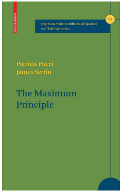
We celebrate Patrizia's birthday

- At least my 6th visit to Perugia: Enjoyed many deep and inspiring discussions with Patrizia.
- Patrizia and James (who became 85 last November! Best wishes!!) wrote a couple of papers on polyharmonic critical growth problems. Very interesting conjecture on critical dimensions . . .
- Concerning maximum principles for second order elliptic equations they contributed a lot and know (almost?) everything:



- My contribution to this birthday party: Higher order elliptic problems. No maximum principle, of course, but s.th. persists.
- Report on recent progress in a long ongoing project.

Up to bounded smooth corrections, biharmonic Green functions are positive

Hans-Christoph Grunau, Frédéric Robert, Guido Sweers

Otto von Guericke-Universität, Magdeburg
Université de Nancy 1, Universität zu Köln

Workshop on Nonlinear Partial Differential Equations
Perugia, May 28 – June 1, 2012

The clamped plate equation

The clamped plate equation (more precisely: clamped plate boundary value problem)

$$\text{(CPE)} \qquad \begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = |\nabla u| = 0 & \text{on } \partial\Omega \end{cases}$$

is an important model and prototype for higher order boundary value problems.

Linear theory – estimates, existence, regularity – well understood:
Agmon – Douglis – Nirenberg,
Krasovskiĭ – Maz'ya – Nazarov – Jerison – Kenig – Pipher – Verchota.

Nonlinear questions by far less well understood, even:
What about comparison principles, i.e. positivity preserving,

$$f \geq 0 \Rightarrow u \geq 0??$$

(Non-) Positivity issues

$$\text{(CPE)} \qquad \begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = |\nabla u| = 0 & \text{on } \partial\Omega \end{cases}$$

has no strong maximum principle: $x \mapsto \pm|x|^2$.

When thinking about a possible positivity preserving property, one observes:

(CPE) cannot reasonably be written as a 2nd order system.

No iterative arguments.

Duffin – Coffman – Garabedian – Seif – Osher – Kozlov – Kondratiev – Maz'ya – Shapiro – Tegmark – Hedenmalm (1948–today):

In general, $f \geq 0 \not\Rightarrow u \geq 0$. Even in mildly eccentric ellipses.

On the other hand; engineers' reaction: In smoothly bounded "real" clamped plates, one will not be able to observe this change of sign.

Positivity issues

Let $G := G_\Omega := G_{\Omega,\mathrm{CPE}}$ denote the Green function for

$$(\mathrm{CPE}) \qquad \begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = |\nabla u| = 0 & \text{on } \partial\Omega \end{cases}$$

so that

$$u(x) = \int_\Omega G_{\Omega,\mathrm{CPE}}(x,y) f(y)\, dy.$$

On the other hand: Boggio, 1905, (unit–) ball $B = B_1(0) \subset \mathbb{R}^n$:

$$G_{B,\mathrm{CPE}}(x,y) = c_n |x-y|^{4-n} \int_1^{\sqrt{1+\frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}}} (v^2-1) v^{1-n}\, dv > 0.$$

Even more, G.-Sweers (1997):

$$G_{B,\mathrm{CPE}}(x,y) \sim H_B(x,y).$$

Positivity issues

$$G_{B,\mathrm{CPE}}(x,y) \sim H_B(x,y),$$

where with $d(x) := \mathrm{dist}(x, \partial\Omega)$:

$$H_\Omega(x,y) := \begin{cases} |x-y|^{4-n} \min\left\{1, \dfrac{d(x)^2 d(y)^2}{|x-y|^4}\right\}, & \text{if } n > 4; \\ \log\left(1 + \dfrac{d(x)^2 d(y)^2}{|x-y|^4}\right), & \text{if } n = 4; \\ \left(d(x)d(y)\right)^{2-\frac{n}{2}} \min\left\{1, \dfrac{d(x)^{\frac{n}{2}} d(y)^{\frac{n}{2}}}{|x-y|^n}\right\}, & \text{if } n < 4. \end{cases}$$

This allows for a perturbation theory of positivity.

Green function estimates

- Krasovskiĭ (1967) & Dall'Acqua–Sweers (2004), in any bounded smooth domain Ω:

$$|G_{\Omega,\mathrm{CPE}}(x,y)| \leq C(\Omega) H_\Omega(x,y),$$

  no distinction between $G^+_{\Omega,\mathrm{CPE}}$ and $G^-_{\Omega,\mathrm{CPE}}$.
- Dall'Acqua–Meister–Sweers (2005): If $n = 2$, $G^-_{\Omega,\mathrm{CPE}}$ is smooth and smaller than $G^+_{\Omega,\mathrm{CPE}}$.
- Dall'Acqua–Meister–Sweers ($n = 2$), G.-Robert ($n \geq 3$, 2010), using previous work by Nehari and G.-Sweers:

Theorem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded smooth domain. There exists $\delta(\Omega) > 0$ s.th.:

$$x, y \in \Omega, \quad x \neq y, \quad |x - y| < \delta$$

$$\Rightarrow \quad G_{\Omega,CPE}(x,y) > 0.$$

Around the pole: Positivity! Uniformly!

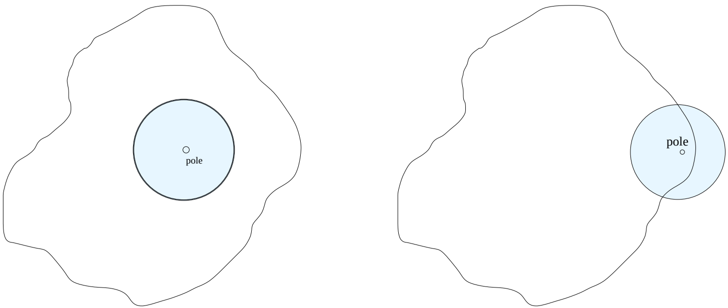


Figure: Uniform positivity around the pole

Recall:

$$H_\Omega(x,y) := \begin{cases} |x-y|^{4-n} \min\left\{1, \dfrac{d(x)^2 d(y)^2}{|x-y|^4}\right\}, & \text{if } n > 4; \\ \log\left(1 + \dfrac{d(x)^2 d(y)^2}{|x-y|^4}\right), & \text{if } n = 4; \\ \Big(d(x)d(y)\Big)^{2-\frac{n}{2}} \min\left\{1, \dfrac{d(x)^{\frac{n}{2}} d(y)^{\frac{n}{2}}}{|x-y|^n}\right\}, & \text{if } n < 4. \end{cases}$$

Conclusion: Refined Green function estimates

Combine Dall'Acqua, G., Krasovskiĭ, Meister, Robert, Sweers:

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. There exist (large) constants $C_1 = C_1(\Omega) > 0$, $C_2 = C_2(\Omega) > 0$ such that in $\Omega \times \Omega$:

$$0 \leq G_{\Omega,CPE}(x,y) + C_1 d(x)^2 d(y)^2 \leq C_2 H_\Omega(x,y).$$

Garabedian's example: Correction term qualitatively optimal.

A suitable rank-one-correction of the Green operator is positivity preserving!

Refined Green function estimates as if there were almost a maximum principle

G.-Robert-Sweers (2011):

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. There exist (large) constants $C_1 = C_1(\Omega) > 0$, $C_2 = C_2(\Omega) > 0$ such that in $\Omega \times \Omega$:

$$\frac{1}{C_2} H_\Omega(x,y) \le G_{\Omega,CPE}(x,y) + C_1 d(x)^2 d(y)^2 \le C_2 H_\Omega(x,y).$$

Up to the bounded smooth correction term $C_1 d(x)^2 d(y)^2$, the CPE-Green function in bounded smooth domains behaves exactly analogously to the second order Laplacian Green function ...
as if there were maximum principles, Harnack inequalities, etc. ...
A suitable rank-one-correction of the 4th order Green operator behaves completely analogously to the 2nd order Green operator!

Local estimates from below

Only estimate from below remains to be proved.
Discuss only the more difficult case that x, y are closer to the boundary than to each other:

$$x, y \in \Omega, x \neq y, |x - y| > \delta_n \max\{d(x), d(y)\} \qquad (1)$$

for some number $\delta_n > 0$.

Lemma

For each $x_0 \in \overline{\Omega}$ there exists $r = r_{x_0} > 0$ and $C = C_{x_0} > 0$ such that for all $x, y \in \overline{\Omega} \cap B_r(x_0)$ subject to (1) one has

$$G_\Omega(x, y) \geq C|x - y|^{-n} d(x)^2 d(y)^2. \qquad (2)$$

Sketch of proof, I.

Here only generic case $n > 4$; $n = 2, 3, 4$ are more involved.
Need to consider only $x_0 \in \partial\Omega$. Achieve $x_0 = 0$, $\vec{e}_1$ exterior unit normal.
Assume by contradiction:
Find sequences $x_k, y_k \in \Omega \cap B_{1/k}(0)$ with $x_k \neq y_k$,

$$G_{\Omega,\mathrm{CPE}}(x_k, y_k) < \frac{1}{k}|x_k - y_k|^{-n} d(x_k)^2 d(y_k)^2.$$

Have by assumption:

$$|x_k - y_k| > \delta_n \max\{d(x_k), d(y_k)\}$$

and $0 = \lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k \in \partial\Omega$.

Sketch of proof, II.

$\tilde{x}_k \in \partial\Omega$: closest boundary point to x_k.
We introduce the rescaled biharmonic Green's functions,
$\tilde{x}_k \to$ origin, $|x_k - y_k| \to$ length unit.

$$G_k(\xi, \eta) := |x_k - y_k|^{n-4} G_{\Omega,\mathrm{CPE}}(\tilde{x}_k + |x_k - y_k|\xi, \tilde{x}_k + |x_k - y_k|\eta)$$

for

$$\xi, \eta \in \Omega_k := \frac{1}{|x_k - y_k|}\left(-\tilde{x}_k + \Omega\right).$$

$\tilde{x}_k \to 0$, normal in 0 at $\partial\Omega_k$ becomes vertical,

$$\Omega_k \to \mathcal{H} := \{x : x_1 < 0\} \text{ locally uniformly for } \ k \to \infty.$$

Sketch of proof, III.

Compactness for $(G_k)_{k\in\mathbb{N}}$ by Krasovskiĭ's estimates:

$$|G_{\Omega,\mathrm{CPE}}(x,y)| \le C_0|x-y|^{4-n}.$$

These estimates yield for any $\xi,\eta\in\mathcal{H}$ and k large enough

$$|G_k(\xi,\eta)| \le C_0|\xi-\eta|^{4-n} \text{ uniformly in } k.$$

Elliptic estimates: locally uniformly in $\mathcal{H}\times\mathcal{H}$

$$G_k(\xi,\eta)\to G_{\mathcal{H},\Delta^2}(\xi,\eta)=\frac{1}{4ne_n}|\xi-\eta|^{4-n}\int_1^{|\xi^*-\eta|/|\xi-\eta|}(v^2-1)v^{1-n}\,dv,$$

where $\xi^*=(-\xi_1,\xi_2,\dots,\xi_n)$.

Sketch of proof, IV

Consider

$$\xi_k = \frac{1}{|x_k - y_k|}(x_k - \tilde{x}_k), \quad \eta_k = \frac{1}{|x_k - y_k|}(y_k - \tilde{x}_k),$$

so that by assumption

$$G_k(\xi_k, \eta_k) = |x_k - y_k|^{n-4} G_\Omega(x_k, y_k) < \frac{1}{k}|x_k - y_k|^{-4} d(x_k)^2 d(y_k)^2. \tag{3}$$

Assumption $|x_k - y_k| \geq \delta_n \max\{d(x_k), d(y_k)\}$ gives

$$|\xi_k| = \frac{d(x_k)}{|x_k - y_k|} \leq \frac{1}{\delta_n}, \quad |\xi_k - \eta_k| = 1.$$

Select a subsequence:
$\xi, \eta \in \overline{\mathcal{H}}$ with $\xi = \lim_{k\to\infty} \xi_k$, $\eta = \lim_{k\to\infty} \eta_k$.

Sketch of proof, V. Conclusion.

In view of $|\xi - \eta| = 1$ and the convergence of $G_k \to G_{\mathcal{H}}$, Boggio's explicit formula for $G_{\mathcal{H}}$ yields existence of a positive constant $\sigma > 0$ such that for k large enough:

$$\begin{aligned} G_k(\xi_k, \eta_k) &\geq \sigma \operatorname{dist}(\xi_k, \partial\Omega_k)^2 \operatorname{dist}(\eta_k, \partial\Omega_k)^2 \\ &= \sigma \left(\frac{d(x_k)}{|x_k - y_k|} \right)^2 \left(\frac{d(y_k)}{|x_k - y_k|} \right)^2 \\ &= \sigma |x_k - y_k|^{-4} d(x_k)^2 d(y_k)^2. \end{aligned}$$

This contradicts (3).

Further remarks

- Proof is concluded by compactness argument.
- This and having used an argument by contradiction shows that the constants in

$$\frac{1}{C_2} H_\Omega(x,y) \le G_{\Omega,\mathrm{CPE}}(x,y) + C_1 d(x)^2 d(y)^2 \le C_2 H_\Omega(x,y)$$

  are not explicit.
- Adding uniformly bounded lower order perturbations of (CPE) doesn't change the result.
- The negative part of the Green functions for $\Delta^2 + \lambda$ with $\lambda \to \infty$ is not uniformly bounded.
- Applicability to really nonlinear problems??
- However, this "almost positivity" seems to be a characteristic feature also for nonlinear problems: numerical evidence.

Peculiarities of $n = 2$, i.e. $<$ order or equation -1

- Here, the case $|x - y| \leq \delta_n \max\{d(x), d(y)\}$, x, y close to the boundary is not simpler and requires a different blow up.
  No whole space blow-up!
- But $\infty > G_{\Omega,\mathrm{CPE}}(x, x) > 0$.
- Half space Green function , $n = 2$:

$$
\begin{aligned}
& G_{\mathcal{H},\mathrm{CPE}}(x, y) \\
&= \frac{1}{16\pi} \left(|x - y|^2 \log |x - y|^2 - |x - y|^2 \log |x^* - y|^2 \right. \\
&\quad \left. + |x^* - y|^2 - |x - y|^2 \right) \\
&\leq C(1 + |x|^2 + |y|^2)(1 + \log(2 + |x|) + \log(2 + |y|)).
\end{aligned}
$$

- Thanks to its symmetry this growth condition ensures uniqueness of $G_{\mathcal{H},\mathrm{CPE}}$.
- Term $c(y)x_1^2$ would result in a term $c\, x_1^2 y_1^2$.