On the existence of Hermitian-harmonic maps from complete Hermitian to complete Riemannian manifolds *

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July 8, 2004

Dedicated to Prof. E. Heinz on the occasion of his 80th birthday

Abstract

On non-Kähler manifolds the notion of harmonic maps is modified to that of Hermitian harmonic maps in order to be compatible with the complex structure. The resulting semilinear elliptic system is *not* in divergence form.

The case of noncompact complete preimage and target manifolds is considered. We give conditions for existence and uniqueness of Hermitian-harmonic maps and solutions of the corresponding parabolic system, which observe the non-divergence form of the underlying equations. Numerous examples illustrate the theoretical results and the fundamental difference to harmonic maps.

1 Introduction

Let M be a Hermitian manifold of complex dimension m with Hermitian metric $\left(\gamma_{\alpha\,\bar\beta}(z)\right)_{\alpha,\beta=1,\dots,m}$ and let N be a Riemannian manifold of real dimension n with metric $\left(g_{j\,k}(x)\right)_{j,k=1,\dots,n}$ and the Levi-Civita-connection, which in local coordinates is given by means of the Christoffel symbols $\Gamma^{j}_{k\,\ell}(x)$. We look for Hermitian harmonic maps $u: M \to N$, which are defined as solutions of the semilinear elliptic system

$$\gamma^{\alpha\bar\beta}\left(\frac{\partial^2 u^\ell}{\partial z^\alpha \partial z^{\bar\beta}} + \Gamma^\ell_{jk}\frac{\partial u^j}{\partial z^\alpha}\frac{\partial u^k}{\partial z^{\bar\beta}}\right) = 0, \qquad \ell = 1,\dots,n. \tag{1}$$

We focus on the case, where the Hermitian manifold is *not* Kähler, and where the system (1) is *not* in divergence form. This system was studied first by Jost and Yau [JY]: As they explain, in contrast with the harmonic map system, this system is compatible with the holomorphic structure on M. They obtain beside others existence and uniqueness results, which cover the Dirichlet problem for (1) on compact preimage manifolds with boundary. Subsequent work of Chen [Ch] covers the case of *target manifolds* with boundary. Extensions of existence and uniqueness results for the Dirichlet problem as obtained in the work of Jost and Yau [JY] to noncompact complete preimage manifolds were first considered by Lei Ni [LN]. He requires the bilinear form corresponding to the "holomorphic Laplace operator" for *functions* $u: M \to \mathbb{R}$

$$-\tilde\Delta u = -4\gamma^{\alpha\bar\beta}\frac{\partial^2 u}{\partial z^\alpha \partial z^{\bar\beta}}, \tag{2}$$

*Support from the research focus "Globale Methoden in der komplexen Geometrie" under the auspices of "Deutsche Forschungsgemeinschaft" is gratefully acknowledged

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to be bounded from below by a positive multiple of $\int_M u^2$. Such a condition is adequate – although very restrictive – in the selfadjoint setting, but does not really seem to fit in the nonselfadjoint framework on non-Kähler manifolds.

We impose an invertibility condition on the holomorphic Laplace operator between suitably chosen function spaces, see Assumption 1 below. These function spaces are defined in terms of decay conditions at "infinity". The preimage and image spaces for the solution operator for the holomorphic Laplacian may be chosen different, and hence our condition is very flexible and applies to many different situations. Even in the selfadjoint setting of harmonic maps this condition still applies, when 0 may be a singular value of the Laplace-Beltrami operator. In this sense, the present note also extends work of Li and Tam [LT].

For an extensive discussion we refer to Subsection 2.3 below. There it will become clear that this invertibility condition is even weaker than assuming that 0 is not a spectral value for the holomorphic Laplace operator, defined as a closed unbounded operator in one fixed function space.

The holomorphic Laplace operator coincides with the usual Laplace operator if and only if the manifold M is Kähler. That means that we focus on the case, where the holomorphic Laplacian is *not* selfadjoint.

Further we have to assume that there is an initial mapping $h : M \to N$, such that the Hermitian-harmonic differential operator, applied to h, decays suitably at ∞. Then we can show the existence of a Hermitian-harmonic map $u : M \to N$, which is homotopic to h and which approaches h at ∞. This main result is contained in Subsection 2.2

In [LT] examples of harmonic diffeomorphisms are given which are homotopic to the identity. One might expect to see similar examples for Hermitian harmonic maps. In Section 2.4 we prove that in a series of manifolds including the ones in [LT], it is not possible for the identity to satisfy the decay condition mentioned above. We believe that in those cases there do not exist Hermitian harmonic diffeomorphisms homotopic to the identity. It is not only in this respect that the complex structure of the preimage manifold and the nonselfadjoint principal part of the elliptic system complicate the construction of relevant examples. In future work we want to study the question whether the notion of Hermitian harmonic map may be modified in order to resolve the incompatibilities between the complex structure on the preimage and the Riemannian structure on the target manifold.

Originally, existence of harmonic as well as of Hermitian-harmonic maps was proved via the seeming detour of the corresponding parabolic equations. The reason is the lack of compactness properties of the underlying elliptic systems. That this approach works out also for non divergence form systems with a nonlinearity quadratic in the gradient, was observed first in [vW]. In [JY], the parabolic method was applied to the study of Hermitian-harmonic maps, and the required stability and convergence properties in C^0-norms were found. In the present paper, as well as in [LN], the exhaustion procedure will work directly on the elliptic level. Nevertheless it is interesting to know, whether solutions to (1) may be obtained as limits for $t \to \infty$ of the corresponding parabolic system also in our noncompact situation. This question is addressed and answered in Section 3. To ensure convergence we need to impose a decay condition on the linear heat operator

$$\left(\frac{\partial}{\partial t} - \tilde{\Delta}\right) u,$$

which is related to the invertibility condition for the holomorphic Laplace operator. This decay condition is discussed and illustrated in Subsection 3.2 with help of the same series of examples as for the elliptic system.

2 The elliptic Hermitian-harmonic map system

2.1 Preliminaries

In this section, after explaining the notation, we collect some basic results from the fundamental papers on Hermitian-harmonic maps by Jost and Yau [JY] and Lei Ni [LN].

First we specify and explain our notation. Let M be a complete Hermitian manifold of complex dimension m with Hermitian metric

$$\left(\gamma_{\alpha\,\bar{\beta}}(z)\right)_{\alpha,\beta=1,\ldots,m}$$

in local coordinates. By $\gamma^{\alpha\,\bar{\beta}}$ we denote the transposed inverse matrix

$$\sum_{\sigma=1,\ldots,m}\left(\gamma^{\alpha\,\bar{\sigma}}\gamma_{\beta\,\bar{\sigma}}(z)\right)=\delta^{\alpha}_{\beta}.$$

With respect to this metric, the length of a holomorphic tangential vector $w=\left(w^1,\ldots,w^m\right)$ at $z\in M$ in local coordinates is given by

$$\|w\|^2=\sum_{\alpha,\beta=1,\ldots,m} w^{\alpha}\gamma_{\alpha\,\bar{\beta}}(z)\bar{w}^{\beta}.$$

Furthermore, let N be a complete Riemannian manifold of real dimension n with metric

$$\left(g_{j\,k}\right)_{j,k=1,\ldots,n}$$

in local coordinates, its inverse

$$\sum_{\ell=1,\ldots,n} g_{j\,\ell}g^{\ell\,k}=\delta^k_j$$

and the Christoffel symbols

$$\Gamma^j_{k\,\ell}=\frac{1}{2}\sum_{s=1}^{n}g^{js}\left(\frac{\partial g_{\ell s}}{\partial x^k}+\frac{\partial g_{sk}}{\partial x^\ell}-\frac{\partial g_{k\ell}}{\partial x^s}\right).$$

While on the target manifold N, we consider the Levi-Civita connection of the metric, we choose a different connection on the preimage manifold M. We choose a suitable holomorphic torsion free connection such that the “holomorphic Laplace operator” takes the form as above in (2).

Further we need to define the tension field for any smooth map $u: M\to N$ according to the chosen connections

$$(\sigma(u))^\ell \;:=\; \gamma^{\alpha\bar{\beta}}\left(\frac{\partial^2 u^\ell}{\partial z^\alpha\partial z^{\bar{\beta}}}+\Gamma^\ell_{jk}\frac{\partial u^j}{\partial z^\alpha}\frac{\partial u^k}{\partial z^{\bar{\beta}}}\right),\qquad \ell=1,\ldots,n. \tag{3}$$

The first result we need to mention concerns the energy density function $e(u)$, which for any smooth map $u: M\to N$ is defined in local coordinates as follows

$$e(u):=\left(g_{jk}\circ u\right)\gamma^{\alpha\bar{\beta}}\frac{\partial u^j}{\partial z^\alpha}\frac{\partial u^k}{\partial z^{\bar{\beta}}}. \tag{4}$$

If we assume $u: M\to N$ to be a Hermitian harmonic map and N to have nonpositive sectional curvature, then according to [JY, p. 225, formula (5)], for any relatively compact open set $\Omega\subset M$ we have the following differential inequality

$$-\tilde{\Delta}e(u)\le C(\Omega)e(u). \tag{5}$$

The constant C is expected to blow up in general, when Ω is approaching M. For the reader’s convenience we sketch the proof of (5) in Appendix B.

One should observe that by the Hopf-Rinow-theorem (see e.g. [A, 1.37]) the compact subsets of M are precisely the bounded closed sets.

The next important result is due to Lei Ni [LN, Corollary 3.5]. For this we need first to explain the geodesic homotopy distance between two smooth homotopic maps u and $v : M \to N$. Let us recall a result of von Mangoldt-Hadamard-Cartan. Fix a homotopy H between u and v, then, since the target manifold is nonpositively curved, for any $z \in M$ there is precisely one geodesic arc connecting $u(z)$ and $v(z)$ in the same homotopy class as the original arc given by H. Moreover this geodesic arc is length minimizing. See e.g. [J2, Lemma 8.7.1]. The geodesic homotopy distance

$$\rho := \rho(z) := \rho(u(z), v(z))$$

is defined as the length of this geodesic arc.

According to [LN, Corollary 3.5], ρ satisfies the following fundamental differential inequality:

$$-\tilde{\Delta}\rho \leq 4\left(\|\sigma(u)\| + \|\sigma(v)\|\right). \tag{6}$$

In the next section, we will construct Hermitian-harmonic maps by an exhaustion procedure and by solving a boundary value problem for (1) on compact submanifolds of M. The above estimate will turn out to be essential for getting first estimates for the approximate solutions to (1).

2.2 Existence and uniqueness results

We first introduce spaces of suitably decaying functions (at “infinity”), which are adequate in our nonselfadjoint and noncompact framework.

Definition 1. *For $\mu > 0$, let*

$$\begin{aligned} C^0_\mu(M) \quad := \quad & \{f : M \to \mathbb{R};\ f \text{ is continuous and} \\ & \text{there exists } z_0 \in M \text{ and a constant } C = C(f) \text{ such that } |f(z)| \leq C\ (1 + d(z, z_0))^{-\mu}\}. \end{aligned} \tag{7}$$

Assumption 1 (Invertibility of the holomorphic Laplace operator).
We assume that there exist positive numbers $\mu, \mu' > 0$ such that for every $f \in C^0_\mu(M)$, there exists precisely one solution $u \in C^0_{\mu'}(M)$ of

$$-\tilde{\Delta}u = f \text{ in } M.$$

Theorem 1 (Existence and uniqueness of Hermitian harmonic maps). *Assume that M is a noncompact complete Hermitian manifold such that for the holomorphic Laplace operator $-\tilde{\Delta}$ on M, the Assumption 1 is satisfied with positive numbers $\mu, \mu' > 0$. Further let N be a complete Riemannian manifold with nonpositive sectional curvature and $h : M \to N$ a smooth map with $\|\sigma(h)\| \in C^0_\mu(M)$.*

Then there exists a Hermitian harmonic map $u : M \to N$, which is homotopic to h. Moreover, if ρ denotes the homotopy distance between u and h, we have $\rho \in C^0_{\mu'}(M)$. Finally, in this class, the solution is unique.

Proof. The fundamental idea is as in the paper [LN]. Here, however, we replace the “selfadjoint” tools by the appropriate nonselfadjoint analogues. Let $(\Omega_k)_{k\in\mathbb{N}}$ be a compact smooth exhaustion of M. According to Theorem 6 of the paper [JY] by J. Jost and S.-T. Yau, there exist solutions $u_k : \Omega_k \to N$ of the Dirichlet problems

$$\begin{cases} \sigma(u_k) = 0 & \text{in } \Omega_k, \\ u_k = h & \text{on } \partial\Omega_k, \\ u_k \text{ homotopic to } h, & \text{with respect to } \partial\Omega_k. \end{cases} \tag{8}$$

In order to show convergence of (u_k) to a Hermitian harmonic map $u : M \to N$, it is enough to prove local boundedness of the energy density functions $e(u_k)$. As in [LN] we start with global bounds for the homotopy distances ρ_k between u_k and h and $\rho_{k,\ell}$ between u_k and u_ℓ.

We first introduce a comparison function, the existence of which is ensured by Assumption 1: Since $\|\sigma(h)\|$ is assumed to be in $C^0_\mu(M)$, we find a smooth function $V \in C^0_{\mu'}(M)$, such that

$$-\tilde{\Delta} V = 4\|\sigma(h)\| \text{ in } M. \tag{9}$$

In particular, $V(z)$ decays uniformly, as $d(z, z_0) \to \infty$. Together with the strong maximum principle, which can be easily proven by passing to local coordinates and by exploiting the connectedness of M, this gives first $V \geq 0$ and then by repeating the argument:

$$V > 0. \tag{10}$$

By (6), the coincidence of ρ_k and h on $\partial\Omega_k$ and (10), we find the following inequalities for the homotopy distance ρ_k between u_k and h:

$$\begin{aligned} -\tilde{\Delta}\rho_k &\leq 4\|\sigma(h)\| = -\tilde{\Delta} V \text{ in } \Omega_k, \\ \rho_k|\partial\Omega_k &= 0 < V|\partial\Omega_k. \end{aligned}$$

From the maximum principle, we get the uniform bound:

$$0 \leq \rho_k \leq V, \tag{11}$$

where $V \in C^0_{\mu'}(M)$ is the comparison function, introduced in (9) above.

In a second step we will exploit the differential inequality (5) for the energy density $e(u_k)$ of the approximate Hermitian harmonic maps u_k.

We take a local L^1-bound for $e(u_k)$ from [LN, pp. 344/345]: For some fixed $z_0 \in M$ and any $R > 0$, we have with a suitable constant

$$\int_{B_R(z_0)} e(u_k) \leq C. \tag{12}$$

This bound holds true also in our situation since we have shown the maximum bound (11) for ρ_k above.

Eventually from this local L^1-bound (12), we get local L^∞-bounds by making use of the local maximum principle [GT, Theorem 9.20] for elliptic operators, which are not in divergence form. First we work in sufficiently small open sets of M, where simply one chart is sufficient. The holomorphic Laplace operator in these local coordinates satisfies the assumptions of the local maximum principle and we exploit the differential inequality

$$-\tilde{\Delta} e(u_k) \leq C_{\text{loc}} e(u_k).$$

See (5); the constant can be found at least on compact subsets of M. Second, since by the Hopf-Rinow theorem (see e.g. [A, 1.37]), all the $\overline{\Omega}_\ell$ are compact, we get there with help of a bootstrapping argument uniform $C^{2,\alpha}$-bounds and hence convergence to a smooth solution u of the Hermitian harmonic map system (1).

It is easy to see that u and h are homotopic. For this purpose we extend $u_k : \Omega_k \to N$ by h to a continuous mapping $\tilde{u}_k : M \to N$. Further let $\tilde{u}_0 := h$. Obviously, $\tilde{u}_k$ and $\tilde{u}_{k+1}$ are homotopic; for $k \in \mathbb{N}_0$ let $H_k : \left[\frac{1}{k+2}, \frac{1}{k+1}\right] \times M \to N$ be continuous with $H_k\left(\frac{1}{k+1}, \,.\,\right) = \tilde{u}_k$ and $H_k\left(\frac{1}{k+2}, \,.\,\right) = \tilde{u}_{k+1}$. Defining

$$\begin{aligned} H &: [0,1] \times M \to N, \\ H(t, \,.\,) &= \begin{cases} H_k(t, \,.\,), & \text{if } t \in \left[\frac{1}{k+2}, \frac{1}{k+1}\right], \\ u, & \text{if } t = 0, \end{cases} \end{aligned}$$

we get a homotopy between u and h.

We conclude from (11) and locally uniform convergence that $0 \le \rho \le V$ and hence $\rho \in C^0_{\mu'}(M)$.

Finally we prove uniqueness of the solution u with the mentioned properties. Let $\tilde{u} : M \longrightarrow N$ be an arbitrary Hermitian-harmonic map of class $C^0_\mu(M)$ homotopic to h, such that $\rho(\tilde{u}, h) \in C^0_{\mu'}(M)$. By (6) we know

$$-\tilde{\Delta}\rho(u, \tilde{u}) \le 0$$

and, by the previous arguments, that

$$0 \le \rho(u, \tilde{u}) \le \rho(u, h) + \rho(\tilde{u}, h) \in C^0_{\mu'}(M).$$

Hence for every $\varepsilon > 0$ outside a sufficiently large ball $B_R(z_0)$ around an arbitrary $z_0 \in M$ we have

$$\rho(u, \tilde{u}) \le \varepsilon.$$

By the maximum principle this implies $\rho(u, \tilde{u}) \le \varepsilon$ on all of M for every $\varepsilon > 0$ and hence $\rho(u, \tilde{u}) = 0$. This implies $u = \tilde{u}$. □

2.3 Examples

First, with help of some examples, we want to discuss the invertibility condition on the holomorphic Laplace operator, i.e. Assumption 1. We are basing our first examples on the following simple result:

Lemma 1. *Let $n > 2$, $\alpha \in \left(0, \frac{n}{2} - 1\right)$. Then, for every continuous $f : \mathbb{R}^n \to \mathbb{R}$ with $|f(x)| \le C\,(1 + |x|^2)^{-\alpha-1}$, we find precisely one strong solution $u : \mathbb{R}^n \to \mathbb{R}$ of*

$$-\Delta u = f \text{ in } \mathbb{R}^n, \tag{13}$$

such that

$$|u(x)| \le C\,(1 + |x|^2)^{-\alpha}.$$

Proof. We define

$$v(x) := (1 + |x|^2)^{-\alpha}$$

as a barrier function and calculate:

$$\begin{aligned} -\Delta v(x) &= 2\alpha n(1 + |x|^2)^{-\alpha-1} - 4\alpha(\alpha+1)|x|^2(1 + |x|^2)^{-\alpha-2} \\ &\ge c_{n,\alpha}(1 + |x|^2)^{-\alpha-1}; \end{aligned}$$

where the positive constant $c_{n,\alpha}$ is given by

$$c_{n,\alpha} = 4\alpha\left(\frac{n}{2} - (\alpha+1)\right).$$

In order to find a solution to (13), we solve the corresponding Dirichlet problems with homogeneous boundary data on the balls B_k around the origin with radius k. Since a suitable multiple of v will serve as a barrier function for the approximate solutions $|u_k|$, after selecting a suitable subsequence we will have local convergence in C^0 and weakly in $W^{2,p}$ for arbitrarily large p towards an entire solution of (13), obeying the same bound $C\,v(x)$.

Uniqueness is immediate from Liouville's theorem. □

Example 1. *Let us consider $M = \mathbb{C}^m$, $m \geq 2$, with the standard euclidean metric, so that the holomorphic Laplacian is also the standard one: Δ_e. Then, according to the previous lemma, Assumption 1 is satisfied with any $\mu \in (2, 2m)$ and $\mu' = \mu - 2$.*

In this example, the holomorphic Laplace operator is selfadjoint. Although we do not focus on this case here, this observation shows: Even if 0 is a singular value of the Laplace operator, our invertibility assumption may still be satisfied.

In order to cover also nonselfadjoint examples, we would like to equip $M = \mathbb{C}^m$, $m \geq 2$ with the conformal metric

$$\gamma_{\alpha\,\bar\beta}(z) = (1+|z|^2)^{-1}\delta_{\alpha\,\beta}.$$

The holomorphic Laplace operator then becomes

$$-\tilde\Delta = -(1+|z|^2)\Delta_e$$

with Δ_e being the euclidean Laplace operator. In $L^2\left(\mathbb{C}^m, \gamma\left(\frac{i}{2}\right)^m (d\,z_1 \wedge d\,\bar z_1) \wedge \ldots \wedge (d\,z_m \wedge d\,\bar z_m)\right) = L^2\left(\mathbb{C}^m, \left((1+|z|^2)^{-m}\left(\frac{i}{2}\right)^m (d\,z_1 \wedge d\,\bar z_1) \wedge \ldots \wedge (d\,z_m \wedge d\,\bar z_m)\right)\right)$, the holomorphic Laplacian is not selfadjoint.

Since the Kähler form is given by

$$\omega = \frac{i}{2}(1+|z|^2)^{-1}\sum dz_\alpha \wedge d\bar z_\alpha$$

we compute

$$d\omega = \frac{i}{2}(1+|z|^2)^{-2}\sum_{\alpha,\beta} 2(\bar{z_\beta} dz_\beta \wedge dz_\alpha \wedge d\bar{z_\alpha} + z_\beta dz_\alpha \wedge d\bar{z_\alpha} \wedge d\bar{z_\beta}) \neq 0.$$

This means that (M,γ) is not a Kähler manifold, what is important, since otherwise Hermitian-harmonic maps are harmonic.

Again, Lemma 1 shows, that for any smooth f with $|f(z)| \leq C(1+|z|^2)^{-\alpha}$, $\alpha \in (0, m-1)$, we find a solution u of

$$-\tilde\Delta u = f \quad \text{in } \mathbb{C}^m$$

with $|u(z)| \leq C(1+|z|^2)^{-\alpha}$. However, with this metric we have

$$d(z,0) \sim \log(1+|z|^2), \qquad |z| \sim \exp(d(z,0)) - 1.$$

This example doesn't fall under our formulation of Assumption 1. However it shows that the choice of the metric

$$\gamma_{\alpha\,\bar\beta}(z) = (1+|z|^2)^{-1}\delta_{\alpha\,\beta}.$$

and of the corresponding holomorphic Laplace operator

$$-\tilde\Delta = -(1+|z|^2)\Delta_e$$

may be reasonable. Since $\log(1+|z|^2) \sim d(0,z)$, where $|z|$ is the euclidean norm and $d(z,0)$ the distance in our metric to the origin, we should find a refinement of Lemma 1, which involves logarithmic terms:

Lemma 2. *Let the dimension be $n > 2$ and let $\alpha > 0$ be a real number. Then for every $f : \mathbb{R}^n \to \mathbb{R}$ with $|f(x)| \leq C\,\left(\log(2+|x|^2)\right)^{-\alpha-1}$, we find precisely one solution $u : \mathbb{R}^n \to \mathbb{R}$ of*

$$-(1+|x|^2)\Delta u = f \text{ in } \mathbb{R}^n, \tag{14}$$

such that

$$|u(x)| \leq C\,\left(\log(2+|x|^2)\right)^{-\alpha}.$$

Proof. Similarly as in the proof of Lemma 1 we look for a suitable comparison function. First let us work with an auxiliary number $A \geq 2$, which will be fixed in the course of the following calculations. We define

$$v(x) := \left(\log(A+|x|^2)\right)^{-\alpha}$$

and calculate:

$$\begin{aligned}
-\Delta v(x) &= 2\alpha \left(\log(A+|x|^2)\right)^{-\alpha-1} \left(\frac{n}{A+|x|^2} - 2\frac{|x|^2}{(A+|x|^2)^2}\right) \\
&\quad -4\alpha(\alpha+1)\left(\log(A+|x|^2)\right)^{-\alpha-2} \frac{|x|^2}{(A+|x|^2)^2} \\
&\geq 2\alpha(n-2)\left(\log(A+|x|^2)\right)^{-\alpha-1} \frac{1}{A+|x|^2}\left\{1-2\frac{\alpha+1}{n-2}\left(\log(A+|x|^2)\right)^{-1}\right\} \\
&\geq \frac{\alpha(n-2)}{A}\left(\log(A+|x|^2)\right)^{-\alpha-1} \frac{1}{1+|x|^2},
\end{aligned}$$

provided A is chosen large enough in dependence on $\alpha > 0$ and $n > 2$. As in the proof of Lemma 1, we have now: For every continuous function f with $|f(x)| \leq C\left(\log(A+|x|^2)\right)^{-\alpha-1}$ we have precisely one solution u of $-(1+|x|^2)\Delta u(x) = f(x)$ in $\mathbb{R}^n$ with $|u(x)| \leq C\left(\log(A+|x|^2)\right)^{-\alpha}$. But since the strictly positive function $(0,\infty) \ni r \mapsto \log(2+r^2)/\log(A+r^2)$ is bounded from above and below, this immediately gives the statement of our lemma. □

Example 2. *Let $M = \mathbb{C}^m$, $m \geq 2$ be equipped with the conformal metric*

$$\gamma_{\alpha\bar{\beta}}(z) = (1+|z|^2)^{-1}\delta_{\alpha\beta},$$

such that the holomorphic Laplace operator is

$$-\tilde{\Delta} = -(1+|z|^2)\Delta_e$$

with Δ_e being the euclidean Laplace operator. Then $-\tilde{\Delta}$ satisfies the invertibility condition Assumption 1 with any $\mu > 1$ and $\mu' = \mu - 1$.

The second purpose of this subsection is to discuss the decay condition on $\|\sigma(h)\| \in C^0_\mu(M)$. For this we construct some prototype manifolds M and N and suitable "initial maps" $h : M \to N$.

Example 3. On $\mathbb{R}^2$, the rotational symmetric metric $g_0 = dr^2 + (r^2+r^4)d\phi^2$ has strictly negative curvature. If we now choose $N = \mathbb{R}^2 \times \mathbb{R}^2$ with the metric $g = pr_1^* g_0 + pr_2^* g_0$. where $pr_i : N \longrightarrow \mathbb{R}^2$ denotes the projections onto the i-th copy of $\mathbb{R}^2$, then (N, g) has nonpositive sectional curvature.

As for the manifold M we first choose $\tilde{M} = \mathbb{C}^2$ with the Hermitian metric

$$\tilde{\gamma} = \frac{1}{1+|z|^2}(dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2).$$

Then it is easy to see that the geodesic length $d(z,0) \sim \log(1+|z|^2)$.

Now $M := \tilde{M} \setminus B_1(0)$ shall be regarded as a manifold with boundary $\partial B_1(0)$. The proof of the theorem works in the same way for this M where additionally $u = h$ on $\partial B_1(0)$ can be satisfied.

If we define $h : M \longrightarrow N$ via

$$h(z) = \frac{z}{1+|z|^2},$$

the norm of the tension field $\|\sigma(h)\|$ can be computed to be

$$\|\sigma(h)\| = \frac{|z|(7+2|z|^2)}{2(1+|z|^2)^2} \leq \frac{7|z|}{2(1+|z|^2)},$$

and hence $\|\sigma(h)\| \in C^0_\mu(M)$ for every $\mu > 0$.

Applying Example 2 for $\mu > 1$ yields by Theorem 1 a Hermitian-harmonic map $u : M \longrightarrow N$ homotopic to h with $u = h$ at $\partial B_1(0)$ and approaching 0 at infinity. In particular, u is not a constant map.

Example 4. Let $M = (\mathbb{S}^1)^{2m-1} \times (-1,1)$, $m \geq 2$, be equipped with the following complex structure: Denote $H := \{z \in \mathbb{C} | \, |\Im(z)| < 1\}$ and take $(\mathbb{C}^{m-1} \times H)/\Gamma \cong M$, where Γ is the cartesian lattice of rank $2m-1$.

For the choice of the metric, denote by s the noncompact parameter with range $s \in (-1,1)$. Then the metric

$$\tilde{\gamma}_{\alpha\bar{\beta}} := f(s)\delta_{\alpha\bar{\beta}}$$

in cartesian coordinates is Γ-invariant and not Kähler unless f is constant. We denote by γ the induced metric on M. We choose $\delta > 0$ and

$$f(s) := \begin{cases} \delta^2(1-|s|)^{-2\delta-2} & \text{for } 1/2 < |s| < 1, \\ a(s) & \text{for } |s| \leq 1/2, \end{cases}$$

such that $a(s) > 0$ for all $|s| \leq 1/2$ and $f \in C^\infty(-1,1)$.

Since for $|s|$ close to 1 one has $d(z,0) \sim (1-|s|)^{-\delta} - 1 := \tilde{d}(s)$, the metric γ is complete.

We can prove that $-\tilde{\Delta}(1+\tilde{d})^{-\mu'} > C(1+\tilde{d})^{-\mu'-2}$ as long as $\delta\mu' < 1$ and $|s| > 1/2$: Since

$$(1+\tilde{d}(s))^{-\mu'} = (1-|s|)^{\delta\mu'},$$

we get even

$$\begin{aligned} -\tilde{\Delta}(1+\tilde{d}(s))^{-\mu'} &= -\frac{1}{\delta^2}(1-|s|)^{2\delta+2}\frac{\partial^2}{\partial s^2}(1-|s|)^{\delta\mu'} \\ &= \frac{\mu'(1-\delta\mu')}{\delta}(1-|s|)^{2\delta+2}(1-|s|)^{\delta\mu'-2} \\ &= \frac{\mu'(1-\delta\mu')}{\delta}(1-|s|)^{\delta(\mu'+2)} \\ &= \frac{\mu'(1-\delta\mu')}{\delta}(1+\tilde{d}(s))^{-\mu'-2}. \end{aligned}$$

Now we remark that $b(s) = 1 + \varepsilon(1/4 - s^2)$ satisfies $-\tilde{\Delta}b > 0$ for every $\varepsilon > 0$. We define

$$v(s) = \begin{cases} (1+\tilde{d}(s))^{-\mu'} & \text{for } |s| > 1/2, \\ (1+\tilde{d}(1/2))^{-\mu'}b(s) & \text{for } |s| \leq 1/2. \end{cases}$$

Then we compute for $\phi \in C_0^\infty(M), \phi \geq 0$

$$\begin{aligned} \int \left(-\tilde{\Delta}^*\phi\right) v\, f^m\, dx &\geq \left(\phi f^{m-1}\right)(1/2)\left(\frac{\partial v}{\partial s}\left(\frac{1}{2}-0\right) - \frac{\partial v}{\partial s}\left(\frac{1}{2}+0\right)\right) \\ &\quad + \left(\phi f^{m-1}\right)(-1/2)\left(\frac{\partial v}{\partial s}\left(-\frac{1}{2}-0\right) - \frac{\partial v}{\partial s}\left(-\frac{1}{2}+0\right)\right) \\ &= \left(\left(\phi f^{m-1}\right)(1/2) + \left(\phi f^{m-1}\right)(-1/2)\right)\left(-\varepsilon + \mu'\frac{\tilde{d}'(1/2)}{1+\tilde{d}(1/2)}\right)\left(1+\tilde{d}(1/2)\right)^{-\mu'} \\ &\geq 0 \end{aligned}$$

for ε sufficiently small. Hence $v(s) \in C^0_{\mu'}(M)$ is a supersolution and we proceed like before to prove the validity of Assumption 1 with $\mu' = \mu - 2$, $\mu > 2$ and $\delta(\mu - 2) < 1$. Hence we have proved:

Lemma 3. *Let $M := (\mathbb{S}^1)^{2m-1} \times (-1, 1)$ be like in Example 4. If $\delta > 0, \mu > 2$ and $\delta(\mu - 2) < 1$, then Assumption 1 is valid with $\mu' := \mu - 2$.*

Now we construct a starting map h. The idea is to fix the values in both infinite edges and to interpolate such that $\|\sigma(h)\| \in C^0_\mu(M)$.

For this purpose denote $N = B_1(0) \subset \mathbb{R}^n$ equipped with the Poincaré metric $g = \frac{1}{(1-r^2)^2}\delta_{ij}$.

Proposition 1. *If M is like in Example 4, N is the unit ball with the Poincaré metric, and $\tilde{h} : \mathbb{C}^{m-1} \times H \longrightarrow \mathbb{R}^n$ a Γ-invariant C^2-map with bounded first and second derivatives and the image $\tilde{h}(\mathbb{C}^{m-1} \times H) \subset N$ being precompact in N, then there is a Hermitian-harmonic map $u : M \longrightarrow N$ homotopic to the quotient map $h : M \longrightarrow N$.*

Proof. It suffices to prove that $\|\sigma(h)\| \in C^0_\mu(M)$ for some $\mu > 2$. For this purpose we choose $2 < \mu < 2+\frac{1}{\delta}$. Then Lemma 3 shows that Assumption 1 is valid. We note that $|\Gamma^l_{jk}| \le \frac{r}{1-r^2} \le \frac{1}{1-r^2}$ for the given Poincaré metric, which is an easy calculation. By assumption,

$$\left|\frac{\partial^2}{\partial z^\alpha \partial z^{\bar\beta}}\tilde{h}^j\right| \le C_1, \quad \left|\frac{\partial}{\partial z^\alpha}\tilde{h}^j\right| \le C_1,$$

and $r(x) := |\tilde{h}(x)| \le q < 1$ in $\mathbb{C}^{m-1} \times H$. Now we can estimate

$$\begin{aligned}
\|\sigma(h)\|^2 &\le C_2 \frac{(1-|s|)^{4+4\delta}}{(1-r^2)^2}\left(C_1 + \frac{C_3}{(1-r^2)}\right)^2 \\
&\le C_4(1-|s|)^{4+4\delta} \\
&\le C_4(1-|s|)^{2\delta\mu} \\
&= C_4(1+\tilde{d}(s))^{-2\mu},
\end{aligned}$$

if $2 < \mu \le 2 + \frac{2}{\delta}$. By our choice even $2 < \mu < 2 + \frac{1}{\delta}$ holds. □

2.4 Negative Results

Harmonic maps sometimes may be thought of as diffeomorphisms or deformations of the identity in an appropriate setting. If, for example, $M = N$ is the unit ball, N equipped with the Poincaré metric and M with a slightly perturbed Poincaré metric, in [LT] it is proved that there is a harmonic map $u : M \longrightarrow N$ homotopic to the identity. This suggests to choose h as an identity map and to use Theorem 1 in order to obtain a Hermitian-harmonic map homotopic to h. This idea fails in many examples, in particular, if $M = N$ is the unit ball with the Poincaré metric. We will prove that in this case the assumptions of Theorem 1 are not satisfied. We have to leave open whether there are Hermitian-harmonic maps homotopic to the identity.

Since we are now going to inquire into rotational symmetric metrics, let us collect some basic knowledge.

Lemma 4. *Let $B_r(0) \subset \mathbb{R}^k$ be equipped with a rotational symmetric Riemannian metric γ. Let $x \in B_r(0)$ and Γ be a geodesic connecting x and 0. Then Γ is a line segment.*

Proof. First we note that by rotational symmetry the geodesic equations tell us that the line segment between x and 0 is a geodesic. Now take $y \in \Gamma$ near 0 such that there is only one geodesic through y and 0. This has to be the line segment connecting y and 0. Since the line through y and 0 is the unique geodesic with tangent direction $\Gamma'(y)$ in y, we conclude that Γ is the line segment between x and 0. □

If $0 \notin \Omega$ we obtain a somewhat weaker result:

Lemma 5. *Let $I \subset \mathbb{R}^+$ be an open interval and $I \times S^{k-1} \cong \Omega \subset \mathbb{R}^k$ be an annulus equipped with a rotational symmetric Riemannian metric γ of the form $\gamma = pr_1^*\gamma_r + pr_2^*\gamma_\phi$ ('polar block form'). Let $x, y \in M$ be collinear with 0. Then the shortest geodesic between x and y is a line segment.*

Proof. By assumption,

$$\gamma = a(r)dr^2 + b_{ij}(r)d\phi_i d\phi_j,$$

with $a > 0, (b_{ij}) > 0$. If $\Gamma : [0,1] \longrightarrow \Omega$ is a path connecting x and y, then

$$l(\Gamma) = \int_0^1 \sqrt{a\left(\frac{d\Gamma_r}{ds}\right)^2 + b_{ij}\frac{d\Gamma_{\phi_i}}{ds}\frac{d\Gamma_{\phi_j}}{ds}}ds \geq \int_0^1 \sqrt{a\left(\frac{d\Gamma_r}{ds}\right)^2}ds = l(L),$$

if L denotes the line segment between x and y. □

Remark 1. *Note that the polar block form condition of Lemma 5 is satisfied if γ is conformal to the euclidean metric.*

First we show the positive result that the Poincaré ball is within the range of Assumption 1:

Example 5. *Let $M = D^4 := \{z \in \mathbb{C}^2| \quad |z| < 1\}$ equipped with the Poincaré metric $\gamma := \frac{4}{(1-|z|^2)^2}\delta_{\alpha\bar{\beta}}$. Then Assumption 1 is valid for $\mu > 1$ and $\mu' := \mu - 1$.*

Proof. Since γ is rotational symmetric, Lemma 4 states that geodesics through 0 are lines, hence the distance function is given by

$$d(0,z) = \int_0^{|z|} \frac{2}{1-t^2}dt = 2\text{artanh}(|z|).$$

For a rotational symmetric function $f(r)$ on $\mathbb{R}^4$ the ordinary Laplacian is given by

$$\Delta f = \left(\frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r}\right)f.$$

So we compute

$$\begin{aligned}\Delta(A + 2\text{artanh}(r))^{-\mu'} &= \frac{4\mu'(\mu'+1)}{(1-r^2)^2}(A + 2\text{artanh}(r))^{-(\mu'+2)} - \frac{4\mu' r}{(1-r^2)^2}(A + 2\text{artanh}(r))^{-(\mu'+1)} \\ &\quad - \frac{6\mu'}{r(1-r^2)}(A + 2\text{artanh}(r))^{-(\mu'+1)}\end{aligned}$$

and hence

$$\begin{aligned}-\tilde{\Delta}(A + 2\text{artanh}(r))^{-\mu'} &= -\frac{1}{4}(1-r^2)^2\Delta(A + 2\text{artanh}(r))^{-\mu'} \\ &= -\mu'(\mu'+1)(A + 2\text{artanh}(r))^{-(\mu'+2)} + \mu' r(A + 2\text{artanh}(r))^{-(\mu'+1)} \\ &\quad + \frac{3\mu'(1-r^2)}{2r}(A + 2\text{artanh}(r))^{-(\mu'+1)}.\end{aligned}$$

Elementary calculations show that the coefficient of $(A + 2\text{artanh}(r))^{-(\mu'+1)}$ is strictly decreasing,

$$\frac{\partial}{\partial r}\left(\mu' r + \frac{3\mu'(1-r^2)}{2r}\right) < 0$$

and hence

$$\mu' r + \frac{3\mu'(1-r^2)}{2r} > \mu'.$$

If we now choose $A > \mu' + 1$, then

$$-\tilde{\Delta}(A + 2\text{artanh}(r))^{-\mu'} > C(A + 2\text{artanh}(r))^{-\mu'-1},$$

with $C := \frac{A-\mu'-1}{A}$.

With arguments as above this implies that Assumption (1) is satisfied for any $\mu > 1$ and $\mu' := \mu - 1$. $\square$

Now we turn our attention to the norm of the tension field. We will see that this is the crucial point.

Proposition 2. *If M is a complex manifold with Hermitian metrics γ and $\tilde{\gamma}$, and if $id : (M,\gamma) \longrightarrow (M,\tilde{\gamma})$ denotes the identity map, then we define*

$$A^{\varepsilon} := \frac{1}{2}\tilde{\gamma}^{\varepsilon\bar{\delta}}\gamma^{\alpha\bar{\beta}}\left(\tilde{\gamma}_{\alpha\bar{\delta},\bar{\beta}} - \tilde{\gamma}_{\alpha\bar{\beta},\bar{\delta}}\right).$$

With this vector field given we obtain

$$\|\sigma(id)\|^2 = \gamma_{\varepsilon\bar{\phi}}A^{\varepsilon}\overline{A^{\phi}}.$$

This formula simplifies in the conformal case:

Proposition 3. *If $\gamma = f\delta_{\alpha\bar{\beta}}, \tilde{\gamma} = \tilde{f}\delta_{\alpha\bar{\beta}}$, with smooth real valued positive functions $f, \tilde{f}$, then $\|\sigma(id)\| = \frac{m-1}{2f}\left|\nabla\sqrt{\tilde{f}}\right|$. In particular, if $\gamma = \tilde{\gamma} = f\delta_{\alpha\bar{\beta}}$, then $\|\sigma(id)\| = \frac{(m-1)}{2}\left|\nabla\frac{1}{\sqrt{f}}\right|$.*

Example 6. *In particular, if $M = B = B_1(0) \subset \mathbb{C}^m$ and $\gamma = \tilde{\gamma} = \frac{4}{(1-|z|^2)^2}\delta_{\alpha\bar{\beta}}$ is the Poincaré metric, then*

$$\|\sigma(id)\| = \frac{(m-1)}{2}|z|.$$

So $\|\sigma(id)\| \notin C^0_{\mu}(M)$ for the Poincaré case, we do not even have decay to zero. We set this result in a more general framework now.

Let $\Omega \subset \mathbb{C}^m$ be a rotational symmetric domain, equipped with a rotational symmetric metric γ, which obeys the polar block form condition of Lemma 5, if $0 \notin \Omega$. Let $S \subset \Omega$ be a sphere centered in 0 with radius r_0. Then, for $r \geq 0$ choose $x \in \Omega$ with $|x| = r$ and define

$$D(r) := \text{dist}(x, S).$$

Obviously, $D(r)$ is well-defined and for any fixed $s \in S$ the function $D(r)$ obeys

$$D(r) \leq d(x,s) \leq D(r) + C,$$

where C depends only on $b_{ij}(r_0)$. Hence statements about growth of $d(x,s)$ are equivalent to those about $D(r)$ and independent of the choices of S and s. A calculation similar to that in the proof of Lemma 5 shows that dist(x,S) is realized by the segment of the line containing x and 0. If $0 \in \Omega$ we set $r_0 := 0$, i.e. $D(r) = d(r,0)$.

Proposition 4. *Let $\Omega \subset \mathbb{C}^m$ be equipped with a rotational symmetric, complete metric γ conformal to the euclidean such that $\|\sigma(id)\| \in C^0_\lambda(\Omega)$ for $\lambda > 1$. Then $D(r)$ has linear growth. In particular, $\Omega = \mathbb{C}^m$.*

Proof. We denote $\gamma = f\delta_{\alpha\bar\beta}$ and abbreviate $h := \frac{1}{\sqrt{f}}$. We calculate

$$\|\sigma(id)\| = (m-1)|h'| = (m-1)\left|\left(\frac{1}{D'}\right)'\right| = (m-1)\left|\frac{D''}{(D')^2}\right| < CD^{-\lambda}$$

for $D \gg 0$. Since D is strictly increasing for $r > r_0$, this implies for $D \gg 0$

$$\left|\frac{D''}{D'}\right| < C_1 D^{-\lambda} D'.$$

Integration yields

$$\text{Var}\,(\ln D', [r_0+\varepsilon, r]) < C_3 - C_2 D^{1-\lambda}.$$

If $r > r_0 + \varepsilon$ increases, Var $(\ln D', [r_0+\varepsilon, r])$ is increasing, as well as $D(r)$, hence

$$\text{Var}\,(\ln D', \{r > r_0 + \varepsilon\}) \leq C_3.$$

This implies

$$0 < C_4 < D'(r) < C_5$$

for all $r \geq r_0$ and hence

$$C_4 r + C_6 < D(r) < C_5 r + C_7.$$

Very similar arguments apply for $r < r_0$, if $0 \notin \Omega$. Hence $0 \in \Omega$ and $\Omega = \mathbb{C}^m$. $\square$

Proposition 5. *Any rotational symmetric Hermitian metric γ on $\mathbb{C}^m$, which has nonpositive sectional curvature is either euclidean or $D(r) := d(0, r)$ has superlinear growth.*

Proof. First, we reduce to the case $m = 1$: If M has nonpositive sectional curvature and E is a complex plane through 0, then $M \cap E$ has also nonpositive sectional curvature. On the other hand, if $\gamma|_{M\cap E}$ is euclidean or has superlinear growth for some plane E containing 0, we conclude by the rotational symmetry that this holds also for γ. So we assume now $m = 1$.

With notation as above, we compute for $\gamma = f(r)dz \otimes d\bar z = \phi(r^2)dz \otimes d\bar z$ and $s := r^2$

$$\phi(s)R_{1212} = -\left[2s\left(\phi''(s)\phi(s) - (\phi'(s))^2\right) + 2\phi'(s)\phi(s)\right].$$

This implies

$$2\phi^2(s(\ln\phi)'' + (\ln\phi)') \geq 0.$$

We abbreviate $(\ln\phi)' =: \psi$. Since $R = -\frac{\Delta \ln f}{2f}$ the maximum principle implies that $f(r)$ is increasing and hence $f'(r) \geq 0$ and also $\phi'(s) \geq 0$. So we conclude

$$\psi \geq 0.$$

We claim that $\psi(s) > 0$ for $s > 0$ unless $\psi \equiv 0$. To prove this we assume that there are $0 < s_1 < s_2$ such that $\psi(s_1) > 0$ and $\psi(s_2) = 0$. Then

$$0 \leq \int_{s_1}^{s_2} (s\psi'(s) + \psi(s))ds = s_2\psi(s_2) - s_1\psi(s_1) = -s_1\psi(s_1) < 0,$$

what is a contradiction. Hence $\psi(s) > 0$ for all $s > 0$ or $\psi \equiv 0$. The last case is the euclidean case.

So we assume $\psi(s) > 0$ for $s > 0$. Let us fix some $s_0 > 0$.
Then we compute for all $s > s_0$:

$$
\begin{aligned}
s\psi' + \psi \geq 0 \quad &\Longleftrightarrow \quad \frac{\psi'}{\psi} \geq -\frac{1}{s} \\
&\Rightarrow \quad \ln \psi \geq C_1 - \ln s \\
&\Rightarrow \quad (\ln \phi)' \geq \frac{C_2}{s} \text{ with } C_2 > 0 \\
&\Rightarrow \quad \phi(s) \geq C_3 s^{C_2} \text{ with } C_3 > 0 \\
&\Rightarrow \quad f(r) \geq C_3 r^{2C_2} \\
&\Rightarrow \quad D(r) \geq C_4 + C_5 r^{1+C_2} \text{ with } C_2, C_5 > 0.
\end{aligned}
$$

In the computations we always integrate from s_0 to s. □

Corollary 2. *If $M \subset \mathbb{C}^m$ allows for a rotational symmetric complete metric conformal to the euclidean with nonpositive sectional curvature and $\|\sigma(id)\| \in C^0_\lambda(M)$ for some $\lambda > 1$, then $M = \mathbb{C}^m$ and the metric is the euclidean metric multiplied with a constant. In particular, $\sigma(id) \equiv 0$.*

These results illustrate that there is no obvious example, where the identity map id may serve as initial map h.

In order to construct nontrivial Hermitian-harmonic maps, one might look for manifolds with *two* infinite ends as in Example 4 above. However, if one wants to choose $M = N$ in this case, one has to observe the following obstruction:

Remark 2. $N = \mathbb{R}^n \setminus \{0\}$ *does not admit a complete, nonpositively curved metric for $n \geq 3$. If we would have a nonpositively curved metric, the Cartan-Hadamard-theorem would imply that $\mathbb{R}^n$ is the universal cover of N. Since N is simply connected for $n \geq 3$, N would have to be isomorphic to $\mathbb{R}^n$. But since $\pi_{n-1}(\mathbb{R}^n) = 0$ and $\pi_{n-1}(N) = \mathbb{Z}$, this is not the case.*

3 The corresponding parabolic system

3.1 Existence and convergence results

Originally in the fundamental work of Jost and Yau [JY], as in many contributions to the harmonic map system, existence results were obtained via the seeming detour of the corresponding heat system

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \sigma(u) & \text{on } (0,\infty) \times M, \\
u(0) = h, & \\
u \sim h & \text{at infinity, homotopic to each other.}
\end{cases} \tag{15}
$$

The initial map is chosen as in Theorem 1, and here, the notation “initial map” as well as the homotopy between h and u become more transparent.

Similarly as in Theorem 1 we get existence of a global solution to (15) and also convergence to *some smooth map* $u : M \to N$ for a sequence $t_k \to \infty$ by means of an exhaustion procedure.

Theorem 3 (Global Existence). *Assume that M is a noncompact complete Hermitian manifold such that for the holomorphic Laplace operator $-\tilde{\Delta}$ on M, the Assumption 1 is satisfied with positive numbers $\mu, \mu' > 0$. Further let N be a complete Riemannian manifold with nonpositive sectional curvature and $h : M \to N$ a smooth map with $\|\sigma(h)\| \in C^0_\mu(M)$.*

Then there exists a global smooth solution $u : [0,\infty) \times M \to N$ to (15) such that $u(t,\,.\,)$ is for every $t \geq 0$ homotopic to h. For the homotopy distance $\rho(t,\,.\,)$ between $h(\,.\,)$ and $u(t,\,.\,)$, we have that $\rho(t) \in C^0_{\mu'}(M)$ uniformly in t.

Moreover there exists a sequence $t_k \to \infty$ such that $u(t_k,\,.\,)$ converges to a smooth map $u : M \to N$, being homotopic to h and converging to h at "infinity".

Proof. As above, let Ω_k be a compact exhaustion of M. According to [JY, Proof of Theorem 6] , there exist smooth solutions $u_k : [0,\infty) \times \Omega_k \to N$ of the initial boundary value problems

$$\begin{cases} (u_k)_t - \sigma(u_k) = 0 & \text{in } [0,\infty) \times \Omega_k, \\ u_k(t,x) = h(x) & \text{for } (t,x) \in [0,\infty) \times \partial\Omega_k, \\ u_k(0,\,.\,) = h & \text{on } \Omega_k, \\ u_k \text{ homotopic to } h, & \text{with respect to } \partial\Omega_k. \end{cases} \tag{16}$$

We first need to show that $\left\|\frac{\partial u_k}{\partial t}(t,x)\right\|$ are uniformly bounded. For this purpose we note that according to formula [LN, (6.2)], $\left\|\frac{\partial u}{\partial t}(t,x)\right\|$ satisfies an initial boundary value problem for the following differential inequality

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{1}{4}\tilde{\Delta}\right)\left\|\dfrac{\partial u_k}{\partial t}(t,x)\right\| \leq 0 & \text{in } [0,\infty) \times \Omega_k, \\ \left\|\dfrac{\partial u_k}{\partial t}(t,x)\right\| = 0 & \text{for } (t,x) \in [0,\infty) \times \partial\Omega_k, \\ \left\|\dfrac{\partial u_k}{\partial t}(0,x)\right\| = \|\sigma(h)(x)\| & \text{for } x \in \Omega_k. \end{cases} \tag{17}$$

By means of the parabolic maximum principle, we conclude that for any k and all $(t,x) \in [0,\infty) \times \Omega_k$ we have

$$\left\|\frac{\partial u_k}{\partial t}(t,x)\right\| \leq \max_{x\in M} \|\sigma(h(x))\|. \tag{18}$$

Next we need a L^∞ bound for ρ_k on any $[0,T] \times \Omega_k$, where again $\rho_k(t,x)$ denotes the homotopy distance between $u_k(t,x)$ and $h(x)$. For this purpose we again introduce a nonnegative barrier function $V \in C^0_{\mu'}(M)$, such that

$$-\tilde{\Delta}V = 4\|\sigma(h)\| \text{ in } M. \tag{19}$$

Furthermore we note that the crucial estimate (6) generalizes to smooth time dependent maps $u, v : [0,\infty) \times M \to N$ as follows:

$$\left(\frac{\partial}{\partial t} - \frac{1}{4}\tilde{\Delta}\right)\rho(u,v) \leq \left\|\frac{\partial u}{\partial t} - \sigma(u)\right\| + \left\|\frac{\partial v}{\partial t} - \sigma(v)\right\|. \tag{20}$$

We conclude

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{1}{4}\tilde{\Delta}\right)\rho_k \leq \|\sigma(h)\| \leq \left(\dfrac{\partial}{\partial t} - \dfrac{1}{4}\tilde{\Delta}\right)V & \text{in } [0,\infty) \times \Omega_k, \\ \rho_k(t,x) = 0 \leq V(x) & \text{for } (t,x) \in [0,\infty) \times \partial\Omega_k, \\ \rho_k(0,x) = 0 \leq V(x) & \text{for } x \in \Omega_k. \end{cases} \tag{21}$$

By means of the parabolic maximum principle, we get for all $(t,x) \in [0,\infty) \times \Omega_k$

$$\rho_k(t,x) \le V(x), \tag{22}$$

i.e. the desired uniform L^∞ bound for the ρ_k and hence for the $\rho_{kj} = \rho(u_k, u_j)$. Analyzing and correcting carefully the argument given in [LN, pp. 351–352], one obtains for any $T > 0$ and any relative compact $\Omega \subset M$ a uniform (in k and T) bound for

$$\int_T^{T+2} \int_\Omega e(u_k).$$

From this we want to deduce a local maximum bound for $e(u_k)$, which by means of standard linear parabolic theory will allow to pass to the limit and to obtain a *global* smooth solution to (15). For this purpose we may assume that Ω is contained in one single coordinate chart, and we take from [LN, (6.5)] that $e(u_k)$ satisfies a differential inequality of the form

$$\left(\frac{\partial}{\partial t} - \frac{1}{4}\tilde{\Delta} + C(\Omega)\right) e(u_k) \le 0. \tag{23}$$

The constant $C(\Omega)$ may be suitably chosen independently of k. Here we have to apply the local maximum principle for parabolic operators not in divergence form, which can be adapted from [L, Theorem 7.21]. (For an extensive discussion we refer to Proposition 7 in the appendix.) This gives a bound for $\max_{[T+1,T+2]\times B} e(u_k)$ for sufficiently small balls contained in Ω, which depends on $\max e(h)$, B, Ω and the L^1-bound on $e(u_k)$, but not on T. Hence we have found a maximum bound for the gradient, which is local in space, but *global in time*.

Homotopy between u and h is shown as in the proof of Theorem 1. Moreover we note that the homotopy distance is also bounded by $V \in C^0_{\mu'}(M)$:

$$\rho(t,x) := \rho(u(t,x), h(x)) \le V(x). \tag{24}$$

The stated convergence now follows from the (uniform in time, local in space) boundedness of $e(u_k)$ and the global boundedness of $\left\|\frac{\partial u}{\partial t}(t,x)\right\|$ by standard linear parabolic theory. □

Next, we want to prove that this global solution converges to a (stationary) Hermitian-harmonic map $u : M \to N$. Here it seems that we need something stronger than Assumption 1. As additional hypothesis we formulate:

Assumption 2 (Decay properties).
We assume that there exists a positive number $\mu > 0$ such that for every $\varphi \in C^0_\mu(M)$, we have decay of $\max v(t,\,.\,)$ towards 0 for every bounded solution v of the initial value problem for the heat equation with the holomorphic Laplace operator and φ as initial datum. Moreover we assume that the solution of the initial value problem is unique in the class of all uniformly bounded functions on $[0,T] \times M$.

The formulation of Assumption 2 suggests the use of a comparison function, what is a great difficulty for arbitrary metrics. Hence it would be more adequate to find a spectral reformulation. This is aimed at by the following Lemma. We denote $S_{\phi_1,\phi_2} := \{r\exp(i\phi) | \phi_1 \le \phi \le \phi_2\}$ and $C^0_b(M) := C^0_{\mu=0}(M)$ for the bounded continuous functions in order to avoid confusion with compactly supported functions.

Lemma 6. *Assume that $u_t - \tilde{\Delta} u = 0$ has a unique bounded solution for every initial datum $\varphi \in C^0_\mu(M)$ and moreover*

$$\|(-\tilde{\Delta} + \lambda)^{-1}\|_{B(C^0_\mu(M), C^0_b(M))} \le C$$

uniformly for all $\lambda \in S_{\phi_1,\phi_2}$ for certain $\pi/2 < \phi_1 < \pi, -\pi/2 > \phi_2 > -\pi$. Then Assumption 2 holds.

Proof. The key ingredient is the keyhole integral. Let $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ be a path in $\mathbb{C}$ such that

$$\begin{aligned}\Gamma_1 &:= \{r\exp(i\phi_1)|r \in [1;\infty)\}\\ \Gamma_2 &:= \{\exp(i\phi)|\phi \in [\phi_1;\phi_2]\}\\ \Gamma_3 &:= \{r\exp(i\phi_2)|r \in [1;\infty)\}.\end{aligned}$$

Then we can define a semigroup (cf. e.g. [Fr, Part II])

$$\exp(\tilde{\Delta}t) := \frac{1}{2\pi i}\int_\Gamma \exp(\lambda t)(-\tilde{\Delta}+\lambda)^{-1}d\lambda$$

as operator in $B(C^0_\mu(M), C^0_b(M))$, since we assumed the uniform boundedness of $(-\tilde{\Delta}+\lambda)^{-1}$. It has the property

$$\frac{d}{dt}(\exp(\tilde{\Delta}t)u) = \tilde{\Delta}\exp(\tilde{\Delta}t)u.$$

Cauchy's integral formula implies that integration over Γ yields the same as integration over Γ/t for $t > 0$. Hence

$$\begin{aligned}\|\exp(\tilde{\Delta}t)\| &= \frac{1}{2\pi}\left\|\int_{\Gamma/t}\exp(\lambda t)\left(-\tilde{\Delta}+\lambda\right)^{-1}d\lambda\right\|\\ &= \frac{1}{2\pi t}\left\|\int_\Gamma \exp(\lambda)\left(-\tilde{\Delta}+\frac{\lambda}{t}\right)^{-1}d\lambda\right\|\\ &\le \frac{1}{2\pi t}\int_\Gamma |\exp(\lambda)|\left\|\left(-\tilde{\Delta}+\frac{\lambda}{t}\right)^{-1}\right\|\,|d\lambda|\\ &\le \frac{C}{2\pi t}\int_\Gamma |\exp(\lambda)|\,|d\lambda|\\ &= \frac{C'}{t},\end{aligned}$$

since $\cos(\phi_1) < 0$ and $\cos(\phi_2) < 0$.

Hence the unique bounded solution $v := \exp(\tilde{\Delta}t)\varphi$ satisfies

$$\max_{x\in M} v(t,x) \le \frac{C'\|\varphi\|}{t}.$$

So Assumption 2 is valid. □

The meaning and relevance of Assumption 2 will be extensively discussed in the examples in subsection 3.2 below. With help of this assumption, we may now state:

Theorem 4 (Convergence to a Hermitian-harmonic map).
Let the assumptions of Theorem 3 be satisfied as well as Assumption 2 with the same μ as in Assumption 1. Then, for the solution $u(t,\,.)$ of the time dependent Hermitian-harmonic map system (15), there exists a sequence $t_k \to \infty$ such that $u(t_k,\,.)$ converges to a Hermitian harmonic map $u : M \to N$.

Proof. It remains to show a decay result for $\max_M \left\|\frac{\partial u}{\partial t}\right\|$. The latter is achieved by means of the differential inequality

$$\begin{cases}\left(\dfrac{\partial}{\partial t}-\dfrac{1}{4}\tilde{\Delta}\right)\left\|\dfrac{\partial u}{\partial t}(t,x)\right\| \le 0 & \text{in } [0,\infty)\times M,\\ \left\|\dfrac{\partial u}{\partial t}(0,x)\right\| = \|\sigma(h)(x)\| & \text{for } x \in M.\end{cases} \tag{25}$$

Since $\left\|\frac{\partial u}{\partial t}\right\|$ is uniformly bounded, Assumption 2 gives decay to 0, as $t \to \infty$. $\square$

3.2 Examples

In the remainder, we show that Assumption 2 is likewise satisfied in all the examples treated above in Subsection 2.3.

Example 7. *Let $M = \mathbb{C}^m$ and $\gamma = \delta_{\alpha\bar{\beta}}$ be the euclidean metric. Then Assumption 2 holds true.*

Proof. Assume $\varphi \in C^0_\mu(M)$ with $\mu \le 2m$, for simplicity we specialize to $|\varphi(y)| < (1+|y|)^{-2\alpha m-\varepsilon}$ with $\alpha \in (0;1]$ and $\varepsilon > 0$ such that $2\alpha m + \varepsilon = \mu$. Since the fundamental solution of the heat equation is

$$\gamma(t,x) = C_0 t^{-m} \exp\left(-\frac{|x|^2}{4t}\right),$$

the solution $v(t,x)$ of the initial value problem with φ as initial datum is given by

$$\begin{aligned}
|v(t,x)| &= C_0 \left| t^{-m} \int \exp\left(-\frac{|x-y|^2}{4t}\right) \varphi(y) dy \right| \\
&\le C_0 t^{-m} \left(\int \exp\left(-\frac{|x-y|^2}{4t}\right)^{\frac{1}{1-\alpha}} dy \right)^{1-\alpha} \left(\int |\varphi(y)|^{\frac{1}{\alpha}} dy \right)^{\alpha} \\
&\le C_1(\varphi) t^{-m} \left(\int \exp\left(-\frac{|x-y|^2}{4t}\right)^{\frac{1}{1-\alpha}} dy \right)^{1-\alpha}, \\
&\quad \text{since } |\varphi|^{\frac{1}{\alpha}} \text{ is integrable,} \\
&\le C_1(\varphi) t^{-m} \left(\int \exp\left(-\frac{|x-y|^2}{4t}\right) dy \right)^{1-\alpha} \\
&= C_2(\varphi) t^{-m} \left(t^m \int \exp(-|z|^2) dz \right)^{1-\alpha} \\
&= C_3(\varphi) t^{-\alpha m}.
\end{aligned}$$

If $\mu > 2m$, then φ is integrable and hence

$$|v(t,x)| \le C_4 t^{-m}.$$

This proves the validity of Assumption 2 for the euclidean case for every $\mu > 0$. $\square$

To be able to treat the case of Example 2, we have to formulate a maximum principle for the corresponding Laplace operator.

Lemma 7. *Let $M = \mathbb{C}^m$ and $\gamma = (1+r^2)^{-1}\delta_{\alpha\bar{\beta}}$. If $u : M \times [0;T] \longrightarrow \mathbb{R}$ is bounded and satisfies $(-\tilde{\Delta} + \frac{\partial}{\partial t})u \ge 0, u(x,0) \ge 0$, then $u \ge 0$.*

Proof. We imitate the proof of the euclidean case like given in [Di, V,4,Thm4.1]. We take the function

$$v(x,t) := (1+|x|^2)\exp(4mt) \ge 1 + |x|^2,$$

which also satisfies the heat equation, i.e. $(-\tilde{\Delta} + \frac{\partial}{\partial t})v = 0$. For the function

$$w_\varepsilon := u + \varepsilon v$$

we hence get

$$(-\tilde{\Delta}+\frac{\partial}{\partial t})w_{\varepsilon}\geq 0, w_{\varepsilon}(x,0)\geq 0$$

and $w_\varepsilon(x,t)\geq 0$ outside a compact set $K_\varepsilon\times[0;T]\subset M\times[0;T]$. Now using the parabolic maximum principle for the bounded domain $K_\varepsilon\times[0;T]$ we see that $w_\varepsilon\geq 0$ everywhere. Hence $u(x,t)=\lim_{\varepsilon\longrightarrow 0}w_\varepsilon(x,t)\geq 0$. □

Now we are able to continue Example 2.

Example 8. *Let us consider the conformal metric $\gamma_{\alpha\bar{\beta}}=(1+r^2)^{-1}\delta_{\alpha\bar{\beta}}$ on $M=\mathbb{C}^m$, $m\geq 2$. Then again Assumption 2 holds true.*

Proof. Recall that the geodesic length $d(0,x)\sim\ln(1+r^2)$. The decay of the solution of the corresponding heat equation is proven by comparison to a test function. Let us consider

$$w(t,x):=(A+\ln(1+r^2)+t)^{-\mu}.$$

Now we verify

$$\left(-\tilde{\Delta}+\frac{\partial}{\partial t}\right)w\geq 0,$$

if A is chosen big enough, given μ and m.

We compute

$$\begin{aligned}\left(-\tilde{\Delta}+\frac{\partial}{\partial t}\right)w &= \mu\Big(((4m-5)A-4\mu-4)r^2+(4m-5)r^2\ln(1+r^2)+(4m-5)r^2t\\ &\quad+(4m-1)A+(4m-1)\ln(1+r^2)+(4m-1)t\Big)\\ &\quad\cdot\left(1+\ln(1+r^2)+t\right)^{-\mu-2}\left(1+r^2\right)^{-1}\end{aligned}$$

Obviously, $\left(-\tilde{\Delta}+\frac{\partial}{\partial t}\right)w>0$, if

$$(4m-5)A-4\mu-4\geq 0.$$

So, given μ and $m\geq 2$ we choose $A\geq 1$ such that this inequality is satisfied.

Since for an exact bounded solution with $|v(0,x)|\in C^0_\mu(M)$ we have $|v(0,x)|\leq Cw(0,x)$, the parabolic maximum principle stated in Lemma 7 proves

$$|v(t,x)|\leq Cw(t,x)\leq Ct^{-\mu}.$$

The uniqueness in the class of bounded solutions immediately follows by Lemma 7 and the observation that u is a solution of the heat equation with zero initial data if and only if $-u$ is.

Hence Assumption 2 is valid for all $\mu>0$. □

Finally, we come back to Example 4. Again we first have to prove a maximum principle.

Lemma 8. *Let $M=(\mathbb{S}^1)^{2m-1}\times(-1,1)$ with $\gamma=\delta^2(1-|s|)^{-2\delta-2}\delta_{\alpha\bar{\beta}}$ for $1/2<|s|<1$ like in Example 4. If $u:M\times[0;T]\longrightarrow\mathbb{R}$ is bounded and satisfies $(-\tilde{\Delta}+\frac{\partial}{\partial t})u\geq 0$ and $u(x,0)\geq 0$, then $u\geq 0$.*

Proof. Like in the proof of Lemma 7 it is sufficient to construct a supersolution $v(s,t)$ such that $\inf_{t\in[0;T]} v(s,t) \longrightarrow \infty$ for $s \longrightarrow \pm 1$. The choice

$$\tilde{v}(s,t) := (1 - \log(1-|s|)) \exp(\frac{t}{\delta^2})$$

works for $|s| > \frac{1}{2}$. Since for $|s| \le \frac{1}{2}$ there exists $C = C(T) > 0$ such that

$$\left(-\tilde{\Delta} + \frac{\partial}{\partial t}\right) \tilde{v} \ge -C,$$

the function $v(s,t) := \tilde{v}(s,t) + Ct$ is a supersolution satisfying the required conditions. □

Example 9. *Let M be like in Example 4. Then Assumption 2 is valid.*

Proof. Let us define $(a)_i := a(a+1)\cdot \ldots \cdot (a+i-1), (a)_0 := 1$ for real a and integer i. Denote the Kummer function

$$F(a,b,z) := {}_1F_1(a,b,z) = \sum_{i=0}^{\infty} \frac{(a)_i}{(b)_i i!} z^i,$$

which is convergent for all z, if b is not a negative integer. We will make use of the following properties:

$$\begin{aligned}
0 &= z\frac{\partial^2}{\partial z^2}F(a,b,z) + (b-z)\frac{\partial}{\partial z}F(a,b,z) - aF(a,b,z) && (26)\\
F(a,b,z) &= \frac{\Gamma(b)}{\Gamma(a)}\exp(z)z^{a-b}(1+O(|z|^{-1})) \text{ if } \Re(z) > 0 && (27)\\
F(a,b,z) &= \exp(z)F(b-a,b,-z) && (28)\\
aF(a+1,b,z) &= aF(a,b,z) + z\frac{\partial}{\partial z}F(a,b,z). && (29)
\end{aligned}$$

All these properties can be found in [AS] as $13.1.1, 13.1.4, 13.1.27, 13.4.10$.

As a comparison function we choose

$$w(s,t) := t^c F\left(-c, 1+\frac{1}{2\delta}, -\frac{1}{4}(1-|s|)^{-2\delta}t^{-1}\right),$$

where we choose $\max(-\frac{1}{2\delta}, -\frac{\mu}{2}) < c < 0$. By the first Kummer transformation (28) this becomes

$$w(s,t) = t^c \exp(-\frac{1}{4}(1-|s|)^{-2\delta}t^{-1}) F\left(1+\frac{1}{2\delta}+c, 1+\frac{1}{2\delta}, \frac{1}{4}(1-|s|)^{-2\delta}t^{-1}\right),$$

and now we see easily that $w(s,t) > 0$. By (27) for fixed s and $t \longrightarrow 0$ the function $w(s,t)$ is continuously extendable to $t = 0$ and

$$w(s,0) = C(1-|s|)^{-2c\delta} > C(1-|s|)^{\delta\mu} = C(1+\tilde{d}(s))^{-\mu}.$$

By a simple calculation using (26) we can see that $w(s,t)$ is an exact solution to

$$\left(-\delta^{-2}(1-|s|)^{2\delta+2}\Delta + \frac{\partial}{\partial t}\right) w(s,t) = 0 \tag{30}$$

on M outside $s=0$. Since this is not yet the original equation and w is singular in $s=0$ we have to do some more calculations. First we note that by (29) and (28)

$$\begin{aligned}\frac{\partial}{\partial t}w(s,t) &= ct^{c-1}F\left(-c+1,1+\frac{1}{2\delta},-\frac{1}{4}(1-|s|)^{-2\delta}t^{-1}\right)\\ &= ct^{c-1}\exp(-\frac{1}{4}(1-|s|)^{-2\delta}t^{-1})F\left(\frac{1}{2\delta}+c,1+\frac{1}{2\delta},\frac{1}{4}(1-|s|)^{-2\delta}t^{-1}\right)\\ &< 0,\end{aligned}$$

since $c<0, \frac{1}{2\delta}+c>0$ and $F(a,b,z)$ is positive, if $a,b,z>0$. From this and the fact that $w(s,t)$ is a solution of (30) we deduce that

$$-\Delta w(s,t)>0$$

for all $0\neq s,t>0$. If we determine $C>0$ such that $Ca(s)<\delta^2(1-|s|)^{-2\delta-2}$ for $|s|\leq\frac{1}{2}$ then $\tilde{w}(s,t):=w(s,Ct)$ satisfies

$$\left(-\tilde{\Delta}+\frac{\partial}{\partial t}\right)\tilde{w}(s,t)\geq 0$$

for $s\neq 0$. So, for the sake of simplicity let us assume that $C=1$.

Now we are considering $s>0$. Then, again by (29) and (28)

$$\begin{aligned}\frac{\partial}{\partial s}w(s,t) &= -2\delta ct^{c}(1-s)^{-1}\left(F\left(-c+1,1+\frac{1}{2\delta},-\frac{1}{4}(1-|s|)^{-2\delta}t^{-1}\right)\right.\\ &\quad\left.-F\left(-c,1+\frac{1}{2\delta},-\frac{1}{4}(1-|s|)^{-2\delta}t^{-1}\right)\right)\\ &= 2\delta ct^{c}(1-s)^{-1}\exp(-\frac{1}{4}(1-s)^{-2\delta}t^{-1})\left(F\left(1+\frac{1}{2\delta}+c,1+\frac{1}{2\delta},\frac{1}{4}(1-s)^{-2\delta}t^{-1}\right)\right.\\ &\quad\left.-F\left(\frac{1}{2\delta}+c,1+\frac{1}{2\delta},\frac{1}{4}(1-s)^{-2\delta}t^{-1}\right)\right)\\ &< 0\end{aligned}$$

again by $c<0$ and the property that $F(a,b,z)>F(a',b,z)$ if $a>a'>0,b,z>0$.

Finally, with this in mind we are going to prove that $(-\tilde{\Delta}+\frac{\partial}{\partial t})w(s,t)\geq 0$ in a weak sense. We recall the definition of the conformal factors f and a resp. in Example 4. We compute for $\phi(s,t)\in C_0^\infty\left(M\times(0,T)\right),\phi\geq 0$

$$\begin{aligned}&\int_M\int_{(0,T)}w(s,t)\left(\left(-\tilde{\Delta}^*-\frac{\partial}{\partial t}\right)\phi(s,t)\right)f^m\,dtdx\\ &\geq\ a(0)^{m-1}\int_0^T\left(\frac{\partial w}{\partial s}(0-0,t)-\frac{\partial w}{\partial s}(0+0,t)\right)\phi(0,t)dt\geq 0.\end{aligned}$$

Moreover, since $\frac{\partial}{\partial s}w(s,t)<0$ for $s>0$, we see that

$$w(s,t)\leq w(0,t)=t^cF\left(-c,1+\frac{1}{2\delta},-\frac{1}{4}t^{-1}\right)<2t^c\longrightarrow 0$$

for $t\gg 1$.

Using the maximum principle stated in Lemma 8 yields the statement. $\square$

A A general local parabolic maximum principle

For the reader's convenience we shall outline the derivation of a local parabolic maximum principle, which is even more general than we need it in the proof of Theorem 3. Of particular interest is the dependence of the estimation constants among others on the elliptic operator and the size and shape of the domains. To a large extent, we follow [L, Ch. VII].

Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ be a domain. We denote the coordinates $X = (x,t) \in \mathbb{R}^n \times \mathbb{R}$ (resp. $Y = (y,s)$). In this section we consider the operator

$$Lu := -u_t + a^{ij} D_{ij} u + b^i D_i u + cu$$

with real valued bounded measurable coefficients. Moreover we assume the symmetric matrix (a^{ij}) to be positive semidefinite. We abbreviate $\mathcal{D} := \det \left(a^{ij}\right)$ and $\mathcal{D}^* := \mathcal{D}^{\frac{1}{n+1}}$. Furthermore, $\Lambda(X)$ denotes the maximal and $\lambda(X)$ the minimal eigenvalue of $a^{ij}(X)$. The function u is considered in $u \in W^{2,1}_{n+1,loc}(\Omega) \cap C^0(\overline{\Omega})$. As usual we denote by $\mathcal{P}\Omega$ the parabolic boundary and by $\mathcal{B}\Omega$ the bottom of the domain Ω.

We define the upper contact set $E(u)$ to be the set of all $X \in \Omega \setminus \mathcal{P}\Omega$ such that there exists $\xi \in \mathbb{R}^n$ such that

$$u(X) + \xi(y-x) \geq u(Y) \tag{31}$$

for all Y with $s \leq t$. This implies $u_t \geq 0, -D^2 u \geq 0$ on $E(u)$.

If $\Omega = B_R \times (0,T)$, we write $E^+(u)$ for the subset of $E(u)$ in which $u > 0$ and

$$R|\xi| < u(X) - \xi \cdot x < \sup_\Omega \frac{u^+}{2}. \tag{32}$$

Similarly, we denote by $\Sigma(u)$ the set of all $\Xi = (\xi, h) \in \mathbb{R}^{n+1}$ such that

$$R|\xi| < h < \sup_\Omega \frac{u^+}{2}. \tag{33}$$

First we quote the global version of a maximum principle involving L^p-norms.

Proposition 6 [L, Theorem 7.1]. *Let $\Omega = B_R \times (0,T)$ and $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$ satisfying $Lu \geq f$ with $c \leq k$ in Ω, where k is a nonnegative constant. Then*

$$\sup_\Omega u \leq \exp(kT) \left(\sup_{\mathcal{P}\Omega} u^+ + c_1(n) B_0 R^{\frac{n}{n+1}} \left\| \frac{f}{\mathcal{D}^*} \right\|_{n+1, E^+(w)} \right),$$

with $B_0 := R^{-1} \| \frac{b}{\mathcal{D}^*} \|^{n+1}_{n+1,E^+(w)} + 1$ and $w(x,t) := \exp(-kt)u - \sup_{\mathcal{P}\Omega} (\exp(-k\,.\,)u^+)$.

Our goal is to prove the local counterpart of the preceding result. The crucial point will be to estimate $\sup u$ by $\|f\|_{n+1}$ and the weakest possible "norm" of u. Since we will argue by means of a scaling argument in the next proof, let us consider the degrees of the coefficients with respect to the two-parameter group $\mathbb{R}^2 \cong (x \mapsto kx, t \mapsto lt)$. A simple calculation shows:

$$\begin{aligned} \deg R &= (-1,0) \\ \deg T &= (0,-1) \\ \deg a^{ij} &= (-2,1) \\ \deg b &= (-1,1) \\ \deg c &= (0,1) \\ \deg f &= (0,1) \end{aligned}$$

For the following result, cf. [L, Theorem 7.21].

Proposition 7 (Local parabolic maximum principle). *Let $\Omega = B_R \times (-T, 0)$ and $u \in W^{2,1}_{n+1,loc}(\Omega) \cap C^0(\overline{\Omega})$ satisfying $Lu \geq f$. Assume further that*

$$\lambda \geq \lambda_0 > 0, \quad \Lambda \leq \Lambda_0, \quad |b| \leq B, \quad c \leq c_0.$$

Then for any $p > 0$ and $0 < \rho < 1$ there exists C depending only on p, ρ and

$$(\lambda_0^{-n} T R^{-2})^{\frac{1}{n+1}} (c_0 R^2 + BR + \Lambda_0 + R^2 T^{-1})$$

such that

$$\sup_{\rho\Omega} u \leq C \left(|\Omega|^{-\frac{1}{p}} \|u^+\|_p + (TR^{-1})^{\frac{n}{n+1}} \|f\|_{n+1} \right).$$

Proof. By approximation, we may assume that $u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega})$. We note that both sides of the claimed inequality are invariant under the scaling

$$x \mapsto lx, t \mapsto kt.$$

Hence it suffices to prove the theorem for $\Omega = Q(1) = B_1 \times [-1, 0]$.

For this purpose we define $\zeta := (1 - |x|^2)^+(1 + t)^+$ and $\eta := \zeta^q$ for $q > 2$. We define the operator P as principal part of L by

$$Pv := -v_t + a^{ij} D_{ij} v.$$

We will apply it to $v = \eta u$. This yields

$$Pv \geq \eta f - \eta(b^i D_i u + cu) + uP\eta + 2a^{ij} D_i u D_j \eta.$$

We will calculate the terms separately. First we note that by Cauchy's inequality

$$|Dv| \leq \frac{v}{1 - |x|} \text{ on } E^+(v). \tag{34}$$

From this it is easy to see that on $E^+(v)$

$$|Du| \leq 2(1 + q) \frac{v}{\zeta \eta}. \tag{35}$$

In order to compute $uP\eta$ – again on $E^+(v)$ – we first note that $\eta_t \leq q\frac{\eta}{\zeta}$. Next we use $(a^{ij}) \geq 0$ to conclude

$$a^{ij} D_{ij} \eta \geq -2q \left(\operatorname{tr}(a^{ij}) \right) \frac{\eta}{\zeta},$$

hence

$$uP\eta \geq -q(1 + 2\operatorname{tr} a^{ij}) \frac{v}{\zeta} \geq -q(1 + 2n\Lambda) \frac{v}{\zeta}. \tag{36}$$

Finally, we have to compute $a^{ij} D_i u D_j \eta$. This splits up into the sum $\frac{a^{ij}}{\eta} D_i v D_j \eta - \frac{a^{ij}}{\eta^2} v D_i \eta D_j \eta$. For the first summand we obtain on $E^+(v)$ using (34)

$$\left| \frac{a^{ij}}{\eta} D_i v D_j \eta \right| \leq 4q\Lambda \frac{v}{\zeta^2}.$$

The second summand can be estimated

$$\left|\frac{a^{ij}}{\eta^2} v D_i\eta D_j\eta\right| \le \frac{\Lambda v|D\eta|^2}{\eta^2} \le 4q^2\Lambda\frac{v}{\zeta^2},$$

hence

$$a^{ij}D_iuD_j\eta \ge -4q\Lambda(1+q)\frac{v}{\zeta^2}. \tag{37}$$

Adding up (35),(36) and (37) yields

$$\begin{aligned} Pv &\ge \eta f - v\zeta^{-2}\left(c\zeta^2 + (2(1+q)|b| + q(1+2n\Lambda))\,\zeta + 8q(1+2q)\Lambda\right) \\ &\ge \eta f - \tilde{C}v\zeta^{-2}\mathcal{D}^* \end{aligned} \tag{38}$$

on $E^+(v)$, where $\tilde{C}$ can be chosen as

$$\tilde{C} := 2q(4(1+q)+n)\,\lambda_0^{-n/(n+1)}\,(c_0 + B + \Lambda_0 + 1).$$

Note that the unique homogenization of C to an element of degree $(0,0)$ (in (k,l)) gives the form mentioned in the theorem. Since $v=0$ on $\mathcal{P}\Omega$ and P is an operator with $b=c=0$ we can apply Proposition 6 with $w=v$ and $k=0$. This yields with $c_1 = c_1(n)$:

$$\sup_\Omega v \le c_1\left(\left\|\frac{f}{\mathcal{D}^*}\right\|_{n+1} + \tilde{C}\|v\zeta^{-2}\|_{n+1}\right) \le c_1\tilde{C}(\|f\|_{n+1} + \|v\zeta^{-2}\|_{n+1}).$$

If $p > n+1$, we use Hölder's inequality, $v\zeta^{-2} \le u^+$ and $\eta \ge (1-\rho)^{2q}$ on $\rho\Omega$ to conclude the claim of the theorem. Here we may choose $q=2$.

If $p \le n+1$, we note that $v\zeta^{-2} = u^{\frac{2}{q}}v^{1-\frac{2}{q}}$. We choose $q = \frac{2(n+1)}{p}$ and compute

$$\|v\zeta^{-2}\| \le (\sup_\Omega v)^{1-\frac{p}{n+1}}\|u^+\|_p^{\frac{p}{n+1}} \le \varepsilon\sup_\Omega v + c_2(n,p)\varepsilon^{1-\frac{n+1}{p}}\|u^+\|_p$$

by Young's inequality. Now we choose $\varepsilon := (2c_1\tilde{C})^{-1}$ and proceed like before. □

B The energy differential inequality

If $u : M \longrightarrow N$ satisfies the Hermitian harmonic system and N has nonpositive sectional curvature, in [JY] an energy inequality is given which we are using several times. This is why we sketch the proof here.

Proposition 8. *If N has nonpositive sectional curvature, then for every relatively compact open set $\Omega \subset M$ there exists a constant $C(\Omega)$ such that on Ω*

$$-\tilde{\Delta}e(u) \le C(\Omega)e(u).$$

Proof. First we fix $x \in M$ and choose coordinates such that in x resp. $u(x)$

$$\gamma_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}, \quad g_{ij} = \delta_{ij}, \quad g_{ij,k} = 0.$$

With these choices the left hand side becomes

$$\begin{aligned}
\tilde{\Delta} e(u) &= \sum_j \sum_\delta \Big(\gamma^{\alpha\bar\beta}_{,\delta\bar\delta} u^j_{,\alpha} u^j_{,\bar\beta} && (39)\\
&\quad + \left[\gamma^{\alpha\bar\beta}_{,\delta} u^j_{,\alpha\bar\delta} u^j_{,\bar\beta} + \gamma^{\alpha\bar\beta}_{,\delta} u^j_{,\alpha} u^j_{,\bar\beta\bar\delta} \right] && (40)\\
&\quad + \left[\gamma^{\alpha\bar\beta}_{,\bar\delta} u^j_{,\alpha\delta} u^j_{,\bar\beta} + \gamma^{\alpha\bar\beta}_{,\bar\delta} u^j_{,\alpha} u^j_{,\bar\beta\delta} \right] && (41)\\
&\quad + \sum_\alpha \left[u^j_{,\alpha\delta} u^j_{,\bar\alpha\bar\delta} + u^j_{,\alpha\bar\delta} u^j_{,\bar\alpha\delta} \right] && (42)\\
&\quad + \sum_\alpha \left[u^j_{,\alpha\delta\bar\delta} u^j_{,\bar\alpha} + u^j_{,\alpha} u^j_{,\bar\alpha\delta\bar\delta} \right] \Big) && (43)\\
&\quad + \sum_{\alpha,\delta} (g_{ij,kl} \circ u) u^k_{,\delta} u^l_{,\bar\delta} u^i_{,\alpha} u^j_{,\bar\alpha}. && (44)
\end{aligned}$$

For (39) we obtain the inequality

$$\left| \sum_j \sum_\delta \gamma^{\alpha\bar\beta}_{,\delta\bar\delta} u^j_{,\alpha} u^j_{,\bar\beta} \right| \le \Lambda e(u),$$

where Λ can be estimated by bounds of terms of $\gamma_{\alpha\bar\beta}$ and their second derivatives.

In a similar way we can estimate (40) by

$$\left| \sum_j \sum_\delta \gamma^{\alpha\bar\beta}_{,\delta} u^j_{,\alpha\bar\delta} u^j_{,\bar\beta} + \gamma^{\alpha\bar\beta}_{,\delta} u^j_{,\alpha} u^j_{,\bar\beta\bar\delta} \right| \le mC(\varepsilon |D^2 u|^2 + \frac{4m}{\varepsilon} e(u))$$

for any $\varepsilon > 0$. The constant depends on $\gamma^{\alpha\bar\beta}$ and their first derivatives. The same holds true for (41).

It is not hard to see that (42) equals

$$\sum_j \sum_{\alpha,\delta} \left(u^j_{,\alpha\delta} u^j_{,\bar\alpha\bar\delta} + u^j_{,\alpha\bar\delta} u^j_{,\bar\alpha\delta} \right) = \frac{1}{8} |D^2 u|^2.$$

For $(43)+(44)$ we have to use the Hermitian harmonic map system and the nonpositivity of the sectional curvature to conclude that

$$\begin{aligned}
&\sum_{\alpha,\delta} \left(\sum_j \left[u^j_{,\alpha\delta\bar\delta} u^j_{,\bar\alpha} + u^j_{,\alpha} u^j_{,\bar\alpha\delta\bar\delta} \right] + (g_{ij,kl} \circ u) u^k_{,\delta} u^l_{,\bar\delta} u^i_{,\alpha} u^j_{,\bar\alpha} \right) \\
&= \quad -2 \sum_{\alpha,\delta} R_{ijkl} (u^i_{,\alpha} u^j_{,\delta} u^k_{,\bar\alpha} u^l_{,\bar\delta} + u^i_{,\bar\alpha} u^j_{,\delta} u^k_{,\alpha} u^l_{,\bar\delta}) \ge 0.
\end{aligned}$$

Putting all together yields

$$-\tilde{\Delta} e(u) \le \Lambda e(u) + 2mC \left(\varepsilon |D^2 u|^2 + \frac{4m}{\varepsilon} e(u) \right) - \frac{1}{8} |D^2 u|^2.$$

Choosing $\varepsilon \le (16mC)^{-1}$ yields the claimed inequality. □

Acknowledgment. We are grateful to Wolf von Wahl (University of Bayreuth) for his suggestion to investigate Hermitian-harmonic maps on noncompact manifolds.

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