ for $0 < x < \delta$ leads to a vanishing boundary term in the application of the partial integration above.

Next we define the Lebesgue integrable function $\eta : (0,\infty) \to \mathbb{R}$ by

$$\eta(x) := \psi(x) - \chi_{(1,\infty)}(x)\cdot\left(x + \frac{1}{2}\,\log\frac{x^2}{x^2-1} - \log(2\pi)\right)\,, \tag{5.12}$$

where $\chi_{(1,\infty)}$ is the characteristic function of the interval $(1,\infty)$. Due to Theorem (4.7) and (5.6) the Mellin transform of η is for $\mathrm{Re}(s) > 1$

$$\int_0^\infty \frac{\eta(y)}{y^{s+1}}\,dy = \int_1^\infty \frac{\eta(y)}{y^{s+1}}\,dy = -\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1}} \frac{1}{\rho(s-\rho)}\,. \tag{5.13}$$

We define for $\varepsilon > 0$ and all $n \in \mathbb{N}_0$ the regularizations $R_\varepsilon^{(n)} : (0,\infty) \to \mathbb{R}$ of the function η by the recurrence relations

$$R_\varepsilon^{(0)}\eta := \eta\,, \quad R_\varepsilon^{(n+1)}\eta := R_\varepsilon\left(R_\varepsilon^{(n)}\eta\right)\,. \tag{5.14}$$

From (5.11) and (5.13) we obtain for all $\varepsilon > 0$ and all $n \in \mathbb{N}_0$

$$\int_0^\infty \frac{\left(R_\varepsilon^{(n)}\eta\right)(x)}{x^{s+1}}\,dx = -\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1}} \frac{1}{\rho(s-\rho)} \left(\frac{\sinh(\frac{s\,\varepsilon}{2})}{(\frac{s\,\varepsilon}{2})}\right)^n\,. \tag{5.15}$$

In the sequel we assume that $n \geq 1$. On the right hand side of (5.15) the Mellin inversion integral formula can be applied termwise due to $n \geq 1$ to recover $\left(R_\varepsilon^{(n)}\eta\right)(x)$ according to

$$\left(R_\varepsilon^{(n)}\eta\right)(x) = -\frac{1}{2\pi i}\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1}} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{\rho(s-\rho)} \left(\frac{\sinh(\frac{s\,\varepsilon}{2})}{(\frac{s\,\varepsilon}{2})}\right)^n ds\,. \tag{5.16}$$

The summation of the integrals is performed with respect to the nontrivial zeros of the ζ-function, where the ordering of the terms is arbitrary thanks to the absolute convergence. Here σ is any fixed real number larger than 1.

Each single integral in (5.16) can be calculated if we replace for $T > 0$ the integration path from $\sigma - i\infty$ to $\sigma + i\infty$ by the closed rectangular path Γ_T ranging from $\sigma - iT$ to $\sigma + iT$, from $\sigma + iT$ to $\sigma - T + iT$, from $\sigma - T + iT$ to $\sigma - T - iT$ and from $\sigma - T - iT$ to $\sigma - iT$. If $x > 1$ and if $\varepsilon > 0$ is small enough such that $\log x > \frac{n\,\varepsilon}{2}$, then we obtain by Cauchy's theorem

$$\begin{gathered}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{\rho(s-\rho)} \left(\frac{\sinh(\frac{s\,\varepsilon}{2})}{(\frac{s\,\varepsilon}{2})}\right)^n ds = \\ \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\Gamma_T} \frac{x^s}{\rho\,(s-\rho)} \left(\frac{\sinh(\frac{s\,\varepsilon}{2})}{(\frac{s\,\varepsilon}{2})}\right)^n ds = \frac{x^\rho}{\rho}\left(\frac{\sinh(\frac{\rho\,\varepsilon}{2})}{(\frac{\rho\,\varepsilon}{2})}\right)^n\,.\end{gathered} \tag{5.17}$$

For $\log x > \frac{n\,\varepsilon}{2}$ there results the formula

$$\left(R_\varepsilon^{(n)}\eta\right)(x) = -\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1}} \frac{x^\rho}{\rho}\left(\frac{\sinh(\frac{\rho\,\varepsilon}{2})}{(\frac{\rho\,\varepsilon}{2})}\right)^n\,. \tag{5.18}$$

We conclude at each point $x > 1$ of continuity for all $n \geq 1$ that

$$\psi(x) = x - \lim_{\varepsilon \to 0} \sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1}} \frac{x^\rho}{\rho} \left(\frac{\sinh(\frac{\rho\,\varepsilon}{2})}{(\frac{\rho\,\varepsilon}{2})} \right)^n + \frac{1}{2} \log \frac{x^2}{x^2-1} - \log(2\pi) . \quad (5.19)$$

The regularizations (5.14) can also be rewritten by the n-fold integral

$$\left(R_\varepsilon^{(n)} \eta\right)(x) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{+\frac{1}{2}} \eta\left(x \, \exp(\varepsilon\,(u_1 + ... + u_n))\right) \, du_1 ... du_n . \quad (5.20)$$

Using the n-th order cardinal B-splines $N_n(u)$,

$$N_1(u) := \chi_{(0,1)}(u) , \qquad N_{n+1}(u) := \int_0^1 N_n(u-v)\, dv\,, \;\; n \geq 1\,, \quad (5.21)$$

we obtain that

$$\left(R_\varepsilon^{(n)} \eta\right)(x) = \int_{-\frac{n}{2}}^{+\frac{n}{2}} N_n(u + \frac{n}{2})\, \eta(x \, \exp(\varepsilon\, u))\, du\,. \quad (5.22)$$

The n-th order cardinal B-spline $N_n(u)$ has support $[0, n]$, is normalized, has center $\frac{n}{2}$ and variance $n/12$, i.e.

$$\int_0^n N_n(u)\, du = 1\,, \;\; \int_0^n u\, N_n(u)\, du = \frac{n}{2}\,, \;\; \int_0^n (u - \frac{n}{2})^2 \, N_n(u)\, du = \frac{n}{12}\,, \quad (5.23)$$

and by the Central Limit Theorem we have with uniform convergence in u

$$\lim_{n \to \infty} \sqrt{\frac{n}{12}}\, N_n(\sqrt{\frac{n}{12}}\, u + \frac{n}{2}) = \frac{1}{\sqrt{2\,\pi}} \, \exp\left(-\frac{u^2}{2}\right) . \quad (5.24)$$

For more details about cardinal B-splines see the textbook of Chui [4].

Thus the regularizations (5.14) are related to the Gaussian regularization

$$(G_\delta\, \eta)(x) := \frac{1}{\sqrt{2\pi\delta}} \int_{-\infty}^{+\infty} \exp\left(-\frac{u^2}{2\,\delta}\right) \eta\left(xe^u\right)\, du \quad (5.25)$$

for a constant parameter $\delta > 0$ with its Mellin transform

$$\int_0^{+\infty} \frac{(G_\delta\, \eta)(x)}{x^{s+1}}\, dx = -\exp\left(\frac{\delta}{2}\, s^2\right) \sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1}} \frac{1}{\rho(s-\rho)}\,. \tag{5.26}$$

Here we will not prove (5.26), which holds for $\mathrm{Re}(s) > 1$, because we will obtain a generalization of this result in Theorem (5.6).

To see the relation between $G_\delta\, \eta$ and the cardinal B-splines we define for given $n \geq 1$ and given $\delta > 0$

$$\varepsilon_n := \sqrt{\frac{12\,\delta}{n}}\,. \tag{5.27}$$

Then we obtain with (5.24) and (5.22) that

$$\begin{aligned}(G_\delta\, \eta)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{v^2}{2}\right)\, \eta\left(xe^{\sqrt{\delta}\, v}\right)\, dv \\ &= \lim_{n\to\infty} \sqrt{\frac{n}{12}} \int_{-\infty}^{+\infty} N_n(\sqrt{\frac{n}{12}} v + \frac{n}{2})\, \eta(xe^{\sqrt{\delta}\, v})\, dv \\ &= \lim_{n\to\infty} (R^{(n)}_{\varepsilon_n} \eta)(x)\,. \end{aligned} \tag{5.28}$$

Note that $(R^{(n)}_{\varepsilon}\, \eta)(x)$ in (5.14) is for $x \gg 1$ a mean value of $\psi(x)-x+\log(2\pi)$. In the Gaussian limit $n \to \infty$ there results

$$\lim_{n\to\infty} \left(\frac{\sinh(\frac{\rho\,\varepsilon_n}{2})}{(\frac{\rho\,\varepsilon_n}{2})}\right)^n = \exp\left(\frac{\delta}{2}\, \rho^2\right)\,, \tag{5.29}$$

which can be seen by Taylor expansion of $\frac{\sinh z}{z}$. However, we cannot use (5.18) and (5.29) in order to express $(G_\delta\, \eta)(x)$ in terms of the zeros of the zeta function due to the restriction $\log x - \frac{n\varepsilon_n}{2} > 0$. Fortunately this is not necessary because we can termwise apply the Mellin inversion formula in Theorem (5.1)(b) on the Mellin transform (5.26). For this purpose we need the following

Lemma (5.4) *For $x, \delta > 0$, $\rho \in \mathbb{C}$ and $\alpha \in \mathbb{R}$ with $\mathrm{Re}(\rho) < \alpha$ we have*

$$f_\delta(x,\rho) := -\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^{s-\rho}\, \exp\left(\frac{\delta}{2}\, s^2\right)}{\rho(s-\rho)}\, ds = -\frac{\exp\left(\frac{\delta}{2}\, \rho^2\right)}{\rho}\, \Phi\left(\frac{\rho\,\delta + \log x}{\sqrt{\delta}}\right)$$

with the complex Gaussian error function $\Phi(z) := \int\limits_0^\infty \frac{e^{-\frac{(z-u)^2}{2}}}{\sqrt{2\pi}}\, du\,,\ z \in \mathbb{C}\,.$

Proof: First we form the derivative

$$\frac{\partial f_\delta}{\partial x}(x,\rho) = \frac{x^{-\rho-1}\exp\left(-\frac{\log^2 x}{2\delta}\right)}{-2\pi i\rho} \int\limits_{\alpha-i\infty}^{\alpha+i\infty} \exp\left(\frac{\delta}{2}\left(s+\frac{\log x}{\delta}\right)^2\right) ds\,, \quad (5.30)$$

and conclude that the integral on the right hand side in (5.30) is defined for $\alpha \in \mathbb{R}$ and independent on α by Cauchy's Theorem and the rapid decay of the integrand. We put $\alpha := -\frac{\log x}{\delta}$ and obtain

$$\frac{\partial f_\delta}{\partial x}(x,\rho) = -\frac{1}{\rho}\frac{x^{-\rho-1}}{\sqrt{2\pi\delta}} \exp\left(-\frac{\log^2 x}{2\delta}\right). \quad (5.31)$$

On the other hand we have

$$\lim_{x\to 0^+} f_\delta(x,\rho) = 0\,, \quad (5.32)$$

such that

$$f_\delta(x,\rho) = -\frac{1}{\rho}\int\limits_0^x \frac{t^{-\rho-1}}{\sqrt{2\pi\delta}} \exp\left(-\frac{\log^2 t}{2\delta}\right) dt\,. \quad (5.33)$$

Using the substitution $u = \log t$ in (5.33) we can easily evaluate the resulting integral in order to obtain Lemma (5.4). ■

There results the following

Theorem (5.5)

(a) *For the Gaussian regularization of* η *we have*

$$(G_\delta\,\eta)(x) = -\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1}} \frac{x^\rho}{\rho}\exp\left(\frac{\delta}{2}\rho^2\right)\Phi\left(\frac{\rho\,\delta+\log x}{\sqrt{\delta}}\right).$$

(b) *If we define for* $x > 0$ *and* $\beta \in \mathbb{C}$ *the quantity*

$$H_\delta(x,\beta) := \exp\left(\frac{\delta}{2}\beta^2\right)\left(1-\Phi\left(\frac{\beta\,\delta+\log x}{\sqrt{\delta}}\right)\right),$$

then we obtain for $\mathrm{Re}(\beta) \geq 0$ *and* $\beta \neq 0$ *the estimate*

$$|H_\delta(x,\beta)| \leq \frac{2\,x^{-\mathrm{Re}(\beta)}}{|\beta|}\frac{1}{\sqrt{2\pi\delta}}\exp\left(-\frac{\log^2 x}{2\,\delta}\right).$$

Proof: Part (a) results from Lemma (5.4), the Mellin inversion formula and part (b). In order to show part (b) we conclude for $\beta \neq 0$

$$H_\delta(x,\beta) = \exp\left(\frac{\delta}{2}\,\beta^2\right) \left(1 - \int_0^\infty \frac{\exp\left(-\frac{1}{2}\left(\frac{\beta\,\delta + \log x}{\sqrt{\delta}} - u\right)^2\right)}{\sqrt{2\pi}}\, du\right)$$

$$= x^{-\beta}\, \exp\left(\frac{\delta}{2}\,\beta^2 + \beta\,\log x\right) \int_0^\infty \frac{\exp\left(-\frac{1}{2}\left(\frac{\beta\,\delta + \log x}{\sqrt{\delta}} + v\right)^2\right)}{\sqrt{2\pi}}\, dv$$

$$= x^{-\beta} \int_0^\infty \frac{\exp\left(-\frac{1}{2}\left(v + \frac{\log x}{\sqrt{\delta}}\right)^2\right)}{\sqrt{2\pi}}\, \exp\left(-\beta\,\sqrt{\delta}\,v\right)\, dv$$

$$= \frac{x^{-\beta}}{\beta}\,\frac{1}{\sqrt{2\pi\delta}} \int_0^\infty \left(1 - e^{-\beta\,\sqrt{\delta}\,v}\right) \left(v + \frac{\log x}{\sqrt{\delta}}\right) \exp\left(-\frac{1}{2}\left(v + \frac{\log x}{\sqrt{\delta}}\right)^2\right)\, dv\,. \tag{5.34}$$

From this equality we finally obtain for $\mathrm{Re}(\beta) \geq 0$ due to $v \geq 0$ that

$$|H_\delta(x,\beta)| \leq \frac{x^{-\mathrm{Re}(\beta)}}{|\beta|}\,\frac{1}{\sqrt{2\pi\delta}} \int_0^\infty 2\,\left(v + \frac{\log x}{\sqrt{\delta}}\right) \exp\left(-\frac{1}{2}\left(v + \frac{\log x}{\sqrt{\delta}}\right)^2\right)\, dv$$

$$= \frac{2\,x^{-\mathrm{Re}(\beta)}}{|\beta|}\,\frac{1}{\sqrt{2\pi\delta}}\,\exp\left(-\frac{\log^2 x}{2\,\delta}\right)\,. \qquad \blacksquare$$

Note that the last inequality is in perfect agreement with the asymptotic prediction (5.29) from the Central Limit Theorem, because it implies

$$\left| - \sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1}} \frac{x^\rho}{\rho}\,\exp\left(\frac{\delta}{2}\,\rho^2\right) - (G_\delta\,\eta)(x) \right| \leq \frac{c}{\sqrt{2\pi\delta}}\,\exp\left(-\frac{\log^2 x}{2\,\delta}\right) \tag{5.35}$$

for a constant $c > 0$ neither depending on $x > 0$ nor depending on $\delta > 0$. Thus the counterpart of (5.19) reads in the Gaussian limit

$$\psi(x) = x - \lim_{\delta \to 0} \sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1}} \frac{x^\rho}{\rho}\,\exp\left(\frac{\delta}{2}\,\rho^2\right) + \frac{1}{2}\,\log\frac{x^2}{x^2-1} - \log(2\pi)\,. \tag{5.36}$$

Theorem (5.5) can be generalized if we introduce for $x, \delta > 0$ and $\lambda \in \mathbb{C}$ the following integral transform of η,

$$(G_{\delta,\lambda}\,\eta)(x) := \frac{1}{\sqrt{2\pi\delta}} \int\limits_{-\infty}^{+\infty} e^{-\lambda\, u} \exp\left(-\frac{u^2}{2\,\delta}\right) \eta\,(xe^u)\; du\,. \tag{5.37}$$

For $\lambda = 0$ it reduces to $G_\delta\,\eta$. We obtain

Theorem (5.6)

(a) *For $Re(s) > 1$ we have the following Mellin transform of $G_{\delta,\lambda}\,\eta$*

$$\int\limits_0^\infty \frac{(G_{\delta,\lambda}\,\eta)(x)}{x^{s+1}}\, dx = -\exp\left(\frac{\delta}{2}\,(s-\lambda)^2\right) \sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1}} \frac{1}{\rho(s-\rho)}\,.$$

(b) *The representation of $G_{\delta,\lambda}\,\eta$ in terms of the nontrivial zeros of the ζ function reads*

$$(G_{\delta,\lambda}\,\eta)(x) = -\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1}} \frac{x^\rho}{\rho} \exp\left(\frac{\delta}{2}\,(\rho-\lambda)^2\right) \Phi\left(\frac{(\rho-\lambda)\,\delta + \log x}{\sqrt{\delta}}\right)\,.$$

Proof: For part (a) we conclude that

$$\int\limits_0^\infty \frac{(G_{\delta,\lambda}\,\eta)(x)}{x^{s+1}}\, dx = \int\limits_{-\infty}^{+\infty} \frac{e^{-\lambda\, u}}{\sqrt{2\pi\delta}} \exp\left(-\frac{u^2}{2\,\delta}\right) \left(\int\limits_0^\infty \frac{\eta\,(xe^u)}{x^{s+1}}\, dx\right) du\,. \tag{5.38}$$

On the inner integral we apply the substitution $y = x\,e^u$ with $dx/x = dy/y$, $x^{-s} = e^{us}\,y^{-s}$ and obtain

$$\int\limits_0^\infty \frac{(G_{\delta,\lambda}\,\eta)(x)}{x^s}\, \frac{dx}{x} = \int\limits_{-\infty}^{+\infty} \frac{e^{(s-\lambda)\,u}}{\sqrt{2\pi\delta}} \exp\left(-\frac{u^2}{2\,\delta}\right) \left(\int\limits_0^\infty \frac{\eta(y)}{y^s}\, \frac{dy}{y}\right) du\,. \tag{5.39}$$

From (5.13) we obtain that

$$\begin{aligned} \int\limits_0^\infty \frac{(G_{\delta,\lambda}\,\eta)(x)}{x^{s+1}}\, dx &= -\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1}} \frac{1}{\rho\,(s-\rho)} \int\limits_{-\infty}^{+\infty} \frac{e^{(s-\lambda)\,u}}{\sqrt{2\pi\delta}} \exp\left(-\frac{u^2}{2\,\delta}\right)\, du \\ &= -\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1}} \frac{1}{\rho\,(s-\rho)} \exp\left(\frac{\delta\,(s-\lambda)^2}{2}\right)\,. \end{aligned} \tag{5.40}$$

In order to prove part (b) we will directly apply the Mellin transform on the desired representation, which will also lead to an alternative proof of part (a) in Theorem (5.5). We obtain for each fixed ρ that

$$
\begin{aligned}
&- \frac{e^{\frac{\delta}{2}(\rho-\lambda)^2}}{\rho} \int_0^\infty x^{\rho-s-1}\,\Phi\left(\frac{\delta\,(\rho-\lambda)+\log x}{\sqrt{\delta}}\right)\,dx \\
&= \frac{e^{\frac{\delta}{2}(\rho-\lambda)^2}}{\rho\,\sqrt{2\pi\delta}} \int_0^\infty \frac{x^{\rho-s}}{\rho-s}\,\exp\left\{-\frac{1}{2}\left((\rho-\lambda)\sqrt{\delta}+\frac{\log x}{\sqrt{\delta}}\right)^2\right\}\,\frac{dx}{x} \\
&= \frac{1}{\rho\,(\rho-s)\,\sqrt{2\pi\delta}} \int_0^\infty x^{\lambda-s}\,\exp\left(-\frac{\log^2 x}{2\,\delta}\right)\,\frac{dx}{x} \\
&= \frac{1}{\rho\,(\rho-s)\,\sqrt{2\pi\delta}} \int_{-\infty}^\infty e^{u(\lambda-s)}\,e^{-\frac{u^2}{2\,\delta}}\,du \\
&= -\frac{1}{\rho\,(s-\rho)}\,\exp\left(\frac{\delta\,(s-\lambda)^2}{2}\right)\,. \qquad (5.41)
\end{aligned}
$$

Thus we have shown Theorem (5.6). ∎

Remark: The series in Theorem (5.6)(b) is absolute convergent due to (5.34) and $|1-\exp(-\beta\sqrt{\delta}v)| \le 1+\exp(-\mathrm{Re}(\beta)\sqrt{\delta}v)$ with $\beta := \rho-\lambda$. The integral transform in (5.37) shows a strong relationship to the windowed Fourier transforms. It permits the study of the local behaviour of $\eta(x)$ and $\psi(x)$ in terms of the nontrivial roots of the ζ function. The function $\eta(x)$ and its regularizations describe the fluctuations of the von Mangoldt function ψ, and these fluctuations are caused by the nontrivial zeros of the zeta function.

These results could be obtained from Euler's and Hadamard's product decomposition, without making use of the von Mangoldt formula. However, we can employ (5.7) in order to derive a further explicite formula which gives another explanation for the asymptotic behaviour of (5.35).

We start for real $x>0$ with the definition of the absolute convergent series

$$
g(x) := -\sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1}} \frac{x^\rho}{\rho\,(1-\rho)}\,, \qquad (5.42)
$$

which represents a continuous function for positive x. We determine g by performing for $0<x<1$ its termwise derivative and making use of the functional equation $\xi(s) = \xi(1-s)$ as follows

$$g'(x) = -\lim_{T\to\infty} \sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|\le T}} \frac{x^{\rho-1}}{1-\rho} = -\lim_{T\to\infty} \sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|\le T}} \frac{\left(\frac{1}{x}\right)^{1-\rho}}{1-\rho} = \eta(\frac{1}{x})\,. \qquad (5.43)$$

For $x = 1$ we obtain

$$g(1) = -\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1}} \frac{1}{\rho(1-\rho)} = 2\left(\log(2\pi) - 1\right) - (\gamma + \log\pi)\,, \qquad (5.44)$$

and thus for $0 < x < 1$

$$g(x) = g(1) + \int_1^x \eta\left(\frac{1}{t}\right)\,dt\,. \qquad (5.45)$$

We conclude for all $x > 0$ the convergence of

$$\tilde{\eta}(x) := -\lim_{T\to\infty} \sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|\le T}} \frac{x^{\rho}}{\rho} \qquad (5.46)$$

by addying for $0 < x < 1$ termwise the series for $g(x)$ and von Mangoldt's series for $-x\eta\left(\frac{1}{x}\right)$,

$$g(x) - x\eta\left(\frac{1}{x}\right) = \lim_{T\to\infty} \sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|\le T}} \frac{x^{\rho}\cdot(\rho-1)}{\rho(1-\rho)} = \tilde{\eta}(x)\,. \qquad (5.47)$$

Like η, the function $\tilde{\eta}$ also describes the fluctuations of von Mangoldt's ψ function and also has a logarithmic singularity at $x = 1$. Note $\tilde{\eta}(x) = \eta(x)$ for all $x > 1$. But the following "reflection formula" for $\tilde{\eta}$ is valid for all $x > 0$

$$\tilde{\eta}(x) + x\,\tilde{\eta}\left(\frac{1}{x}\right) = 2\,\tilde{\eta}(1) + \int_1^x \tilde{\eta}\left(\frac{1}{t}\right)\,dt\,. \qquad (5.48)$$

Here we have $2\,\tilde{\eta}(1) = g(1)$. Instead of (5.37) we consider now for $x, \delta > 0$ and $\lambda \in \mathbb{C}$ the integral transform

$$(G_{\delta,\lambda}\,\tilde{\eta})(x) := \frac{1}{\sqrt{2\pi\delta}} \int_{-\infty}^{+\infty} e^{-\lambda\,u} \exp\left(-\frac{u^2}{2\,\delta}\right)\,\tilde{\eta}\left(xe^{u}\right)\,du\,. \qquad (5.49)$$

Then the following Theorem gives a very simple explicit formula by making essentially use of Theorem (5.6)(b):

Theorem (5.7) *The representation of $G_{\delta,\lambda}\,\tilde{\eta}$ in terms of the nontrivial zeros of the ζ function reads*

$$(G_{\delta,\lambda}\,\tilde{\eta})(x) = -\sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1}} \frac{x^{\rho}}{\rho} \exp\left(\frac{\delta}{2}\,(\rho-\lambda)^2\right) .$$

Proof: This results formally by termwise integration from (5.46) and (5.49). For a rigorous derivation we split the integral (5.49) into two parts, integrating from $-\log x$ to ∞ and from $-\infty$ to $-\log x$, respectively. The first one is just $(G_{\delta,\lambda}\,\eta)(x)$ in Theorem (5.6)(b), and the second one can also be evaluated due to Theorem (5.6)(b) by using the relation (5.47), the integral substitution $x\,e^u = \frac{1}{x}\,e^{-v}$ and the symmetry relations $\Phi(z) + \Phi(-z) = 1$, $\xi(s) = \xi(1-s)$. ∎

Both definitions of the fluctuation functions η and $\tilde{\eta}$ are useful for different purposes. The function η has the advantage that its Mellin transform is well defined, whereas the representation formula for the regularization of $\tilde{\eta}$ in Theorem (5.7) has a very simple structure.

6 Riemann's formula for $\pi_*(x)$

In terms of Stieltjes differentials there holds the following relation between von Mangoldt's function $\psi(x)$ and Riemann's function $\pi_*(x)$, by Riemann denoted by $f(x)$,

$$d\pi_*(x)\log x \,=\, d\psi(x)\,. \tag{6.1}$$

This relation has the interesting consequence

$$\int_1^x \frac{\pi_*(t)}{t}\,dt = \pi_*(x)\log x - \psi(x)\,, \tag{6.2}$$

which is valid for all $x > 1$. Using this identity and partial integration we rewrite the integral

$$\int_1^\infty \frac{\pi_*(x)}{x^{s+1}}\,dx = \int_1^\infty \frac{\pi_*(x)}{x}\cdot\frac{dx}{x^s} = s\int_1^\infty \left(\int_1^x \frac{\pi_*(t)}{t}\,dt\right) x^{-(s+1)}\,dx\,,$$

and obtain in view of (6.2) and (2.3)

$$\int_1^\infty \frac{\pi_*(x)}{x^{s+1}}\,dx = s\int_1^\infty \frac{\pi_*(x)\log x - \psi(x)}{x^{s+1}}\,dx \tag{6.3}$$

as well as

$$\zeta(s) = \exp\left(s^2\int_1^\infty \frac{\pi_*(x)\log x - \psi(x)}{x^{s+1}}\,dx\right). \tag{6.4}$$

Assume that $\rho \in \mathbb{C}\setminus[0,\infty)$ and $\mathrm{Re}(\rho) < \mathrm{Re}(s)$. Then we define for $x > 1$

$$\varphi_\rho(x) := \gamma + \log(-\rho) + \log(\log x) + \mathrm{Ei}_0(\rho\log x)\,, \tag{6.5}$$

and obtain from Theorem (4.3) by partial integration analogous as for the derivation of (6.4) the two relations

$$(1-\frac{s}{\rho})\,\exp(\frac{s}{\rho}) = \exp\left(-s^2\int_1^\infty \frac{\varphi_\rho(x)\log x - \frac{x^\rho}{\rho}}{x^{s+1}}\,dx\right) \tag{6.6}$$

as well as

$$\frac{\exp(-s)}{s-1} = \exp\left(s^2\int_1^\infty \frac{\mathrm{Li}(x)\log x - x}{x^{s+1}}\,dx\right) \tag{6.7}$$

with the logarithmic integral $\mathrm{Li}(x) := \mathrm{Ei}(\log x)$. The next equation follows from the product representation of $1/\Gamma(s/2+1)$ and Theorem (4.4)(b) if we calculate the sum of the expressions in (6.6) with respect to all trivial roots $\rho = -2n$ of the ζ-function, namely

$$\frac{\exp(-s\,\frac{\gamma}{2})}{\Gamma(\frac{s}{2}+1)} = \exp\left(s^2\int_1^\infty \frac{\left(\log x\cdot\int_x^\infty \frac{dt}{t(t^2-1)\log t} - \frac{1}{2}\log\frac{x^2}{x^2-1}\right)}{x^{s+1}}\,dx\right). \tag{6.8}$$

Finally we have due to Lemma (4.2)(a) the identity

$$\frac{\exp(s\,\log(2\pi))}{2} = \exp\left(s^2\int_1^\infty \frac{-(\log 2)\log x + \log(2\pi)}{x^{s+1}}\,dx\right). \tag{6.9}$$

The last four representations are valid for all complex s with $\mathrm{Re}(s) > 1$, where all the integrals are well defined in the Lebesgue sense. Especially the term $\log x$ under the integral in equation (6.8) is very important since it eliminates the pole singularity at $x = 1$ of the expression

$$\int_x^\infty \frac{dt}{t(t^2-1)\log t}\,.$$

Now we determine the product of the expressions (6.6) with respect to all nontrivial zeros ρ of the ζ-function and obtain

$$\prod_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1}} \left(\left(1-\frac{s}{\rho}\right)\,\exp(\frac{s}{\rho})\right) = 2\,\xi(s)\,\pi^{s/2}\,\exp\left(s\left(1+\frac{\gamma}{2}-\log(2\pi)\right)\right)\,. \tag{6.10}$$

We note that due Theorem (4.5)(b) the following series converges pointwise and absolutely for all $x > 1$ according to

$$\lim_{T\to\infty} \sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|\le T}} \left(\varphi_\rho(x)\log x - \frac{x^\rho}{\rho}\right) =$$

$$\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1}} \left(\big(\mathrm{Ei}(\rho\log x) - i\pi\,\mathrm{sign}(\mathrm{Im}\rho)\big)\,\log x - \frac{x^\rho}{\rho}\right) =$$

$$\log x\int_0^\infty \sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1}} \frac{x^\rho\,e^{-u}}{(\rho\,(\log x)-u)^2}\,du\,. \tag{6.11}$$

For $x > 1$ we denote the expression on the right hand side in (6.11) by $\Delta(x)$. We first conclude that $\Delta(x)/x^{s+1}$ is Lebesgue integrable on each x-interval (x_0,∞) with $x_0 > 1$ for fixed s with $\mathrm{Re}(s) > 1$.

Assume for a while we know that $\Delta(x)/x^{s+1}$ is Lebesgue integrable on the whole x-interval $(1,\infty)$. The product of the four expressions (6.7)-(6.10) gives $\zeta(s)$ due to equation (5.5). We compare this product with the representation (6.4) for $\zeta(s)$ to conclude from our assumption and from von Mangoldt's formula for $\psi(x)$ with (6.11) and Theorem (5.1)(a) that Riemann's formula for $\pi_*(x)$ is also valid, namely

$$\pi_*(x) = \mathrm{Li}(x) - \lim_{T\to\infty} \sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|\le T}} \mathrm{Ei}(\rho \log x) + \int\limits_x^\infty \frac{dt}{t(t^2-1)\log t} - \log 2 \quad (6.12)$$

for all $x > 1$ such that $x \neq p^\alpha$ for all prime numbers $p \geq 2$ and all integer exponents $\alpha \geq 1$. We have already shown the pointwise convergence of the right hand side in (6.12) for these values of x. It only remains to prove for our assumption that $\Delta(x)/x^{s+1}$ is Lebesgue integrable on a finite interval $(1, x_0)$ for a constant $x_0 > 1$ and all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

According to Lemma (5.3) we will choose x_0 with $1 < x_0 < e$ sufficiently small such that we can find constants $c_1, c_2 > 0$ with

$$\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|\le 1/\log x}} \frac{1}{|Im\,\rho|} \leq c_1 \left(\log\left(\frac{1}{\log x}\right)\right)^2 \qquad \forall x \in (1, x_0)\,, \qquad (6.13)$$

$$\sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|>1/\log x}} \frac{1}{|Im\,\rho|^2} \leq c_2\,(\log x)\, \log\left(\frac{1}{\log x}\right) \qquad \forall x \in (1, x_0)\,. \qquad (6.14)$$

Then we define the functions $\Delta_1, \Delta_2 : (1, x_0) \to \mathbb{R}$ by

$$\Delta_1(x) := \log x \int\limits_0^\infty \sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|\le 1/\log x}} \frac{x^\rho\, e^{-u}\, du}{(\rho\,(\log x) - u)^2}\,, \qquad (6.15)$$

$$\Delta_2(x) := \log x \int\limits_0^\infty \sum_{\substack{\rho\,:\,\zeta(\rho)=0\\ 0<Re\,\rho<1\\ |Im\,\rho|>1/\log x}} \frac{x^\rho\, e^{-u}\, du}{(\rho\,(\log x) - u)^2}\,. \qquad (6.16)$$

Now we use again Theorem (4.5)(b) in order to conclude for all $x \in (1, x_0)$

$$|\Delta_1(x)| \leq \sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1 \\ |Im\,\rho|\leq 1/\log x}} \left\{ \log x \, |\varphi_\rho(x)| + \frac{x^{\mathrm{Re}\rho}}{|\rho|} \right\} \leq$$

$$\sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1 \\ |Im\,\rho|\leq 1/\log x}} \left\{ \log x \, \frac{x^{\mathrm{Re}\rho}}{|(\log x)\,\mathrm{Im}\rho|} + \frac{x^{\mathrm{Re}\rho}}{|\rho|} \right\} \leq$$

$$\sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1 \\ |Im\,\rho|\leq 1/\log x}} 2\, c_1 x \left(\log\left(\frac{1}{\log x} \right) \right)^2 . \tag{6.17}$$

It follows that $\Delta_1(x)/x^{s+1}$ is Lebesgue integrable on $(1, x_0)$, and in the same way we also conclude that $\Delta_2(x)/x^{s+1}$ is Lebesgue integrable on $(1, x_0)$,

$$|\Delta_2(x)| \leq \log x \sum_{\substack{\rho\,:\,\zeta(\rho)=0 \\ 0<Re\,\rho<1 \\ |Im\,\rho|>1/\log x}} \frac{x^{\mathrm{Re}\rho}}{|(\log x)\,\mathrm{Im}\rho|^2} \leq c_2\, x \, \log\left(\frac{1}{\log x} \right) . \tag{6.18}$$

Since $\Delta(x) = \Delta_1(x) + \Delta_2(x)$ for all $x \in (1, x_0)$, we have established Riemann's prime number formula (6.12).

Formula (6.12) is usually formulated in a wrong way, where the expressions $\mathrm{Ei}(\rho \log x)$ are replaced by $\mathrm{Li}(x^\rho)$. For real $\beta > 0$ we have indeed that $\mathrm{Ei}(\beta \log x) = \mathrm{Li}(x^\beta)$. The expression $\mathrm{Ei}(\beta \log x)$ is analytic with respect to β in the half plane $\mathrm{Re}(\beta) > 0$, but **not** so $\mathrm{Li}(x^\beta)$, $x > 1$ a fixed parameter. This is shown in the following two figures, where the section

$$S := \{ \beta = \frac{1}{2} + it \,|\, -10 \leq t \leq 10 \}$$

of the critical line is mapped by $\mathrm{Ei}(\beta \log x)$ as well as by $\mathrm{Li}(x^\beta)$ for $x := 1000$.

The Ei-curve in Figure 1 indicates the correct convergence to the limits $\pm i\pi$. On the other hand, the Li-curve in Figure 2 has a jump at its boundary points when x^β crosses the negative real axis for $\mathrm{Im}(\beta) = \pm \frac{(2k+1)\pi}{\log x}$ and an integer number $k \in \mathbb{Z}$, and x^β describes a circle with radius $\sqrt{x}$. This shortcoming is independent on the branch of the logarithm, and we conclude that the analytic continuation principle cannot be applied here.

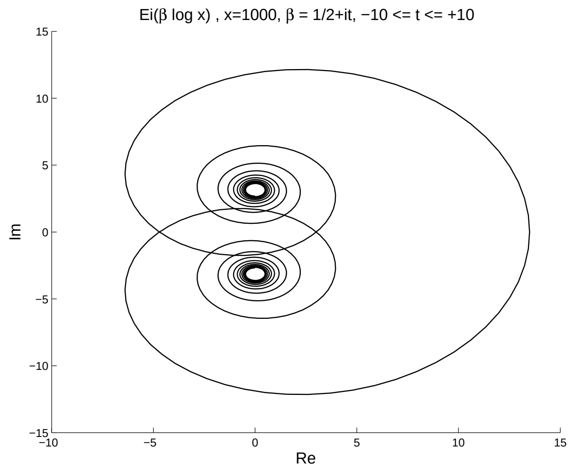

Figure 1: The correct Ei-curve

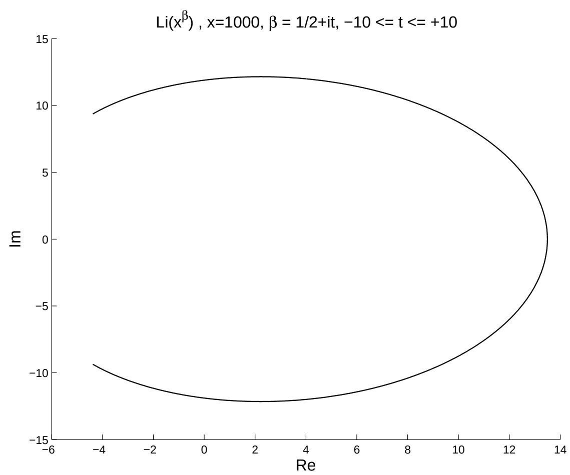

Figure 2: The incorrect Li-curve

This may be disregarded for the following reasons. First of all numerical calculations of $\pi_*(x)$ are performed due to the correct expression in (6.12) or due to correct approximations of (6.12). For numerical aspects of explicit prime number formulas and their relations to $\pi(x)$ and $\pi_*(x)$ we also refer to the book of Riesel [13] as well as to Lagarias, Miller, Odlyzko [8]. A second reason may be that Riemann's formulas for $\pi_*(x)$ or $\pi(x)$ are often neglected nowadays since they are equivalent to the simplier von Mangoldt formula for $\psi(x)$. We have used Hadamards theorem for entire functions, Theorem (5.2) for the disribution of the zeros of $\xi(s)$ and von Mangoldt's formula for $\psi(x)$, which are derived in the textbooks of Edwards [6] and Ingham [7].

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