Boundary Value Problems for the Willmore Functional

Hans-Christoph Grunau

Otto–von–Guericke–Universität

Magdeburg

Marburg, November 25, 2015

Coauthors

- Anna Dall'Acqua (Ulm)
- Klaus Deckelnick (Magdeburg)
- Steffen Fröhlich (Mainz)
- Matthias Röger (Dortmund)
- Numerics: Friedhelm Schieweck (Magdeburg)

The Willmore functional

For sufficiently smooth (2D) surfaces $S \subset \mathbb{R}^3$ (with or without boundary) we define the Willmore functional

$$W(S) := \frac{1}{4} \int_S H^2 \, dS.$$

with mean curvature

$$H(x) := \kappa_1(x) + \kappa_2(x),$$

the sum of the principal curvatures.
Smooth critical points of the Willmore functional are called Willmore surfaces and solve the Willmore equation

$$\Delta_S H + 2H\Big(\frac{1}{4}H^2 - K\Big) = 0 \quad \text{on } S.$$

Quasilinear, fourth order, elliptic, but not uniformly.

Characteristic features: Conformal invariance & lack of comparison principle

- The Willmore functional: a cousin of the area functional?
  Of course, minimal surfaces (like catenoids) are Willmore surfaces. Some of what is explained in the last part is reminiscent of Giusti's (and other's) BV-approach to minimal graphs.
- However, there are characteristic differences:
- Conformal invariance: If $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ is conformal (i.e. a Möbius transformation), then $W(S) = W(\Phi(S))$. In particular: Scaling invariance.
- Compact Willmore surfaces without boundary exist: e.g. the sphere.
- Willmore functional involves second derivatives: No Stampacchia-tricks! No comparison principles!
- In general, no a-priori-estimates known for solutions of (the Dirichlet problem for) the Willmore equation.

Closed surfaces

Quite some interesting results are known about closed Willmore surfaces (i.e. embedded, without boundary).

- Gauß-Bonnet + elementary differential geometry: If $S \subset \mathbb{R}^3$ is a closed smooth surface, then
$$W(S) \geq 4\pi$$
with equality iff S a (round) sphere.
- Li-Yau inequality: If $S \subset \mathbb{R}^3$ is a closed smooth surface, $\Psi : S \to \mathbb{R}^3$ an immersion of "multiplicity" k, then
$$W(\Psi(S)) \geq k \cdot 4\pi.$$
- Mayer-Simonett, Kuwert-Schätzle: Willmore flow.
- Leon Simon, Bauer-Kuwert: For any $g \in \mathbb{N}$ we have a genus g Willmore minimiser.
- Marques-Neves (Willmore conjecture): If $g \geq 1$ then
$$W(S) \geq 2\pi^2$$
with equality iff S is the stereographic projection of the Clifford torus.

The Dirichlet problem

Now, consider immersions $S \subset \mathbb{R}^3$ with boundary $\partial S \neq \emptyset$.
Here much less is known:

- Find Willmore immersions S or, even better, Willmore minimisers S such that:
- ∂S is prescribed (fixed),
- the tangential planes along ∂S are prescribed (fixed) as well.

In this general setting only one existence result:

> *Reiner Schätzle, The Willmore boundary problem,* Calc. Var. Partial Differential Equations, **37**, *275–302, 2010.*

Parametric/GMT–approach: Branched Willmore minimisers in $\mathbb{S}^3 \cong \mathbb{R}^3 \cup \{\infty\}$!
In general, little geometric information. These solutions may not be embedded, will in general not be (global) graphs or may even contain ∞. No a-priori-bounds.

The Dirichlet problem

Our approach: Try to find solutions / minimisers where we have and are able to uncover more precise geometric information.

To this end we work in restricted classes of admissible (comparison) surfaces:

- Assuming invariance w.r.t. translations: Graphs over 1D intervals.
- Assuming axial symmetry: Surfaces of revolutions. Profile curves graphs over 1D intervals.
- Assuming projectability: Graphs over general 2D domains.

Minimisation procedure of the Willmore functional is discussed:

As degree of symmetry decreases, the difficulties become more serious and, as for the third topic, the results more restricted. As for Willmore graphs over general 2D domains our contribution has to be considered as a first of ?? steps.

Assuming invariance w.r.t. translations: 1D

Look for $u : [0,1] \to \mathbb{R}$, solution of

$$\frac{1}{\sqrt{1+u'(x)^2}} \frac{d}{dx} \left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}} \right) + \frac{1}{2}\kappa^3(x) = 0, \quad x \in (0,1),$$

here κ curvature of the unknown graph of u:

$$\kappa(x) = \frac{d}{dx} \left(\frac{u'(x)}{\sqrt{1+u'(x)^2}} \right) = \frac{u''(x)}{(1+u'(x)^2)^{3/2}}.$$

Dirichlet boundary conditions:

$$u(0) = u(1) = 0, \quad u'(0) = \beta_0,\ u'(1) = -\beta_1.$$

Nontrivial data for $u(0), u(1)$ are a nontrivial issue!
Solutions may not be graphs.

An auxiliary function, already known to Euler

Observation, cf. also Euler: If $u : [0, 1] \to \mathbb{R}$ solves the 1D-Willmore (elastica) equation:

$$\frac{1}{\sqrt{1 + u'(x)^2}} \frac{d}{dx} \left(\frac{\kappa'(x)}{\sqrt{1 + u'(x)^2}} \right) + \frac{1}{2} \kappa^3(x) = 0, \quad x \in (0, 1),$$

then

$$v(x) := \kappa(x) \left(1 + u'(x)^2\right)^{1/4}$$

solves an ODE of second order without term of order zero:

$$-\left(a(x) v'(x)\right)' + b(x) v'(x) = 0.$$

When seeking symmetric solutions w.r.t. $x = \frac{1}{2}$, i.e. $u(x) = u(1 - x)$, hence $v(x) = v(1 - x)$, then v constant by the maximum principle:

$$\kappa(x) \left(1 + u'(x)^2\right)^{1/4} \equiv -c.$$

Symmetric solutions

Assume for simplicity that $\beta = \beta_0 = \beta_1$.

For solving $\kappa(x)\left(1+u'(x)^2\right)^{1/4} \equiv -c$ explicitly, consider

$$G:\mathbb{R}\to\left(-\frac{c_0}{2},\,\frac{c_0}{2}\right), \qquad G(s):=\int_0^s \frac{1}{\left(1+\tau^2\right)^{5/4}}\,d\tau,$$

$c_0 = B(\frac{1}{2},\frac{3}{4}) = 2.396\ldots$, inverse function $G^{-1}:\left(-\frac{c_0}{2},\,\frac{c_0}{2}\right)\to\mathbb{R}$.

For smooth symmetric solutions there exists $c\in(-c_0,c_0)$:

$$\forall x\in[0,1]: \qquad u'(x) = G^{-1}\left(\frac{c}{2}-cx\right).$$

And $u(0)=u(1)=0$ gives:

$$u(x) = \frac{2}{c\sqrt[4]{1+G^{-1}\left(\frac{c}{2}-cx\right)^2}} - \frac{2}{c\sqrt[4]{1+G^{-1}\left(\frac{c}{2}\right)^2}} \quad (c\neq 0).$$

It remains to solve for $\beta \overset{!}{=} u'(0) = -u'(1) = G^{-1}\left(\frac{c}{2}\right)$.

Existence for the Dirichlet problem

Theorem

For any $\beta \in \mathbb{R}$ there exists a, in the class of graphs unique Willmore-energy minimising, smooth solution $u : [0,1] \to \mathbb{R}$ to the Dirichlet problem:

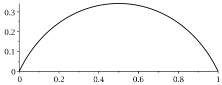
$$\frac{1}{\sqrt{1+u'(x)^2}}\frac{d}{dx}\left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right)+\frac{1}{2}\kappa^3(x)=0, \quad x \in (0,1),$$

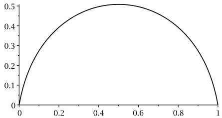
$$u(0)=u(1)=0, \quad u'(0)=-u'(1)=\beta.$$

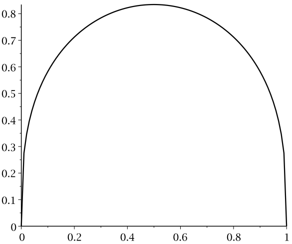
Remark

- No uniqueness of solutions to the boundary value problem in the class of *smooth curves*.
- Existence of a smooth graph solution for nonsymmetric data $u(0)=u(1)=0, \quad u'(0)=\beta_0,\ u'(1)=-\beta_1$.

Solutions to the symmetric Dirichlet problem







Remark. Observe that all symmetric solutions are uniformly bounded by $\frac{2}{c_0} = 0.8346\ldots$.

Existence, horizontal clamping

Surfaces of revolution S:

$$(x, \varphi) \mapsto (x, u(x)\cos\varphi, u(x)\sin\varphi),\ x \in [-1,1],\ \varphi \in [0, 2\pi],$$

with sufficiently smooth $u : [-1,1] \to (0, \infty)$.

Theorem

$\forall \alpha > 0\ \exists u \in C^\infty([-1,1],(0,\infty))$ *s.th.:*

$$\begin{cases} \Delta_S H + 2H(\frac{1}{4}H^2 - K) = 0 & \textit{in } (-1,1), \\ u(\pm 1) = \alpha, \qquad u'(\pm 1) = 0. \end{cases}$$

u is even, in this class energy minimising,
$\forall x \in [0,1] : u'(x) \le 0,\ 0 \le x + u(x)u'(x)$

$$\forall x \in [-1,1] : \quad \alpha \le u(x) \le \sqrt{\alpha^2 + 1 - x^2}, \quad |u'(x)| \le \frac{1}{\alpha}.$$

Background: Direct methods in the calculus of variations

- In order to minimise a functional $W : X \to \mathbb{R}$
- on a Banach space $(X, \|\,.\,\|_X)$,
- which is bounded from below, one tries to:
- Construct minimising sequences which are bounded w.r.t. to the norm $\|\,.\,\|_X$.
- Apply local compactness in order to achieve convergence in a weaker sense Y.
- Prove lower semicontinuity of $\|\,.\,\|_X$ and W w.r.t. Y-convergence.
- Achieve a minimum of W which obeys the same bound as the minimising sequence.

Geometric background

Mean curvature

$$H = -\frac{u''(x)}{(1+u'(x)^2)^{3/2}} + \frac{1}{u(x)\sqrt{1+u'(x)^2}};$$

Willmore functional on surface of revolution

$$\begin{aligned} W(S) &= \frac{1}{4}\int_S H^2\, dS \\ &= \frac{\pi}{2}\int_{-1}^{1}\left(\frac{1}{u(x)\sqrt{1+u'(x)^2}} - \frac{u''(x)}{(1+u'(x)^2)^{3/2}}\right)^2 \\ &\qquad\qquad \cdot u(x)\sqrt{1+u'(x)^2}\, dx. \end{aligned}$$

Hyperbolic geometry

Hyperbolic half plane $\mathbb{R}^2_+ := \{(x,y) : y > 0\}$, equipped with the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

Hyperbolic curvature of $[-1,1] \ni x \mapsto (x, u(x)) \in \mathbb{R}^2_+$:

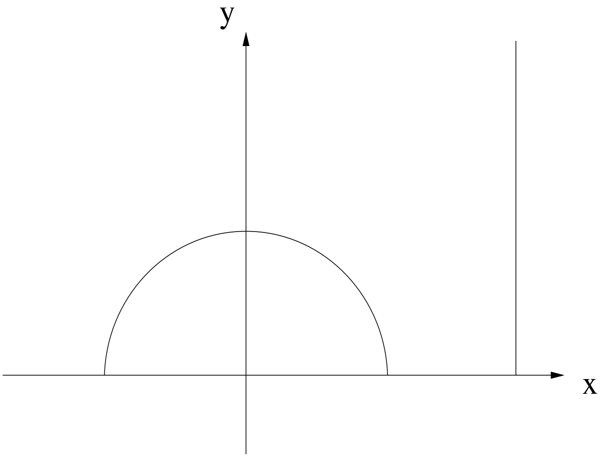
$$\kappa(x) = \frac{u(x)u''(x)}{(1+u'(x)^2)^{3/2}} + \frac{1}{\sqrt{1+u'(x)^2}}.$$

Hyperbolic Willmore functional

Bryant, Griffiths, Pinkall, Langer, Singer, ...

$$
\begin{aligned}
\hat{W}(u) &:= \int_{-1}^{1} \kappa(x)^2 \, ds(x) = \int_{-1}^{1} \kappa(x)^2 \frac{\sqrt{1+u'(x)^2}}{u(x)} \, dx \\
&= \int_{-1}^{1} \left(\frac{u''}{(1+u'^2)^{3/2}} - \frac{1}{u\sqrt{1+u'^2}} \right)^2 u\sqrt{1+u'^2} \, dx \\
&\quad +4 \int_{-1}^{1} \frac{u''(x)}{(1+u'(x)^2)^{3/2}} \, dx \\
&= \frac{2}{\pi} \int_S \frac{1}{4} H^2 \, dS + 4 \left[\frac{u'(x)}{\sqrt{1+u'(x)^2}} \right]_{-1}^{1}, \\
W(S) &= \frac{\pi}{2} \hat{W}(u) - 2\pi \left[\frac{u'(x)}{\sqrt{1+u'(x)^2}} \right]_{-1}^{1}.
\end{aligned}
$$

Hyperbolic geodesics



Minimising sequences

For $\alpha \in (0, \infty)$ we define:

$N_\alpha := \{v \in C^{1,1}([-1, 1]), v \text{ even, positive}, v(1) = \alpha, v'(1) = 0\}$

and

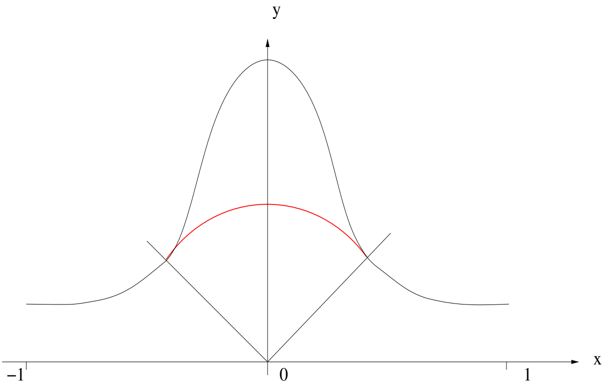
$$M_\alpha := \inf\{\hat{W}(v) : v \in N_\alpha\}.$$

For fixed α, consider minimising sequence

$$(v_{\alpha,k})_{k\in\mathbb{N}} \subset N_\alpha,$$

each $v_{\alpha,k}$ only finitely many critical points.

Condition $0 \leq x + v(x)v'(x)$ $(x \geq 0)$ on minimising sequence



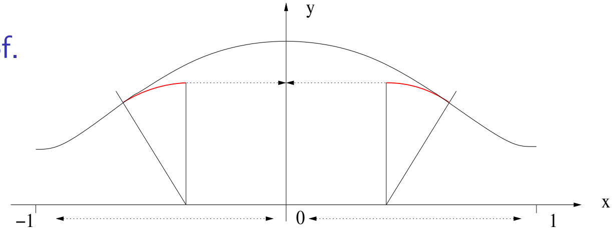
How to shorten the curve and to decrease the energy

Lemma

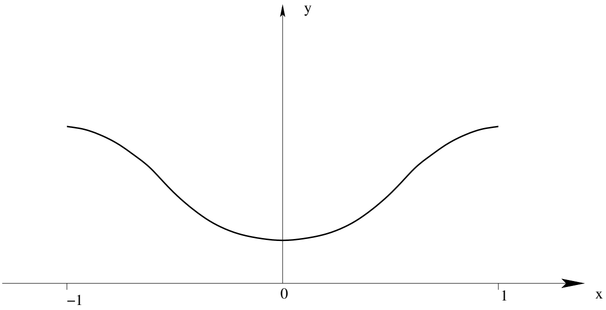
Assume $v \in N_\alpha$ has only finitely many critical points and that $v'(x) \leq 0$ for all $x \in [0,1]$. Then, $\forall \rho \in (0,1]$, there exists a positive symmetric function $v_\rho \in N_\alpha([-\rho,\rho])$ such that $v_\rho'(x) \leq 0$ for all $x \in [0,1]$, v_ρ at most as many critical points as v and

$$\int_{-\rho}^{\rho} \kappa[v_\rho]^2 \, ds[v_\rho] \leq \hat{W}(v).$$

Proof.

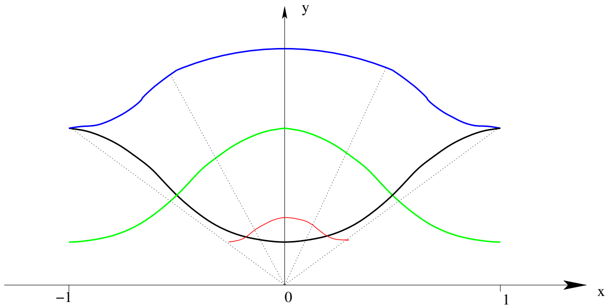


How to avoid



on minimising sequences?

Cut, interchange, shorten, scale



Conclusions

With these techniques:

Theorem

$$\alpha \leq \alpha' \Rightarrow M_\alpha \geq M_{\alpha'}.$$

Theorem

On minimising sequence $(v_{\alpha,k})_{k\in\mathbb{N}}$ we may achieve:

$$\forall x \in [0,1] : v'_{\alpha,k}(x) \leq 0, \quad x + v_{\alpha,k}(x) v'_{\alpha,k}(x) \geq 0.$$

Yields C^1- and weak H^2-compactness, a minimum, C^∞-smooth.

Numerically calculated solutions

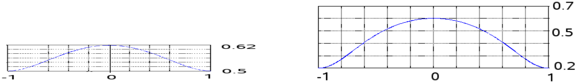


Figure: $\alpha = 0.5$ (left) and $\alpha = 0.2$ (right)

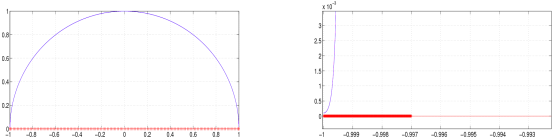


Figure: $\alpha = 0.0001$

The Willmore functional for 2D-graphs

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and $\varphi : \overline{\Omega} \to \mathbb{R}$ a smooth boundary datum.
We consider the minimisation of the Willmore functional

$$W(u) = \frac{1}{4} \int_\Omega H^2 \sqrt{1 + |\nabla u|^2}\, dx$$

for graphs $u : \overline{\Omega} \to \mathbb{R}$
with mean curvature

$$H = \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)$$

(sum of principal curvatures, w.r.t. upper unit normal)
subject to Dirichlet boundary conditions, i.e. in the class

$$\mathcal{M} := \{ u \in H^2(\Omega) : (u - \varphi) \in H^2_0(\Omega) \}.$$

Small data results: J.C.C. Nitsche.

Area and diameter bounds

Recall $\mathcal{M} = \{u \in H^2(\Omega) : (u - \varphi) \in H_0^2(\Omega)\}$.
First main step:

Theorem

There exists a constant $C = C(\Omega, \|\varphi\|_{W^{2,1}(\partial\Omega)})$ such that for any $u \in \mathcal{M}$ we have

$$\sup_{x\in\Omega} |u(x)| + \int_\Omega \sqrt{1 + |\nabla u(x)|^2}\, dx \le C\big(W(u)^2 + 1\big).$$

Key ingredients:

- Lemma by Leon Simon (1993), which estimates the diameter in terms of the boundary data and the L^1-norm (!) of the length of the second fundamental form.
- Integrate

$$\int_\Omega u\, H\, dx$$

by parts.

In general, no *better* estimates!

- Previous result requires to work with "smooth" graphs.
- In general, no $W^{1,p}$-estimates with $p > 1$ in terms of the Willmore energy.
- Think of a bowler hat, smooth as a surface, but with arbitrarily steep profile curve: $u \notin W^{1,p}(\Omega)$ but $W(u) < \infty$.
- This means that the area and diameter bounds determine $BV \cap L^\infty(\Omega)$ as solution class!
- $BV(\Omega)$ is a sort of a "weak-$*$-closure" of $W^{1,1}(\Omega)$.
- Willmore functional is not defined there.
- Consider there the L^1-lower semicontinuous envelope of the Willmore functional.
- What do these functionals have to do with each other?
- How are the boundary conditions then encoded?

A lower semicontinuity result in $\mathcal{M}$

Recall $\mathcal{M} = \{u \in H^2(\Omega) : (u - \varphi) \in H_0^2(\Omega)\}$.

Theorem

Let $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ be a given sequence that satisfies

$$\liminf_{k\to\infty} W(u_k) < \infty.$$

Then there exists a $u \in BV(\Omega) \cap L^\infty(\Omega)$ such that up to selecting a subsequence

$$u_k \to u \text{ in } L^1(\Omega) \quad (k \to \infty).$$

If in addition $u \in H^2(\Omega)$ then

$$W(u) \leq \liminf_{k\to\infty} W(u_k).$$

In particular, the Willmore functional is L^1-lower semicontinuous in $\mathcal{M}$: The second main step!
Proof uses the area and diameter bounds.

A lower semicontinuous extension of the Willmore functional

Similar approach already used by Ambrosio, Bellettini, Dal Maso & coworkers.

Recall $\mathcal{M} = \{u \in H^2(\Omega) : (u - \varphi) \in H_0^2(\Omega)\}$.

Definition (L^1-Lower semicontinuous envelope)

$$\overline{W} : L^1(\Omega) \to [0, \infty],$$
$$\overline{W}(u) := \inf\{\liminf_{k\to\infty} W(u_k) : \mathcal{M} \ni u_k \to u \text{ in } L^1(\Omega)\}.$$

A more explicit characterisation of $\overline{W}$ is often out of reach. The previous lower semicontinuity result, however, implies:

Theorem

For $u \in \mathcal{M}$ one has $\overline{W}(u) = W(u)$.

This means: The L^1-lower semicontinuous envelope $\overline{W}$ is indeed the largest possible L^1-*lower semicontinuous extension* of W to $L^1(\Omega)$.

Existence of a minimiser

Theorem

There exists a function $u \in BV(\Omega) \cap L^\infty(\Omega)$ such that

$$\forall v \in L^1(\Omega): \quad \overline{W}(u) \leq \overline{W}(v).$$

Attainment of the Dirichlet boundary conditions

- $u = \varphi$ in $L^1(\partial\Omega)$ in the sense of traces.
- Lebesgue decomposition $\nabla u = \nabla^s u + \nabla^a u$.
- $|\nabla^s u|(\partial\Omega) = 0$.
- $\nabla^a u$ has an approximately continuous representative which is well defined $\mathcal{H}^1$-almost everywhere on $\partial\Omega$ and coincides with $\nabla\varphi$.