Local eventual positivity for a biharmonic heat equation in $\mathbb{R}^n$*

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Abstract

We study the positivity preserving property for the Cauchy problem for the linear fourth order heat equation. Although the complete positivity preserving property fails, we show that it holds eventually on compact sets.

1 Introduction

For the biharmonic heat equation,

$$u_t + \Delta^2 u = 0 \qquad t > 0,$$

either in $\mathbb{R}^n$ or in a bounded smooth domain (then complemented with suitable boundary conditions) it is known that no positivity preserving of the solution with respect to the initial datum holds true. In general, one even has to expect instantaneous change of sign, see e.g. [2], which is a property of the differential equation and can be observed independently of a possible choice of boundary data.

On the other hand, there exist bounded domains with boundary conditions such that the corresponding elliptic first eigenvalue is simple and the first eigenfunction is of fixed sign and displays a nondegenerate behaviour at the boundary, see [6] and references therein. In these domains and for each positive initial datum, in dependence of this datum the solution of the initial boundary value problem is eventually positive. In general, this positivity comes up almost immediately, since the higher modes (in the expansion with respect to the eigenfunctions) decay much faster than the fundamental mode.

It is the goal of the present note to show a related result for the Cauchy problem, i.e. in the case when the domain is the whole $\mathbb{R}^n$. Here $n \in \mathbb{N}$.

Consider the problem

$$\begin{cases} u_t + \Delta^2 u = 0 & \text{in } \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times [0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n \ , \end{cases} \qquad (1)$$

where u_0 is a continuous initial datum. We prove

Theorem 1. *Assume that $0 \not\equiv u_0 \geq 0$ is continuous and has compact support in $\mathbb{R}^n$. Let $u = u(x,t)$ denote the corresponding bounded strong solution of (1). Then,*

(i) for any compact set $K \subset \mathbb{R}^n$ there exists $T_K = T_K(u_0) > 0$ such that $u(x,t) > 0$ for all $x \in K$ and $t \geq T_K$;

(ii) there exists $\tau = \tau(u_0) > 0$ such that for all $t > \tau$ there exists $x_t \in \mathbb{R}^n$ such that $u(x_t,t) < 0$.

The trivial example $u_0 \equiv 1$ shows that, at least for statement (ii), the compact support assumption cannot be dropped. By Theorem 1 we see that *negativity for (1) exists in general and goes to infinity.* We believe that a similar phenomenon might also be observed in nonlinear equations like e.g. those considered in [5]; for the elliptic counterpart, see [4]. Such a result, combined with [5, Theorem 2] would allow to prove *blow up* results for certain Cauchy problems with odd nonlinearities. This observation was our original motivation for this paper. As far as *existence* questions for nonlinear problems are concerned, a way out of the lack of positivity preserving is the positive majorizing self similar kernel introduced by Galaktionov–Pohožaev [3].

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2 Proof of Theorem 1

The kernel of the linear operator $v \mapsto v_t + \Delta^2 v$ in $\mathbb{R}^n$ is given by

$$\begin{gathered} b(x,t) = \frac{f(\eta)}{t^{n/4}} \ , \qquad \eta = \frac{x}{t^{1/4}} \ , \\ f(\eta) = \omega_0 |\eta|^{1-n} \int_0^\infty e^{-s^4} (|\eta| s)^{n/2} J_{(n-2)/2}(|\eta| s)\, ds \ , \end{gathered} \tag{2}$$

where J_m denotes the m-th Bessel function and ω_0 a suitable constant such that

$$\int_{\mathbb{R}^n} f(\eta)\, d\eta = 1.$$

The Fourier transform of b is easily obtained explicitly. Transforming this term back, writing the integral in spherical coordinates and applying [1, (4.9.12)] yields (2). With the change of variables $\sigma = |x|s/t^{1/4}$, the kernel $b(t) = b(x,t)$ reads

$$b(x,t) = \frac{\omega_0}{|x|^n} \int_0^\infty \exp\left(-\frac{t}{|x|^4}\sigma^4\right) \sigma^{n/2} J_{(n-2)/2}(\sigma)\, d\sigma \ .$$

We define

$$H(\sigma) = \omega_0 \sqrt{\sigma} J_{(n-2)/2}(\sigma), \tag{3}$$

since this function has suitable monotonicity properties, which we will explain and use below. So, b may be rewritten as

$$b(x,t) = |x|^{-n} \int_0^\infty \exp\left(-\frac{t}{|x|^4}\,\sigma^4\right) \sigma^{(n-1)/2} H(\sigma)\, d\sigma.$$

Therefore, since u_0 is bounded, the (bounded strong) solution u of (1) satisfies the integral equation

$$u(x,t) = b(t) * u_0 = \int_{\mathbb{R}^n} \frac{u_0(x-y)}{|y|^n} \int_0^\infty \exp\left(-\frac{t}{|y|^4}\,\sigma^4\right) \sigma^{(n-1)/2} H(\sigma)\, d\sigma\, dy \ . \tag{4}$$

We first prove the following

Lemma 1. *There exists $T_0 > 0$ such that*

$$\int_0^\infty e^{-T\sigma^4} \sigma^{(n-1)/2} H(\sigma)\, d\sigma > 0 \qquad \textit{for all } T > T_0 \ .$$

Proof. As for the properties of the Bessel type function H, we refer to [1, Chapter 4], in particular Sections 4.14 and 4.15. We denote by P_k (resp. N_k) the successive intervals, where H is positive (resp. negative) so that we have

$$[0,\infty) = \bigcup_{k=1}^{\infty} \left(\overline{P_k} \cup \overline{N_k}\right).$$

Moreover, we write

$$\alpha_k := \sup P_k = \inf N_k, \qquad \gamma_k := \sup N_k = \inf P_{k+1} \qquad (k \in \mathbb{N})$$

so that α_k and γ_k are the zeros of the Bessel function $J_{(n-2)/2}$.

Consider the function $g_T(s) := s^{(n-1)/2} e^{-Ts^4}$. A simple computation shows that

$$s \mapsto g_T(s) \qquad \text{is strictly decreasing over } \left[\left(\frac{n-1}{8T}\right)^{1/4}, +\infty\right) \ . \tag{5}$$

To start, just fix T sufficiently large so that

$$\left(\frac{n-1}{8T}\right)^{1/4} < \frac{\alpha_1}{2} \ . \tag{6}$$

If (6) holds, then, in view of (5), $s \mapsto g_T(s)$ is decreasing in particular over $[\gamma_1, \infty)$ and we have

$$\begin{aligned} \int_{\gamma_1}^{\infty} g_T(\sigma)H(\sigma)\,d\sigma &= \sum_{k=2}^{\infty} \left(\int_{P_k} g_T(\sigma)H(\sigma)\,d\sigma + \int_{N_k} g_T(\sigma)H(\sigma)\,d\sigma \right) \\ &> \sum_{k=2}^{\infty} g_T(\alpha_k) \left(\int_{P_k} H(\sigma)\,d\sigma + \int_{N_k} H(\sigma)\,d\sigma \right) \geq 0. \end{aligned} \tag{7}$$

In the last step we applied the Lorch-Szegö Theorem [1, Corollary 4.15.2], which applies even with strict inequality, provided $(n-2)/2 > 1/2$, i.e. if $n > 3$. If $n = 3$, one simply has $H(\sigma) = \omega_0\sqrt{\frac{2}{\pi}}\sin(\sigma)$ and the last inequality in (7) becomes an equality. Equality also holds if $n = 1$ when $H(\sigma) = \omega_0\sqrt{\frac{2}{\pi}}\cos(\sigma)$ while we postpone the case $n = 2$ to the end of this proof.

Moreover, we have in view of (6)

$$\begin{aligned} \int_0^{\gamma_1} g_T(\sigma)H(\sigma)\,d\sigma &> \int_{\sqrt[4]{(n-1)/(8T)}}^{\alpha_1} g_T(\sigma)H(\sigma)\,d\sigma + \int_{\alpha_1}^{\gamma_1} g_T(\sigma)H(\sigma)\,d\sigma \\ &> g_T\left(\frac{\alpha_1}{2}\right) \int_{\sqrt[4]{(n-1)/(8T)}}^{\alpha_1/2} H(\sigma)\,d\sigma - g_T(\alpha_1) \left| \int_{\alpha_1}^{\gamma_1} H(\sigma)\,d\sigma \right| \\ &> 0 \end{aligned}$$

thanks to the exponential factor in g_T, provided T is chosen large enough. This enables us to conclude that

$$\int_0^{\gamma_1} g_T(\sigma)H(\sigma)\,d\sigma > 0$$

for sufficiently large T, say $T \geq T_0$. Combining this inequality with (7) proves the statement, if $n \neq 2$.

If $n = 2$, according to [1, (4.8.5)], for σ large one has that asymptotically

$$H(\sigma) = \omega_0\sqrt{\frac{2}{\pi}}\cos\left(\sigma - \frac{\pi}{4}\right) + O\left(\frac{1}{\sigma}\right).$$

For each pair (P_k, N_k), one has to proceed as for (P_1, N_1) in the case $n \neq 2$. Thanks to the asymptotic expansion of H, this argument becomes uniform in k, if T is chosen large enough. □

Proof of (i) in Theorem 1. We take T_0 from Lemma 1 and for all $(x,t) \in \mathbb{R}^{n+1}_+$ we decompose the integral in (4) as follows

$$u(x,t) = \int_{|y|>(t/T_0)^{1/4}} \int_0^{\infty} + \int_{|y|<(t/T_0)^{1/4}} \int_0^{\infty} . \tag{8}$$

From Lemma 1 we infer at once that

$$\int_{|y|<(t/T_0)^{1/4}} \int_0^{\infty} \geq 0\,, \tag{9}$$

while for x in a fixed compact set and t large enough, one even has strict positivity. Moreover, taking $R > 0$ sufficiently large so that $\mathrm{spt}(u_0) \subset B_R$ (the ball of radius R), we may observe that

$$u_0(x-y) \neq 0 \quad \Longrightarrow \quad |x-y| < R \quad \Longrightarrow \quad |x| > |y| - R > \left(\frac{t}{T_0}\right)^{1/4} - R$$

since $|y| > \left(\frac{t}{T_0}\right)^{1/4}$ in the first integral in (8). Therefore

$$|x| \leq \left(\frac{t}{T_0}\right)^{1/4} - R \quad \Longrightarrow \quad u_0(x-y) = 0$$

in the first integral in (8). Together with (9), this shows that

$$u(x,t) \geq 0 \qquad \text{for all } |x| \leq \left(\frac{t}{T_0}\right)^{1/4} - R$$

and even strict positivity for large enough t. Finally, fix a compact set $K \subset \mathbb{R}^n$ and let $T_K > 0$ be sufficiently large so that strict positivity holds in (9) and that

$$K \subset \left\{ x \in \mathbb{R}^n;\ |x| \leq \left(\frac{T_K}{T_0}\right)^{1/4} - R \right\} .$$

Then, $u(x,t) > 0$ for all $x \in K$ and $t \geq T_K$, which is precisely statement (i) of Theorem 1. □

Proof of (ii) *in Theorem 1.* According to [2], b and hence f in (2) are sign changing. Denote by N the open (radially symmetric) region of negativity of f, $N := \{z \in \mathbb{R}^n;\ f(z) < 0\}$. As t increases, the set $t^{1/4}N := \{z \in \mathbb{R}^n;\ f(\frac{z}{t^{1/4}}) < 0\}$ becomes larger and larger. Take t sufficiently large (say $t > \tau$) so that

$$B_t := \{x \in \mathbb{R}^n;\ x - z \in t^{1/4}N \quad \text{for all} \quad z \in \mathrm{spt}(u_0)\} \neq \emptyset.$$

For any $t > \tau$ choose $x_t \in B_t$ and let $C(x_t) := \{y \in \mathbb{R}^n;\ x_t - y \in \mathrm{spt}(u_0)\}$. Next, rewrite (4) as

$$u(x,t) = t^{-n/4} \int_{\mathbb{R}^n} u_0(x-y)\, f\left(\frac{y}{t^{1/4}}\right) dy$$

so that

$$u(x_t,t) = t^{-n/4} \int_{C(x_t)} u_0(x_t-y)\, f\left(\frac{y}{t^{1/4}}\right) dy\,. \tag{10}$$

Note that if $y \in C(x_t)$, then there exists $z \in \mathrm{spt}(u_0)$ such that $y = x_t - z$. But since $x_t \in B_t$, this implies $y \in t^{1/4}N$, namely $f(\frac{y}{t^{1/4}}) < 0$. Since for a.e. $y \in C(x_t)$ we also have $u_0(x_t - y) > 0$, by using (10) we deduce that $u(x_t,t) < 0$, which proves statement (ii) of Theorem 1. □

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