Polyharmonic Dirichlet Problems: Positivity, Critical Exponents and Critical Dimensions

English summary of the author's "Habilitationsschrift",
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by

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Preface

In this summary we want to present the main results which are contained in the author's "Habilitationsschrift" (dissertation). It has been written in German and has been submitted to the Faculty of Mathematics and Physics of the University of Bayreuth. Moreover we want to comment on these results, sketch some of the ideas how to prove them and give some background information.

For the reader's convenience we keep the numbering of the lemmas (Hilfssatz), theorems (Satz), corollaries (Folgerung), definitions (Definition) and equations of the German original version. Further we refer to its bibliography (Literaturverzeichnis).

Some of the results have been obtained in collaboration with Guido Sweers of Delft (in particular Theorems 1.1 and 1.26) and with Francisco Bernis of Madrid (Part b of Theorem 3.2). Parts of this dissertation are based on the papers [BerG], [Gr2], [Gr3], [GS1], [GS2], [GS3].

I am grateful to my colleagues Dr. R. Kaiser, Dr. B. Schmitt and Prof. M. Wiegner for numerous interesting and stimulating discussions. I owe a special gratitude to my academic teacher Prof. W. von Wahl for his permanent support.

0 Introduction

We deal with higher order elliptic Dirichlet problems

$$(1)\quad \begin{cases} Lu := (-\Delta)^m u + f(x,u,\ldots,D^{2m-1}u) = 0 & \text{in } \Omega, \\ \left(\dfrac{\partial}{\partial\nu}\right)^j u|\partial\Omega = \varphi_j & \text{for } j = 0,\ldots,m-1. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ is a sufficiently smooth bounded domain, ν its unit outward normal. In the major part of the present dissertation we confine ourselves to the polyharmonic principal part $(-\Delta)^m$. Only Theorem 1.26 deals also with more general highest order terms. The lower order term f may be linear or nonlinear, but in the nonlinear case we still have to restrict f to depend on x and u only.

Sometimes for higher order boundary value problems ($m > 1$) similar results hold as for those of second order. But in most cases generalizations from $m = 1$ to $m > 1$ are either hard to obtain, true only in modified form, not yet known or even false.

E.g. the Schauder estimates for linear higher order equations may be developed analogously to the second order case, see [ADN]. Maximum estimates for linear equations are also available but much more difficult to prove. Moreover for a certain given boundary value problem the fulfillment of their assumptions is not so easy to check, see [Ag]. Strong maximum principles are obviously false, and only little is known about comparison principles. In nonlinear theory even the derivation of maximum estimates is in general a very hard and, in many cases, unsolved problem.

In this dissertation we want to investigate common features of some corresponding higher and second order Dirichlet problems as well as differences between them. This gives a better understanding of some features of second order theory too: In which cases does ellipticity alone suffice? And which results or techniques do additionally require "$m = 1$"? Moreover for some phenomena the elucidation of their dependence on m may permit their distinct interpretation.

In Chapter 1 we discuss positivity properties of or comparison principles for higher order linear Dirichlet problems. In the theory of second order elliptic equations, linear as well as nonlinear, maximum and comparison principles and their variants have proved to be extremely powerful and efficient devices. So, for a better understanding of higher order elliptic equations it is an obvious step to investigate to what extent similar results do exist there.

As the simple polyharmonic function $x \mapsto -|x|^2$ demonstrates maximum principles cannot hold in the higher order case $m > 1$. But it's reasonable to ask, whether in any domain and for any elliptic operator positive data (right-hand side or boundary data) yield positive solutions. It is known [Bo2] that there is a comparison principle for the higher order prototype

$$\begin{cases} (-\Delta)^m u = f & \text{in } B, \\ D^\alpha u | \partial B = 0 & \text{for } |\alpha| \leq m-1, \end{cases}$$

i.e. in this example $0 \not\equiv f \geq 0$ implies $u > 0$. Here $B \subset \mathbb{R}^n$ is the unit ball, $\alpha \in \mathbb{N}_0^n$ denotes a multiindex, $D^\alpha = \prod_{i=1}^n \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}$. It should be noted that the Dirichlet boundary data prevent us from considering boundary value problems like (1) as systems of second order equations, although the principal part $(-\Delta)^m$ looks like a power of second order differential operators. We will show that comparison principles continue to hold for suitable perturbations of the polyharmonic prototype, see Theorems 1.1 and 1.26. On the other hand numerous counterexamples deny in general our positivity question. So, in spite of some similarities there are significant differences between second order ($m = 1$) and higher order ($m > 1$) theory of positivity.

Chapters 2 and 3 are devoted to the semilinear polyharmonic eigenvalue problem

$$(5)\qquad \begin{cases} (-\Delta)^m u = \lambda u + |u|^{s-1}u, \quad u \not\equiv 0 & \text{in } \Omega, \\ D^\alpha u|\partial\Omega = 0 & \text{for } |\alpha| \le m-1. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2m,\gamma}$-smooth domain, $n > 2m$, $\lambda \in \mathbb{R}$; $s = (n+2m)/(n-2m)$ is the critical (Sobolev) exponent. From second order theory (see [BrN], [R], [CFP], [AS], [ABP] et al.) it may be expected that (due to the critical growth of the nonlinearity) the behaviour of (5) strongly depends on the dimension n. It will be interesting to see how this dependence on n in turn depends on $2m$, the order of problem (5). Chapter 2 deals with existence and Chapter 3 with nonexistence. Irrespective of some technical difficulties the existence theory may be developed according to the second order case, as long as solutions without special properties (e.g. positivity, change of sign, number of zeros in the radially symmetric situation, asymmetrical solutions) are considered, see Theorem 2.8. Things change completely as soon as special properties are required: E.g. as far as the author knows nothing is known about radial solutions with prescribed numbers of zeros (if $\Omega = B$). And it's even a very hard problem to seek *positive* solutions and to distinguish them as the simplest nontrivial solutions: Making use of the results of Chapter 1 we can show the existence of positive solutions only if we restrict the domain to be a ball: $\Omega = B$. See Theorem 2.5.

The study of nonexistence phenomena of problem (5, $m > 1$), again due to the criticality of the exponent s, has been initiated by Pucci and Serrin [PS1], [PS3]. They formulated an interesting conjecture concerning special nonexistence results in certain, so called *critical dimensions*. In Chapter 3 we prove some special cases of this conjecture, cf. Theorem 3.2, and the conjecture in general under the additional assumption on the solution u to be positive, cf. Theorem 3.1.

1 Positivity

To explain some characteristics of the positivity behaviour of higher order Dirichlet problems we consider first an important prototype, the clamped plate equation:

$$(2)\qquad \begin{cases} \Delta^2 u = f(x) & \text{in } \Omega, \\ u|\partial\Omega = \dfrac{\partial u}{\partial \nu}|\partial\Omega = 0. \end{cases}$$

Here $\Omega \subset \mathbb{R}^2$ is the shape of the plate, f the pushing and u the bending. We would like to know for which shapes of the plates upwards pushing does imply upwards bending, i.e. $0 \not\equiv f \ge 0 \Rightarrow u > 0$?

Let $G_{2,\Omega}$ denote the corresponding Green function, then we may ask ("almost equivalently") for which domains $G_{2,\Omega}$ is positive? Hadamard [H2, 1908] and also Boggio, as Hadamard mentioned, believed that, by physical intuition, there is no serious doubt that the Green function should be positive at least in convex domains.

Boggio [Bo2] could prove this conjecture in balls, also in $\mathbb{R}^n$, and also for arbitrary powers $(-\Delta)^m$: he calculated the Green function $G_{m,n} := G_{m,B}$ in the unit ball $B \subset \mathbb{R}^n$

explicitly. As far as the author knows no further positive result has been proven since then.

On the other hand after 1949 numerous counterexamples ([Cof], [CD], [Du], [Ga1], [KKM], [Loe], [Osh], [ST], [Sz]) disproved Hadamard's positivity conjecture. The most striking examples were found by Coffman, Duffin and Garabedian: They could show change of sign of Green's function $G_{2,\Omega}$ in ellipses with ratio of half axes ≈ 2 ([Ga1], [Ga2, p. 275], for an elementary proof see [ST]) as well as in squares [CD]. That means that neither in arbitrarily smooth uniformly convex nor in rather symmetric domains Green's function needs to be positive.

Here we will show that Green's function of problem (2) remains positive in domains Ω that are close to the unit disk $B \subset \mathbb{R}^2$ in a suitable sense. Together with the counterexamples just mentioned this may be interpreted in the way that it is the deviation of $\partial\Omega$ from the constant "flexure" of the circumference ∂B that determines the positivity behaviour of the Green function $G_{2,\Omega}$. More generally, in two dimensions we are able to treat Dirichlet problems of the form:

$$(6)\qquad \begin{cases} Lu := \left(-\sum_{i,j=1}^{2} a_{ij}(x) \dfrac{\partial^2}{\partial x_i \partial x_j} \right)^m u + \sum_{|\alpha| \le 2m-1} b_\alpha(x) D^\alpha u = f & \text{in } \Omega, \\ D^\alpha u | \partial\Omega = 0 & \text{for } |\alpha| \le m-1. \end{cases}$$

Here we assume the domain $\Omega \subset \mathbb{R}^2$ to be "close" to the disk B, the leading coefficients a_{ij} to be "close" to δ_{ij} and the lower order coefficients b_α to be "small", see Theorem 1.26.

In higher dimensions $n > 3$ we do not have such strong devices as in two dimensions like conformal mapping or reduction to canonical form. Consequently our results are more restricted: only lower order perturbations of the prototype $((-\Delta)^m,\ B \subset \mathbb{R}^n)$ are admitted. That means in Theorem 1.1 we prove positivity for Dirichlet problems

$$(8)\qquad \begin{cases} Lu := (-\Delta)^m u + \sum_{|\alpha| \le 2m-1} b_\alpha(x) D^\alpha u = f & \text{in } B, \\ D^\alpha u | \partial B = 0 & \text{for } |\alpha| \le m-1. \end{cases}$$

with "small" coefficients b_α; $B \subset \mathbb{R}^n$ is the unit ball. Although Theorem 1.26 seems to be much more general than Theorem 1.1, the latter is the crucial step in the proof of the first.

If in (8) we remove the smallness assumptions on the coefficients b_α we are still able to prove a *local* maximum principle for differential **in**equalities, which is true also in arbitrary domains Ω, see Theorem 1.30. This local maximum principle may be applied to semilinear problems, cf. Corollary 1.31.

It's another interesting question to ask for the role of nontrivial Dirichlet boundary data with regard to the positivity of the solution. Again we look at first at the clamped plate equation:

$$(7)\qquad \begin{cases} \Delta^2 u = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^n, \\ u|\partial\Omega = \psi, \qquad -\dfrac{\partial u}{\partial \nu}|\partial\Omega = \varphi. \end{cases}$$

One could think that at least in the unit ball $\Omega = B \subset \mathbb{R}^n$ nonnegative data $\psi \geq 0$, $\varphi \geq 0$ yield a nonnegative solution $u \geq 0$. Actually this is true with respect to φ in any dimension and with respect to ψ, if $n \leq 4$. But for $n \geq 5$ the corresponding integral kernel changes sign! See [Nic, p. 34]. A perturbation theory of positivity, similar to that for problems (6) and (8), can be developed with respect to φ in the special case $\psi = 0$. This can also be generalized to equations of arbitrary order, see Corollaries 1.19 and 1.20. With respect to ψ we can prove only a rather restricted perturbation result in dimensions $n = 1, 2$ and 3, cf. Theorem 1.25.

1.1 Arbitrary Dimensions

A reasonable framework for our positivity results in order to avoid unnecessarily strong assumptions on the coefficients is L^p-theory. For existence and regularity we refer to [ADN].

Theorem 1.1. *There exists $\varepsilon_0 = \varepsilon_0(m,n) > 0$ such that the following holds:*

Let $b_\alpha \in C^0(\overline{B})$ satisfy the smallness condition $\|b_\alpha\|_{C^0(\overline{B})} \leq \varepsilon_0$, $|\alpha| \leq 2m-1$. Then for every $f \in L^p(B)$, $1 < p < \infty$, there exists a solution $u \in W^{2m,p}(B) \cap W_0^{m,p}(B)$ of the Dirichlet problem (8). Moreover if $0 \not\equiv f \geq 0$ then the solution is strictly positive:

$$u > 0 \text{ in } B.$$

Let $G_{m,n} : \overline{B} \times \overline{B} \to \mathbb{R} \cup \{\infty\}$ denote the Green function for $(-\Delta)^m$ under homogeneous Dirichlet boundary conditions and

$$\mathcal{G}_{m,n} : L^p(B) \to W^{2m,p}(B) \cap W_0^{m,p}(B), \quad (\mathcal{G}_{m,n} f)(x) = \int_B G_{m,n}(x,y) f(y)\, dy,$$

the corresponding Green operator. Starting point of our proof of Theorem 1.1 is Boggio's explicit formula for $G_{m,n}$:

Lemma 1.2 (Boggio [Bo2, p.126]).

$$(10) \quad G_{m,n}(x,y) = k_{m,n} \, |x-y|^{2m-n} \int_1^{\left||x|y - \frac{x}{|x|}\right| / |x-y|} (v^2-1)^{m-1} v^{1-n}\, dv;$$

$k_{m,n} > 0$ is a positive (well known) constant factor.

With help of the simple formula

$$(11) \quad \left| |x|y - \frac{x}{|x|} \right|^2 - |x-y|^2 = (1-|x|^2)(1-|y|^2) > 0,$$

positivity of Green's function $G_{m,n} > 0$ is immediate. Moreover from formula (10) we may derive appropriate estimates of the Green function and its derivatives, see Theorems 1.4 and 1.5 below. Before doing this, in order to simplify our statements, we define some convenient notation:

Definition 1.1. *a) For $x, y \in B$:*

(12) $d(x) := 1 - |x|,$

(13) $[XY] := [YX] := \left| |x|y - \frac{x}{|x|} \right| = \left| |y|x - \frac{y}{|y|} \right|.$

b) Let $N \subset \mathbb{R}^k$, $f, g : N \to [0, \infty]$. We call

(14) $f \sim g,$

iff there is a constant $C > 0$ such that for $t \in N$:

$$\frac{1}{C} f(t) \leq g(t) \leq C f(t).$$

Further we say

(15) $f \preceq g,$

iff there is a constant $C > 0$ such that for $t \in N$:

$$f(t) \leq C g(t).$$

The following lemma characterises the crucial distinction between the cases "x and y are closer to the boundary ∂B than to each other" and vice versa.

Lemma 1.3. *If $|x - y| \geq \frac{1}{2}[XY]$ then:*

(16) $d(x)d(y) \leq 3|x - y|^2,$

(17) $\max\{d(x), d(y)\} \leq 3|x - y|.$

If $|x - y| \leq \frac{1}{2}[XY]$ then:

(18) $\frac{3}{4}|x - y|^2 \leq \frac{3}{16}[XY]^2 \leq d(x)d(y),$

(19) $\frac{1}{4}d(x) \leq d(y) \leq 4d(x),$

(20) $|x - y| \leq 3 \min\{d(x), d(y)\},$

(21) $[XY] \leq 5 \min\{d(x), d(y)\}.$

Moreover for every $x, y \in B$ we have:

(22) $d(x) \leq [XY], \qquad d(y) \leq [XY].$

Theorem 1.4 (Two-sided estimates of the Green function).
On $B \times B$ we have:

$$(23) \quad G_{m,n}(x, y) \sim \begin{cases} |x - y|^{2m-n} \min\left\{1, \frac{d(x)^m d(y)^m}{|x - y|^{2m}}\right\}, & \text{if } n > 2m; \\ \log\left(1 + \frac{d(x)^m d(y)^m}{|x - y|^{2m}}\right), & \text{if } n = 2m; \\ d(x)^{m-\frac{n}{2}} d(y)^{m-\frac{n}{2}} \min\left\{1, \frac{d(x)^{\frac{n}{2}} d(y)^{\frac{n}{2}}}{|x - y|^n}\right\}, & \text{if } n < 2m. \end{cases}$$

Theorem 1.5 (Estimates of the derivatives of the Green function).
Let $\alpha \in \mathbb{N}_0^n$ be a multiindex. Then on $B \times B$ we have,
a) if $|\alpha| \geq 2m-n$ and n odd, or if $|\alpha| > 2m-n$ and n even:

$$(24)\quad |D_x^\alpha G_{m,n}(x,y)| \preceq \begin{cases} |x-y|^{2m-n-|\alpha|} \min\left\{1, \dfrac{d(x)^{m-|\alpha|}d(y)^m}{|x-y|^{2m-|\alpha|}}\right\} & \text{for } |\alpha| \leq m, \\ |x-y|^{2m-n-|\alpha|} \min\left\{1, \dfrac{d(y)^m}{|x-y|^m}\right\} & \text{for } |\alpha| \geq m; \end{cases}$$

b) if $|\alpha| = 2m-n$ and n even:

$$(25)\quad |D_x^\alpha G_{m,n}(x,y)| \preceq \begin{cases} \log\left(2+\dfrac{d(y)}{|x-y|}\right) \min\left\{1, \dfrac{d(x)^{m-|\alpha|}d(y)^m}{|x-y|^{2m-|\alpha|}}\right\} & \text{for } |\alpha| \leq m, \\ \log\left(2+\dfrac{d(y)}{|x-y|}\right) \min\left\{1, \dfrac{d(y)^m}{|x-y|^m}\right\} & \text{for } |\alpha| \geq m; \end{cases}$$

c) if $|\alpha| \leq 2m-n$ and n odd, or if $|\alpha| < 2m-n$ and n even:

$$(26)\quad |D_x^\alpha G_{m,n}(x,y)| \preceq \begin{cases} d(x)^{m-\frac{n}{2}-|\alpha|}d(y)^{m-\frac{n}{2}} \min\left\{1, \dfrac{d(x)^{\frac{n}{2}}d(y)^{\frac{n}{2}}}{|x-y|^n}\right\} & \text{for } |\alpha| \leq m-\frac{n}{2}, \\ d(y)^{2m-n-|\alpha|} \min\left\{1, \dfrac{d(x)^{m-|\alpha|}d(y)^{n-m+|\alpha|}}{|x-y|^n}\right\} & \text{for } m-\frac{n}{2} \leq |\alpha| \leq m, \\ d(y)^{2m-n-|\alpha|} \min\left\{1, \dfrac{d(y)^{n-m+|\alpha|}}{|x-y|^{n-m+|\alpha|}}\right\} & \text{for } |\alpha| \geq m. \end{cases}$$

Remarks. 1) The Green function for the Laplacian ($m=1, n>2$) satisfies the estimates above in arbitrary bounded $C^{2,\gamma}$-smooth domains, see e.g. [Wid]. This result is proven with help of general maximum principles and Harnack's inequality. For higher order equations we proceed just in the opposite way: We deduce the estimates above from Boggio's explicit formula and, in turn, make use of them to prove some comparison principles.
2) In what follows the estimates of $G_{m,n}$ *from below* are crucial.

Remark on the proofs of Theorems 1.4 and 1.5.

The distinction of the cases "$|x-y| \geq \frac{1}{2}[XY]$" and "$|x-y| \leq \frac{1}{2}[XY]$" is essential.

The representation (10) of the Green function $G_{m,n}$ is appropriate to show Theorem 1.4, and Theorem 1.5 in the case $|x-y| \geq \frac{1}{2}[XY]$, i.e. the boundary behaviour of the derivatives of $G_{m,n}$. In order to prove Theorem 1.5 in the case $|x-y| \leq \frac{1}{2}[XY]$ (near the singularity) it is suitable to first carry out the integration in formula (10) and then to differentiate.

In the proof of Theorem 1.4 the key observation is as follows:

$$G_{m,n}(x,y) \sim \begin{cases} |x-y|^{2m-n} \displaystyle\int\limits_1^{[XY]/|x-y|} v^{2m-n-1}\,dv, & \text{if } |x-y| \le \frac{1}{2}[XY], \\ |x-y|^{2m-n} \displaystyle\int\limits_1^{[XY]/|x-y|} (v^2-1)^{m-1}\,v\,dv, & \text{if } |x-y| \ge \frac{1}{2}[XY]. \end{cases}$$

In the proof of Theorem 1.5 we observe first that the transformation $s = 1 - \frac{1}{v^2}$ points out the boundary behaviour of $G_{m,n}$ more clearly:

$$(32) \quad G_{m,n}(x,y) = \frac{k_{m,n}}{2}\,|x-y|^{2m-n} \int\limits_0^{A_{x,y}} s^{m-1}(1-s)^{\frac{n}{2}-m-1}\,ds,$$

where

$$(34) \quad A_{x,y} := \frac{[XY]^2 - |x-y|^2}{[XY]^2} = \frac{(1-|x|^2)(1-|y|^2)}{[XY]^2} \sim \frac{d(x)d(y)}{[XY]^2}.$$

By means of a general product and chain rule we obtain the claim of Theorem 1.5 in the case $|x-y| \ge \frac{1}{2}[XY]$.

If $|x-y| \le \frac{1}{2}[XY]$ we use the integrated version of (10):

$$(39) \quad G_{m,n}(x,y) = \begin{cases} c_m|x-y|^{2m-n} + \displaystyle\sum_{j=0}^{m-1} c_j [XY]^{2m-n-2j}|x-y|^{2j}, \\ \qquad \text{if } n > 2m \text{ or } n \text{ odd}, \\ c_m|x-y|^{2m-n} \log\dfrac{[XY]}{|x-y|} + \displaystyle\sum_{j=0}^{m-1} c_j [XY]^{2m-n-2j}|x-y|^{2j}, \\ \qquad \text{if } n \le 2m \text{ and } n \text{ even}, \end{cases}$$

with appropriate constant factors $c_j = c_j(m,n) \in \mathbb{R}$. We observe that $|x-y|^{2j}$ is a polynomial. Negative powers of $|x-y|$ arise at most from the differentiation of $|x-y|^{2m-n}$ or $|x-y|^{2m-n}\log\left([XY]/|x-y|\right)$, resp. For details we refer to the calculations in the German original version. ■

Theorem 1.1 will be proven by means of a Neumann series. For this purpose we need to estimate iterated Green operators by the Green operator itself. In terms of Green functions we have to show the following 3-G-type result.

Theorem 1.8 (3-G-Theorem).
Let $\alpha \in \mathbb{N}_0^n$ be a multiindex. Then on $B \times B \times B$ we have:

$$(41)\quad \frac{G_{m,n}(x,z)\ |D_z^\alpha G_{m,n}(z,y)|}{G_{m,n}(x,y)} \preceq \begin{cases} |x-z|^{2m-n-|\alpha|} + |y-z|^{2m-n-|\alpha|}, \\ \qquad \text{if } |\alpha| > 2m-n, \\ \log\left(\dfrac{3}{|x-z|}\right) + \log\left(\dfrac{3}{|y-z|}\right), \\ \qquad \text{if } |\alpha| = 2m-n \text{ and } n \text{ even}, \\ 1, \qquad \text{if } |\alpha| < 2m-n \text{ or} \\ \qquad \text{if } |\alpha| = 2m-n \text{ and } n \text{ odd}. \end{cases}$$

The proof is crucially based on the Green function estimates (Theorems 1.4 and 1.5) and a number of technical inequalities and equivalencies, see Lemmas 1.9-1.12 of the German original version.

Sketch of proof of the positivity result, Theorem 1.1.

For details we refer to Section 1.1.3 of the German version.

We rewrite the Dirichlet problem (8) in the form:

$$(56)\quad \begin{cases} ((-\Delta)^m + \mathcal{A})\, u = f & \text{in } B, \\ D^\alpha u|\partial B = 0 & \text{for } |\alpha| \le m-1, \end{cases}$$

where

$$\mathcal{A}u := \sum_{|\alpha| \le 2m-1} b_\alpha(\,.\,) D^\alpha u, \quad b_\alpha \in C^0(\overline{B}).$$

For any $1 < p < \infty$, $\mathcal{A}$ is considered as bounded linear operator $W^{2m,p}(B) \cap W_0^{m,p}(B) \to L^p(B)$. The Green operator $\mathcal{G}_{m,n} : L^p(B) \to W^{2m,p}(B) \cap W_0^{m,p}(B)$ for the polyharmonic Dirichlet problem ((56) with $\mathcal{A} = 0$) has been already defined on p. 5.

Let $\varepsilon := \max_{|\alpha| \le 2m-1} \|b_\alpha\|_{C^0(\overline{B})}$. For $\varepsilon \ge 0$ small enough

$$(\mathcal{I} + \mathcal{G}_{m,n}\mathcal{A}) : W^{2m,p}(B) \cap W_0^{m,p}(B) \to W^{2m,p}(B) \cap W_0^{m,p}(B)$$

has a bounded inverse. For every $f \in L^p(B)$ the Dirichlet problem (56) has a unique solution:

$$(57)\quad u = (\mathcal{I} + \mathcal{G}_{m,n}\mathcal{A})^{-1}\, \mathcal{G}_{m,n} f \in W^{2m,p}(B) \cap W_0^{m,p}(B).$$

The Green operator for the Dirichlet problem (56) is given by

$$\mathcal{G}_{m,n,\mathcal{A}} = (\mathcal{I} + \mathcal{G}_{m,n}\mathcal{A})^{-1}\, \mathcal{G}_{m,n} = \mathcal{G}_{m,n} + \sum_{i=1}^{\infty} (-1)^i\, (\mathcal{G}_{m,n}\mathcal{A})^i\, \mathcal{G}_{m,n}.$$

Moreover we have the corresponding Green function

$$(61)\quad G_{m,n,\mathcal{A}}(x,y) := \sum_{i=0}^{\infty} G^{(i)}(x,y),$$

where

$$\begin{aligned}
G^{(0)}(x,y) &:= G_{m,n}(x,y),\\
G^{(i)}(x,y) &:= (-1)^i \int_B \cdots \int_B G_{m,n}(x,z_1)\,(\mathcal{A}_{z_1} G_{m,n}(z_1,z_2)) \ldots\\
&\qquad \ldots (\mathcal{A}_{z_i} G_{m,n}(z_i,y))\; d(z_1,\ldots,z_i).
\end{aligned}$$

By means of the 3-G-Theorem 1.8 we find a number $M = M(m,n) > 0$ such that

$$\int_B \frac{G_{m,n}(x,z)\,|\mathcal{A}_z G_{m,n}(z,y)|}{G_{m,n}(x,y)} \leq \varepsilon M.$$

As a consequence we may estimate the terms $G^{(i)}$ as follows:

$$\begin{aligned}
\left|G^{(i)}(x,y)\right| &= \Bigg| \int_B \cdots \int_B \frac{G_{m,n}(x,z_1)\,(\mathcal{A}_{z_1} G_{m,n}(z_1,z_2))}{G_{m,n}(x,z_2)} \cdot\\
&\qquad \cdot \frac{G_{m,n}(x,z_2)\,(\mathcal{A}_{z_2} G_{m,n}(z_2,z_3))}{G_{m,n}(x,z_3)} \cdots\\
&\quad \cdots \frac{G_{m,n}(x,z_i)\,(\mathcal{A}_{z_i} G_{m,n}(z_i,y))}{G_{m,n}(x,y)} \cdot G_{m,n}(x,y) d(z_1,\ldots,z_i) \Bigg|\\
&\leq G_{m,n}(x,y) \cdot \prod_{j=1}^{i} \sup_{\xi,\eta\in B} \int_B \frac{G_{m,n}(\xi,z_j)\,\left|\mathcal{A}_{z_j} G_{m,n}(z_j,\eta)\right|}{G_{m,n}(\xi,\eta)}\, dz_j\\
&\leq (\varepsilon M)^i\, G_{m,n}(x,y).
\end{aligned}$$

For $\varepsilon < \frac{1}{4M}$ we come up with

$$\begin{aligned}
\frac{2}{3} G_{m,n}(x,y) &\leq \frac{1-2\varepsilon M}{1-\varepsilon M} G_{m,n}(x,y) \leq G_{m,n,\mathcal{A}}(x,y)\\
&\leq \frac{1}{1-\varepsilon M} G_{m,n}(x,y) \leq \frac{4}{3} G_{m,n}(x,y).
\end{aligned}$$

To sum up we find some $\varepsilon_0 = \varepsilon_0(m,n) > 0$ and some $C > 0$ such that for $0 \leq \varepsilon \leq \varepsilon_0$ we have:

(58) $\displaystyle \frac{1}{C}\, \mathcal{G}_{m,n} \leq \mathcal{G}_{m,n,\mathcal{A}} \leq C\, \mathcal{G}_{m,n},$

(59) $\displaystyle \frac{1}{C}\, G_{m,n}(x,y) \leq G_{m,n,\mathcal{A}}(x,y) \leq C\, G_{m,n}(x,y)$ on $B \times B$.

∎

In (58) we used the definition:

Definition 1.2. *Let $\mathcal{S}, \mathcal{T} : L^p(B) \to L^p(B)$. We write*

$$\mathcal{S} \geq \mathcal{T},$$

iff for every $f \in L^p(B)$, $f \geq 0$, it holds:

$$\mathcal{S} f \geq \mathcal{T} f.$$

We may supplement Theorem 1.1 considerably if we confine ourselves to perturbations of order zero:

$$(63)\quad \begin{cases} (-\Delta)^m u + b(x)u = f & \text{in } \Omega, \\ D^\alpha u|\partial\Omega = 0 & \text{for } |\alpha| \leq m-1. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded, $C^{2m,\gamma}$-smooth domain, $b \in C^0(\overline{\Omega})$. We denote by $\mathcal{G}_{m,\Omega,b}$ the corresponding Green operator (if it exists). We show the compatibility of the ordering "$\leq$" for the coefficients b with the positivity of the Green operator $\mathcal{G}_{m,\Omega,b} \geq 0$, see Theorem 1.15 of the German version. Furthermore, if $m > 1$, we prove the necessity of a kind of smallness condition on the coefficient b in order to have positivity $\mathcal{G}_{m,\Omega,b} \geq 0$, see Theorem 1.16 of the German version. As an immediate consequence we have:

Corollary 1.17. *Let $m > 1$. The parabolic initial boundary value problem:*

$$(65)\quad \begin{cases} u_t + (-\Delta)^m u = 0 & \textit{in } (0,\infty) \times \Omega, \\ D_x^\alpha u(t,\,.\,)|\partial\Omega = 0 & \textit{for } |\alpha| \leq m-1, \quad t \in (0,\infty), \\ u(0,\,.\,) = \varphi \geq 0 & \textit{in } \Omega, \end{cases}$$

does in general not preserve the positivity of the initial value φ.

The positivity properties of Dirichlet problem (63) may be summarized as follows:

Corollary 1.18. *Let $m > 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain such that the (Dirichlet-) Green function for $(-\Delta)^m$ is positive in $\Omega \times \Omega$. Let $\Lambda_{m,1}$ denote the first Dirichlet eigenvalue of $(-\Delta)^m$ in Ω.*

Then there exist numbers $b_c, \tilde{b}_c \in [0,\infty)$, $b_c \geq \tilde{b}_c$, such that for $b \in C^0(\overline{\Omega})$ we have:
a) If $b > b_c$ on $\overline{\Omega}$, then $\mathcal{G}_{m,\Omega,b}$ does not preserve positivity:

$$(66)\quad \exists\, 0 \not\equiv f \geq 0 : \quad \mathcal{G}_{m,\Omega,b} f \not\geq 0.$$

On the other hand we have:

$$(67)\quad \forall\, 0 \not\equiv f \geq 0 : \quad \mathcal{G}_{m,\Omega,b} f \not\leq 0,$$
$$(68)\quad \exists\, 0 \not\equiv f \geq 0 : \quad \mathcal{G}_{m,\Omega,b} f \geq 0.$$

b) If $-\Lambda_{m,1} < b \leq b_c$ or $-\Lambda_{m,1} < b < \tilde{b}_c$ resp. then $\mathcal{G}_{m,\Omega,b}$ is positivity preserving or strongly positivity preserving resp., i.e.:

$$(69)\quad \forall\, 0 \not\equiv f \geq 0 : \quad \mathcal{G}_{m,\Omega,b} f \geq 0 \textit{ or } \mathcal{G}_{m,\Omega,b} f > 0 \textit{ in } \Omega \textit{ resp.}$$

c) If $b = -\Lambda_{m,1}$ and $0 \not\equiv f \geq 0$ then (63) has no solution.
d) If $b < -\Lambda_{m,1}$ then (63) kills positivity: If $0 \not\equiv f \geq 0$ and u is a solution to (63) then: $u \not\geq 0$ in Ω.

Remark. Case a) does not occur in second order equations. This different behaviour may be responsible for the difficulties in classical solvability of semilinear boundary value problems of higher order. Cf. Corollary 1.31 below.

The rest of this section is devoted to the role of nonhomogeneous boundary data with regard to the sign of the solution. As already mentioned in the introduction this problem is rather subtle: In general we cannot expect that fixed sign of any particular Dirichlet datum leads to fixed sign of the solution. It seems that a perturbation theory of positivity (analogous to that above with regard to the right-hand side) exists in general only for the Dirichlet datum of highest order. Only in a special (biharmonic) case we are able to treat the Dirichlet datum of lowest order too, see Theorem 1.25 below. So, we will first consider the following boundary value problem:

$$(70)\quad \begin{cases} ((-\Delta)^m + \mathcal{A})\, u = f & \text{in } B, \\ D^\alpha u | \partial B = 0 & \text{for } |\alpha| \le m-2, \\ \begin{cases} -\dfrac{\partial}{\partial \nu} \Delta^{(m/2)-1} u | \partial B = \varphi, & \text{if } m \text{ is even}, \\ \Delta^{(m-1)/2} u | \partial B = \varphi, & \text{if } m \text{ is odd}. \end{cases} \end{cases}$$

Here $f \in C^0(\overline{B})$, $\varphi \in C^0(\partial B)$ and

$$(71)\quad \mathcal{A} = \sum_{|\alpha| \le 2m-1} b_\alpha(\,.\,) D^\alpha, \qquad b_\alpha \in C^{|\alpha|}(\overline{B}),$$

is a sufficiently small lower order perturbation. For existence of solutions $u \in W^{2m,p}_{\mathrm{loc}}(B) \cap C^{m-1}(\overline{B})$, $p > 1$, we refer to the local L^p-theory in [ADN, §15] and to the maximum estimates of Agmon [Ag]. The latter already require the strong regularity assumptions on the coefficients b_α.

Corollary 1.19. *There exists $\varepsilon_0 = \varepsilon_0(m,n) > 0$ such that we have:*

If for all $|\alpha| \le 2m-1$ the smallness condition $\|b_\alpha\|_{C^{|\alpha|}(\overline{B})} \le \varepsilon_0$ is fulfilled, then for every $f \in C^0(\overline{B})$, $\varphi \in C^0(\partial B)$ there exists a solution $u \in W^{2m,p}_{\mathrm{loc}}(B) \cap C^{m-1}(\overline{B})$, $1 < p < \infty$, to the Dirichlet problem (70). Moreover $f \ge 0$, $\varphi \ge 0$, $f \not\equiv 0$ or $\varphi \not\equiv 0$ implies $u > 0$.

If $m = 1$ we recover a special form of the strong maximum principle for second order elliptic equations. The next result, in some sense dual to the previous one, may be viewed as a higher order analogue to the Hopf boundary lemma.

Corollary 1.20. *Assume $b_\alpha \in C^0(\overline{B})$, $|\alpha| \le 2m-1$. There exists $\varepsilon_0 = \varepsilon_0(m,n) > 0$ such that the following holds:*

If $\|b_\alpha\|_{C^0(\overline{B})} \le \varepsilon_0$, $|\alpha| \le 2m-1$, then for every $f \in C^0(\overline{B})$ the Dirichlet problem (8) has a solution $u \in W^{2m,p}(B) \cap C^{2m-1}(\overline{B})$, $p > 1$ arbitrary. Moreover $0 \not\equiv f \ge 0$ implies $u > 0$ in B and for every $x \in \partial B$:

$$(72)\quad \begin{cases} \Delta^{(m/2)} u(x) > 0, & \textit{if } m \textit{ even}, \\ -\dfrac{\partial}{\partial \nu} \Delta^{(m-1)/2} u(x) > 0, & \textit{if } m \textit{ odd}. \end{cases}$$

The common key point in the proof of both propositions is the observation that the corresponding Green function $G_{m,n,\mathcal{A}}$ vanishes (in both variables) on ∂B precisely of order m, cf. Theorem 1.5 and estimate (59). Further in the proof of Corollary 1.19 we observe that the Poisson kernel for φ is given by

$$\begin{cases} \Delta_y^{m/2} G_{m,n,\mathcal{A}}(x,y) & \text{if } m \text{ even}, \\ \left(-\dfrac{\partial}{\partial \nu_y}\Delta_y^{(m-1)/2}\right) G_{m,n,\mathcal{A}}(x,y) & \text{if } m \text{ odd; } x \in B,\ y \in \partial B. \end{cases}$$

The requisite smoothness of Green's function is shown in Lemmas 1.21 and 1.22.

We conclude this section with some remarks concerning the boundary datum $u|\partial B$ in perturbed biharmonic Dirichlet problems.

Theorem 1.25. *Let $n = 1, 2$ or 3. Then there exists $\varepsilon_0 = \varepsilon_0(n) > 0$ such that we have:*
Assume that the coefficients $b_\alpha \in C^{|\alpha|}(\overline{B})$, $|\alpha| \le 2$, satisfy the smallness condition $\|b_\alpha\|_{C^{|\alpha|}(\overline{B})} \le \varepsilon_0$. Then for every $f \in C^0(\overline{B})$, $\psi \in C^1(\partial B)$, $\varphi \in C^0(\partial B)$ the Dirichlet problem

$$(84)\quad \begin{cases} \Delta^2 u + \sum\limits_{|\alpha|\le 2} b_\alpha(x) D^\alpha u = f \ \text{ in } B, \\ u|\partial B = \psi, \qquad -\dfrac{\partial u}{\partial \nu}|\partial B = \varphi, \end{cases}$$

has a solution $u \in W^{4,p}_{\mathrm{loc}}(B) \cap C^1(\overline{B})$, $p > 1$. Moreover $f \ge 0$, $\psi \ge 0$, $\varphi \ge 0$, $f \not\equiv 0$ or $\psi \not\equiv 0$ or $\varphi \not\equiv 0$ implies $u > 0$ in B.

In the proof we may assume $f = 0$ and $\varphi = 0$. In the unperturbed case ($b_\alpha = 0$) from [Nic, p. 34] we explicitly know the Poisson kernel for ψ:

$$(77)\quad K_n(x,y) := \frac{1}{2\omega_n}\frac{\left(1-|x|^2\right)^2}{|x-y|^{n+2}}\left\{n(1-x\cdot y)-(n-2)|x-y|^2\right\},\ x\in B,\ y\in\partial B.$$

If $n \ge 5$, K_n changes sign, but if $n \le 4$ we have:

Lemma 1.23. *For $n \le 4$, $x \in B$, $y \in \partial B$ we have $K_n(x,y) > 0$. Moreover, on $B \times \partial B$ there holds, if $n = 1, 2, 3$:*

$$(79)\quad K_n(x,y)\begin{cases} \preceq |x-y|^{-n-1}\, d(x)^2, \\ \succeq |x-y|^{-n}\, d(x)^2, \end{cases}$$

and if $n = 4$:

$$(80)\quad K_4(x,y) \sim |x-y|^{-6}\, d(x)^3.$$

Again 3-G-type estimates can be shown, which are uniformly integrable only for $n = 1, 2, 3$. The degeneracy of K_4 near ∂B yields an unbounded factor, see Lemma 1.24. The restriction of the admissible lower order terms, $|\alpha| \le 2$, in Theorem 1.25 is a consequence of the different estimates of K_n from below and above in (79).

These observations suggest the conjecture:

Loss of positivity occurs via degeneracy on the boundary.

1.2 Results in two dimensions

As mentioned at the beginning of this chapter, in two dimensions also perturbations of highest order of the polyharmonic prototype may be considered:

$$(86)\quad \begin{cases} Lu = f & \text{in } \Omega, \\ D^\alpha u | \partial\Omega = 0 & \text{for } |\alpha| \le m-1, \end{cases}$$

with

$$(87)\quad Lu := \left(-\sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \right)^m u + \sum_{|\alpha| \le 2m-1} b_\alpha(x) D^\alpha u,$$

$a_{ij} = a_{ji} \in C^{2m-1,\gamma}(\overline{\Omega})$, $b_\alpha \in C^{0,\gamma}(\overline{\Omega})$. First we define an appropriate notion of closeness for domains and operators.

Definition 1.3. *We assume that Ω^*, Ω are bounded, $C^{k,\gamma}$-smooth domains. Let $\varepsilon > 0$. We call Ω ε-close to Ω^* in $C^{k,\gamma}$-sense, iff there exists a $C^{k,\gamma}$-mapping $g : \overline{\Omega^*} \to \overline{\Omega}$ such that $g(\overline{\Omega^*}) = \overline{\Omega}$ and*

$$\|g - Id\|_{C^{k,\gamma}(\overline{\Omega^*})} \le \varepsilon.$$

Remark. If $k \ge 1$, Ω^* convex and ε sufficiently small, then $g^{-1} \in C^{k,\gamma}(\overline{\Omega})$ exists and $\|g^{-1} - Id\|_{C^{k,\gamma}(\overline{\Omega})} = \mathcal{O}(\varepsilon)$.

Definition 1.4. *Let $\varepsilon > 0$ and L as in (87). We call the operator L ε-close to $(-\Delta)^m$ in $C^{k,\gamma}$-sense, if additionally $a_{ij} \in C^{k,\gamma}(\overline{\Omega})$ (if necessary) and*

$$\|a_{ij} - \delta_{ij}\|_{C^{k,\gamma}(\overline{\Omega})} \le \varepsilon,$$

$$\|b_\alpha\|_{C^0(\overline{\Omega})} \le \varepsilon \text{ for } |\alpha| \le 2m-1.$$

Remark. If $\varepsilon \ge 0$ is small, then L is uniformly elliptic.

The following result is a two dimensional generalization of Theorem 1.1.

Theorem 1.26. *There exists $\varepsilon_0 = \varepsilon_0(m) > 0$ such that, for $0 \le \varepsilon \le \varepsilon_0$, we have:*

We assume that the bounded, $C^{2m,\gamma}$-smooth domain Ω is ε-close to the unit disk B in C^{2m}-sense and that the differential operator L is ε-close to $(-\Delta)^m$ in $C^{2m-1,\gamma}$-sense. Then for every $f \in C^{0,\gamma}(\overline{\Omega})$ there exists a solution $u \in C^{2m,\gamma}(\overline{\Omega})$ to the Dirichlet problem (86) which, in case of nonnegative right-hand side $0 \not\equiv f \ge 0$, is strictly positive:

$$u > 0 \text{ in } \Omega.$$

Remark. Let $E_{a,b}$ be an ellipse with half axes $a, b > 0$. In case of small eccentricity, i.e. $\frac{a}{b} \approx 1$, Green's function for Δ^2 in $E_{a,b}$ is positive, while according to the example of Garabedian [Ga1], in case of large eccentricity, e.g. $\frac{a}{b} \approx 2$, it changes sign.

Sketch of proof of Theorem 1.26. With help of appropriate mappings we want to establish the situation of Theorem 1.1.
1. First we consider the special case $a_{ij} = \delta_{ij}$; i.e. the principal part $(-\Delta)^m$. If ε is small enough, Ω is simply connected. By the Riemann mapping theorem and its supplement of Kellogg-Warschawski (see [Pom]) we find a biholomorphic mapping $h : B \to \Omega$, $h \in C^{2m,\gamma}(\overline{B})$, $h^{-1} \in C^{2m,\gamma}(\overline{\Omega})$. We "pull back" the Dirichlet problem (86) to the disk: We set $v : \overline{B} \to \mathbb{R}$, $v(x) := u(h(x))$; by virtue of $\Delta v(x) = \frac{1}{2} |\nabla h(x)|^2 ((\Delta u) \circ h)(x)$ we have

$$\begin{cases} \left(-\dfrac{2}{|\nabla h|^2}\Delta\right)^m v + \displaystyle\sum_{|\alpha| \le 2m-1} \hat{b}_\alpha D^\alpha v = f \circ h & \text{in } B, \\ D^\alpha v | \partial B = 0 & \text{for } |\alpha| \le m-1, \end{cases}$$

with suitable coefficients $\hat{b}_\alpha \in C^0(\overline{B})$. Additionally we assume

$$\|h - Id\|_{C^{2m-1}(\overline{B})} = \mathcal{O}(\varepsilon).$$

Then for ε small enough, v is solution of a Dirichlet problem that satisfies the assumptions of Theorem 1.1.
2. In order to complete the proof in the special case "$a_{ij} = \delta_{ij}$" it remains to show that there are biholomorphic mappings as above, which in particular satisfy

$$\|h - Id\|_{C^{2m-1}(\overline{B})} = \mathcal{O}(\varepsilon).$$

By means of the Green function $G_{1,\Omega}$ of the Laplacian in Ω the inverse mapping $h^{-1} : \overline{\Omega} \to \overline{B}$ may be defined as follows (see [Cou], [Swe2, Sect. 4.2]):

$$w(x) := 2\pi G_{1,\Omega}(x, 0),$$

$$w^*(x) := \int_{1/2}^{x} \left(-\frac{\partial}{\partial \xi_2} w(\xi)\, d\xi_1 + \frac{\partial}{\partial \xi_1} w(\xi)\, d\xi_2 \right),$$

$$h^{-1}(x) := \exp\left(-w(x) - i w^*(x)\right), \quad x \in \overline{\Omega}.$$

The integral is taken with respect to any curve from $\frac{1}{2}$ to x in $\Omega \setminus \{0\}$ and is well defined up to multiples of 2π. Employing Schauder estimates for norms of high order for the "regular part" of Green's function

$$r(x) := w(x) + \log|x| = 2\pi G_{1,\Omega}(x, 0) + \log|x|$$

we show

$$\|r\|_{C^{2m-1}(\overline{\Omega})} = \mathcal{O}(\varepsilon). \tag{88}$$

It follows

$$\|h^{-1} - Id\|_{C^{2m-1}(\overline{\Omega})} = \mathcal{O}(\varepsilon), \qquad \Rightarrow \|h - Id\|_{C^{2m-1}(\overline{B})} = \mathcal{O}(\varepsilon).$$

3. Finally with help of a suitable mapping

(94) $\Phi = (\varphi, \psi): \quad \Omega \to \Phi(\Omega) = \Omega^*$

the principal part

$$\left(- \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \right)^m u$$

can be reduced to canonical form:

$$\left(-A(\xi)\Delta - B_1(\xi)\frac{\partial}{\partial \xi_1} - B_2(\xi)\frac{\partial}{\partial \xi_2} \right)^m v; \quad v(\xi) := u\left(\Phi^{-1}(\xi)\right).$$

The second component ψ of the mapping Φ is determined as solution of the following Dirichlet problem of second order:

$$(97) \quad \begin{cases} \dfrac{\partial}{\partial x_1}\left(\dfrac{a_{11}\psi_{x_1} + a_{12}\psi_{x_2}}{\sqrt{a_{11}a_{22} - a_{12}^2}} \right) + \dfrac{\partial}{\partial x_2}\left(\dfrac{a_{21}\psi_{x_1} + a_{22}\psi_{x_2}}{\sqrt{a_{11}a_{22} - a_{12}^2}} \right) = 0 & \text{in } \Omega, \\ \psi(x) = x_2 & \text{on } \partial\Omega. \end{cases}$$

The first component φ is then defined by the Beltrami system

$$(93) \quad \frac{\partial \varphi}{\partial x_1} = \frac{a_{21}\psi_{x_1} + a_{22}\psi_{x_2}}{\sqrt{a_{11}a_{22} - a_{12}^2}}, \quad \frac{\partial \varphi}{\partial x_2} = -\frac{a_{11}\psi_{x_1} + a_{12}\psi_{x_2}}{\sqrt{a_{11}a_{22} - a_{12}^2}}$$

and by $\varphi(0) = 0$. Again by means of Schauder estimates for norms of high order we show that C^{2m}-closeness of Ω to B and $C^{2m-1,\gamma}$-closeness of L to $(-\Delta)^m$ uniformly imply C^{2m}-closeness of $\Omega^* = \Phi(\Omega)$ to B and appropriate smallness of the additional coefficients. Hence, the result for the special case "$a_{ij} = \delta_{ij}$" applies, and the proof is complete. ∎

Remark. Similarly the Corollaries 1.19 and 1.20 concerning the boundary behaviour of solutions can be generalized to domains, close to B, and to principal parts, close to $(-\Delta)^m$.

1.3 A local maximum principle for differential inequalities

At the expense of the global character of the estimates the smallness assumptions on the coefficients can be removed from Theorem 1.1.

In this section we consider bounded domains Ω of class $C^{2m,\gamma}$. Further we assume that the differential operator

$$(102) \; Lu := \left(- \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \right)^m u + \sum_{|\alpha| \le 2m-1} b_\alpha(\,.\,) D^\alpha u$$

has constant leading coefficients $a_{ij} = a_{ji}$, satisfying the ellipticity condition

$$\exists 0 < \lambda \le \Lambda \quad \forall \xi \in \mathbb{R}^n: \quad \lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2,$$

and smooth lower order coefficients

$$b_\alpha \in C^{|\alpha|,\gamma}(\overline{\Omega}).$$

Under these assumptions we have:

Theorem 1.30. *Let $q \geq 1$, $q > \frac{n}{2m}$. Let $K \subset \Omega$ be a compact subset.*

Then there exists a constant $C = C\Big(n, m, \lambda, \Lambda, q, \max_{|\alpha|\leq 2m-1} \|b_\alpha\|_{C^{|\alpha|}(\overline{\Omega})}, \operatorname{dist}(K, \partial\Omega)\Big)$ such that every subsolution $u \in C^{2m}(\overline{\Omega})$, $f \in C^0(\overline{\Omega})$ to the differential inequality

$$Lu \leq f$$

satisfies the local maximum estimate:

$$\sup_K u \leq C \left\{ \|f^+\|_{L^q(\Omega)} + \|u\|_{W^{m-1,1}(\Omega)} \right\}. \tag{103}$$

This result can be applied to semilinear equations:

Corollary 1.31. *In addition to Theorem 1.30, we assume L to be positive definite in $H_0^m(\Omega)$: $\int_\Omega Lu \cdot u \, dx \geq \frac{1}{C}\|u\|^2_{H_0^m}$ for $u \in C^{2m}(\overline{\Omega}) \cap H_0^m(\Omega)$. Furthermore let $g \in C^{0,\gamma}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ be a nonlinearity, which satisfies the sign condition $g(x,t) \cdot t \geq 0$, $x \in \overline{\Omega}$, $t \in \mathbb{R}$, and whose negative part (or positive part) grows at most linearly: $g(t) \geq -C(1 + |t|)$, $t \leq 0$.*

Then for every $f \in C^{0,\gamma}(\overline{\Omega})$ there exists a solution $u \in C^{2m,\gamma}(\Omega) \cap H_0^m(\Omega)$ to the Dirichlet problem

$$\begin{cases} Lu(x) + g(x, u(x)) = f(x) & \text{in } \Omega, \\ D^\alpha u | \partial\Omega = 0 & \text{for } |\alpha| \leq m - 1. \end{cases}$$

This existence theorem still is far from the generality of the corresponding second order result, where beside Hölder continuity only the sign condition is imposed on g, cf. [GT], and also of the corresponding fourth order result [To], where g is additionally supposed to be C^1 and monotone.

On the other hand in large dimensions $n > 2m$ the existence of classical solutions to semilinear equations of arbitrary order $2m$ is usually shown under "controllable growth conditions" on g:

$$|g(t)| \leq C \; (1 + |t|)^{(n+2m)/(n-2m)},$$

see [Lu], [Wa1], [Wa2]. For $n \nearrow \infty$ this is only little more than linear growth, and the assumptions of Corollary 1.31 seem to be considerably weaker.

2 Semilinear eigenvalue problems: Existence results

We consider the semilinear polyharmonic eigenvalue problem

$$(107)\quad \begin{cases} (-\Delta)^m u = \lambda u + |u|^{p-1}u,\ u \not\equiv 0 & \text{in } \Omega, \\ D^\alpha u|\partial\Omega = 0 & \text{for } |\alpha| \le m-1. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $C^{2m,\gamma}$, $n > 2m$, $\lambda \in \mathbb{R}$, $1 < p \le s$; $s = (n+2m)/(n-2m)$ is the critical (Sobolev-) exponent.

We will show the existence of nontrivial solutions with help of variational methods. Solutions to (107) are critical points of the functional E_λ on $H_0^m(\Omega)$

$$\begin{aligned} E_\lambda(u) &:= \frac{1}{2} S_\lambda(u) - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx; \\ S_\lambda(u) &:= \begin{cases} \displaystyle\int_\Omega \left(\left(\Delta^{m/2}u\right)^2 - \lambda u^2 \right) dx, & \text{if } m \text{ even}, \\ \displaystyle\int_\Omega \left(\left|\nabla\Delta^{(m-1)/2}u\right|^2 - \lambda u^2 \right) dx, & \text{if } m \text{ odd}. \end{cases} \end{aligned}$$

The simplest nontrivial solutions can also be sought by constrained minimization:

$$S_\lambda(v) \stackrel{!}{=} \min \quad \text{on} \quad \tilde{N} := \left\{ v \in H_0^m(\Omega) : \int_\Omega |v|^{p+1}\,dx = 1 \right\}.$$

In the subcritical case, i.e. $1 < p < s$, we see by an obvious simplification of the proof of Lemma 2.9 that E_λ satisfies a global Palais-Smale condition. A general mountain pass lemma, cf. Lemma 2.10, readily yields the existence of infinitely many solutions for any $\lambda \in \mathbb{R}$. So, in what follows we will concentrate on the more interesting critical case. But in higher order equations there is at least one question which is more difficult to answer in the subcritical than in the critical case: the existence of *positive* solutions. Keeping the positivity results of Chapter 1 in view we have to restrict ourselves to balls $\Omega = B$. Then in the critical case we can show an "optimal" result, cf. Theorems 2.5 and 3.1, while in the subcritical case we have to leave open whether the "ground state" (the simplest solution, found by constrained minimization) remains positive as $\lambda \searrow -\infty$, cf. Theorem 2.4.

In the critical case $p = s$ the global Palais-Smale condition is violated. This loss of compactness is reflected by nonexistence theorems. E.g. if Ω is starshaped then according to Pohožaev [Poh] ($m = 1$) and Pucci, Serrin [PS1] ($m > 1$) problem (107) with $p = s$ has a nontrivial solution at most for

$$(108)\quad \lambda \ge 0.$$

In balls $\Omega = B$ further nonexistence phenomena can be shown; Chapter 3 is devoted to this subject.

Since the fundamental works of Brezis, Coron, Nirenberg ([BrCN], [BrN]) et al. it has been realized that functionals with critical growth have local compactness properties. This can also be shown in our situation, cf. Lemmas 2.6 and 2.9 below. The ensuing existence results are presented in Theorem 2.5 (positive solution in a ball) and Theorem 2.8 (any nontrivial solution).

2.1 Positive solutions: "Ground states"

In this section we seek solutions by constrained *minimization* (i.e. ground states). As $u \mapsto |u|$ maps $H_0^1(\Omega) \to H_0^1(\Omega)$ and $\|u\|_{H_0^1(\Omega)} = \| \, |u| \, \|_{H_0^1(\Omega)}$ in the second order problem a ground state obviously can assumed to be positive. In higher order problems this simple trick fails completely:

$$m > 1: \quad u \in H_0^m \not\Rightarrow |u| \in H_0^m.$$

Instead we consider a modified Dirichlet problem

$$(110) \begin{cases} (-\Delta)^m u - \lambda u = |u|^p, \ u \not\equiv 0 & \text{in } B, \\ D^\alpha u|\partial B = 0 & \text{for } |\alpha| \le m-1, \end{cases}$$

and employ the positivity results of Chapter 1: Theorem 1.1 and Corollary 1.18. Up to now these positivity results necessitate to restrict Ω in this section to be the (unit) ball B. (Nevertheless we believe that also small perturbations of B are admissible.) We have a negative number $\tilde{\lambda}_c < 0$ such that at least for $\tilde{\lambda}_c < \lambda < \Lambda_{m,1}$ (the first Dirichlet eigenvalue of $(-\Delta)^m$) every nontrivial solution to (110) is strictly positive. Hence, it solves the original problem (107) too.

As $\Omega = B$ we may consider positive solutions that are radially symmetric. The following lemma shows that, if $\lambda \ge 0$, they are also monotone, see also [Sor, Prop. 1].

Lemma 2.3. *Let $u \in C^{2m}(\overline{B})$ be radially symmetric, $u = u(r)$, $r = |x|$. We assume:*

$$\begin{cases} (-\Delta)^m u > 0 & \text{in } B, \\ \left(\dfrac{d}{dr}\right)^j u|\partial B = 0 & \text{for } j = 0, \dots, m-1. \end{cases}$$

Then $u > 0$ in B, and $u = u(r)$ strictly decreases in $r \in (0,1)$.

Idea of proof. Mean value theorem and counting zeros. ■

Theorem 2.4 (Subcritical growth). *Let $n, m \in \mathbb{N}$, $p \in (1, s)$ with $s = (n+2m)/(n-2m)$ if $n > 2m$ and $p \in (1, \infty)$ if $1 \le n \le 2m$. We assume that the domain is the unit ball: $\Omega = B \subset \mathbb{R}^n$. Let $\Lambda_{m,1}$ denote the first Dirichlet eigenvalue of $(-\Delta)^m$ in B.*

Then for every $\lambda < \Lambda_{m,1}$ the Dirichlet problem (107) has a nontrivial radial solution. Moreover there exists a negative number $\tilde{\lambda}_c < 0$, $\tilde{\lambda}_c = \tilde{\lambda}_c(m,n)$, such that for $\lambda \in (\tilde{\lambda}_c, \Lambda_{m,1})$ we even have a positive radial solution $u \in C^\infty(B) \cap C^{2m+1}(\overline{B})$:

$$(111) \begin{cases} (-\Delta)^m u = \lambda u + u^p & \text{in } B, \\ u > 0 & \text{in } B, \\ D^\alpha u|\partial B = 0 & \text{for } |\alpha| \le m-1. \end{cases}$$

If $\lambda \ge 0$ then $u = u(r)$ strictly decreases in $r = |x| \in (0,1)$.

Sketch of proof. As S_λ is coercive for $\lambda < \Lambda_{m,1}$ the first statement follows immediately by constrained minimization and by the Rellich-Kondrachov theorem. In order to prove the second statement concerning positive solutions we consider the modified Dirichlet problem (110) and, hence, the modified minimization problem:

$$S_\lambda(v) \stackrel{!}{=} \min \qquad \text{on} \qquad N = \left\{ v \in H_0^m(B) : \int_B |v|^p v\, dx = 1 \right\}.$$

Monotonicity follows from Lemma 2.3. ■

Remarks. 1) For $\lambda \geq \Lambda_{m,1}$ problem (111) has no solution. This follows readily from the existence of a positive first eigenfunction, cf. [PS3, p. 69].
2) It remains open whether $\tilde{\lambda}_c$ could be chosen as $-\infty$.
3) Theorem 2.4 follows from [Da1] and [Osw], if $\lambda = 0, m = 2$, and from [Sor], if $0 \leq \lambda < \Lambda_{m,1}$, $m \geq 2$.
4) The assumption that the domain be a ball is not needed but for the positivity result. We have a solution to (111) in any domain and for any λ where the Green operator $\mathcal{G}_{m,\Omega,-\lambda}$ exists and is positivity preserving. This is e.g. true in two dimensional domains Ω, which are C^{2m}-close to the unit disk, and $\lambda \in (\tilde{\lambda}_c, \Lambda_{m,1})$ for some $\tilde{\lambda}_c = \tilde{\lambda}_c(m, \Omega) < 0$. See Theorem 1.26 and Corollary 1.18.

Theorem 2.5 (Critical growth). *Let $n > 2m$, $s = (n + 2m)/(n - 2m)$, $B \subset \mathbb{R}^n$ the unit ball. Let $\Lambda_{m,1}$ denote the first Dirichlet eigenvalue of $(-\Delta)^m$ in B.*
a) If $n \geq 4m$, then for every $\lambda \in (0, \Lambda_{m,1})$ there exists a radial solution $u \in C^\infty(B) \cap C^{2m+1}(\overline{B})$ to the following Dirichlet problem:

$$(117) \begin{cases} (-\Delta)^m u = \lambda u + u^s & \text{in } B, \\ u > 0 & \text{in } B, \\ D^\alpha u | \partial B = 0 & \text{for } |\alpha| \leq m - 1. \end{cases}$$

$u = u(r)$ strictly decreases in $r = |x| \in (0, 1)$.
b) If $2m + 1 \leq n \leq 4m - 1$, then there exists a $\bar{\Lambda} = \bar{\Lambda}(n, m) \in (0, \Lambda_{m,1})$ such that for every $\lambda \in (\bar{\Lambda}, \Lambda_{m,1})$ the Dirichlet problem (117) has a solution u as above.

Remark. As in Remark 1 above for $\lambda \geq \Lambda_{m,1}$ we have no solution to (117). As for the special role of the dimensions $n = 2m + 1, \ldots, 4m - 1$, see Theorems 3.1 and 3.2.

Like in Theorem 2.4 we want to solve (117) by constrained minimization:

$$S_\lambda(v) \stackrel{!}{=} \min \qquad \text{on} \qquad N = \left\{ v \in H_0^m(B) : \int_B |v|^s v\, dx = 1 \right\}.$$

The major difficulty is to show appropriate compactness properties of this variational problem. As in the fundamental work of Brezis and Nirenberg [BrN] concerning $m = 1$ the optimal Sobolev constant

$$(118)\ S := \inf_{v \in D^{m,2} \setminus \{0\}} \frac{\tilde{S}_0(v)}{\|v\|^2_{L^{s+1}}} > 0$$

determines the energy range, where compactness holds. Here

$$\tilde{S}_0(v) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} \left(\Delta^{m/2} v\right)^2 \, dx, & \text{if } m \text{ even}, \\ \displaystyle\int_{\mathbb{R}^n} \left|\nabla \Delta^{(m-1)/2} v\right|^2 \, dx, & \text{if } m \text{ odd}, \end{cases}$$

and $D^{m,2}$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\tilde{S}_0(\,.\,)^{1/2}$.

Lemma 2.6. *Let $\lambda < \Lambda_{m,1}$, $(v_k) \subset H_0^m(B) \cap N$ a minimizing sequence for $S_\lambda(\,.\,)|N$. We assume that*

$$S_\lambda := \inf_{v \in N} S_\lambda(v) < S.$$

Then $S_\lambda(\,.\,)|N$ attains its infimum in some $v_0 \in H_0^m(B) \cap N$. Moreover, after selecting a subsequence we have strong convergence: $v_k \to v_0$ in $H_0^m(B)$.

The assumption $S_\lambda < S$ actually can be verified for certain ranges of λ.

Lemma 2.7. *a) $S = S_0 := \inf_{v \in N} S_0(v)$.*
b) If $n \geq 4m$ then for every $\lambda > 0$: $S_\lambda < S$.
c) If $2m + 1 \leq n \leq 4m - 1$ then there exists a $\bar{\Lambda} = \bar{\Lambda}(n, m) < \Lambda_{m,1}$ such that for $\lambda > \bar{\Lambda}$ we have: $S_\lambda < S$.

The proof rests on a work of Swanson [Swa], where minimizers for the optimal Sobolev constant in $D^{m,2}$ are determined explicitly. We crucially employ the positivity and radial symmetry of these minimizers, they do not notice the modification in the constraint N.

Lemmas 2.6, 2.7 and Luckhaus's regularity result [Lu] yield the proof of Theorem 2.5.

2.2 Beyond the first eigenvalue

Capozzi, Fortunato, Palmieri [CFP] studied the corresponding second order semilinear eigenvalue problem with critical growth, i.e. (107) with $m = 1$ and $p = s$. They showed that if $n \geq 5$ for every $\lambda > 0$ there is a nontrivial solution, while if $n = 4$, $\lambda > 0$ one should additionally assume that λ is not a Dirichlet eigenvalue of $-\Delta$. Cf. the remark below on p. 22.

In this section we show that this result carries over to equations of arbitrary order. Let $\Lambda_{m,1} \leq \Lambda_{m,2} \leq \ldots \leq \Lambda_{m,j} \leq \ldots$ denote the Dirichlet eigenvalues of $(-\Delta)^m$ in Ω.

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{2m,\gamma}$, $n > 2m$, $p = s = (n + 2m)/(n - 2m)$.*
a) If $n > (\sqrt{8} + 2)m$ then for every $\lambda > 0$ the Dirichlet problem (107) admits a nontrivial solution $u \in C^{2m,\gamma}(\overline{\Omega})$.
b) If $4m \leq n < (\sqrt{8} + 2)m$ then for every $\lambda > 0$, $\lambda \notin \{\Lambda_{m,j} : j \in \mathbb{N}\}$, there exists a solution as above.
c) If $2m + 1 \leq n \leq 4m - 1$ then there exists a $\lambda_0 = \lambda_0(m, n, \Omega) \in (0, \Lambda_{m,1})$ such that for $\lambda \in (\Lambda_{m,j} - \lambda_0, \Lambda_{m,j})$, $j \in \mathbb{N}$, we have a solution as above.

Remark. The number $(\sqrt{8}+2)m$ does not look very natural. Capozzi, Fortunato, Palmieri [CFP] state that also for $n=4$ ($m=1$) they have solutions on the eigenvalues $\Lambda_{m,j}$, but their proof doesn't seem to be correct in this case. Actually if $4m \leq n < (\sqrt{8}+2)m$, $\lambda = \Lambda_{m,j}$ due to the localization of minimizers for the optimal Sobolev constant S, there is some loss of information so that we miss the compactness threshold $\frac{m}{n}S^{n/2m}$.

On the other hand Atkinson, Brezis, Peletier [ABP] study for $m=1$ the asymptotic behaviour of branches of radially symmetric solutions in $\Omega = B$: if $n=4$ the solution branch, bifurcating from the branch of trivial solutions in a certain (radial) eigenvalue, approaches the neighbouring (radial) eigenvalue as $\max |u(\lambda)| \nearrow \infty$. It remains open whether the branch passes this eigenvalue or not.

So it would be an interesting question whether the number $(\sqrt{8}+2)m$ is in deed technical and artificial, or whether there are perhaps nonexistence results on eigenvalues if $n < (\sqrt{8}+2)m$.

Solutions to (107) are critical points of the functional E_λ which is defined on p. 18. First of all we notice that this functional satisfies a *local Palais-Smale condition.* Again the optimal Sobolev constant S plays a critical role.

Lemma 2.9. *Let $(u_k) \subset H_0^m$ be a Palais-Smale sequence for E_λ, i.e. $\lim_{k\to\infty} E_\lambda(u_k)$ exists and $dE_\lambda(u_k) \to 0$ strongly in H^{-m}. Further we assume that $\lim_{k\to\infty} E_\lambda(u_k) < \frac{m}{n}S^{n/2m}$. Then there is a strongly convergent subsequence: $u_{k_\nu} \to u_0$ in H_0^m for some $u_0 \in H_0^m$.*

The appropriate device from the Calculus of Variations is a general mountain pass lemma, which we quote from a work of Bartolo, Benci, Fortunato [BBF, Theorem 2.4].

Lemma 2.10. *Let H be a real Hilbert space and $E : H \to \mathbb{R}$ a continuously Fréchet-differentiable functional. We assume that E satisfies a local Palais-Smale condition, i.e. there is a $c_0 > 0$ such that every sequence $(u_k) \subset H$ with $\lim_{k\to\infty} E(u_k) < c_0$ and $dE(u_k) \to 0$ in H^* has a strongly convergent subsequence. Further we assume:*
i) $E(0) = 0$.
ii) E is even: for all $u \in H$ one has $E(u) = E(-u)$.
iii) There exist two closed linear subspaces V^+, $V^- \subset H$ and positive numbers ρ, $\delta > 0$ such that:

$$(146)\quad \sup_{u\in V^-} E(u) < c_0,$$

$$(147)\quad E(u) \geq \delta \text{ for all } u \in V^+ \text{ with } \|u\|_H = \rho,$$

$$(148)\quad \operatorname{codim} V^+ < \infty.$$

Then E has (at least)

$$(149)\quad \ell = \dim (V^+ \cap V^-) - \operatorname{codim} (V^+ + V^-)$$

pairs of nontrivial critical points.

It remains to construct the spaces V^+ and V^- in such a way that they have nonvoid intersection and that in particular (146) is satisfied. This is done with help of appropriate eigenfunctions of $(-\Delta)^m$ and localized minimizing sequences for the optimal Sobolev constant S. For the technical details we refer to the German version.

3 Semilinear eigenvalue problems: Non-existence

In this chapter we want to discuss some nonexistence results for the semilinear polyharmonic eigenvalue problem of Chapter 2:

$$(163)\quad \begin{cases} (-\Delta)^m u = \lambda u + |u|^{s-1}u, \ u \not\equiv 0 & \text{in } B, \\ D^\alpha u|\partial B = 0 & \text{for } |\alpha| \le m-1. \end{cases}$$

Again $B \subset \mathbb{R}^n$ is the unit ball, $n > 2m$, $s = (n+2m)/(n-2m)$, $\lambda \in \mathbb{R}$. These results reflect loss of compactness of the corresponding variational problems with critical growth.

First we want to mention a work of Pucci and Serrin [PS1] where problem (163) is considered in bounded smooth starshaped domains Ω (instead of B). Employing the testing functions $\sum_{i=1}^n x_i u_{x_i}$ and u itself they deduce a “Pohožaev identity”:

$$(164)\ 2m\lambda \int_\Omega u^2\,dx = \begin{cases} \displaystyle\int_{\partial\Omega} \left(\Delta^{m/2}u(x)\right)^2 (x\cdot\nu)\,d\omega(x), & \text{if } m \text{ even}, \\ \displaystyle\int_{\partial\Omega} \left(\frac{\partial}{\partial\nu}\Delta^{(m-1)/2}u(x)\right)^2 (x\cdot\nu)\,d\omega(x), & \text{if } m \text{ odd}. \end{cases}$$

Here Ω is starshaped with respect to the origin, ν is the unit outward normal of $\partial\Omega$ and $u \in C^{2m}(\overline{\Omega})$ solves (163). From $x \cdot \nu \ge 0$ on $\partial\Omega$ and $u \not\equiv 0$ we conclude:

$$(165)\ \lambda \ge 0.$$

In second order equations we have the stronger necessary condition $\lambda > 0$. Actually the case $\lambda = 0$ is difficult to handle if $m \ge 2$. Even in the ball only partial results are known. We will discuss this subject in Section 3.3.

The major part of this chapter is devoted to the critical role of the dimensions $n = 2m+1, \dots, 4m-1$, which is suggested by Theorem 2.5. The study of this subject was initiated by Brezis, Nirenberg [BrN] ($m = 1$) and Pucci, Serrin [PS3] ($m > 1$). In order to have a suggestive name for those dimensions where the semilinear eigenvalue problem (163) behaves critically we define according to Pucci and Serrin [PS3]:

Definition 0.1. *The dimension n is called* **critical** *(with respect to the boundary value problem (163)) iff there is a positive bound $\Lambda > 0$ such that a necessary condition for a nontrivial radial solution to (163) to exist is*

$$\lambda > \Lambda.$$

Pucci and Serrin seem to have expected an existence result like Theorem 2.5. They conjecture [PS3]:

Conjecture of Pucci and Serrin. *The critical dimensions for the boundary value problem (163) are precisely* $n = 2m+1, \ldots, 4m-1$.

Brezis and Nirenberg [BrN] proved this conjecture for $m = 1$ and Pucci and Serrin [PS3] for $m = 2$. Moreover the latter found that for any $m \in \mathbb{N}$ the dimension $n = 2m+1$ is critical. In the following section we will prove a weakened version of the Pucci-Serrin conjecture: we exclude the existence of *positive* radial solutions. In Section 3.2 we prove the original Pucci-Serrin conjecture for $m = 3$ and 4. Further for every $m \geq 3$ we show that $n = 2m+1$, $2m+2$, $2m+3$, $2m+4$ and $2m+5$ are critical. In Subsection 3.2.6 of the German version we comment on substantial difficulties in the full proof of the Pucci-Serrin conjecture and even in the proof of further critical dimensions.

3.1 The Pucci-Serrin conjecture, I: Nonexistence of positive radial solutions

The following theorem is the counterpart of Theorem 2.5 and shows that the latter is optimal in so far as $\bar{\Lambda}$ cannot be replaced by 0 if $n = 2m+1, \ldots, 4m-1$.

Theorem 3.1. *Let* $m \in \mathbb{N}$, $n \in \{2m+1, \ldots, 4m-1\}$, $s = (n+2m)/(n-2m)$, $B \subset \mathbb{R}^n$ *the unit ball.*

Then there exists a positive number $\tilde{\Lambda} = \tilde{\Lambda}(n,m) > 0$ *such that*

(166) $\lambda > \tilde{\Lambda}$

is a necessary condition for the existence of a positive radial solution to the following Dirichlet problem:

$$(167) \begin{cases} (-\Delta)^m u = \lambda u + u^s & \text{in } B, \\ u > 0 & \text{in } B, \\ D^\alpha u | \partial B = 0 & \text{for } |\alpha| \leq m-1. \end{cases}$$

Sketch of proof. We assume that there exists a positive radial solution $u \in C^{2m}(\overline{B})$ to (167). By (165) we have $\lambda \geq 0$. We introduce the weight function $w(r) := (1-r^2)^{m-1}$, $r = |x|$. Integration by parts yields

$$\int_B w\,(-\Delta)^m u\,dx = \begin{cases} C(n,m) \int_{\partial B} \left(\Delta^{m/2} u\right) d\omega, & \text{if } m \text{ even}, \\ C(n,m) \int_{\partial B} \left(-\frac{d}{dr} \Delta^{(m-1)/2} u\right) d\omega, & \text{if } m \text{ odd}. \end{cases}$$

With help of the Cauchy-Schwarz inequality and the Pohožaev identity (164) we find:

$$(169) \left(\int_B w\,(-\Delta)^m u\,dx\right)^2 \leq C(n,m) \cdot \lambda \int_B u^2\,dx.$$

Because $u = u(r)$ and consequently $((-\Delta)^m u)(r)$ strictly decreases in $r \in (0,1)$ the term $\int_B w(-\Delta)^m u\,dx = \int_B (1-r^2)^{m-1}(-\Delta)^m u\,dx$ is equivalent to the L^1-norm of $(-\Delta)^m u$:

$$(170)\ 0 < \int_B (-\Delta)^m u\,dx \le C(n,m) \int_B w\,(-\Delta)^m u\,dx.$$

By a duality argument and general elliptic estimates [ADN], for $n < 4m$ there holds:

$$\|u\|_{L^2} \le C\|(-\Delta)^m u\|_{L^1} = C\int_B (-\Delta)^m u\,dx.$$

Together with (170) and (169) this yields

$$\lambda \ge \frac{1}{C(n,m)} > 0$$

as stated. ■

3.2 The Pucci-Serrin conjecture, II: Nonexistence of radial solutions, Critical dimensions

It is a much more difficult problem to show nonexistence of any radial solution to (163). In general the Pucci-Serrin conjecture is still open. But if the order of the equation or the dimension considered is not too large we can prove that for λ close to zero the semilinear eigenvalue problem (163) has no (nontrivial) radial solution.

Theorem 3.2. *If $m \in \{1,2,3,4\}$ the dimensions $n = 2m+1,\dots,4m-1$ are critical. If $m \ge 5$ there are at least five critical dimensions, namely $n = 2m+1,\dots,2m+5$.*

More precisely, we have: Let $m,n \in \mathbb{N}$, $n > 2m$, $s = (n+2m)/(n-2m)$, $\lambda \in \mathbb{R}$, $B \subset \mathbb{R}^n$ the unit ball. We assume that there exists a nontrivial radial solution $u \in C^{2m}(\overline{B})$ of the following semilinear polyharmonic Dirichlet problem:

$$(171)\ \begin{cases} (-\Delta)^m u = \lambda u + |u|^{s-1}u & \text{in } B, \\ D^\alpha u|\partial B = 0 & \text{for } |\alpha| \le m-1. \end{cases}$$

Then we have,
a) if $m \ge 1$, $n = 2m+1$:

$$(172)\ \lambda > \frac{1}{2}\Big((2m+2)-n\Big)\Big(n+(2m-2)\Big)\Lambda_{m-1,1};$$

b) if $m \ge 2$, $n = 2m+2, 2m+3$:

$$(173)\ \lambda > \frac{1}{4}\Big((2m+4)-n\Big)\Big(n-2m\Big)\Big(n+2m\Big)\Big(n+(2m-4)\Big)\Lambda_{m-2,1};$$

c) if $m \ge 3$, $n = 2m+4, 2m+5$:

$$(174)\ \begin{aligned} \lambda > \frac{3}{32}&\Big((2m+6)-n\Big)\Big(n-(2m+2)\Big)\Big(n-(2m-2)\Big) \\ &\cdot\Big(n+(2m+2)\Big)\Big(n+(2m-2)\Big)\Big(n+(2m-6)\Big)\Lambda_{m-3,1}; \end{aligned}$$

d) if $m = 4$, $n = 14$*:*

(175) $\lambda > 111\,600$;

and if $m = 4$, $n = 15$*:*

(176) $\lambda > \dfrac{40\,883\,535}{128}$.

Remark. Part a) and the special case $m = 2$ of Part b) have already been proven by Pucci and Serrin [PS3].

Remarks on the proof of Theorem 3.2. We suppose that $u \in C^{2m}(\overline{B})$ is a nontrivial radial solution to (171). Then it holds necessarily $\lambda \geq 0$. We use a testing function which has been introduced by Pucci and Serrin [PS3]:

$$
\begin{aligned}
&h = \nabla\varphi_b \cdot \nabla u + a_b \cdot u, \\
&\varphi_b(r) = \frac{r^2}{2} - \frac{r^{b+2}}{b+2}, \quad a_b(r) = \mu r^b + \frac{n-2m}{2}, \quad r = |x|,
\end{aligned} \tag{177}
$$

$\mu \in \mathbb{R}$, $b \in 2\mathbb{N}$. They put $b = 2$ to prove a) and $b = 4$ to prove b) for $m = 2$. So, for arbitrary m one could hope that the choices $b \in 2\mathbb{N}$, $b = 4, \ldots, 2m$ yield the proof that the dimensions $n = 2m + b - 3$, $2m + b - 2$, $2m + b - 1$ are critical.

In a first step the product of the differential equation (171) and the testing function (177) has to be reduced in an inequality between weighted seminorms of u. This step involves extensive and combinatorically very difficult computations. But the formulation of the corresponding Lemmas 3.5, 3.6 and 3.7 (see the German version) suggests that these difficulties could be overcome also for arbitrarily large m and b. In the special case $m \geq 3$, $b = 6$, $\mu = 7m - (n+6)/2$ we come up with the following inequality:

$$
\begin{aligned}
m\lambda \int_B u^2\,dx > {} & \frac{m}{2}\Big\{35(2m+1)(2m-1) - 3(n-1)(n-3)\Big\} \int_B r^4 \left|\nabla\Delta^{(m/2)-1}u\right|^2 dx \\
& -m\Big\{21(2m+1)(2m-1)(m+1)(m-1) \\
& \quad +35(2m+1)(2m-1)(n-1) + (11m^2-5)(n-1)(n-3) \\
& \quad -3(n+1)(n-1)(n-3)\Big\} \int_B r^2 \left(\Delta^{(m/2)-1}u\right)^2 dx \\
& +\frac{1}{2}(m+2)m(m-2)\Big\{3(2m+1)(2m-1)(m+1)(m-1) \\
& \quad +3(11m^2-2)(n-1)(n-3) - (n+1)(n-1)(n-3)(n-5)\Big\} \\
& \qquad \cdot \int_B \left|\nabla\Delta^{(m/2)-2}u\right|^2 dx \\
=: {} & c_3 \int_B r^4 \left|\nabla\Delta^{(m/2)-1}u\right|^2 dx - c_4 \int_B r^2 \left(\Delta^{(m/2)-1}u\right)^2 dx \\
& +c_5 \int_B \left|\nabla\Delta^{(m/2)-2}u\right|^2 dx,
\end{aligned} \tag{185}
$$

if m is even, and

$$
\begin{aligned}
m\lambda \int_B u^2\, dx > \frac{m}{2}&\Big\{35(2m+1)(2m-1) - 3(n-1)(n-3)\Big\} \int_B r^4 \left(\Delta^{(m-1)/2} u\right)^2\, dx \\
&-(m+1)m(m-1)\Big\{21(2m+1)(2m-1) + 11(n-1)(n-3)\Big\} \\
&\qquad\qquad\qquad \cdot \int_B r^2 \left|\nabla \Delta^{(m-3)/2} u\right|^2\, dx \\
&+\frac{1}{2}(m+1)m(m-1)\Big\{3(2m+1)(2m-1)(m+2)(m-2) \\
&\quad +42(2m+1)(2m-1)(n-1) + 3(11m^2-3)(n-1)(n-3) \\
&\quad +22(n+1)(n-1)(n-3) - (n+1)(n-1)(n-3)(n-5)\Big\} \\
&\qquad\qquad\qquad \cdot \int_B \left(\Delta^{(m-3)/2} u\right)^2\, dx \\
=: \quad & c_8 \int_B r^4 \left(\Delta^{(m-1)/2} u\right)^2\, dx - c_9 \int_B r^2 \left|\nabla \Delta^{(m-3)/2} u\right|^2\, dx \\
&+c_{10} \int_B \left(\Delta^{(m-3)/2} u\right)^2\, dx,
\end{aligned} \tag{189}
$$

if m is odd. For $n = 2m+4,\ 2m+5$ all coefficients are positive: $c_3, c_4, c_5, c_8, c_9, c_{10} > 0$.

In a second step we have to try to reduce further inequalities like (185) and (189) in inequalities of the form $m\lambda \int_B u^2\, dx > \frac{1}{C} \int_B u^2\, dx$. A first attempt would be to apply weighted embedding inequalities.

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, C^2-smooth domain, $b \in 2\mathbb{N}$.*
a) For every $v \in C^1(\overline{\Omega})$ with $v|\partial\Omega = 0$ we have:

$$
\int_\Omega r^b \left|\nabla v\right|^2\, dx \geq \frac{1}{4}(n+b-2)^2 \int_\Omega r^{b-2} v^2\, dx. \tag{190}
$$

b) We assume further $n > b$, $v \in C^2(\overline{\Omega})$ with $v|\partial\Omega = 0$ or $\nabla v|\partial\Omega = 0$:

$$
\int_\Omega r^b \left(\Delta v\right)^2\, dx \geq \frac{1}{4}(n-b)^2 \int_\Omega r^{b-2} \left|\nabla v\right|^2\, dx. \tag{191}
$$

Application of these inequalities to (185) and (189) gives a negative coefficient in front of the highest order terms $\displaystyle\int_B r^2 \left(\Delta^{(m/2)-1} u\right)^2\, dx$ and $\displaystyle\int_B r^2 \left|\nabla \Delta^{(m-3)/2} u\right|^2\, dx$ respectively. Hence, we have to interpolate the negative term in (185) and (189) resp. between the positive terms.

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, C^2-smooth domain, $b \in 2\mathbb{N}$. We assume that $v \in C^3(\overline{\Omega})$ and $\nabla v|\partial\Omega = 0$.*
a) Then for every $\varepsilon > 0$ one has:

$$
\begin{aligned}
\int_\Omega r^b \left(\Delta v\right)^2\, dx \quad \leq \quad & \varepsilon \int_\Omega r^{b+2} \left|\nabla \Delta v\right|^2\, dx \\
&+\left\{\frac{1}{4\varepsilon} - \frac{b}{2}(n-b)\right\} \int_\Omega r^{b-2} \left|\nabla v\right|^2\, dx.
\end{aligned} \tag{193}
$$

b) If additionally $n > b$ and $\varepsilon \geq \frac{2}{(n+b)(n-b)}$ we even have:

$$
(194)\quad \begin{aligned} \int_\Omega r^b (\Delta v)^2\, dx \;\; &\leq \;\; \varepsilon \int_\Omega r^{b+2} |\nabla \Delta v|^2\, dx \\ &+ \frac{1}{4}(n-b)^2 \left\{ 1 - \frac{\varepsilon}{4}(n+b)^2 \right\} \int_\Omega r^{b-2} |\nabla v|^2\, dx. \end{aligned}
$$

Theorem 3.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, C^2-smooth domain, $v \in C^2(\overline{\Omega})$, $v|\partial\Omega = 0$, $b \in 2\mathbb{N}$.*
a) For every $\varepsilon > 0$ there holds:

$$
(195)\quad \begin{aligned} \int_\Omega r^b |\nabla v|^2\, dx \;\; &\leq \;\; \varepsilon \int_\Omega r^{b+2} (\Delta v)^2\, dx \\ &+ \left\{ \frac{1}{4\varepsilon} + \frac{b}{2}(n+b-2) \right\} \int_\Omega r^{b-2} v^2\, dx. \end{aligned}
$$

b) If additionally $n > b+2$ and $\varepsilon \geq \frac{2}{(n+b-2)(n-b-2)}$ then we have the stronger inequality:

$$
(196)\quad \begin{aligned} \int_\Omega r^b |\nabla v|^2\, dx \;\; &\leq \;\; \varepsilon \int_\Omega r^{b+2} (\Delta v)^2\, dx \\ &+ \frac{1}{4}(n+b-2)^2 \left\{ 1 - \frac{\varepsilon}{4}(n-b-2)^2 \right\} \int_\Omega r^{b-2} v^2\, dx. \end{aligned}
$$

If possible one should apply the "optimal" parts b) of the theorems above. In order to estimate (185) and (189) from below we put $\varepsilon = \frac{c_3}{c_4}$ and $\varepsilon = \frac{c_8}{c_9}$ resp. Actually if $n = 2m+4$ or $n = 2m+5$ then this ε satisfies the admissibility criterion $\varepsilon \geq \frac{2}{(n+2)(n-2)}$ or $\varepsilon \geq \frac{2}{n(n-4)}$ resp. After some tedious calculations Claim c) of Theorem 3.2 follows. ■

In general the methods presented above already fail if $b = 8$, $m \geq 4$, because the required ε is too small for "optimal" interpolation and too much information is lost. These difficulties are explained in Subsection 3.2.6 of the German version.

Only in eighth order equations, by a suitable choice of the parameter μ in (177), we can prove that $n = 2m+6 = 14$ and $n = 2m+7 = 15$ are also critical.

It seems that for a full proof of the Pucci-Serrin conjecture other devices and presumably other testing functions will be needed.

3.3 Remarks on the marginal case $\lambda = 0$

The problem, whether

$$
(207)\quad \begin{cases} (-\Delta)^m u = |u|^{s-1}u, \quad u \not\equiv 0 & \text{in } B, \\ D^\alpha u|\partial B = 0 & \text{for } |\alpha| \leq m-1, \end{cases}
$$

has a nontrivial solution or not, seems to be as difficult as the Pucci-Serrin conjecture. Yet only partial results are known.

Solutions of *minimum type*, i.e. minimizers of $v \mapsto \frac{S_0(v)}{\|v\|^2_{L^{s+1}}}$, cannot exist: Due to scaling invariance and unique continuation (see [Pro]) the optimal Sobolev constant S is not attained in $H_0^m(B)$.

By virtue of the Pohožaev identity (164) and the positivity of Green's function $G_{m,n}$ nonexistence of *positive* solutions was shown by Soranzo [Sor].

We would like to exclude the existence of *radial* solutions, too. We employ similar techniques as in the proof of Theorem 3.2. Like there we only have a rather restricted nonexistence result.

Theorem 3.11. *Let $m = 2$ or $m = 3$, $n > 2m$, $s = (n+2m)/(n-2m)$, $B \subset \mathbb{R}^n$ the unit ball.*

We assume that $u \in C^{2m}(\overline{B})$ is a radial solution to

$$\left\{ \begin{array}{ll} (-\Delta)^m u = |u|^{s-1} u & \text{in } B, \\ D^\alpha u | \partial B = 0 & \text{for } |\alpha| \leq m-1. \end{array} \right. \tag{210}$$

Then we have $u(x) \equiv 0$ in B.

This result indicates that in the semilinear polyharmonic eigenvalue problem (107) still a lot could be done:

- (Non-) existence in the marginal case $\lambda = 0$.
- The Pucci-Serrin conjecture.
- Optimal values for $\bar{\Lambda}$ and $\tilde{\Lambda}$, $n = 2m+1, \ldots, 4m-1$, in Theorems 2.5 and 3.1 respectively.
- In balls is there a relation between positivity and radial symmetry (cf. [GNN])?
- In which domains do exist positive solutions?
- Multiplicity results.
- Asymptotic and global behaviour of solution branches.
- In which dimension does the transition to generic behaviour occur?