Some fourth order differential equations related to differential geometry

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- Rough approximation for bending energy of a thin elastic plate under orthogonal load f

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$\longrightarrow$ Linear plate equation

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- Differential geometry: Looking for conformal metrics with certain curvature properties

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- More realistic measure for bending energy of a thin elastic plate under orthogonal load f

$$\int_{\text{graph }[u]} \left((H[u])^2 - \frac{f\,u}{\sqrt{1+|\nabla u|^2}}\right)\,dS$$

$\longrightarrow$ Willmore equation

Linear plate equation

Given $\Omega \subset \mathbb{R}^n$, $f : \overline{\Omega} \to \mathbb{R}$
look for $u : \overline{\Omega} \to \mathbb{R}$ as solution of the
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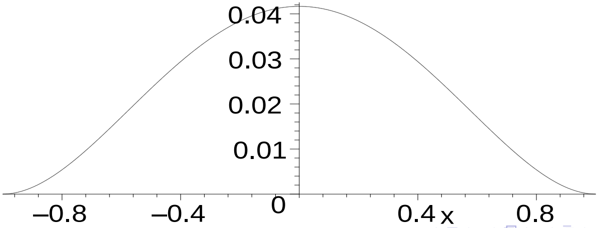
plus boundary conditions: Dirichlet. Clamped plate.

$$u = |\nabla u| = 0 \text{ on } \partial\Omega.$$

Example: $f \equiv 1$.



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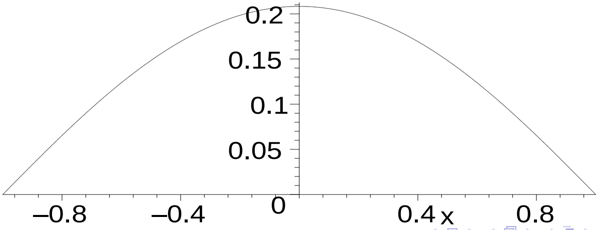

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elliptic differential inequalities:
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Fundamental for nonlinear problems.

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Remark.
Linear existence and regularity theory: o.k. since 1959.

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$$P_4^n := \Delta^2 + \sum_{i,j=1}^{n} \nabla_i \big(a_n R \delta_{ij} - b_n R_{ij} \big) \nabla_j + \frac{n-4}{2} Q_4^n$$

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Paneitz equation: $P_4^n u = \dfrac{n-4}{2} Q u^{\frac{n+4}{n-4}}.$

Q curvature

Consider here four dimensional manifold.
Gauß–Bonnet–formula:

$$\int_{\mathcal{M}} \left(Q + \frac{1}{8}|W|^2 \right) \, dS = 4\pi^2 \chi(\mathcal{M}).$$

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Existence of conformal metrics with constant Q–curvature.
Existence of conformal metrics with prescribed Q–curvature.

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Two dimensional

$$\Delta_{\mathcal{M}} H + 2H(H^2 - K) = 0 \quad \text{on } \mathcal{M}.$$

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Boundary value problems???

One dimensional, in what follows:

Look for $u : [0,1] \to \mathbb{R}$, solution of

$$\frac{1}{\sqrt{1+u'(x)^2}} \frac{d}{dx} \left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}} \right) + \frac{1}{2}\kappa^3(x) = 0, \quad x \in (0,1),$$

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Boundary value problems: Navier.

$$u(0)=u(1)=0, \quad \kappa(0)=-\alpha_0,\ \kappa(1)=-\alpha_1.$$

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Boundary value problems: Dirichlet.

$$u(0) = u(1) = 0, \quad u'(0) = \beta_0,\ u'(1) = -\beta_1,$$

Comparison principles?

Dirichlet problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = |\nabla u| = 0 & \text{on } \partial\Omega. \end{cases}$$

Question: $f \geq 0 \Rightarrow u \geq 0$?
Boggio–Hadamard–conjecture.

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Equivalent: Positivity of the Green function
Boggio, 1905, (unit–) ball $B = B_1(0) \subset \mathbb{R}^n$

$$G_{B,\Delta^2}(x,y) = c_n \int_1^{\sqrt{1+\frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}}} (v^2-1)v^{1-n}\, dv > 0.$$

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General domains: Numerous counterexamples, 1949 ... 2000

Positivity for optimal solutions

Typical question: Is first eigenfunction

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Problem:

$$v \in H_0^2(B) \not\Rightarrow |v| \in H_0^2(B)!$$

Way out?

Moreau decomposition, abstract setting

Theorem

Let $\mathscr{H}$ be a Hilbert space, $\mathscr{K}$ a closed convex cone;

$$\mathscr{K}^* := \{h \in \mathscr{H};\ \forall g \in \mathscr{K}:\ (g,h) \leq 0\}$$

the dual. Then: $\forall f \in \mathscr{H}$ one has precisely one pair

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Proof. As in the classical projection theorem

$$\|f - f_1\|^2 = \min_{g \in \mathscr{K}} \|f - g\|^2.$$

Variational principle, parallelogram identity. □

Moreau decomposition, concrete setting in $H_0^2(B)$

Scalar product: $(g,h) := \int_B \Delta g \Delta h \, dx$

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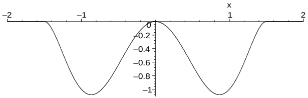

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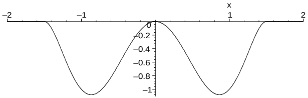

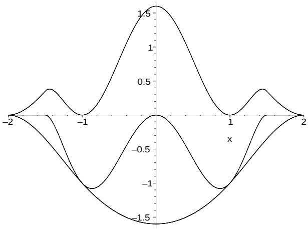

Eigenvalue problem for the clamped plate

Lemma

Let φ eigenfunction w.r.t. eigenvalue λ_1 of clamped plate over B:

$$\lambda_1 = \min_{v \in H_0^2(B)} \frac{\int_B (\Delta v)^2 \, dx}{\int_B v^2 \, dx} = \frac{\int_B (\Delta \varphi)^2 \, dx}{\int_B \varphi^2 \, dx}.$$

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Proof. Assume, φ changes sign.
Moreau decomposition: $\varphi = \varphi_1 + \varphi_2$,

$$\int_B \Delta\varphi_1 \Delta\varphi_2 \, dx, \quad 0 \not\equiv \varphi_1 \geq 0, \quad \varphi_2 < 0.$$

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a contradiction! □

Euclidean background metric

Let $\Omega \subset \mathbb{R}^n$, $n > 4$. Look for $u > 0$:

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plus suitable boundary conditions. Then:

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$$\mathscr{D}^2 (\mathbb{R}^n) \hookrightarrow L^{2n/(n-4)} (\mathbb{R}^n)\,; \qquad S := \inf \frac{\|\Delta u\|_{L^2}^2}{\|u\|_{L^{2n/(n-4)}}^2}.$$

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$$\mathscr{D}^2(\mathbb{R}^n) \hookrightarrow L^{2n/(n-4)}(\mathbb{R}^n); \qquad S := \inf \frac{\|\Delta u\|_{L^2}^2}{\|u\|_{L^{2n/(n-4)}}^2}.$$

Minima attained, are positive solutions of

$$\Delta^2 u = |u|^{8/(n-4)} u \text{ in } \mathbb{R}^n \tag{P}$$

Doubling of energy of sign changing solutions

Lemma

Let $u \in \mathscr{D}^2(\mathbb{R}^n)$ *be a sign changing solution of*

$$\Delta^2 u = |u|^{8/(n-4)} u \text{ in } \mathbb{R}^n. \tag{P}$$

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Proof. Similar as in the linear eigenvalue problem:

$$\mathscr{K} = \left\{ g \in \mathscr{D}^2(\mathbb{R}^n) : \ g \geq 0 \right\},$$

$$\mathscr{K}^* = \left\{ h \in \mathscr{D}^2(\mathbb{R}^n) : \quad \forall g \in \mathscr{K} : \quad \int_{\mathbb{R}^n} \Delta g \cdot \Delta h \, dx \leq 0 \quad \right\} \subset -\mathscr{K}.$$

Moreau decomposition

$$u = u_1 + u_2,\ u_1 \in \mathscr{K} \setminus \{0\},\ u_2 \in \mathscr{K}^* \setminus \{0\},\ \int_{\mathbb{R}^n} \Delta u_1 \cdot \Delta u_2 \, dx = 0.$$

Moreau decomposition

$u = u_1 + u_2$, $u_1 \in \mathscr{K} \setminus \{0\}$, $u_2 \in \mathscr{K}^* \setminus \{0\}$, $\int_{\mathbb{R}^n} \Delta u_1 \cdot \Delta u_2 \, dx = 0$.
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and since both $u_i \not\equiv 0$:

$$\|u_i\|^2_{L^{2n/(n-4)}} \ge S^{(n-4)/4}.$$

Previous slide:

$$\|u_i\|^2_{L^{2n/(n-4)}} \geq S^{(n-4)/4}.$$

Combining this once more with the PDE:

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&\geq \left(S^{n/4} + S^{n/4}\right)^{4/n} \geq 2^{4/n} S. \qquad \square
\end{aligned}$$

Navier boundary value problem

Differential equation:

$$\frac{1}{\sqrt{1+u'(x)^2}}\frac{d}{dx}\left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right)+\frac{1}{2}\kappa^3(x)=0, \quad x\in(0,1),$$

Navier boundary data:

$$u(0)=u(1)=0, \quad \kappa(0)=-\alpha,\ \kappa(1)=-\alpha.$$

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Observation, cf. also Euler:

$$v(x):=\kappa(x)\left(1+u'(x)^2\right)^{1/4}$$

solves ODE of second order without term of order zero::

$$-\left(a(x)v'(x)\right)'+b(x)v'(x)=0.$$

Symmetric solutions

Then $\kappa(x)\left(1+u'(x)^2\right)^{1/4} \equiv -c$.

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inverse function $G^{-1}:\left(-\frac{c_0}{2},\frac{c_0}{2}\right)\to\mathbb{R}$.

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inverse function $G^{-1}:\left(-\frac{c_0}{2},\frac{c_0}{2}\right)\to\mathbb{R}$.
For smooth symmetric solutions there exists $c\in(-c_0,c_0)$:

$$\forall x\in[0,1]:\qquad u'(x)=G^{-1}\left(\frac{c}{2}-cx\right).$$

$$\kappa(x)=-\frac{c}{\sqrt[4]{1+G^{-1}\left(\frac{c}{2}-cx\right)^2}}.$$

And $u(0)=u(1)=0$ gives:

$$u(x)=\frac{2}{c\sqrt[4]{1+G^{-1}\left(\frac{c}{2}-cx\right)^2}}-\frac{2}{c\sqrt[4]{1+G^{-1}\left(\frac{c}{2}\right)^2}}\quad (c\neq 0).$$

Attaining the boundary data

To solve:

$$\alpha \stackrel{!}{=} -\kappa_c(0) = \frac{c}{\sqrt[4]{1 + G^{-1}\left(\frac{c}{2}\right)^2}} =: h(c).$$

Image of h $=$ set of admissible boundary data for α.

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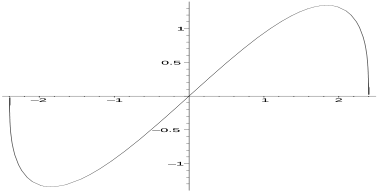

Figure: Admissible boundary data α, depending on the parameter c

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Theorem

One has $\alpha_{\text{max}} = 1.343799725\ldots$ such that for $0 < |\alpha| < \alpha_{\text{max}}$ the Navier boundary value problem has precisely two solutions among all smooth symmetric graphs.
For $|\alpha| = \alpha_{\text{max}}$: precisely one such solution.
For $\alpha = 0$ only the trivial solution.
For $|\alpha| > \alpha_{\text{max}}$: no such solution.

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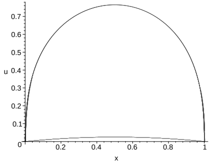

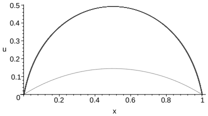

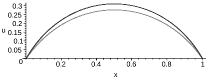

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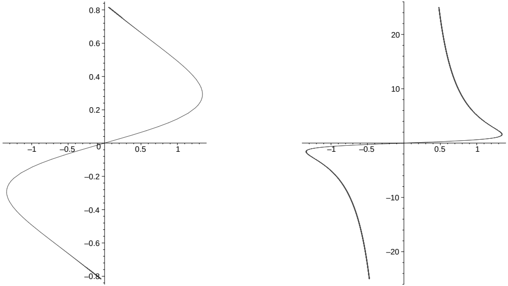

Figure: Bifurcation diagrams for the Navier problem

Navier boundary value problem

Differential equation:

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Idea: Reduce to symmetric case!

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Idea: Reduce to symmetric case!
Take large solution U_0 for $\alpha=0$,
extend oddly,
consider suitable parts, rotate, rescale.

Construction of nonsymmetric solutions

κ_0 curvature function of U_0,
choose intersection points $-1 < x_0 < x_1 < 1$

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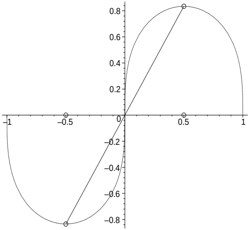

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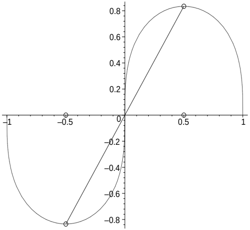

Consider U_0 as graph (check!!)
over line through $(x_0, U_0(x_0))$ and $x_1, U_0(x_1))$.

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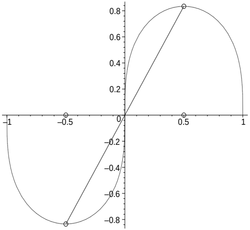

Length of connecting line:

$$L(x_0, x_1) := \sqrt{(x_1 - x_0)^2 + (U_0(x_1) - U_0(x_0))^2},$$

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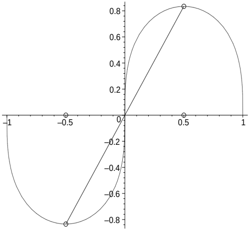
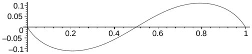


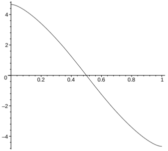
After rotation and rescaling: boundary curvature to be attained on $[0, 1]$:

$$L(x_0, x_1)\kappa_0(x_0), \qquad L(x_0, x_1)\kappa_0(x_1).$$

Examples

Intersection points $x_0 = -1/2, x_0 = 1/2$:





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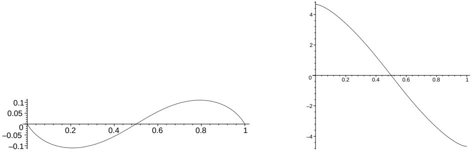
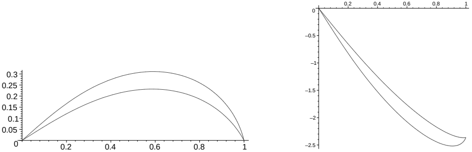


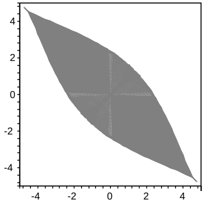
Intersection points $x_0 = 0,\ x_1 = 0.7$ und $x_0 = 0,\ x_1 = 0.53\ldots$:



Existence theorem

Theorem

Let $\mathcal{C} \subset \mathbb{R}^2$:



Then, for any $(\alpha_0, \alpha_1) \in \mathcal{C}$ there is a smooth graph solution of the Navier boundary value problem:

$$\begin{cases} \dfrac{1}{\sqrt{1+u'(x)^2}} \dfrac{d}{dx} \left(\dfrac{\kappa'(x)}{\sqrt{1+u'(x)^2}} \right) + \dfrac{1}{2}\kappa^3(x) = 0, \quad x \in (0,1), \\ u(0) = 0, \qquad u(1) = 0, \qquad \kappa(0) = -\alpha_0, \qquad \kappa(1) = -\alpha_1. \end{cases}$$