OPTIMAL ESTIMATES FROM BELOW FOR GREEN FUNCTIONS OF HIGHER ORDER ELLIPTIC OPERATORS WITH VARIABLE LEADING COEFFICIENTS

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Abstract. Estimates from above *and below* by the same positive prototype function for suitably modified Green functions in bounded smooth domains under Dirichlet boundary conditions for elliptic operators L of higher order $2m \geq 4$ have been shown so long only, when the pricipal part of L is the polyharmonic operator $(-\Delta)^m$. In the present note it is shown that such kind of results still hold, when the Laplacian is replaced by any second order uniformly elliptic operator in divergence form with smooth variable coefficients. For general higher order elliptic operators, whose principal part cannot be written as a power of second order operators, it was recently proved that such kind of results become false in general.
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1. Introduction

In a series of works [3, 4, 6, 7, 9, 13] Dall'Acqua, Meister, Pulst, Robert, Sweers and the present author have studied the question whether (suitably modified) Green functions in bounded smooth domains $\Omega \subset \mathbb{R}^n$ under Dirichlet boundary conditions for elliptic operators L of higher order $2m \geq 4$ may be estimated from above *and below* by the same positive prototype function. Such kind of results may be considered as a kind of substitute or relaxation of the maximum principle which in its strong form is true only for second order operators. Employing perturbative and blow-up arguments, all these works are based on Boggio's explicit formula for the polyharmonic Green function under Dirichlet boundary conditions in balls (see [2, p. 126] and also [5, Lemma 2.27]) and similarly in half spaces (see [5, Remark 2.28]). This does not only show its positivity in these special domains but does also allow for precise estimates from above and below, see [5, Theorem 4.6] and Remark 2 (a) below.

In order to keep this note short and as simple as possible I shall consider only the (simplest) generic case

$$n > 2m.$$

The "small" dimensions $n = 2, \dots, 2m$ require more effort, but related results will hold also there. The most general class of operators, which could be treated so long, was considered by Pulst [13]. He could show that in general bounded smooth domains the estimate (2) below holds for operators of the form

$$L = (-\Delta)^m + \sum_{j=0}^{m-1} \sum_{|\alpha|=|\beta|=j} D^\alpha (a_{\alpha\beta}(\,.\,) D^\beta),$$

where the coefficients obey symmetry, smoothness and coercivity conditions.

The goal here is to show that the same estimates can be proved for a principal part with variable coefficients, as long as this stays in the class of powers of second order elliptic operators.

Theorem 1. *For $n, m \in \mathbb{N}$ with*

$$n > 2m$$

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let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞-smooth domain and L be a uniformly elliptic symmetric operator of the form

$$L = \left(-\sum_{i,j=1}^{n} \partial_i \left(a_{ij}(\,.\,)\partial_j \right) \right)^m .$$

For the coefficients we assume that

$$a_{ij}(\,.\,) = a_{ji}(\,.\,) \in C^\infty(\overline{\Omega})$$

and that there exist numbers

$$0 < \lambda \le \Lambda : \quad \forall x \in \overline{\Omega},\ \forall \xi \in \mathbb{R}^n : \quad \lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2.$$

Let $G_{\Omega,L}$ denote the Green function in Ω for the following Dirichlet problem

$$Lu = f \text{ in } \Omega, \quad u = |\nabla u| = \ldots = |\nabla^{m-1}u| = 0 \text{ on } \partial\Omega. \tag{1}$$

Then there exist constants $c_1 \ge 0$, $c_2 > 0$, depending on the domain Ω and the elliptic operator L, such that we have the following Green function estimate:

$$c_2^{-1}\ H_\Omega(x,y) \le G_{\Omega,L}(x,y) + c_1 d_\Omega(x)^m d_\Omega(y)^m \le c_2\ H_\Omega(x,y) \tag{2}$$

for all $x, y \in \Omega$, where

$$H_\Omega(x,y) := |x-y|^{2m-n} \min\left\{1, \frac{d_\Omega(x)^m d_\Omega(y)^m}{|x-y|^{2m}}\right\} \tag{3}$$

and

$$d_\Omega(x) := \operatorname{dist}(x, \partial\Omega).$$

Remark 2. (a) If $L = (-\Delta)^m$ and Ω is a ball or a half space in $\mathbb{R}^n$, (2) holds true with $c_1 = 0$. This follows from Boggio's formula as already mentioned at the beginning of the introduction. Since $n > 2m$, one has a unique Green function even in the whole space and this is given by the positive fundamental solution $G_{(-\Delta)^m,\mathbb{R}^n}(x,y) = c_{m,n}|x-y|^{2m-n}$ with a suitable positive normalisation constant $c_{m,n} > 0$.

(b) The existence of a Green function is in the case of a principal part with nonconstant coefficients by no means obvious and is due to Krasovskiĭ [11, Theorem 3.3]. Thanks to the symmetry of the operator L, this is symmetric, i.e. $G_{\Omega,L}(x,y) = G_{\Omega,L}(y,x)$.

(c) Krasovskiĭ's work requires high regularity of the coefficients of L and of the domain. To keep the exposition as simple as possible I just assume everything to be C^∞-smooth.

(d) The estimate from above follows from [4, 10, 11], see also [6], so that only the estimate from below has to be proved here.

(e) With the techniques used here and developed in the mentioned works one may admit also lower order terms as long as symmetry and coercivity assumptions are obeyed.

(f) As it was investigated in [8], the situation becomes completely different when L is a general uniformly elliptic operator (even with constant coefficients), which cannot be written as a composition of second order operators, and when the dimension is large, i.e. $n \ge 2m+2$. Here, in general, even the fundamental solution is sign changing near its singularity. This means that close to its singularity the negative part of any corresponding Green function becomes unbounded of the same order of magnitude as the positive part.

(g) For implications of two-sided estimates as in Theorem 1 one may see [14, 15, 16].

An interesting consequence of Theorem 1 is a uniform local positivity result, which was shown in the mentioned works before for the more special class of operators.

Theorem 3. *Let the assumptions of Theorem 1 be satisfied. Then there exist a constant $r_{\Omega,L} > 0$, such that*

$$G_{\Omega,L}(x,y) > 0 \text{ for all } x, y \in \Omega \text{ with } |x-y| < r_{\Omega,L}. \tag{4}$$

2. Local estimates from below

Lemma 4. *Suppose that the assumptions of Theorem 1 are satisfied. Then for each $x_0 \in \overline{\Omega}$ there exists a radius $r = r_{x_0} > 0$ and a constant $C = C_{x_0} > 0$ such that for all $x, y \in \Omega_{x_0,r} := \Omega \cap B_r(x_0)$, $x \neq y$, one has*

$$G_{\Omega,L}(x,y) \geq C H_\Omega(x,y), \tag{5}$$

where H_Ω is defined in (3).

Proof. We assume by contradiction that no such positive numbers exist such that (5) is satisfied. This means that we may find sequences $(x_k)_{k\in\mathbb{N}},\ (y_k)_{k\in\mathbb{N}} \subset \Omega$ with

$$x_k \neq y_k,\ x_k, y_k \to x_0 \tag{6}$$

and

$$G_{\Omega,L}(x_k,y_k) < \frac{1}{k} H_\Omega(x,y). \tag{7}$$

We shall rescale the domain and the Green function such that $|x_k - y_k|$ becomes the new length unit. In the scaling limit we shall end up with a Green function for a constant coefficient elliptic operator in the half space or in the whole space which is known to enjoy an estimate like (5) so that we end up with a contradiction. This key argument relies on knowing the Green functions in these prototype situations explicitly, see Remark 2 (a) above.

To this end we need to distinguish the cases whether the x_k, y_k are as close together as to the boundary or much closer.

First case: The sequence $d_\Omega(x_k)/|x_k - y_k|$ remains bounded.
This means that there exists a constant $c_3 > 0$ such that

$$\frac{d_\Omega(x_k)}{|x_k - y_k|} \leq c_3. \tag{8}$$

Thanks to the triangle inequality $d_\Omega(y_k) \leq d_\Omega(x_k) + |x_k - y_k|$ we also have

$$\frac{d_\Omega(y_k)}{|x_k - y_k|} \leq c_3 + 1. \tag{9}$$

This yields in view of (6) that

$$d_\Omega(x_k), d_\Omega(y_k) \to 0 \text{ and } x_0 \in \partial\Omega$$

and due to our assumption (7) that

$$G_{\Omega,L}(x_k,y_k) < \frac{1}{k} \frac{d_\Omega(x_k)^m d_\Omega(y_k)^m}{|x_k - y_k|^n}. \tag{10}$$

Without loss of generality we may assume that $x_0 = 0$ and that the first unit vector $\vec{e}_1$ is the exterior unit normal to $\partial\Omega$ at x_0.

For k large enough, we may define $\tilde{x}_k \in \partial\Omega$ as the closest boundary point to x_k. We introduce the rescaled Green functions

$$G_k(\xi,\eta) := |x_k - y_k|^{n-2m} G_{\Omega,L}(\tilde{x}_k + |x_k - y_k|\xi, \tilde{x}_k + |x_k - y_k|\eta)$$

for

$$\xi, \eta \in \Omega_k := \frac{1}{|x_k - y_k|} \left(-\tilde{x}_k + \Omega\right).$$

These belong to the elliptic operators

$$L_k = \left(- \sum_{i,j=1}^n \partial_i \left(a_{ij}^{(k)}(\,.\,) \partial_j \right) \right)^m$$

with

$$a_{ij}^{(k)}(\xi) := a_{ij}(\tilde{x}_k + |x_k - y_k|\xi), \tag{11}$$

in Ω_k. This can be seen as follows. For smooth $v : \overline{\Omega_k} \to \mathbb{R}$ we define

$$u : \overline{\Omega} \to \mathbb{R}, \quad u(x) := v\left(\frac{x - \tilde{x}_k}{|x_k - y_k|}\right)$$

and find

$$\partial_i \left(a_{ij}(x)\partial_j u(x)\right) = \frac{1}{|x_k - y_k|^2}\partial_i \left(a_{ij}^{(k)}(\,.\,)\partial_j v(\,.\,)\right)\left(\frac{x - \tilde{x}_k}{|x_k - y_k|}\right),$$
$$Lu(x) = \frac{1}{|x_k - y_k|^{2m}}(L_k v)\left(\frac{x - \tilde{x}_k}{|x_k - y_k|}\right).$$

Since $\tilde{x}_k \to x_0 = 0$, the exterior unit normal at $\partial\Omega$ converges to the first unit vector and so we conclude that

$$\Omega_k \to \mathcal{H} := \{x : x_1 < 0\} \text{ locally uniformly for } \ k \to \infty.$$

In order to understand the limit of the G_k we observe first that Krasovskiĭ's estimates [11, Theorem 3.3] yield uniformly in k that

$$|G_k(\xi, \eta)| \le C|\xi - \eta|^{2m-n}.$$

Employing local elliptic estimates [1] we find a limit

$$\overline{G} : \overline{\mathcal{H}} \times \overline{\mathcal{H}} \to \mathbb{R} \cup \{\infty\}$$

in L^1_{loc} and locally uniformly in $\overline{\mathcal{H}} \times \overline{\mathcal{H}} \setminus \{(\xi, \xi) : \xi \in \overline{\mathcal{H}}\}$. This is symmetric, i.e. $\overline{G}(\xi, \eta) = \overline{G}(\eta, \xi)$, obeys homogeneous Dirichlet boundary conditions on $\partial\mathcal{H}$ and satisfies

$$|\overline{G}(\xi, \eta)| \le C|\xi - \eta|^{2m-n}. \tag{12}$$

We introduce the limit operator

$$L_\infty = \left(-\sum_{i,j=1}^{n} \partial_i \left(a_{ij}^\infty \partial_j\right)\right)^m$$

with the *constant* coefficients

$$a_{ij}^\infty = a_{ij}(x_0).$$

One should observe that this operator can be easily transformed by means of a linear transformation into the polyharmonic $(-\Delta)^m$. To be more precise let $C_1 \in SO(n)$ diagonalise $\left(a_{ij}^\infty\right)_{i,j=1,\dots,n}$. We then introduce a diagonal dilation matrix C_2 with positive diagonal entries such that $(C_1 \cdot C_2)^T \cdot \left(a_{ij}^\infty\right)_{i,j=1,\dots,n} \cdot (C_1 \cdot C_2)$ becomes the n-dimensional identity matrix. Finally we introduce a further "rotation" $C_3 \in SO(n)$ such that $C := C_1 \cdot C_2 \cdot C_3 : \mathcal{H} \to \mathcal{H}$ maps again the half space onto itself. Using this transformation we find that

$$G_{\mathcal{H},L_\infty}(\xi, \eta) = \det(C) G_{\mathcal{H},(-\Delta)^m}(C \cdot \xi, C \cdot \eta).$$

The latter is given by Boggio's formula

$$G_{\mathcal{H},(-\Delta)^m}(\xi, \eta) = k_{m,n}|\xi - \eta|^{2m-n} \int_1^{|\xi^* - \eta|/|\xi - \eta|} (v^2 - 1)^{m-1} v^{1-n}\, dv,$$

where $k_{m,n} > 0$ is a suitable constant and $\xi^* = (-\xi_1, \xi_2, \dots, \xi_n)$. So we conclude from [5, Theorem 4.6] that

$$G_{\mathcal{H},L_\infty}(\xi, \eta) \ge c_4 H_{\mathcal{H}}(\xi, \eta) \tag{13}$$

with a strictly positive constant c_4. $H_{\mathcal{H}}$ is as in (3) with $d_{\mathcal{H}}(\xi) = |\xi_1|$. Since Boggio's formula has the same form in $\mathcal{H}$ as in the ball, (13) is deduced in $\mathcal{H}$ in the same way as in the ball. Moreover, as explained in [6, 7] $G_{\mathcal{H},L_\infty}$ is unique under the assumption

$$|G_{\mathcal{H},L_\infty}(\xi,\eta)| \leq C|\xi-\eta|^{2m-n}.$$

In order to conclude that $\overline{G} = G_{\mathcal{H},L_\infty}$ we only need to show that

$$L_\infty \overline{G}(\xi,\,.\,) = \delta_\xi(\,.\,) \tag{14}$$

in the distributional sense. To this end we consider any function $\psi \in C_0^\infty(\mathcal{H})$ and find from the fact that G_k is the Green function for L_k that one has for any $\xi \in \mathcal{H}$ and k large enough

$$\psi(\xi) = \int_{\mathcal{H}} G_k(\xi,\eta)(L_k\psi)(\eta)\,d\eta.$$

The smoothness assumptions on the coefficients yield that $a_{ij}^{(k)} \to a_{ij}^\infty$ in $C_{loc}^{2m-1}(\mathcal{H})$ and hence

$$L_k\psi \to L_\infty\psi$$

uniformly in $C_0^0(\mathcal{H})$. We come up with

$$\psi(\xi) = \int_{\mathcal{H}} \overline{G}(\xi,\eta)(L_\infty\psi)(\eta)\,d\eta.$$

This shows that $\overline{G}$ is *a* Green function for L_∞ in $\mathcal{H}$, which in view of its uniqueness explained above yields that

$$\overline{G} = G_{\mathcal{H},L_\infty}.$$

Defining

$$\xi_k := \frac{x_k - \tilde{x}_k}{|x_k - y_k|}, \quad \eta_k := \frac{y_k - \tilde{x}_k}{|x_k - y_k|},$$

on the one hand the assumption gives via (10) that

$$G_k(\xi_k,\eta_k) = |x_k-y_k|^{n-2m} G_{\Omega,L}(x_k,y_k) < \frac{1}{k}\frac{d_\Omega(x_k)^m d_\Omega(y_k)^m}{|x_k-y_k|^{2m}}. \tag{15}$$

On the other hand, we have

$$|\xi_k| = \frac{d_\Omega(x_k)}{|x_k-y_k|} \leq c_3, \quad |\xi_k - \eta_k| = 1.$$

After passing to a further subsequence we find $\xi,\eta \in \overline{\mathcal{H}}$ with $\xi = \lim_{k\to\infty}\xi_k$, $\eta = \lim_{k\to\infty}\eta_k$. In view of the local smooth convergence of G_k to the Green function $G_{\mathcal{H},L_\infty}$ we see from (13) that there exists a positive constant $c_5 > 0$ such that for k large enough:

$$\begin{aligned} G_k(\xi_k,\eta_k) &\geq c_5 d_{\Omega_k}(\xi_k)^m d_{\Omega_k}(\eta_k)^m = c_5 \left(\frac{d_\Omega(x_k)}{|x_k-y_k|}\right)^m \left(\frac{d_\Omega(y_k)}{|x_k-y_k|}\right)^m \\ &= c_5 \frac{d_\Omega(x_k)^m d_\Omega(y_k)^m}{|x_k-y_k|^{2m}}. \end{aligned}$$

This contradicts (15) and the proof of the lemma is complete in the first case.

Second case: The sequence $d_\Omega(x_k)/|x_k-y_k|$ becomes unbounded.
The reasoning here is similar to the first case and I outline only the main steps. After passing to a subsequence we may assume that

$$\frac{d_\Omega(x_k)}{|x_k-y_k|} \to \infty.$$

This also implies that

$$\frac{d_\Omega(y_k)}{|x_k-y_k|} \geq \frac{d_\Omega(x_k)}{|x_k-y_k|} - 1 \to \infty.$$

We obtain from the assumption (7) that in this case

$$G_{\Omega,L}(x_k,y_k)<\frac{1}{k}|x_k-y_k|^{2m-n}. \tag{16}$$

We rescale again such that $|x_k-y_k|$ becomes the new length unit; however, here it is the x_k which become the new origins.

$$G_k(\xi,\eta):=|x_k-y_k|^{n-2m}G_{\Omega,L}(x_k+|x_k-y_k|\xi,x_k+|x_k-y_k|\eta)$$

for

$$\xi,\eta\in\Omega_k:=\frac{1}{|x_k-y_k|}\left(-x_k+\Omega\right).$$

These belong to the elliptic operators

$$L_k=\left(-\sum_{i,j=1}^n\partial_i\left(a_{ij}^{(k)}(\,.\,)\partial_j\right)\right)^m$$

with

$$a_{ij}^{(k)}(\xi):=a_{ij}(x_k+|x_k-y_k|\xi), \tag{17}$$

in Ω_k. In view of the assumptions in this case we conclude that

$$\Omega_k\to\mathbb{R}^n.$$

For the sequence G_k we find a limit in L^1_{loc} and locally uniformly in $\mathbb{R}^n\times\mathbb{R}^n\setminus\{(\xi,\xi):\xi\in\mathbb{R}^n\}$, which is symmetric and decays like $(\xi,\eta)\mapsto|\xi-\eta|^{2m-n}$ at infinity. Thanks to the uniqueness of such kind of Green function we conclude that in the sense just described we have

$$G_k(\,.\,,\,.\,)\to G_{\mathbb{R}^n,L_\infty}(\,.\,,\,.\,), \tag{18}$$

where the constant coefficient operator L_∞ is obtained precisely as in the first part. I emphasise that here the assumption $n>2m$ simplifies the proof a lot because the uniqueness conclusion follows directly from Liouville's theorem for polyharmonic operators, cf. e.g. [12, p. 19]. Observe that a linear transformation changes L_∞ into the polyharmonic operator.

Defining

$$\eta_k:=\frac{y_k-x_k}{|x_k-y_k|},$$

our assumption gives via (16) that

$$G_k(0,\eta_k)=|x_k-y_k|^{n-2m}G_{\Omega,L}(x_k,y_k)<\frac{1}{k}. \tag{19}$$

Thanks to $|\eta_k|=1$ we may pass to a subsequence and find $\eta_k\to\eta_\infty$ with $|\eta_\infty|=1$ and

$$G_{\mathbb{R}^n,L_\infty}(0,\eta_\infty)\le 0.$$

We proceed now precisely as explained in some detail above in the first case. After applying a linear transformation we may change L_∞ into $(-\Delta)^m$ whose Green function in $\mathbb{R}^n$ is given by the fundamental solution with zero boundary conditions at infinity, i.e. a positive multiple of $(\xi,\eta)\mapsto|\xi-\eta|^{2m-n}$. We conclude that $G_{\mathbb{R}^n,L_\infty}(0,\eta_\infty)>0$ and achieve a contradiction. So the proof of the lemma is complete also in this case. □

3. Proof of the main results: a compactness argument

3.1. **Proof of Theorem 1.** Applying a compactness argument to

$$\overline{\Omega} = \bigcup_{x_0 \in \overline{\Omega}} \Omega_{x_0, r_{x_0}/2}$$

we see that there exist positive numbers $r > 0$, $c_6 > 0$, such that $|x-y| \leq r$ implies that $G_{\Omega,L}(x,y) \geq c_6 H_\Omega(x,y)$. If $|x-y| \geq r$ we take from [4, 11], cf. also [5], that $G_{\Omega,L}(x,y) \geq -c_7 H_\Omega(x,y)$ so that

$$G_{\Omega,L}(x,y) + 2c_7 H_\Omega(x,y) \geq c_7 H_\Omega(x,y).$$

Since $H_\Omega(x,y) \leq |x-y|^{-n} d_\Omega(x)^m d_\Omega(y)^m \leq r^{-n} d_\Omega(x)^m d_\Omega(y)^m$ we end up with

$$G_{\Omega,L}(x,y) + c_8 d_\Omega(x)^m d_\Omega(y)^m \geq c_7 H_\Omega(x,y)$$

and positive constants $c_7, c_8 > 0$. The proof of Theorem 1 is complete. □

3.2. **Proof of Theorem 3.** This theorem was proved for $L = \Delta^2$ in [6] when $n \geq 3$ and in [5, Theorem 6.15] when $n = 2$. See also [7, Theorem 2] for a unified and simpler proof, which I shall adapt to the present situation.

Case I: $d_\Omega(x) d_\Omega(y) \leq |x-y|^2$. For this situation we have

$$H_\Omega(x,y) = |x-y|^{-n} d_\Omega(x)^m d_\Omega(y)^m. \tag{20}$$

Then there is $c > 0$ such that we find $c_2^{-1} H_\Omega(x,y) \geq 2c_1 d_\Omega(x)^m d_\Omega(y)^m$ for $|x-y| < c$.

Case II: $d_\Omega(x) d_\Omega(y) > |x-y|^2$. Now we have

$$H_\Omega(x,y) = |x-y|^{2m-n}. \tag{21}$$

Since $n > 2m$ by assumption and since $d_\Omega(x) d_\Omega(y)$ is bounded on Ω one finds a constant $c > 0$ such that $c_2^{-1} H_\Omega(x,y) \geq 2c_1 d_\Omega(x)^m d_\Omega(y)^m$ for $|x-y| < c$. The proof of Theorem 3 is complete. □

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