Global solutions for superlinear parabolic equations involving the biharmonic operator for initial data with optimal slow decay*

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Abstract

We are interested in stability/instability of the zero steady state of the superlinear parabolic equation $u_t + \Delta^2 u = |u|^{p-1}u$ in $\mathbb{R}^n \times [0, \infty)$, where the exponent is considered in the "super-Fujita" range $p > 1 + 4/n$. We determine the corresponding limiting growth at infinity for the initial data giving rise to global bounded solutions. In the supercritical case $p > (n + 4)/(n - 4)$ this is related to the asymptotic behaviour of positive steady states, which the authors have recently studied. Moreover, it is shown that the solutions found for the parabolic problem decay to 0 at rate $t^{-1/(p-1)}$.

1 Introduction

In the present paper we study existence and quantitative properties of global solutions of the following Cauchy problem for superlinear parabolic equations with the biharmonic operator as elliptic linear part

$$\begin{cases} u_t + \Delta^2 u = |u|^{p-1}u & \text{in } \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times [0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n \ , \end{cases} \tag{1}$$

where $n \geq 2$, $p > 1 + \frac{4}{n}$ and u_0 is a bounded initial datum with suitable behaviour at ∞. The decay of these solutions with respect to space $|x| \to \infty$ and time $t \to \infty$ is investigated.

Before entering into the details of (1), we recall that the corresponding superlinear *second order* equation

$$\begin{cases} u_t - \Delta u = |u|^{p-1}u & \text{in } \mathbb{R}^{n+1}_+ \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n \end{cases} \tag{2}$$

was intensively studied in [7, 12, 14, 15, 19]. It was discovered by Fujita [7] that the exponent $p = 1 + \frac{2}{n}$ plays a fundamental role for the stability of the trivial solution $u \equiv 0$ of problem (2). It turned out that $1 + \frac{4}{n}$ is the biharmonic analogue of this so-called "Fujita"-exponent. For the present paper, the following result relative to (2) is of particular relevance:

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Proposition 1. [15, Theorem 3.8]
Assume that $n \geq 2$ and $p > 1 + \frac{2}{n}$. There exists $\alpha > 0$ such that if

$$|u_0(x)| \leq \frac{\alpha}{1 + |x|^{2/(p-1)}}$$

then there exists a global strong solution u of (2). Moreover, there exists $A > 0$ such that

$$|u(x,t)| \leq \frac{A}{1 + |x|^{2/(p-1)} + t^{1/(p-1)}} \qquad \text{for all } (x,t) \in \mathbb{R}^{n+1}_+ \ .$$

In fact, a slightly different result was stated in [15] but with the very same arguments employed there, one also readily obtains Proposition 1. Subsequently, Wang [19] performed a more detailed study of (2) and obtained refined stability/instability results with a more precise description of the region of attraction of the trivial solution $u \equiv 0$. In this connection several "critical exponents" turned out to be of importance, being also related to the corresponding elliptic problem. For further results see e.g. [5, 6] and references therein.

Most of the methods employed for the proof of Proposition 1 and related subsequent results are special for second order equations and are in particular basing upon auxiliary functions satisfying suitable differential inequalities and the maximum principle. Such methods do not apply to (1), since not even a comparison principle is available here. The major difficulty is the change of sign of the kernel of the linear operator $v \mapsto v_t + \Delta^2 v$, namely

$$b(x,t) = \frac{f(\eta)}{t^{n/4}} \ , \qquad \eta = \frac{x}{t^{1/4}} \ , \qquad f(\eta) = \omega_0 |\eta|^{1-n} \int_0^\infty e^{-s^4} (|\eta| s)^{n/2} J_{(n-2)/2}(|\eta| s)\, ds \ , \tag{3}$$

where J_m denotes the m-th Bessel function and ω_0 a suitable constant such that

$$\int_{\mathbb{R}^n} f(\eta)\, d\eta = 1.$$

In order to overcome this difficulty, Galaktionov–Pohožaev [9] introduced the following self-similar majorizing kernel associated to (1):

$$\tilde{b}(x,t) := \omega_1 t^{-n/4} \exp\left(-\mu \left(\frac{|x|^4}{t} \right)^{1/3} \right) \tag{4}$$

with suitable constants $\mu, D > 0$ and

$$\omega_1 = \frac{1}{\int_{\mathbb{R}^n} \exp\left(-\mu |y|^{4/3}\right) \, dy}$$

such that

$$|b(x,t)| \leq D \tilde{b}(x,t) \ . \tag{5}$$

Let us now mention some results already known for the parabolic biharmonic problem (1) and the related Cauchy problem

$$\begin{cases} u_t + \Delta^2 u = |u|^p & \text{in } \mathbb{R}^{n+1}_+ \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n \ . \end{cases} \tag{6}$$

In the "sub-Fujita" case $1 < p \leq 1 + \frac{4}{n}$, Egorov et al. [3] show finite time blow up for solutions to (6) provided the initial datum satisfies

$$\int_{\mathbb{R}^n} u_0(x)\, dx \geq 0.$$

Assuming that $u_0 \geq 0$ and that (1) has a positive solution – which is not obvious due to the oscillatory behaviour of the biharmonic "heat kernel" (3) – this would imply finite time blow up also in this situation. For $p > 1 + \frac{4}{n}$, Caristi and Mitidieri [1] obtain global solutions of (6), provided the initial datum u_0 belongs to $L^1(\mathbb{R}^n)$ and obeys the following growth condition at infinity with some constant c_0 and the following smallness condition:

$$0 \leq u_0(x) \leq \min\left\{\alpha, \frac{c_0}{1+|x|^\beta}\right\}, \quad \beta > \frac{4}{p-1}, \quad \alpha \text{ small enough in dependence of } c_0 \text{ and } \beta. \tag{7}$$

This result generalises and extends previous results in e.g. [2] and [3, 9, 10]. In the first work, for initial data being small enough in an L^1-sense, global existence of solutions decaying to 0 at rate $t^{-n/4}$ was proved in a more general setting basing upon linear semigroup theory. In the latter works, for small exponentially fast decaying initial data, the same decay rate $t^{-n/4}$ and, moreover, the asymptotic profile of the solution were calculated by means of perturbation theory. In particular, the time decay is governed by the *linear principal part*. By (18) below, these results also imply the existence of global solutions to (1) being majorised by those to (6).

Caristi and Mitidieri [1] also focussed on blow-up results for the modified equation (6) in the case $p > 1 + \frac{4}{n}$. They proved e.g. finite time blow up for initial data $u_0(x) = \frac{\alpha}{1+|x|^{4/(p-1)}}$ with large enough α. It does not seem to be obvious, however, to conclude blow-up also for (1) from these results.

Several questions were left open by the just mentioned results. In particular, we find most interesting and challenging to find out if β in (7) can be allowed to be $\beta = \frac{4}{p-1}$. In Theorem 1 we show that this is the case, namely if u_0 satisfies (7) for $\beta = \frac{4}{p-1}$ and a sufficiently small α, then the solution to (1) is global. Our proof relies on two crucial estimates (see Propositions 2 and 3 below) which seem to be a new unifying tool for the study of parabolic problems such as (1) whose kernel is sign-changing. In Remark 2 we also explain how our procedure may be (easily) extended to general higher order semilinear parabolic equations such as $u_t + (-\Delta)^k u = |u|^{p-1}u$ with $p > 1 + \frac{2k}{n}$. In some sense, we provide a unifying proof *independent* of the positivity of the kernel which applies for all k, included the "easy" case $k = 1$. Theorem 1 also states that the global solution converges uniformly to the (stable) stationary solution $u \equiv 0$ at a rate of $t^{-1/(p-1)}$, thereby giving the complete extension of Proposition 1 to higher order problems. Let us recall that in [4] stability of the trivial solution was obtained only for fast decay (exponential) initial data: here we extend it to the case of slowly decaying u_0.

In order to show that Theorem 1 is optimal with respect to the asymptotic decay of the initial datum, one should also prove finite time blow up if this assumption is violated. Theorem 2 below *does not* give the complete answer but it gives a strong hint that blow up should occur in finite time for initial data decaying slower than $|x|^{-4/(p-1)}$.

The asymptotic behaviour $|x|^{-4/(p-1)}$ is also important in the corresponding elliptic problem

$$\Delta^2 u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n. \tag{8}$$

One might observe that a suitable multiple of $x \mapsto |x|^{-4/(p-1)}$ is a singular solution to (8) if $p > n/(n-4)$ and also an entire weak solution (in $H^2_{\mathrm{loc}}(\mathbb{R}^n)$) if $p > (n+4)/(n-4)$, i.e. for supercritical exponents. In the latter case, the authors constructed entire regular radial solutions u to (8) and proved that $|x|^{4/(p-1)} \cdot u(x)$ converges to a suitable constant as $|x| \to \infty$, so that the asymptotic behaviour of solutions to (8) is governed by its singular weak entire solution. See [11] and also Corollary 1 below.

2 Results

In Definitions 1 and 2 we distinguish between two kinds of solutions.

Definition 1. We say that u is a **strong solution** of (1) over $[0,T)$ if $u \in C^{4,1}(\mathbb{R}^n \times (0,T))$, if u is bounded on $\mathbb{R}^n \times [0,t]$ for every $t \in (0,T)$, if u solves (in the classical sense) (1) and $\|u(t) - b(t) * u_0\|_\infty \to 0$ as $t \to 0$ (in particular, this implies that $u(x_m, t) \to u_0(x)$ whenever $t \to 0$ and $x_m \to x$ for a.e. $x \in \mathbb{R}^n$). The supremum T^* of all T's for which u is a strong solution of (1) over $[0,T)$ is called **life span** of the strong solution u. We say that u is a **global** strong solution of (1) if $T^* = +\infty$.

We first show the existence of global strong solutions:

Theorem 1. *Assume that $n \geq 2$ and $p > 1 + \frac{4}{n}$. There exists $\alpha > 0$ such that if*

$$|u_0(x)| \leq \frac{\alpha}{1 + |x|^{4/(p-1)}} \tag{9}$$

then there exists a global strong solution u of (1)*. Moreover, there exists $A > 0$ such that*

$$|u(x,t)| \leq \frac{A}{1 + |x|^{4/(p-1)} + t^{1/(p-1)}} \qquad \text{for all } (x,t) \in \mathbb{R}^{n+1}_+ \ . \tag{10}$$

Theorem 1 is somehow optimal. Indeed it was shown in [1] that if α is large and equality holds in (9), then any *positive* solution of (1) blows up in finite time. Moreover, the asymptotic behaviour at infinity of the estimate in (9) is *critical* since in the case $p > (n+4)/(n-4)$, it corresponds to the asymptotic behaviour of positive stationary radially symmetric regular and singular solutions of (1), see [11]. Therefore, Theorem 1 has the straightforward consequence:

Corollary 1. *Assume that $n \geq 5$ and $p > (n+4)/(n-4)$. Let $\overline{u}$ be a stationary positive radially symmetric solution of* (1)*. There exists $\beta > 0$ such that if*

$$|u_0(x)| \leq \beta \overline{u}(x)$$

then the solution u of (1) *is global. Moreover, there exists $A > 0$ such that* (10) *holds.*

The strong solution found in Theorem 1 is globally bounded. We now deal with a weaker notion of solution:

Definition 2. We say that u is a **weak solution** of (1) over $(0,T)$ if $u \in L^p_{\text{loc}}(\mathbb{R}^n \times [0,T))$ and

$$\int_0^T \int_{\mathbb{R}^n} |u|^{p-1} u \phi = \int_0^T \int_{\mathbb{R}^n} u(-\phi_t + \Delta^2 \phi) - \int_{\mathbb{R}^n} u_0 \phi(0) \tag{11}$$

for all $\phi \in C^\infty_c(\mathbb{R}^n \times [0,T))$. The supremum T^* of all T's for which u is a weak solution over $(0,T)$ is called the **life span** of the weak solution. We say that u is a **global** weak solution of (1) if $T^* = +\infty$.

We now introduce a suitable family of test functions.

Definition 3. We say that $v \in \mathcal{H}$ if the following facts occur:

(i) there exist $\psi_1 \in C^\infty_c(\mathbb{R}^n)$ and $\psi_2 \in C^\infty_c(\mathbb{R}_+)$ such that $v(x,t) = \psi_1(x)\psi_2(t)$.

(ii) $\psi_1(0) = \psi_2(0) = 1$ and $v(x,t) > 0$ in the interior of $\text{spt}(v)$, the support of v.

(iii) $\displaystyle\int_{\text{spt}(v)} |v_t|^{p'} |v|^{1-p'} < \infty$ and $\displaystyle\int_{\text{spt}(v)} |\Delta^2 v|^{p'} |v|^{1-p'} < \infty$ where $p' = \frac{p}{p-1}$.

As pointed out in [17] (see also [1]), one has $\mathcal{H} \neq \emptyset$.

Then, we have

Theorem 2. *Assume that* $u_0 \in L^\infty(\mathbb{R}^n)$ *and that*

$$\lambda := \liminf_{|x|\to\infty} |x|^{4/(p-1)} u_0(x) > 0 \ . \tag{12}$$

There exists $\Lambda > 0$ *such that if* $\lambda > \Lambda$, *then any weak solution* u *of (1) with initial datum* u_0 *satisfies one of the two following alternatives:*

(i) *u blows up in finite time, that is,* $T^* < \infty$;

(ii) *for all* $v \in \mathcal{H}$ *and for all* $\gamma < 1$ *we have*

$$\liminf_{R\to\infty} R^{4p/(p-1)} \int_{\mathrm{spt}(v)} \Big(|u^-(Ry, R^4\tau)|^p - \gamma |u^+(Ry, R^4\tau)|^p \Big) v(y,\tau)\, dy d\tau > 0 \ . \tag{13}$$

If (12) *is replaced by* $\limsup_{|x|\to\infty} |x|^{4/(p-1)} u_0(x) < 0$, *then the same statement holds provided one switches* u^+ *and* u^- *in* (13).

Let us also briefly comment on Theorem 2. Under the further assumption that $u_0 \geq 0$ in $\mathbb{R}^n$, a byproduct of the results in [1] shows that any *nonnegative* solution of (1) blows up in finite time. Unfortunately, although we believe that for nonnegative initial data any global solution of (1) is eventually positive, as far as we are aware no condition is known which ensures this property of the solution.

Theorem 2 states that if (12) holds and the solution u is global, then the negative part of u *is not* neglectable with respect to the positive part. In some sense, the solution may be global only if the negative and positive parts are "perfectly balanced". We believe that under further suitable assumptions on u_0 (such as positivity) case (ii) never occurs and we have blow up in finite time.

Remark 1. By direct calculation, one can see that if there exists an entire smooth radial solution v, decaying at infinity, of the equation

$$\Delta^2 v = |v|^{p-1} v + \frac{1}{4} r v_r + \frac{1}{p-1} v \qquad \text{in } \mathbb{R}^n$$

then

$$u(x,t) := (1+t)^{-1/(p-1)} v\left(|x|(1+t)^{-1/4} \right) \ , \qquad (x,t) \in \mathbb{R}^{n+1}_+$$

is a (self-similar) solution of (1). These solutions were studied in some detail in [8] in order to prove non-uniqueness results for (1) with *unbounded* initial data u_0.

3 Proof of Theorem 1

During the proof of Theorem 1 we will need the following crucial statements whose proofs are postponed, respectively, to Sections 6 and 7.

Proposition 2. *Assume that* $n \geq 2$ *and* $p > 1 + 4/n$. *There exists a constant* $C_1 = C_1(n,p,\mu) > 0$ *such that for all* $(x,t) \in \mathbb{R}^{n+1}_+$, *one has:*

$$\omega_1 \int_{\mathbb{R}^n} \exp\left[-\mu \left(\frac{|y|^4}{t} \right)^{1/3} \right] \frac{dy}{t^{n/4}(1+|x-y|^{4/(p-1)})} \leq \frac{C_1}{1+|x|^{4/(p-1)} + t^{1/(p-1)}} \ .$$

Proposition 3. *Assume that $n \geq 2$ and $p > 1 + 4/n$. There exists a constant $C_2 = C_2(n,p,\mu) > 0$ such that for all $(x,t) \in \mathbb{R}^{n+1}_+$, one has:*

$$\omega_1 \int_0^t \int_{\mathbb{R}^n} \exp\left[-\mu\left(\frac{|y|^4}{s}\right)^{1/3}\right] \frac{dy\,ds}{s^{n/4}\left(1+(t-s)^{1/(p-1)}+|x-y|^{4/(p-1)}\right)^p} \leq \frac{C_2}{1+|x|^{4/(p-1)}+t^{1/(p-1)}} \ .$$

The proof of Proposition 3 is quite lengthy and delicate. For this reason, in Section 8 we give a much simpler proof of it under the additional constraints that $n \geq 5$ and $p > \frac{n+4}{n-4}$.

Let D be as in (5). Then, we will prove Theorem 1 by taking

$$\alpha := \frac{1}{2^{p/(p-1)} D^{p/(p-1)} C_1 C_2^{1/(p-1)}} \ . \tag{14}$$

Let $b(t) = b(x,t)$ be the kernel defined in (3) and note that under the assumption (9) we have $u_0 \in L^\infty(\mathbb{R}^n)$. It is then clear that any strong solution u of (1) also satisfies the integral equation

$$u(t) = b(t) * u_0 + \int_0^t b(t-s) * |u(s)|^{p-1} u(s)\,ds \ . \tag{15}$$

Conversely, it is well-known (see e.g. [7, Proposition A4]) that a bounded solution of the integral equation (15) is a strong solution of (1). Uniqueness of strong solutions to (15) follows by the standard contraction mapping principle, see e.g. [13, Section 3.3].

We study (15) by following the approach in [1, 20, 21] combined with the extremely powerful Propositions 2 and 3. Let D be as in (5) and let

$$v_0(x) := D|u_0(x)| \ . \tag{16}$$

Then, consider the equation

$$v(t) = \tilde{b}(t) * v_0 + D\int_0^t \tilde{b}(t-s) * v^p(s)\,ds \ . \tag{17}$$

In view of (5) and the existence proof below, it is clear that as long as $v(t)$ exists, we have a solution u to (15)

$$|u(x,t)| \leq v(x,t) \ . \tag{18}$$

In particular, if we can show that v is globally defined, this will also prove that u is globally defined. To this end, for all $v \in L^\infty(\mathbb{R}^{n+1}_+)$ we define

$$\tilde{B}v(x,t) := \tilde{b}(t) * v_0 + D\int_0^t \tilde{b}(t-s) * v^p(s)\,ds \ . \tag{19}$$

For $T > 0$ and for

$$M := \frac{1}{(2DC_2)^{1/(p-1)}} \tag{20}$$

we introduce the set

$$S_T := \left\{ v \in C(\mathbb{R}^n \times [0,T]);\ 0 \leq v(x,t) \leq \frac{M}{1+|x|^{4/(p-1)}+t^{1/(p-1)}} \right\} \ .$$

It is clear that

$$S_T \text{ is a nonempty, closed and bounded convex subset of } C(\mathbb{R}^n \times [0,T]) \ . \tag{21}$$

We claim that

$$\tilde{B}(S_T) \subset S_T \ . \tag{22}$$

In order to prove (22), take $v \in S_T$ and consider $\tilde{B}v$ as defined in (19). Clearly, $\tilde{B}v(x,t) \geq 0$. Moreover, by (9), (16) and Proposition 2 we have

$$[\tilde{b}(t) * v_0](x) \leq \frac{DC_1\alpha}{1+|x|^{4/(p-1)}+t^{1/(p-1)}} \qquad \text{for all } (x,t) \in \mathbb{R}^{n+1}_+ \ . \tag{23}$$

Furthermore, for $v \in S_T$ we get

$$\int_0^t \tilde{b}(t-s) * v^p(s)\, ds = \omega_1 \int_0^t \int_{\mathbb{R}^n} \exp\left[-\mu\left(\frac{|y|^4}{s}\right)^{1/3}\right] \frac{v^p(x-y,t-s)}{s^{n/4}}\, dy\, ds \leq$$

$$\leq \omega_1 M^p \int_0^t \int_{\mathbb{R}^n} \exp\left[-\mu\left(\frac{|y|^4}{s}\right)^{1/3}\right] \frac{dy\, ds}{s^{n/4}\left(1+(t-s)^{1/(p-1)}+|x-y|^{4/(p-1)}\right)^p} \ .$$

Using Proposition 3 we then obtain

$$\int_0^t \tilde{b}(t-s) * v^p(s)\, ds \leq \frac{C_2 M^p}{1+|x|^{4/(p-1)}+t^{1/(p-1)}} \qquad \text{for all } (x,t) \in \mathbb{R}^{n+1}_+ \ . \tag{24}$$

Inserting (23)-(24) into (19) and recalling (14)-(20), we finally obtain

$$\tilde{B}v(x,t) \leq \frac{M}{1+|x|^{4/(p-1)}+t^{1/(p-1)}} \qquad \text{for all } (x,t) \in \mathbb{R}^{n+1}_+ \ ,$$

which proves (22).

Arguing as in [21, Lemma 3.1] (see also [1, Theorem 2.1]), we can also prove the two following facts:

$$\tilde{B}S_T \text{ is a compact subset (with respect to the } L^\infty\text{-norm) of } S_T \ , \tag{25}$$

$$\tilde{B} \text{ is continuous} \ . \tag{26}$$

In view of (21)-(22)-(25)-(26), we may apply Schauder's fixed point Theorem and infer that $\tilde{B}$ has a fixed point $v_T \in S_T$. For any $T > 0$, we now define

$$V_T(x,t) = \begin{cases} v_T(x,t) & \text{if } t \leq T \\ v_T(x,T) & \text{if } t > T \ . \end{cases}$$

By applying a local version of Ascoli-Arzelà's Theorem as in [1, p.718] (see also [21, p.61]) we infer that, up to a subsequence, V_T converges uniformly on compact subsets of $\mathbb{R}^{n+1}_+$ to a (global) solution of (17). This completes the proof of Theorem 1. $\square$

Remark 2. The technique developed here may also serve to treat Cauchy problems with polyharmonic elliptic principal part

$$\begin{cases} u_t + (-\Delta)^k u = |u|^{p-1}u & \text{in } \mathbb{R}^{n+1}_+ \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n \ . \end{cases} \tag{27}$$

As in the biharmonic case one may consider a majorising kernel associated to (27)

$$\tilde{b}_k(x,t) := \omega_{1,k} t^{-n/2k} \exp\left(-\mu_k \left(\frac{|x|^{2k}}{t}\right)^{1/(2k-1)}\right) .$$

Propositions 2 and 3 continue to hold true provided one assumes $p > 1 + 2k/n$ and provided $\left(\frac{|y|^4}{t}\right)^{1/3}$ is replaced by $\left(\frac{|y|^{2k}}{t}\right)^{1/(2k-1)}$, the exponent $n/4$ by $n/2k$ and the exponent $4/(p-1)$ by $2k/(p-1)$. The proofs of these generalised propositions require almost only straightforward changes except for the end of the proof of Lemma 5 in the case $k = 1$, $n = 2$. In this special case one splits the last integral at $3T^{4/3}$ instead.

So, one may prove for (27) the same result as in Theorem 1 by changing $1 + 4/n$ into $1 + 2k/n$ and $4/(p-1)$ into $2k/(p-1)$. This general result contains also Proposition 1 as a special case thereby providing a unified proof independent of the maximum principle.

4 Proof of Theorem 2

Our proof is obtained by adapting the arguments in [1] to the case of sign-changing solutions. Let $v(x,t) = \psi_1(x)\psi_2(t) \in \mathcal{H}$ and let $K = \mathrm{spt}(\psi_1)$ and $[0,T] = \mathrm{spt}(\psi_2)$. For all $R > 0$ take

$$\phi_R(x,t) := v\left(\frac{x}{R}, \frac{t}{R^4}\right) = \psi_1\left(\frac{x}{R}\right)\psi_2\left(\frac{t}{R^4}\right)$$

as test function in (11). Then, we obtain

$$\int_0^{R^4T}\int_{RK} |u(x,t)|^{p-1}u(x,t)v\left(\frac{x}{R}, \frac{t}{R^4}\right)dxdt + \int_{RK} u_0(x)\psi_1\left(\frac{x}{R}\right)dx =$$

$$= R^{-4}\int_0^{R^4T}\int_{RK} u(x,t)\left[-v_t\left(\frac{x}{R}, \frac{t}{R^4}\right) + \Delta^2 v\left(\frac{x}{R}, \frac{t}{R^4}\right)\right]dxdt =: I \ . \tag{28}$$

In the sequel the c's denote positive constants which may have different values also when they appear in the same formula. We estimate the right hand side I of (28) as follows. Fix $\gamma < 1$ and take $\delta := \frac{1-\gamma}{1+\gamma}$; by Young's inequality we know that there exists $C_\delta > 0$ such that

$$I = R^{-4}\int_0^{R^4T}\int_{RK} u(x,t)\, v^{1/p}\left(\frac{x}{R}, \frac{t}{R^4}\right)\frac{-v_t\left(\frac{x}{R}, \frac{t}{R^4}\right) + \Delta^2 v\left(\frac{x}{R}, \frac{t}{R^4}\right)}{v^{1/p}\left(\frac{x}{R}, \frac{t}{R^4}\right)}dxdt \leq$$

$$\leq \delta\int_0^{R^4T}\int_{RK} |u(x,t)|^p\, v\left(\frac{x}{R}, \frac{t}{R^4}\right)dxdt + \frac{C_\delta}{R^{4p'}}\int_0^{R^4T}\int_{RK}\frac{\left|v_t\left(\frac{x}{R}, \frac{t}{R^4}\right)\right|^{p'} + \left|\Delta^2 v\left(\frac{x}{R}, \frac{t}{R^4}\right)\right|^{p'}}{v^{p'-1}\left(\frac{x}{R}, \frac{t}{R^4}\right)}dxdt.$$

With the change of variables

$$x = Ry \ , \quad t = R^4\tau \ , \tag{29}$$

we finally obtain

$$I \leq \delta\int_0^{R^4T}\int_{RK} |u(x,t)|^p\, v\left(\frac{x}{R}, \frac{t}{R^4}\right)dxdt + C_\delta R^{n-4/(p-1)}\int_0^T\int_K \frac{|v_t(y,\tau)|^{p'} + \left|\Delta^2 v(y,\tau)\right|^{p'}}{v^{p'-1}(y,\tau)}dyd\tau \leq$$

$$\leq \delta\int_0^{R^4T}\int_{RK} |u(x,t)|^p\, v\left(\frac{x}{R}, \frac{t}{R^4}\right)dxdt + cR^{n-4/(p-1)} \ . \tag{30}$$

Next, by (12) we know that there exists $C, \rho > 0$ such that

$$|x|^{4/(p-1)}u_0(x) \geq \frac{\lambda}{2} \qquad \text{if } |x| \geq \rho \ .$$

Let R be sufficiently large so that $B_\rho \subset RK$. For such R we have

$$\int_{RK} u_0(x)\psi_1\left(\frac{x}{R}\right)\,dx \geq -\|\psi_1\|_\infty \int_{B_\rho} |u_0(x)|\,dx + \frac{\lambda}{2}\int_{RK\setminus B_\rho} \psi_1\left(\frac{x}{R}\right)\frac{dx}{|x|^{4/(p-1)}}\ .$$

Clearly, there exist $0 < \alpha < \beta$ such that if R is sufficiently large then $X_R := \{x \in \mathbb{R}^n;\ \alpha R < |x| < \beta R\} \subset RK \setminus B_\rho$. Therefore, the last inequality becomes

$$\int_{RK} u_0(x)\psi_1\left(\frac{x}{R}\right)\,dx \geq -c + \frac{\lambda}{2}\min_{\alpha\leq|y|\leq\beta}\psi_1(y)\int_{X_R}\frac{dx}{|x|^{4/(p-1)}} \geq -c + c\lambda R^{n-4/(p-1)}\ . \tag{31}$$

Summarizing, by using (30) and (31) into (28), we arrive at

$$(1-\delta)\int_0^{R^4T}\int_{RK}|u^+(x,t)|^p v\left(\frac{x}{R},\frac{t}{R^4}\right)dxdt - (1+\delta)\int_0^{R^4T}\int_{RK}|u^-(x,t)|^p v\left(\frac{x}{R},\frac{t}{R^4}\right)dxdt \leq$$
$$\leq cR^{n-4/(p-1)} - c\lambda R^{n-4/(p-1)} + c\ .$$

If we divide by $1+\delta$ and we take λ sufficiently large (say $\lambda \geq \Lambda$), the previous inequality becomes

$$\int_0^{R^4T}\int_{RK}|u^-(x,t)|^p v\left(\frac{x}{R},\frac{t}{R^4}\right)dxdt - \gamma\int_0^{R^4T}\int_{RK}|u^+(x,t)|^p v\left(\frac{x}{R},\frac{t}{R^4}\right)dxdt \geq cR^{n-4/(p-1)}$$

for sufficiently large R. Finally, performing the change of variables (29) and letting $R \to \infty$, proves (13).

Assume now that (12) is replaced by $\limsup_{|x|\to\infty}|x|^{4/(p-1)}u_0(x) < 0$ and consider again (28). In this case, instead of (31) we obtain

$$\int_{RK} u_0(x)\psi_1\left(\frac{x}{R}\right)dx \leq c - c\lambda R^{n-4/(p-1)}\ .$$

Moreover, we may replace (30) with

$$I \geq -\delta\int_0^{R^4T}\int_{RK}|u(x,t)|^p\, v\left(\frac{x}{R},\frac{t}{R^4}\right)dxdt - cR^{n-4/(p-1)}\ .$$

The proof may now be completed as in the case where (12) holds.

5 Some technical results

Here and in all the remaining of the paper, with c we denote positive constants which may have different values also within the same line. Moreover, in order to avoid heavy notations, when we write $1/\alpha\beta$, we mean $\frac{1}{\alpha\beta}$.

For both the proofs of Propositions 2 and 3 we will make use of the following trivial facts:

$$\min\left\{\frac{1}{a},\frac{1}{b}\right\} \leq \frac{2}{a+b} \qquad \text{for all } a,b>0\ , \tag{32}$$

and for all $(m,q) \in [\mathbb{N}\setminus\{0\}]\times(0,+\infty)$ there exist $\gamma_1,\gamma_2 > 0$ such that

$$\gamma_1\left(\sum_{i=1}^m \alpha_i\right)^q \leq \sum_{i=1}^m \alpha_i^q \leq \gamma_2\left(\sum_{i=1}^m \alpha_i\right)^q \qquad \text{for all } \alpha_i \geq 0\ . \tag{33}$$

We now prove five technical statements (uniform bounds for 3-dimensional integrals depending on a parameter $T>0$) which are needed for the proof of Proposition 3. Most delicate are Lemmas 3 and 4, where the essential idea consists in a suitable splitting of the domain of integration. The integrals Γ_i ($i=1,...,5$) below depend on T but, for simplicity, we omit to emphasize this dependence.

Lemma 1. *Assume that $n \geq 2$ and $p > 1+4/n$. Then, there exists $c>0$ such that*

$$\Gamma_1 := \int_0^\infty \frac{1}{(1+w^2)^{n/2}} \int_0^{1/2T} \rho^{n-5} e^{-\mu\rho^{4/3}} \int_0^{T\rho} \frac{\sigma^3 \qquad d\sigma\, d\rho\, dw}{\left(\frac{T^3}{\rho}(T\rho-\sigma)+(\sigma-1)^4+\frac{\sigma^2}{(1+w^2)^2}\right)^{p/(p-1)}} \leq c$$

for all $T>0$.

Proof. First, we note that for the integral considered we have $\sigma \leq T\rho \leq \frac{1}{2}$. Hence, $|\sigma - 1| \geq \frac{1}{2}$ and

$$\Gamma_1 \leq \int_0^\infty \frac{dw}{(1+w^2)^{n/2}} \cdot \int_0^{1/2T} \rho^{n-5} e^{-\mu\rho^{4/3}} \int_0^{T\rho} \frac{T^3\rho^3 \qquad d\sigma\, d\rho}{\left(\frac{T^3}{\rho}(T\rho-\sigma)+\frac{1}{16}\right)^{p/(p-1)}} \leq$$

$$\leq c\int_0^{1/2T} \rho^{n-1} e^{-\mu\rho^{4/3}} \left[\frac{1}{\left(\frac{T^3}{\rho}(T\rho-\sigma)+\frac{1}{16}\right)^{1/(p-1)}}\right]_0^{T\rho} d\rho \leq c\int_0^\infty \rho^{n-1} e^{-\mu\rho^{4/3}}\, d\rho = c$$

and the uniform upper bound follows. □

Lemma 2. *Assume that $n \geq 2$ and $p > 1+4/n$. Then, there exists $c>0$ such that*

$$\Gamma_2 := \int_0^\infty \frac{1}{(1+w^2)^{n/2}} \int_{1/2T}^{3/T} \rho^{n-5} e^{-\mu\rho^{4/3}} \int_0^{T\rho/2} \frac{\sigma^3 \qquad d\sigma\, d\rho\, dw}{\left(\frac{T^3}{\rho}(T\rho-\sigma)+(\sigma-1)^4+\frac{\sigma^2}{(1+w^2)^2}\right)^{p/(p-1)}} \leq c$$

for all $T>0$.

Proof. As long as Γ_2 is involved, we have $\sigma^3 \leq \frac{T^3\rho^3}{8}$. Then,

$$\Gamma_2 \leq cT^3 \int_0^\infty \frac{dw}{(1+w^2)^{n/2}} \cdot \int_{1/2T}^{3/T} \rho^{n-2} e^{-\mu\rho^{4/3}} \int_0^{T\rho/2} \frac{d\sigma\, d\rho}{\left[\frac{T^3}{\rho}(T\rho-\sigma)\right]^{p/(p-1)}} \leq$$

$$\leq c\int_{1/2T}^{3/T} \rho^{n-1} e^{-\mu\rho^{4/3}} \left[\frac{1}{\left(\frac{T^3}{\rho}(T\rho-\sigma)\right)^{1/(p-1)}}\right]_0^{T\rho/2} d\rho \leq \frac{c}{T^{4/(p-1)}} \int_{1/2T}^{3/T} \rho^{n-1} e^{-\mu\rho^{4/3}}\, d\rho =$$

$$[\tau = T\rho] \quad = \frac{c}{T^{n+4/(p-1)}} \int_{1/2}^{3} \tau^{n-1} \exp\left[-\mu\left(\frac{\tau}{T}\right)^{4/3}\right] d\tau =: f(T).$$

It is clear that $f(T)$ is well-defined (finite) for all $T \in (0,\infty)$. Moreover, $f(T) \to 0$ for both $T \to 0$ and $T \to \infty$ so that the uniform upper bound for Γ_2 follows. □

Lemma 3. *Assume that $n \geq 2$ and $p > 1 + 4/n$. Then, there exists $c > 0$ such that*

$$\Gamma_3 := \int_0^\infty \frac{1}{(1+w^2)^{n/2}} \int_{1/2T}^{1/T} \rho^{n-5} e^{-\mu\rho^{4/3}} \int_{T\rho/2}^{T\rho} \frac{\sigma^3 \qquad d\sigma\, d\rho\, dw}{\left(\frac{T^3}{\rho}(T\rho-\sigma) + (\sigma-1)^4 + \frac{\sigma^2}{(1+w^2)^2}\right)^{p/(p-1)}} \leq c$$

for all $T > 0$.

Proof. Since $\sigma \geq \frac{T\rho}{2} \geq \frac{1}{4}$ and $\frac{T^3}{\rho} \geq \frac{1}{8\rho^4}$, Γ_3 converges uniformly in T whenever the integral

$$\int_0^\infty \frac{1}{(1+w^2)^{n/2}} \int_{1/2T}^{1/T} \rho^{n-5} e^{-\mu\rho^{4/3}} \int_{T\rho/2}^{T\rho} \frac{\sigma^3 \qquad d\sigma\, d\rho\, dw}{\left[\frac{T\rho-\sigma}{8\rho^4} + (\sigma-1)^4 + \frac{1}{16(1+w^2)^2}\right]^{p/(p-1)}}$$

does so. Moreover, we remark that

$$\int_0^1 \frac{1}{(1+w^2)^{n/2}} \int_{1/2T}^{1/T} \rho^{n-5} e^{-\mu\rho^{4/3}} \int_{T\rho/2}^{T\rho} \frac{\sigma^3 \qquad d\sigma\, d\rho\, dw}{\left[\frac{T\rho-\sigma}{8\rho^4} + (\sigma-1)^4 + \frac{1}{16(1+w^2)^2}\right]^{p/(p-1)}} \leq c \int_0^\infty \rho^{n-5} e^{-\mu\rho^{4/3}}\, d\rho$$

so that the statement follows if there exists $c > 0$ (independent of T) such that

$$\Gamma_3' := \int_1^\infty \frac{1}{w^n} \int_{1/2T}^{1/T} \rho^{n-5} e^{-\mu\rho^{4/3}} \int_{T\rho/2}^{T\rho} \frac{d\sigma\, d\rho\, dw}{\left[\frac{T\rho-\sigma}{8\rho^4} + (\sigma-1)^4 + \frac{1}{64w^4}\right]^{p/(p-1)}} \leq c \qquad \text{for all } T > 0. \tag{34}$$

Consider the map

$$h(\rho) := \min\{1, \rho^4\} \cdot (1 - T\rho)^4 . \tag{35}$$

If $\rho \in [\frac{1}{2T}, \frac{1}{T}]$, then $T\rho - h(\rho) \geq T\rho - (1-T\rho)^4 > \frac{T\rho}{2}$. Hence, we can split the inner integral in (34) as follows

$$\int_{T\rho/2}^{T\rho} = \int_{T\rho/2}^{T\rho-h(\rho)} + \int_{T\rho-h(\rho)}^{T\rho} =: L_1 + L_2 \tag{36}$$

and we estimate L_1 and L_2. This splitting is the *essential idea* in this proof in order to cover the full super-Fujita range $p > 1 + 4/n$.

First, we note that

$$L_1 \leq \int_{T\rho/2}^{T\rho-h(\rho)} \frac{d\sigma}{\left[\frac{T\rho-\sigma}{8\rho^4} + \frac{1}{64w^4}\right]^{p/(p-1)}} = \left[\frac{8(p-1)\rho^4}{\left[\frac{T\rho-\sigma}{8\rho^4} + \frac{1}{64w^4}\right]^{1/(p-1)}}\right]_{T\rho/2}^{T\rho-h(\rho)} \leq$$

$$\leq \frac{8(p-1)\rho^4}{\left[\frac{h(\rho)}{8\rho^4} + \frac{1}{64w^4}\right]^{1/(p-1)}} \leq \frac{c\,\rho^4}{\left[\frac{h(\rho)}{\rho^4} + \frac{1}{w^4}\right]^{1/(p-1)}} . \tag{37}$$

Next, since $\sigma \leq T\rho \leq 1$, by using (33) we note that

$$L_2 \leq c \int_{T\rho-h(\rho)}^{T\rho} \frac{d\sigma}{\left[1 - \sigma + \frac{1}{2\sqrt{2}w}\right]^{4p/(p-1)}} \leq \frac{c\,h(\rho)}{\left[1 - T\rho + \frac{1}{w}\right]^{4p/(p-1)}} . \tag{38}$$

Inserting (37) and (38) into (36), and recalling the definition of Γ_3' in (34), entails

$$\Gamma_3' \le c\int_{1/2T}^{1/T} \rho^{n-5} e^{-\mu\rho^{4/3}} \int_1^\infty \frac{1}{w^n}\left[\frac{\rho^4}{\left[\frac{h(\rho)}{\rho^4}+\frac{1}{w^4}\right]^{1/(p-1)}} + \frac{h(\rho)}{\left[1-T\rho+\frac{1}{w}\right]^{4p/(p-1)}}\right] dw d\rho \tag{39}$$

and we now estimate the two inner integrals in (39). Since $(1-T\rho)^{-1} \ge 2$, for the first integral we have

$$\int_1^\infty \frac{\rho^4}{\left[\frac{h(\rho)}{\rho^4}+\frac{1}{w^4}\right]^{1/(p-1)}}\frac{dw}{w^n} \le$$

$$\le \rho^4\left(\int_1^{1/(1-T\rho)} w^{4/(p-1)-n}\, dw + \frac{\rho^{4/(p-1)}}{h(\rho)^{1/(p-1)}}\int_{1/(1-T\rho)}^\infty \frac{dw}{w^n}\right) \le$$

$$\le \rho^4\left[\Phi_1(1-T\rho) + \frac{c\,\rho^{4/(p-1)}}{\min\{1,\rho^{4/(p-1)}\}\cdot(1-T\rho)^{4/(p-1)-n+1}}\right] , \tag{40}$$

where (recall that $\frac{4}{p-1} < n$)

$$\Phi_1(s) = c\begin{cases} s^{n-1-4/(p-1)} & \text{if } \ 4/(p-1)-n > -1 \\ |\log s| & \text{if } \ 4/(p-1)-n = -1 \\ 1 & \text{if } \ 4/(p-1)-n < -1\ . \end{cases}$$

For the second inner integral in (39) we have

$$\int_1^\infty \frac{h(\rho)}{\left[1-T\rho+\frac{1}{w}\right]^{4p/(p-1)}}\frac{dw}{w^n} \le$$

$$\le h(\rho)\left(\int_1^{1/(1-T\rho)} w^{4p/(p-1)-n}\, dw + \frac{1}{(1-T\rho)^{4p/(p-1)}}\int_{1/(1-T\rho)}^\infty \frac{dw}{w^n}\right) \le$$

$$\le \min\{1,\rho^4\}\cdot\left(\Phi_2(1-T\rho) + \frac{c}{(1-T\rho)^{4/(p-1)-n+1}}\right) , \tag{41}$$

where

$$\Phi_2(s) = c\begin{cases} s^{n-1-4/(p-1)} & \text{if } \ 4p/(p-1)-n > -1 \\ s^4|\log s| & \text{if } \ 4p/(p-1)-n = -1 \\ s^4 & \text{if } \ 4p/(p-1)-n < -1\ . \end{cases}$$

Inserting (40)-(41) into (39) gives

$$\Gamma_3' \le c\int_{1/2T}^{1/T} \rho^{n-1} e^{-\mu\rho^{4/3}}\left(\Phi_1(1-T\rho) + \frac{c\,\rho^{4/(p-1)}}{\min\{1,\rho^{4/(p-1)}\}\cdot(1-T\rho)^{4/(p-1)-n+1}}\right) d\rho +$$

$$+c\int_{1/2T}^{1/T} \rho^{n-5} e^{-\mu\rho^{4/3}}\cdot\min\{1,\rho^4\}\cdot\left(\Phi_2(1-T\rho) + \frac{c}{(1-T\rho)^{4/(p-1)-n+1}}\right) d\rho =: f(T).$$

Since $4/(p-1)-n+1 < 1$ by the assumption on the exponent being "super-Fujita", the function f is well-defined (finite) for all $T \in (0,\infty)$. In order to study its behaviour as $T \to 0$ and $T \to \infty$, we perform the change of variables $\tau = T\rho$. Then, we get

$$f(T) = \frac{c}{T^n}\int_{1/2}^1 \tau^{n-1}\exp\left[-\mu\Big(\frac{\tau}{T}\Big)^{4/3}\right]\left(\Phi_1(1-\tau) + \frac{c\,\tau^{4/(p-1)}}{\min\{\tau,T\}^{4/(p-1)}\cdot(1-\tau)^{4/(p-1)-n+1}}\right) d\tau +$$

$$+\frac{c}{T^n}\int_{1/2}^1 \tau^{n-5}\exp\left[-\mu\Big(\frac{\tau}{T}\Big)^{4/3}\right]\cdot\min\{T,\tau\}^4\cdot\left(\Phi_2(1-\tau)+\frac{c}{(1-\tau)^{4/(p-1)-n+1}}\right)d\tau.$$

For all $T\ge 1$ we have $\min\{\tau,T\}=\tau$ so that

$$f(T)\le\frac{c}{T^n}\int_{1/2}^1\tau^{n-1}\left(\Phi_1(1-\tau)+\Phi_2(1-\tau)+\frac{c}{(1-\tau)^{4/(p-1)-n+1}}\right)d\tau$$

and it is clear that $f(T)\to 0$ as $T\to\infty$. For all $T\le 1/2$ we have $\min\{\tau,T\}=T$ so that

$$f(T)\le\frac{c\,e^{-cT^{-4/3}}}{T^n}\int_{1/2}^1\tau^{n-1}\left(\Phi_1(1-\tau)+\frac{c\,\tau^{4/(p-1)}}{T^{4/(p-1)}\cdot(1-\tau)^{4/(p-1)-n+1}}\right)d\tau+$$

$$+\frac{c\,e^{-cT^{-4/3}}}{T^{n-4}}\int_{1/2}^1\tau^{n-5}\left(\Phi_2(1-\tau)+\frac{c}{(1-\tau)^{4/(p-1)-n+1}}\right)d\tau$$

and $f(T)\to 0$ as $T\to 0$. This proves (34) and shows that Γ_3 is uniformly bounded. □

Lemma 4. *Assume that $n\ge 2$ and $p>1+4/n$. Then, there exists $c>0$ such that*

$$\Gamma_4:=\int_0^\infty\frac{1}{(1+w^2)^{n/2}}\int_{1/T}^{3/T}\rho^{n-5}e^{-\mu\rho^{4/3}}\int_{T\rho/2}^{T\rho}\frac{\sigma^3\qquad d\sigma\,d\rho\,dw}{\left(\frac{T^3}{\rho}(T\rho-\sigma)+(\sigma-1)^4+\frac{\sigma^2}{(1+w^2)^2}\right)^{p/(p-1)}}\le c$$

for all $T>0$.

Proof. This proof follows the same lines as that of Lemma 3. Let us just briefly sketch it. As for (34), we see that the statement is equivalent to the existence of some $c>0$ such that

$$\Gamma_4':=\int_1^\infty\frac{1}{w^n}\int_{1/T}^{3/T}\rho^{n-5}e^{-\mu\rho^{4/3}}\int_{T\rho/2}^{T\rho}\frac{d\sigma\,d\rho\,dw}{\left[\frac{T\rho-\sigma}{8\rho^4}+(\sigma-1)^4+\frac{1}{64w^4}\right]^{p/(p-1)}}\le c\qquad\text{for all }T>0.\tag{42}$$

Consider again the map h defined in (35). Since $1\le T\rho\le 3$, the following three facts hold

$$T\rho-\frac{h(\rho)}{32}\ge\frac{T\rho}{2}\,,\qquad T\rho-\frac{h(\rho)}{32}\ge 1\,,\qquad T\rho-1-\frac{h(\rho)}{32}\ge\frac{T\rho-1}{2}\,.\tag{43}$$

Instead of (36), the first inequality in (43) suggests to split here the inner integral in (42) as

$$\int_{T\rho/2}^{T\rho}=\int_{T\rho/2}^{T\rho-h(\rho)/32}+\int_{T\rho-h(\rho)/32}^{T\rho}=:M_1+M_2.$$

Arguing as for (37) and (38), we obtain

$$M_1\le\frac{c\,\rho^4}{\left[\frac{h(\rho)}{\rho^4}+\frac{1}{w^4}\right]^{1/(p-1)}}\quad,\qquad M_2\le\frac{c\,h(\rho)}{\left[T\rho-1+\frac{1}{w}\right]^{4p/(p-1)}}\,,$$

where we used both the second and third inequality in (43).

Finally, repeating the arguments relative to (40)-(41), we conclude that

$$\Gamma_4'\le c\int_{1/T}^{3/T}\rho^{n-1}e^{-\mu\rho^{4/3}}\left(\Phi_1(T\rho-1)+\frac{c\,\rho^{4/(p-1)}}{\min\{1,\rho^{4/(p-1)}\}\cdot(T\rho-1)^{4/(p-1)-n+1}}\right)d\rho+$$

$$+c\int_{1/T}^{3/T}\rho^{n-5}e^{-\mu\rho^{4/3}}\cdot\min\{1,\rho^4\}\cdot\left(\Phi_2(T\rho-1)+\frac{c}{(T\rho-1)^{4/(p-1)-n+1}}\right)d\rho.$$

With the change of variables $\tau=T\rho$ one can then show (42), namely uniform (with respect to T) boundedness of Γ_4'. $\square$

Lemma 5. *Assume that $n\geq 2$ and $p>1+4/n$. Then, there exists $c>0$ such that*

$$\Gamma_5:=\int_0^\infty\frac{1}{(1+w^2)^{n/2}}\int_{3/T}^\infty\rho^{n-5}e^{-\mu\rho^{4/3}}\int_0^{T\rho}\frac{\sigma^3\qquad d\sigma\, d\rho\, dw}{\left(\frac{T^3}{\rho}(T\rho-\sigma)+(\sigma-1)^4+\frac{\sigma^2}{(1+w^2)^2}\right)^{p/(p-1)}}\leq c$$

for all $T>0$.

Proof. As long as Γ_5 is involved, we have $T\rho\geq 3$. Hence, we may estimate:

$$\Gamma_5\leq\int_0^\infty\frac{dw}{(1+w^2)^{n/2}}\cdot\int_{3/T}^\infty\rho^{n-5}e^{-\mu\rho^{4/3}}\left(\frac{8\rho^{p/(p-1)}}{T^{3p/(p-1)}}\int_0^2\frac{d\sigma}{(3-\sigma)^{p/(p-1)}}+\int_2^{T\rho}\frac{\sigma^3\ d\sigma}{(\sigma-1)^{4p/(p-1)}}\right)d\rho\leq$$

$$\leq c\int_{3/T}^\infty\rho^{n-5}e^{-\mu\rho^{4/3}}\left(\frac{c\,\rho^{p/(p-1)}}{T^{3p/(p-1)}}+\int_2^\infty\frac{\sigma^3\ d\sigma}{(\sigma-1)^{4p/(p-1)}}\right)d\rho\leq$$

$$[\tau=T\rho]\quad\leq\frac{c}{T^{n-4+4p/(p-1)}}\int_3^\infty\tau^{n-5+p/(p-1)}\exp\left[-\mu\left(\frac{\tau}{T}\right)^{4/3}\right]d\tau+c\int_0^\infty\rho^{n-5}e^{-\mu\rho^{4/3}}\,d\rho\leq$$

$$\leq c+\frac{c}{T^{n+4/(p-1)}}\int_3^\infty\tau^{n-5+p/(p-1)}\exp\left[-\mu\left(\frac{\tau}{T}\right)^{4/3}\right]d\tau\ .$$

Therefore, the statement will follow if we show that the function

$$g(T):=\frac{1}{T^{n+4/(p-1)}}\int_3^\infty\tau^{n-5+p/(p-1)}\exp\left[-\mu\left(\frac{\tau}{T}\right)^{4/3}\right]d\tau$$

remains bounded when T varies over $(0,\infty)$. Since $g(T)$ is finite for every $T\in(0,\infty)$, it suffices to study the limits of $g(T)$ when $T\to 0$ and $T\to\infty$. We first remark that for all $T<1$ we may write

$$g(T)=\frac{1}{T^{n+4/(p-1)}}\int_3^\infty\tau^{n-5+p/(p-1)}\exp\left[-\frac{\mu}{2}\left(\frac{\tau}{T}\right)^{4/3}\right]\cdot\exp\left[-\frac{\mu}{2}\left(\frac{\tau}{T}\right)^{4/3}\right]d\tau\leq$$

$$\leq\frac{\exp\left[-\frac{\mu}{2}\left(\frac{3}{T}\right)^{4/3}\right]}{T^{n+4/(p-1)}}\int_3^\infty\tau^{n-5+p/(p-1)}\cdot\exp\left[-\frac{\mu}{2}\tau^{4/3}\right]d\tau=\frac{c}{T^{n+4/(p-1)}}\cdot\exp\left[-\frac{\mu}{2}\left(\frac{3}{T}\right)^{4/3}\right]$$

and the last term tends to 0 as $T\to 0$. Next, for all $T>1$ we write

$$g(T)=\frac{1}{T^{n+4/(p-1)}}\left(\int_3^{3T^{n/(n-1)}}+\int_{3T^{n/(n-1)}}^\infty\right).$$

Then, by using the two following facts

$$\tau\leq 3T^{n/(n-1)}\implies\frac{1}{T}\leq\frac{c}{\tau^{(n-1)/n}}\qquad,\qquad\tau\geq 3T^{n/(n-1)}\implies\frac{\tau}{T}\geq c\tau^{1/n}\ ,$$

we may estimate $g(T)$ also when $T > 1$:

$$g(T) \leq c\int_3^{3T^{n/(n-1)}} \frac{\tau^{n-5+p/(p-1)}}{\tau^{n-1+4(n-1)/n(p-1)}}\,d\tau + \frac{c}{T^{n+4/(p-1)}}\int_{3T^{n/(n-1)}}^{\infty} \tau^{n-5+p/(p-1)}\exp[-c\tau^{4/3n}]\,d\tau \leq$$

$$\leq c\int_3^{\infty} \frac{d\tau}{\tau^{(3np-4)/n(p-1)}} + \frac{c}{T^{n+4/(p-1)}}\int_3^{\infty} \tau^{n-5+p/(p-1)}\exp[-c\tau^{4/3n}]\,d\tau.$$

Since $3np-4 > n(p-1)$, the above term remains bounded as $T \to \infty$. This completes the proof of the lemma. □

6 Proof of Proposition 2

In order to prove Proposition 2 we introduce the function

$$G_1(x,t) := \int_{\mathbb{R}^n} \exp\left[-\mu\left(\frac{|y|^4}{t}\right)^{1/3}\right] \frac{dy}{t^{n/4}(1+|x-y|^{4/(p-1)})}\ , \qquad (x,t) \in \mathbb{R}^{n+1}_+\ .$$

With the change of variables $y = s^{1/4}z$, we may also rewrite G_1 as

$$G_1(x,t) = \int_{\mathbb{R}^n} \exp\left[-\mu|z|^{4/3}\right] \frac{dz}{(1+|x-t^{1/4}z|^{4/(p-1)})}\ , \qquad (x,t) \in \mathbb{R}^{n+1}_+\ .$$

We first estimate G_1 for small values of $|x|$:

Lemma 6. *Assume that $n \geq 2$ and $p > 1 + 4/n$. There exists a constant $K_1 = K_1(n,p,\mu) > 0$ such that*

$$G_1(x,t) \leq \frac{K_1}{1+|x|^{4/(p-1)}+t^{1/(p-1)}} \qquad \text{for all } |x| \leq 1\ ,\ t \geq 0\ .$$

Proof. For $|x| \leq 1$, we have

$$G_1(x,t) \leq \int_{\mathbb{R}^n} \exp\left[-\mu|z|^{4/3}\right] dz = c \leq \frac{A_1}{1+|x|^{4/(p-1)}} \qquad \text{for all } |x| \leq 1\ ,\ t \geq 0\ , \tag{44}$$

for some $A_1 > 0$. Moreover, with the change of variables $w = z - t^{-1/4}x$ we obtain

$$G_1(x,t) \leq \frac{1}{t^{1/(p-1)}}\left[\int_{|z-t^{-1/4}x|<1} \frac{dz}{|z-t^{-1/4}x|^{4/(p-1)}} + \int_{|z-t^{-1/4}x|>1} \exp\left[-\mu|z|^{4/3}\right] dz\right] \leq$$

$$\leq \frac{1}{t^{1/(p-1)}}\left[\int_{|w|<1} \frac{dw}{|w|^{4/(p-1)}} + \int_{\mathbb{R}^n} \exp\left[-\mu|z|^{4/3}\right] dz\right] = \frac{A_2}{t^{1/(p-1)}} \qquad \text{for all } (x,t) \in \mathbb{R}^{n+1}_+\ , \tag{45}$$

with A_2 being finite because $\frac{4}{p-1} < n$. Clearly, A_2 is independent of x and t.

By combining (44) and (45), for all $|x| \leq 1$ and $t \geq 0$ we obtain

$$G_1(x,t) \leq \min\left\{\frac{A_1}{1+|x|^{4/(p-1)}}, \frac{A_2}{t^{1/(p-1)}}\right\} \leq \max\{A_1, A_2\} \cdot \min\left\{\frac{1}{1+|x|^{4/(p-1)}}, \frac{1}{t^{1/(p-1)}}\right\}\ .$$

By (32), this yields

$$G_1(x,t) \leq \frac{2\cdot\max\{A_1,A_2\}}{1+|x|^{4/(p-1)}+t^{1/(p-1)}} \qquad \text{for all } |x| \leq 1\ ,\ t \geq 0\ ,$$

which proves the statement with $K_1 = 2\cdot\max\{A_1, A_2\}$. □

Assume now that $|x| > 1$. Then, we prove

Lemma 7. *Assume that $n \geq 2$ and $p > 1 + 4/n$. There exists a constant $K_2 = K_2(n,p,\mu) > 0$ such that*

$$G_1(x,t) \leq \frac{K_2}{1 + |x|^{4/(p-1)} + t^{1/(p-1)}} \qquad \textit{for all } |x| > 1 \ , \ t \geq 0 \ .$$

Proof. Put $F(x,t) := (1 + |x|^{4/(p-1)} + t^{1/(p-1)}) \cdot G_1(x,t)$. Then, the statement follows if we show that $F(x,t) \leq K_2$ for all $|x| > 1$ and $t \geq 0$. When $|x| > 1$, we have

$$F(x,t) \leq c \int_{\mathbb{R}^n} \exp\left[-\mu|z|^{4/3}\right] \frac{|x|^{4/(p-1)} + t^{1/(p-1)}}{|x - t^{1/4}z|^{4/(p-1)}} \, dz \ .$$

Therefore, simplifying by $|x|^{4/(p-1)}$ and putting $T := t^{1/4}|x|^{-1}$, we obtain

$$F(x,t) \leq c(1 + T^{4/(p-1)}) \cdot \int_{\mathbb{R}^n} \exp\left[-\mu|z|^{4/3}\right] \frac{dz}{\left|\frac{x}{|x|} - Tz\right|^{4/(p-1)}} \ .$$

At this stage, only the direction $\frac{x}{|x|}$ of x is involved. Hence, with no loss of generality, we may assume that $\frac{x}{|x|} = e_1 = (1, 0, ..., 0)$, the first unit vector of the canonical basis of $\mathbb{R}^n$. In this case, the above estimate reads

$$F(x,t) \leq c(1 + T^{4/(p-1)}) \cdot \int_{\mathbb{R}^n} \exp\left[-\mu|z|^{4/3}\right] \frac{dz}{|e_1 - Tz|^{4/(p-1)}} =: c\Phi(T) \ .$$

Since $\frac{4}{p-1} < n$, the function $\Phi(T)$ is well-defined (finite) for all $T \in (0,\infty)$. In order to prove the uniform boundedness of F, we have to show that $\Phi(T)$ remains bounded in both the cases $T \to 0$ and $T \to \infty$. When $T \to \infty$, we have

$$\Phi(T) = \frac{1 + T^{4/(p-1)}}{T^{4/(p-1)}} \cdot \left[\int_{|z - \frac{e_1}{T}| < 1} \frac{\exp\left[-\mu|z|^{4/3}\right] \, dz}{\left|z - \frac{e_1}{T}\right|^{4/(p-1)}} + \int_{|z - \frac{e_1}{T}| > 1} \frac{\exp\left[-\mu|z|^{4/3}\right] \, dz}{\left|z - \frac{e_1}{T}\right|^{4/(p-1)}}\right] \leq$$

$$\left[w = z - \frac{e_1}{T}\right] \quad \leq \frac{1 + T^{4/(p-1)}}{T^{4/(p-1)}} \cdot \left[\int_{|w| < 1} \frac{dw}{|w|^{4/(p-1)}} + \int_{\mathbb{R}^n} e^{-\mu|z|^{4/3}} \, dz\right] = O(1),$$

because (once more!) $\frac{4}{p-1} < n$. When $T \to 0$, we have

$$\Phi(T) \leq O(1) \left[\int_{|e_1 - Tz| < 1/2} \frac{\exp\left[-\mu|z|^{4/3}\right] \, dz}{|e_1 - Tz|^{4/(p-1)}} + \int_{|e_1 - Tz| > 1/2} \frac{\exp\left[-\mu|z|^{4/3}\right] \, dz}{|e_1 - Tz|^{4/(p-1)}}\right] \leq$$

$$[w = e_1 - Tz] \quad \leq O(1) \left[\frac{\exp[-cT^{-4/3}]}{T^n} \int_{|w| < 1/2} \frac{dw}{|w|^{4/(p-1)}} + c \int_{\mathbb{R}^n} e^{-\mu|z|^{4/3}} \, dz\right] = O(1),$$

where we also used the fact that $|z| > \frac{1}{2T}$ whenever $|e_1 - Tz| < 1/2$. Therefore, $\Phi(T)$ is uniformly bounded on $(0,\infty)$ and the proof of the lemma follows. □

Recalling the definition of G_1, Proposition 2 follows at once from Lemmas 6-7 by taking $C_1 = \omega_1 \cdot \max\{K_1, K_2\}$.

7 Proof of Proposition 3

Our proof of Proposition 3 requires several lemmas. We define the function

$$G_2(x,t) := \int_0^t \int_{\mathbb{R}^n} \exp\left[-\mu\left(\frac{|y|^4}{s}\right)^{1/3}\right] \frac{dy\,ds}{s^{n/4}\left(1+(t-s)^{1/(p-1)}+|x-y|^{4/(p-1)}\right)^p}\ , \qquad (x,t)\in\mathbb{R}^{n+1}_+.$$

With the change of variables $y = t^{1/4}z$, we may also rewrite G_2 as

$$G_2(x,t) = \int_0^t \int_{\mathbb{R}^n} \frac{\exp\left(-\mu|z|^{4/3}\right)}{\left[1+(t-s)^{1/(p-1)}+|x-s^{1/4}z|^{4/(p-1)}\right]^p}\, dz\,ds\ , \qquad (x,t)\in\mathbb{R}^{n+1}_+.$$

We first obtain an upper bound for G_2 in terms of the time variable:

Lemma 8. *Assume that $n\ge 2$ and $p>1+4/n$. There exists a constant $B_1 = B_1(n,p,\mu)>0$ such that*

$$G_2(x,t) \le \frac{B_1}{t^{1/(p-1)}} \qquad \textit{for all } (x,t)\in\mathbb{R}^{n+1}_+\ .$$

Proof. We split the integral as follows:

$$G_2(x,t) = \int_0^t \int_{|z-s^{-1/4}x|<1/2} + \int_0^t \int_{|z-s^{-1/4}x|>1/2} =: I_1 + I_2 \tag{46}$$

and we estimate separately I_1 and I_2. First, note that

$$I_1 \le \int_0^t \int_{|z-s^{-1/4}x|<1/2} \frac{\exp\left(-\mu|z|^{4/3}\right)}{\left[(t-s)^{1/(p-1)}+|x-s^{1/4}z|^{4/(p-1)}\right]^p}\, dz\,ds \le$$

$$[w=z-s^{-1/4}x] \quad \le \int_0^t \int_{|w|<1/2} \frac{dw\,ds}{\left[(t-s)^{1/(p-1)}+s^{1/(p-1)}|w|^{4/(p-1)}\right]^p} \le \tag{47}$$

$$[\text{by (33)}] \quad \le c\int_{|w|<1/2}\int_0^t \frac{ds\,dw}{[t-s+s|w|^4]^{p/(p-1)}} \le \frac{c}{t^{1/(p-1)}}\int_{|w|<1/2} \frac{dw}{(1-|w|^4)|w|^{4/(p-1)}} \le \frac{c}{t^{1/(p-1)}}$$

since $\frac{4}{p-1}<n$ in view of the assumption $p>1+\frac{4}{n}$. Next, we have

$$I_2 \le \int_0^t \int_{|z-s^{-1/4}x|>1/2} \frac{\exp\left(-\mu|z|^{4/3}\right)}{\left[(t-s)^{1/(p-1)}+s^{1/(p-1)}|xs^{-1/4}-z|^{4/(p-1)}\right]^p}\, dz\,ds \le$$

$$\le \int_0^t \int_{|z-s^{-1/4}x|>1/2} \frac{\exp\left(-\mu|z|^{4/3}\right)}{\left[(t-s)^{1/(p-1)}+(\frac{s}{16})^{1/(p-1)}\right]^p}\, dz\,ds \le \tag{48}$$

$$[\text{by (33)}] \quad \le c\int_{\mathbb{R}^n} \exp\left(-\mu|z|^{4/3}\right)\, dz \ \cdot\ \int_0^t \frac{ds}{[16t-15s]^{p/(p-1)}} \le \frac{c}{t^{1/(p-1)}}\ .$$

Combining (46)-(47)-(48) proves the statement. □

Next, we prove an upper bound for G_2 for small values of $|x|$:

Lemma 9. *Assume that $n\ge 2$ and $p>1+4/n$. There exists a constant $B_2 = B_2(n,p,\mu)>0$ such that*

$$G_2(x,t) \le \frac{B_2}{1+|x|^{4/(p-1)}} \qquad \textit{for all } |x|\le 1\ ,\ t\ge 0\ .$$

Proof. Since $|x| \leq 1$, it suffices to show that $G_2(x,t) \leq B_2/2$ for some $B_2 > 0$. This follows by performing the change of variables $\sigma = t - s$ and the following trivial estimate

$$G_2(x,t) \leq \int_{\mathbb{R}^n} e^{-\mu|z|^{4/3}}\, dz \cdot \int_0^\infty \frac{ds}{[1+\sigma^{1/(p-1)}]^p} := \frac{B_2}{2} \ ,$$

the constant B_2 being independent of x and t. □

We now come to the most delicate estimate, an upper bound for G_2 for large values of $|x|$:

Lemma 10. *Assume that $n \geq 2$ and $p > 1 + 4/n$. There exists a constant $B_3 = B_3(n,p,\mu) > 0$ such that*

$$G_2(x,t) \leq \frac{B_3}{1+|x|^{4/(p-1)}} \qquad \textit{for all } |x| > 1\ ,\ t \geq 0\ .$$

Proof. We will show that $F(x,t) := (1+|x|^{4/(p-1)})G_2(x,t) \leq B_3$ for all (x,t) in their ranges. With the change of variables $\sigma = \frac{|z|}{|x|}s^{1/4}$ and by simplifying by $|x|^{4/(p-1)}$ we obtain

$$F(x,t) = 4\int_{\mathbb{R}^n} \frac{\exp[-\mu|z|^{4/3}]}{|z|^4} \int_0^{\frac{|z|}{|x|}t^{1/4}} \frac{(1+|x|^{-4/(p-1)})\,\sigma^3}{\left[|x|^{-4/(p-1)} + \left(\frac{t}{|x|^4} - \frac{\sigma^4}{|z|^4}\right)^{1/(p-1)} + \left|\frac{x}{|x|} - \frac{z}{|z|}\sigma\right|^{4/(p-1)}\right]^p}\, d\sigma\, dz.$$

Put $y := x/|x|$ so that $|y| = 1$ and y defines the direction of x; with no loss of generality we may take $y = e_1 = (1,0,...,0)$, the first unit vector of the canonical basis. Let also $T := \frac{t^{1/4}}{|x|}$. Then, recalling $|x| > 1$, we obtain

$$F(x,t) \leq c\int_{\mathbb{R}^n} \frac{\exp[-\mu|z|^{4/3}]}{|z|^4} \int_0^{|z|T} \frac{\sigma^3 \qquad d\sigma\, dz}{\left(T^4 - \frac{\sigma^4}{|z|^4}\right)^{p/(p-1)} + (\sigma-1)^{4p/(p-1)} + \left[\left(1 - \frac{z_1}{|z|}\right)2\sigma\right]^{2p/(p-1)}}$$

where we used (33) and the following fact:

$$\left|e_1 - \frac{z}{|z|}\sigma\right|^{4/(p-1)} = \left(1 - 2\frac{z_1}{|z|}\sigma + \sigma^2\right)^{2/(p-1)} = \left[(\sigma-1)^2 + \left(1 - \frac{z_1}{|z|}\right)2\sigma\right]^{2/(p-1)} , \qquad z = (z_1,...,z_n).$$

Now put $z' = (z_2,...,z_n) \in \mathbb{R}^{n-1}$ so that $z = (z_1, z')$ and $|z|^2 = z_1^2 + |z'|^2$. Passing to radial coordinates $r := |z'|$ in $\mathbb{R}^{n-1}$ we obtain

$$F(x,t) \leq c\int_0^\infty r^{n-2} \int_{-\infty}^\infty \frac{\exp[-\mu(z_1^2+r^2)^{2/3}]}{(z_1^2+r^2)^2}$$

$$\int_0^{T\sqrt{z_1^2+r^2}} \frac{\sigma^3 \qquad d\sigma\, dz_1\, dr}{\left(T^4 - \frac{\sigma^4}{(z_1^2+r^2)^2}\right)^{p/(p-1)} + (\sigma-1)^{4p/(p-1)} + \left[\left(1 - \frac{z_1}{\sqrt{z_1^2+r^2}}\right)2\sigma\right]^{2p/(p-1)}} \ .$$

With the change of variables $z_1 = rw$ the previous estimate becomes

$$F(x,t) \leq c\int_0^\infty r^{n-5} \int_{-\infty}^\infty \frac{\exp[-\mu r^{4/3}(1+w^2)^{2/3}]}{(1+w^2)^2}$$

$$\int_0^{Tr\sqrt{1+w^2}} \frac{\sigma^3 \qquad d\sigma\, dw\, dr}{\left(T^4 - \frac{\sigma^4}{r^4(1+w^2)^2}\right)^{p/(p-1)} + (\sigma-1)^{4p/(p-1)} + \left[\left(1 - \frac{w}{\sqrt{1+w^2}}\right)2\sigma\right]^{2p/(p-1)}} \ .$$

Making use of the two inequalities

$$1-\frac{w}{\sqrt{1+w^2}}=\frac{\left(\sqrt{1+w^2}+w\right)\left(\sqrt{1+w^2}-w\right)}{\left(\sqrt{1+w^2}+w\right)\sqrt{1+w^2}}\geq\frac{1}{2(1+w^2)}\qquad\text{for all } w\in\mathbb{R}$$

and

$$T^4-\frac{\sigma^4}{r^4(1+w^2)^2}=\frac{1}{r^4(1+w^2)^2}\left(Tr\sqrt{1+w^2}-\sigma\right)\left(Tr\sqrt{1+w^2}+\sigma\right)\left(T^2r^2[1+w^2]+\sigma^2\right)\geq$$

$$\geq\frac{T^3}{r\sqrt{1+w^2}}\left(Tr\sqrt{1+w^2}-\sigma\right)\qquad\text{for all } \sigma\in[0,Tr\sqrt{1+w^2}],$$

we may estimate further the above expression by

$$F(x,t)\leq c\int_0^\infty r^{n-5}\int_0^\infty\frac{\exp[-\mu r^{4/3}(1+w^2)^{2/3}]}{(1+w^2)^2}$$

$$\int_0^{Tr\sqrt{1+w^2}}\frac{\sigma^3\qquad d\sigma\,dw\,dr}{\left(\frac{T^3}{r\sqrt{1+w^2}}\right)^{p/(p-1)}\left(Tr\sqrt{1+w^2}-\sigma\right)^{p/(p-1)}+(\sigma-1)^{4p/(p-1)}+\left(\frac{\sigma}{1+w^2}\right)^{2p/(p-1)}}\,,$$

where the integration with respect to w is now just over $(0,\infty)$ since the integrand is even with respect to w. We now use (33) and we make the further change of variables $r=\rho/\sqrt{1+w^2}$ to obtain

$$F(x,t)\leq c\int_0^\infty\frac{1}{(1+w^2)^{n/2}}\int_0^\infty\rho^{n-5}e^{-\mu\rho^{4/3}}\int_0^{T\rho}\frac{\sigma^3\qquad d\sigma\,d\rho\,dw}{\left(\frac{T^3}{\rho}(T\rho-\sigma)+(\sigma-1)^4+\frac{\sigma^2}{(1+w^2)^2}\right)^{p/(p-1)}}\,.\qquad(49)$$

We split the integral on the right hand side of (49) as follows:

$$\int_0^\infty\int_0^{1/2T}\int_0^{T\rho}+\int_0^\infty\int_{1/2T}^{3/T}\int_0^{T\rho/2}+\int_0^\infty\int_{1/2T}^{1/T}\int_{T\rho/2}^{T\rho}+\int_0^\infty\int_{1/T}^{3/T}\int_{T\rho/2}^{T\rho}+\int_0^\infty\int_{3/T}^\infty\int_0^{T\rho}\,.$$

Then, thanks to Lemmas 1-2-3-4-5 we know that there exists $B_3>0$ such that $F(x,t)\leq B_3$ for all (x,t) in their ranges. This completes the proof of the lemma. □

We may now give the proof of Proposition 3. By Lemmas 8-9-10, for all $(x,t)\in\mathbb{R}^{n+1}_+$ we obtain

$$G_2(x,t)\leq\min\left\{\frac{B_1}{t^{1/(p-1)}}\,,\,\frac{\max\{B_2,B_3\}}{1+|x|^{4/(p-1)}}\right\}\leq\max\{B_1,B_2,B_3\}\cdot\min\left\{\frac{1}{t^{1/(p-1)}}\,,\,\frac{1}{1+|x|^{4/(p-1)}}\right\}\,.$$

Using (32), the last inequality yields

$$G_2(x,t)\leq\frac{2\cdot\max\{B_1,B_2,B_3\}}{1+|x|^{4/(p-1)}+t^{1/(p-1)}}\,.$$

Recalling the definition of G_2, this proves Proposition 3 with $C_2=2\omega_1\cdot\max\{B_1,B_2,B_3\}$.

8 A simple proof of Proposition 3 when $p > \frac{n+4}{n-4}$

In this section we assume that $n \geq 5$, that $p > \frac{n+4}{n-4}$, and we give a proof of Proposition 3 which is simpler (and shorter) than the one in the previous section. This proof bases upon the idea that in a certain sense – for time independent right hand sides – the limiting operator for the majorising kernel $\tilde{b}$ for $t \to \infty$ is just a multiple of the Green operator for the bi-Laplacian in $\mathbb{R}^n$.

We denote by U_0 an entire smooth positive radial (and radially decreasing) solution of

$$\Delta^2 U_0 = U_0^p \qquad \text{in } \mathbb{R}^n, \tag{50}$$

which have been constructed in [11, Theorem 1]. According to [11, Theorem 3], there exists $C_0 > 0$ such that

$$\frac{C_0^{-1}}{1+|x|^{4/(p-1)}} \leq U_0(x) \leq \frac{C_0}{1+|x|^{4/(p-1)}} \qquad \text{for all } x \in \mathbb{R}^n\ . \tag{51}$$

In particular, the convolution of the fundamental solution of Δ^2 with U_0^p is well-defined (the integral exists). Therefore, we may rewrite (50) as an integral equation

$$U_0(x) = \frac{1}{2n(n-2)(n-4)e_n} \int_{\mathbb{R}^n} |x-y|^{4-n} U_0^p(y)\, dy \tag{52}$$

where $e_n = |B_1|$, the measure of the unit ball. Let $G_2 = G_2(x,t)$ denote the function defined in Section 7. Then, by making use of the coordinate transformation $\sigma := \left(\frac{|x-y|^4}{s}\right)^{1/3}$, we may estimate

$$\begin{aligned} G_2(x,t) &\leq \int_0^\infty \int_{\mathbb{R}^n} s^{-n/4} \exp\left[-\mu\left(\frac{|x-y|^4}{s}\right)^{1/3}\right] \frac{dy\, ds}{\left(1+|y|^{4/(p-1)}\right)^p} \\ &\leq C_0^p \int_{\mathbb{R}^n} \int_0^\infty s^{-n/4} \exp\left[-\mu\left(\frac{|x-y|^4}{s}\right)^{1/3}\right] U_0^p(y)\, ds\, dy \\ &= 3C_0^p \int_0^\infty \exp(-\mu\sigma)\sigma^{\frac{3}{4}n-4}\, d\sigma \cdot \int_{\mathbb{R}^n} |x-y|^{4-n} U_0^p(y)\, dy. \end{aligned}$$

We denote

$$\hat{C} = \hat{C}(n,\mu) = \int_0^\infty \exp(-\mu\sigma)\sigma^{\frac{3}{4}n-4}\, d\sigma < \infty,$$

which is finite since $n > 4$. We now proceed by means of (52):

$$\begin{aligned} G_2(x,t) &\leq \left(6n(n-2)(n-4)e_n \hat{C} C_0^p\right) \frac{1}{2n(n-2)(n-4)e_n} \int_{\mathbb{R}^n} |x-y|^{4-n} U_0^p(y)\, dy \\ &= \left(6n(n-2)(n-4)e_n \hat{C} C_0^p\right) U_0(x) \\ &\leq \frac{K_0}{1+|x|^{4/(p-1)}} \qquad \text{for all } (x,t) \in \mathbb{R}^{n+1}_+ \end{aligned}$$

with $K_0 := 6n(n-2)(n-4)e_n \hat{C} C_0^{p+1}$. The just proved inequality, combined with Lemma 8 and (32) concludes the proof of Proposition 3 when $p > \frac{n+4}{n-4}$. □

Remark 3. 1)When $p < \frac{n+4}{n-4}$ the arguments of the present section do not apply since (50) admits no positive solutions, see [16, Theorem 1.4]. On the other hand, if $p = \frac{n+4}{n-4}$ positive entire radial solutions $U_0 = U_0(x)$ of (50) do exist: however, they behave at infinity as $(1+|x|^{n-4})^{-1}$ (see [18]) so that they

do not satisfy the decay estimate (51). For this reason, also in the critical Sobolev case $p = \frac{n+4}{n-4}$, one cannot argue as in this section.
2) Under the assumption $p > (n+4)/(n-4)$, the above proof replaces Lemmas 9 and 10. The proof simplifies because the term $(t-s)^{1/(p-1)}$ in the denominator of G_2 is dropped. This does not seem to be possible when trying to cover the full range $p > 1 + 4/n$.

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