

Critical dimensions and higher order Sobolev inequalities with remainder terms*

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Abstract

Pucci and Serrin [21] conjecture that certain space dimensions behave “critically” in a semilinear polyharmonic eigenvalue problem. Up to now only a considerably weakened version of this conjecture could be shown. We prove that exactly in these dimensions an embedding inequality for higher order Sobolev spaces on bounded domains with an optimal embedding constant may be improved by adding a “linear” remainder term, thereby giving further evidence to the conjecture of Pucci and Serrin from a functional analytic point of view. Thanks to Brezis-Lieb [5] this result is already known for the space H_0^1 in dimension $n = 3$; we extend it to the spaces H_0^K ($K > 1$) in the “presumably” critical dimensions. Crucial tools are positivity results and a decomposition method with respect to dual cones.

1 Sharp higher order Sobolev inequalities and critical dimensions

In a celebrated paper, Pucci-Serrin [21] studied the following critical growth problem for polyharmonic operators

$$\begin{cases} (-\Delta)^K u = \lambda u + |u|^{K^*-2}u & \text{in } \Omega \\ D^k u = 0 & \text{on } \partial\Omega \quad k = 0, \dots, K-1 \end{cases} \quad (1)$$

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where $K \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ ($n \geq 2K + 1$) is an open bounded domain with smooth boundary and $K_* = \frac{2n}{n-2K}$ is the critical Sobolev exponent (the largest q for which the embedding $H_0^K(\Omega) \subset L^q(\Omega)$ is continuous); with this notation $1_* = 2^* = \frac{2n}{n-2}$. Further, $D^k u$ denotes any derivative of u of order k . Pucci-Serrin were interested in existence of nontrivial radial solutions of (1) when $\Omega = B_1$, the unit ball in \mathbb{R}^n ; their interest was motivated by previous results from the famous paper by Brezis-Nirenberg [6] who studied the corresponding problem in the case $K = 1$ and showed a striking difference between the space dimension $n = 3$ and the higher dimensions $n \geq 4$. More precisely, if λ_1 denotes the first eigenvalue of $-\Delta$ in B_1 , it is shown in [6] that (1) admits a positive radial solution if and only if $\lambda \in (0, \lambda_1)$ when $n \geq 4$, while it admits a positive radial solution if and only if $\lambda \in (\lambda_1/4, \lambda_1)$ when $n = 3$. Thus, one interesting problem concerning (1) is to determine the constant $\lambda^* \geq 0$ for which the relation $\lambda > \lambda^*$ is a necessary condition for the existence of a nontrivial radial solution of (1) in B_1 . Pucci-Serrin [21] call *critical dimensions* the space dimensions for which $\lambda^* > 0$ and conjecture, for a given $K \geq 1$, that the critical dimensions are $n = 2K + 1, \dots, 4K - 1$: by the results in [6] this conjecture was already proved for $K = 1$, next it has been proved for $K = 2, 3, 4$ (see [3, 8, 13, 21, 22]); moreover, it is known that for all $K \geq 1$ the dimensions $n \geq 4K$ are not critical, see [14]. Some time later, Grunau [15] defined *weakly critical* dimensions as the space dimensions for which a necessary condition for the existence of a *positive* radial solution of (1) in B_1 is $\lambda \in (\lambda^*, \lambda_1)$ for some $\lambda^* > 0$: he showed that weakly critical dimensions are precisely $n = 2K + 1, \dots, 4K - 1$. So, at least a weakened version of the Pucci-Serrin conjecture could be shown, a full proof of their original conjecture will be considerably more difficult, if possible at all. It remains to understand better the role of these dimensions.

The semilinear eigenvalue problem (1) also has an intrinsic interest in more general domains. By means of a generalized Pohožaev identity, Pucci-Serrin [20] proved that (1) only admits the trivial solution $u \equiv 0$ whenever $\lambda < 0$ and Ω is star-shaped. Moreover, for any bounded smooth domain Ω , it is shown in [8, 12, 14, 22] that nontrivial solutions exist under different assumptions in the cases $n \geq 4K$ and $n < 4K$. This shows that critical dimensions may not only be a phenomenon of radial symmetry in balls (cf. also [6, Theorem 1.2"]); some attempts [16, 18]

were made in order to explain this phenomenon by means of the local summability properties of the fundamental solution of the operators involved.

Motivated by the non-existence results in [6], Brezis-Lieb [5] proved the following sharp Sobolev inequality in bounded open sets for the space H_0^1 :

$$\|f\|_1^2 \geq S|f|_{1*}^2 + C|f|_{\frac{n}{n-2},w}^2 \quad \forall f \in H_0^1(\Omega) . \quad (2)$$

Here $\|f\|_1^2 := \int_{\Omega} |\nabla f|^2 dx$ is the usual H_0^1 -norm, $|\cdot|_q$ denotes the $L^q(\Omega)$ -norm, $q \in [1, \infty]$,

$$S := \inf_{f \in H_0^1 \setminus \{0\}} \frac{\|f\|_1^2}{|f|_{1*}^2}$$

is the optimal Sobolev constant and the positive constant $C = C(\Omega)$ depends on (the diameter of) the set Ω . Further

$$|f|_{q,w} = \sup_{\substack{A \subseteq \Omega \\ |A| > 0}} |A|^{-1/q'} \int_A |f|$$

denotes the standard weak L^q -norm, where q' is the conjugate of $q \in (1, \infty)$.

In what follows we prove an inequality similar to (2) for the higher order Hilbertian Sobolev spaces H_0^K ($K \geq 1$).

We endow $H_0^K(\Omega)$ with the scalar product

$$(f, g)_K = \begin{cases} \int_{\Omega} (\Delta^M f)(\Delta^M g) & \text{if } K = 2M \\ \int_{\Omega} (\nabla \Delta^M f)(\nabla \Delta^M g) & \text{if } K = 2M + 1 \end{cases}$$

and we denote by $\|\cdot\|_K$ the corresponding norm. Consider the best constant of the Sobolev embedding inequality, namely

$$S_{n,K} = \inf_{f \in H_0^K \setminus \{0\}} \frac{\|f\|_K^2}{|f|_{K*}^2}$$

see e.g. [22, 23]; of course S depends on n and on K : however, since no confusion arises, we will simply denote it by S . It is well-known that S is only attained when Ω is (almost) the entire space \mathbb{R}^n .

We have the following generalization of the Brezis-Lieb inequality (2) to the spaces H_0^K :

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $K \geq 1$; then, there exists a constant $C = C(\Omega, K) > 0$ such that*

$$\|f\|_K^2 \geq S|f|_{K^*}^2 + C|f|_{\frac{n}{n-2K}, w}^2 \quad \forall f \in H_0^K(\Omega). \quad (3)$$

This inequality is closely related to critical dimensions by means of the following principle: *on bounded open sets, the Sobolev inequality for the embedding $H_0^K \subset L^{K^*}$ with optimal constant S may be improved by adding an L^2 -norm if and only if the space dimension is weakly critical.* As this theorem does not require any positivity assumption on f , we think that it gives further support to the conjecture of Pucci and Serrin, according to which the dimensions $n = 2K + 1, \dots, 4K - 1$ are critical for (1) not only in the weak sense.

The proof of (3) for positive functions f may be obtained by adjusting the proof in [5], see Section 2; the difficulty arises when we wish to prove it for any f (possibly changing sign) because we can no longer replace f with $|f|$. Instead, we decompose $f = f_1 + f_2$ with respect to the cone of nonnegative functions \mathcal{K} and its dual cone $\mathcal{K}' \subset H_0^K$, the cone of weak subsolutions of the Dirichlet problem for the polyharmonic operator $(-\Delta)^K$: $f_1 \in \mathcal{K}$, $f_2 \in \mathcal{K}'$. In those domains where a comparison for polyharmonic Dirichlet problems holds (e.g. in balls, which is sufficient here), one can even conclude that f_2 has a sign: $f_2 \leq 0$, see Lemma 2 below. This allows us to prove (3) for all $f \in H_0^K(\Omega)$ in Section 3 where we also give further applications of this decomposition method, see Remark 1.

Similar phenomena appear for the quasilinear degenerate elliptic equation

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + |u|^{p^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (4)$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $n > p > 1$ and $p^* = \frac{np}{n-p}$. We refer to [1] for the variational characterization of the first generalized eigenvalue of $-\Delta_p$ and for the proof that it is simple and isolated. In this case, it is known [11] that (4) admits a positive solution for all $\lambda \in (0, \lambda_1)$ whenever $n \geq p^2$ and for any bounded smooth domain Ω : positive solutions of (4) in B_1 are necessarily radial in view of [7]; moreover, extending in an obvious way the definition of Pucci-Serrin, the critical

dimensions are known to be $p < n < p^2$, see [9]. For further results with more general lower order terms see also [2].

In [10] one can find the following result: for any bounded open set $\Omega \subset \mathbb{R}^n$ and for all $q \in [1, \frac{n(p-1)}{n-p})$ there exists a constant $C_q = C_q(n, q, \Omega) > 0$ such that

$$\|f\|_{W_0^{1,p}}^p \geq S|f|_{p^*}^p + C_q|f|_q^p \quad \forall f \in W_0^{1,p}(\Omega); \quad (5)$$

moreover, $C_q \rightarrow 0$ as $q \rightarrow \frac{n(p-1)}{n-p}$. The optimal Sobolev constant here is defined analogously by

$$S = S_{n,p} = \inf_{f \in W_0^{1,p} \setminus \{0\}} \frac{\|f\|_{W_0^{1,p}}^p}{|f|_{p^*}^p}.$$

Then again the inequality (5) is related to the critical dimensions by means of the following principle: the remainder term in the r.h.s. of (5) contains the corresponding “linear” norm L^p if and only if the dimension n is critical.

2 Proof of (3) for positive functions

Let Ω be as in the statement of Theorem 1; then there exists $R > 0$ such that $\Omega \subseteq B_R$ where B_R denotes the ball centered at the origin with radius R . If we extend by 0 in $B_R \setminus \Omega$ any function $f \in H_0^K(\Omega)$ then we see that $H_0^K(\Omega) \subseteq H_0^K(B_R)$: therefore, it suffices to prove Theorem 1 in the case where $\Omega = B_R$ for some $R > 0$.

Consider the convex closed cone of positive functions

$$\mathcal{K} = \{v \in H_0^K(B_R); v(x) \geq 0 \text{ for a.e. } x \in B_R\};$$

we first prove inequality (3) in the case of positive functions f in B_R : this may be obtained by slight modifications of the arguments in [5].

Lemma 1 *There exists a constant $\bar{C} = \bar{C}(R) > 0$ such that*

$$\|f\|_K^2 \geq S|f|_{K^*}^2 + \bar{C}|f|_{\frac{n}{n-2K}, w}^2 \quad \forall f \in \mathcal{K}. \quad (6)$$

Proof. Let $g \in L^\infty(B_R)$, $g \not\equiv 0$ and $g \geq 0$ a.e., and let u be a solution of the problem

$$\begin{cases} (-\Delta)^K u = g & \text{in } B_R \\ D^k u = 0 & \text{on } \partial B_R \quad k = 0, \dots, K-1; \end{cases}$$

then, $u \in H_0^K \cap L^\infty(B_R)$. Take $f \in \mathcal{K} \setminus \{0\}$ and consider the (entire) function

$$\phi = \begin{cases} f - u + |u|_\infty & \text{in } B_R \\ |u|_\infty P_K(\frac{R}{|x|}) & \text{in } B_R^c \end{cases}$$

where P_K is the polynomial of degree $n - K - 1$ whose lowest power is $n - 2K$ and satisfying

$$P_K(1) = 1 \quad P_K^{(k)}(1) = 0 \quad k = 1, \dots, K - 1 ;$$

for instance, if $K = 2$, then $P_2(s) = (n - 3)s^{n-4} - (n - 4)s^{n-3}$.

Note that $\phi \in \mathcal{D}^{K,2}(\mathbb{R}^n)$: hence we may write the Sobolev inequality for ϕ on \mathbb{R}^n and obtain (for some $C = C(n, R) > 0$)

$$\|f - u\|_K^2 + C|u|_\infty^2 \geq S|f|_{K_*}^2$$

where we used the fact that both $f \geq 0$ and $-u + |u|_\infty \geq 0$ a.e. in B_R . Therefore, we obtain

$$\|f\|_K^2 + \|u\|_K^2 - 2(f, u)_K + C|u|_\infty^2 \geq S|f|_{K_*}^2$$

and hence

$$\|f\|_K^2 \geq 2 \int_{B_R} fg - (\|u\|_K^2 + C|u|_\infty^2) + S|f|_{K_*}^2 ;$$

replacing g, u with $\lambda g, \lambda u$ the previous inequality remains true for all $\lambda > 0$: then, taking into account that for all $a \geq 0$ and $b > 0$ we have $\max_{\lambda \geq 0} (a\lambda - b\lambda^2) = \frac{a^2}{4b}$, we obtain

$$\|f\|_K^2 \geq S|f|_{K_*}^2 + \frac{(\int_{B_R} fg)^2}{\|u\|_K^2 + C|u|_\infty^2} . \quad (7)$$

Let $A \subseteq B_R$, $|A| > 0$ and take $g = \mathbf{1}_A$ (the characteristic function of A) in (7); then, we obviously have

$$\int_{B_R} fg = \int_A f . \quad (8)$$

Moreover, if $\gamma = \sqrt{\frac{n}{\omega_n}|A|}$, $\omega_n = \int_{\partial B_1} d\omega(x)$ and Γ_R denotes the Green function corresponding to $(-\Delta)^K$ in B_R , then for a.e. $x \in B_R$ we have by Boggio's explicit formula [4, p. 126] for Γ_R

$$|u(x)| = \left| \int_{B_R} \Gamma_R(x, y) \cdot \mathbf{1}_A(y) dy \right| \leq C(n, K) \int_A |x - y|^{2K-n} dy$$

$$\begin{aligned}
&\leq C(n, K) \int_{B_\gamma(x)} |x - y|^{2K-n} dy \\
&\leq C(n, K) \int_0^\gamma \rho^{n-1} \rho^{2K-n} d\rho = C(n, K) |A|^{2K/n}.
\end{aligned}$$

This proves that

$$|u|_\infty \leq C(n, K) |A|^{2K/n}. \quad (9)$$

Finally, by Hölder and Sobolev inequalities, we have

$$\|u\|_K^2 = (u, u)_K = \int_A u \leq |u|_{K^*} |A|^{(n+2K)/2n} \leq C(n, K) \|u\|_K |A|^{(n+2K)/2n};$$

therefore,

$$\|u\|_K^2 \leq C(n, K) |A|^{1+2K/n}. \quad (10)$$

Since $|A| \leq |B_R| \leq C_n R^n$, from (9) and (10) we get

$$\|u\|_K^2 + C|u|_\infty^2 \leq C(n, K, R) |A|^{4K/n},$$

which, together with (8), may be replaced into (7) to obtain (6) from the arbitrariness of A . \square

3 The decomposition method

Consider now the dual cone of \mathcal{K} , namely

$$\mathcal{K}' = \{w \in H_0^K(B_R); (w, v)_K \leq 0 \forall v \in \mathcal{K}\};$$

in order to prove that (3) holds for any $f \in H_0^K(B_R)$ we first show that \mathcal{K}' is contained in the cone of negative functions:

Lemma 2 *If $w \in \mathcal{K}'$, then $w(x) \leq 0$ for a.e. $x \in B_R$.*

Proof. Let $w \in \mathcal{K}'$; take an arbitrary function $h \in C_0^\infty(B_R) \cap \mathcal{K}$ and consider the problem

$$\begin{cases} (-\Delta)^K v = h & \text{in } B_R \\ D^k v = 0 & \text{on } \partial B_R \quad k = 0, \dots, K-1; \end{cases} \quad (11)$$

by the comparison principle (positivity of Green's function for the Dirichlet problem (11) in balls, cf. [4, p. 126]) we know that $v \in \mathcal{K}$ and therefore

$$0 \geq (w, v)_K = \int_{B_R} w \cdot (-\Delta)^K v = \int_{B_R} wh.$$

We have so proved that $\int wh \leq 0$ for all $h \in C_0^\infty(B_R) \cap \mathcal{K}$; by density we obtain that $\int wh \leq 0$ for all $h \in L^2(B_R)$ such that $h(x) \geq 0$ a.e. and this yields $w(x) \leq 0$ a.e. \square

Now take any function $f \in H_0^K(B_R)$; by the Proposition in [19] we infer that there exists a unique couple $(f_1, f_2) \in \mathcal{K} \times \mathcal{K}'$ such that $f = f_1 + f_2$ and $(f_1, f_2)_K = 0$: then

$$\|f\|_K^2 = \|f_1\|_K^2 + \|f_2\|_K^2. \quad (12)$$

Since $f_1(x) \geq 0$ and $f_2(x) \leq 0$ (by Lemma 2) for a.e. $x \in B_R$ we have $|f_1(x) + f_2(x)| \leq \max\{|f_1(x)|, |f_2(x)|\}$ and

$$|f_1(x) + f_2(x)|^r \leq \max\{|f_1(x)|^r, |f_2(x)|^r\} \leq |f_1(x)|^r + |f_2(x)|^r$$

for all $r > 0$; furthermore, if $r \geq 2$ we obtain

$$|f|_r^2 = \left(\int_{B_R} |f_1 + f_2|^r \right)^{2/r} \leq \left(\int_{B_R} |f_1|^r + \int_{B_R} |f_2|^r \right)^{2/r} \leq |f_1|_r^2 + |f_2|_r^2. \quad (13)$$

By Lemma 1 we know that (6) holds for f_1 and f_2 : therefore, by (12) and (13), we get

$$\begin{aligned} \|f\|_K^2 &= \|f_1\|_K^2 + \|f_2\|_K^2 \geq S(|f_1|_{K_*}^2 + |f_2|_{K_*}^2) + \bar{C}(|f_1|_{\frac{n}{n-2K}, w}^2 + |f_2|_{\frac{n}{n-2K}, w}^2) \\ &\geq S|f|_{K_*}^2 + C|f|_{\frac{n}{n-2K}, w}^2 \end{aligned}$$

with $C = \bar{C}/2$. This proves (3) for all $f \in H_0^K(B_R)$. \square

Remark 1 When dealing with weak solutions of second order elliptic equations one has to work in H_0^1 . Here one usually decomposes $f = f^+ + f^-$, $|f| = f^+ - f^-$, and $f^+, f^-, |f|$ are again functions in H_0^1 . This simple trick and its refinements (truncation on level sets) may be viewed as basis for the Stampacchia maximum principle and even for the de Giorgi-Nash-Moser theory. As mentioned above this decomposition is no longer admissible in higher order Sobolev spaces H_0^K , $K \geq 2$.

In our opinion this is the reason that only little is known about nonlinear higher order equations compared with second order equations.

The decomposition method of Moreau together with the comparison principle of Boggio may in some situations in H_0^K ($K \geq 2$) substitute the decomposition in positive and negative part just mentioned. A first application to a fourth order eigenvalue problem can e.g. be found in [17].

Here we want to present another example. In the previous paper [14], Grunau constructs a positive solution to the semilinear Dirichlet problem (1) in balls B , but he leaves open the question whether this solution may be found (up to scaling) as minimizer of the variational problem

$$S_\lambda = \min_{v \in H_0^K(B) \setminus \{0\}} \frac{\|v\|_K^2 - \lambda|v|_2^2}{|v|_{K^*}^2}$$

where $\lambda \in (0, \lambda_1)$. Now it is easy to show that, at least in a ball, any minimizer (for existence see also [14]) of this problem must be positive. Indeed, assume that u is a minimizer and $u \not\geq 0$: then we decompose as above $u = u_1 + u_2$, $u_1 \in \mathcal{K}$, $u_2 \in \mathcal{K}'$. Again by Lemma 2 we conclude $u_2 \leq 0$, $u_2 \not\equiv 0$. So replacing u with the positive function $u_1 - u_2$ would strictly increase the L^2 -norm in the numerator and the L^{K^*} -norm in the denominator. By orthogonality we have $\|u_1 + u_2\|_K^2 = \|u_1 - u_2\|_K^2$. As $\lambda > 0$ the fraction would strictly decrease, a contradiction!

An even simpler reasoning gives besides the positivity of the first eigenfunction also the simplicity of the first Dirichlet eigenvalue of $(-\Delta)^K$ in balls without referring to Krein-Rutman's or Jentzsch's theorem.

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