In any dimension a “clamped plate” with a uniform weight may change sign*

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Abstract

Positivity preserving properties have been conjectured for the bilaplace Dirichlet problem in many versions. In this note we show that in any dimension there exist bounded smooth domains Ω such that even the solution of $\Delta^2 u = 1$ in Ω with the homogeneous Dirichlet boundary conditions $u = u_\nu = 0$ on $\partial\Omega$ is sign-changing. In two dimensions this corresponds to the Kirchhoff-Love model of a clamped plate with a uniform weight.

1 Introduction

It is well known that for bounded smooth domains $\Omega \subset \mathbb{R}^n$ with outside unit normal ν, the biharmonic boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial\nu} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

is in general not sign preserving unless the domain is a ball or close to a ball, see [2, 10, 11]. In these papers it was shown that the corresponding Green function is positive, which is equivalent with (1) being sign preserving. A first counterexample, which shows that (1) is not positivity preserving on arbitrary domains, is due to Duffin in [5], cf. also [17]. The most striking one, showing sign change of u with a suitable $f \geq 0$ with $\Omega \subset \mathbb{R}^2$ being a mildly eccentric ellipse, was found by Garabedian, see [7]. For a short history of this problem we refer to [8]. The weaker question, whether or not the first eigenfunction is of one sign, has been studied e.g. in [3, 4, 14]. For an overview see also [18]. Although a wider class of domains are allowed for this eigenfunction to be of one sign, on general domains the fixed sign cannot be expected. Some questions on how the sign change of both problems are related are found in [12].

In the present note we consider the apparently still weaker question, whether or not the solution of

$$\begin{cases} \Delta^2 u = 1 & \text{in } \Omega, \\ u = \frac{\partial}{\partial\nu} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

*The authors are grateful to Svitlana Mayboroda for raising this question.

which is (1) with $f = 1$, is positive. This question was raised by Svitlana Mayboroda and for a motivation from an applied point of view see [6]. In the previous note [13] we constructed a counterexample in $\mathbb{R}^2$, which is based on Garabedian's celebrated example [7]. In that note we use the inversion as a particular Möbius transformation and corresponding covariance properties of the biharmonic operator. This note will show by means of an inductive procedure that sign change may occur in any dimension. This generalises and simplifies an approach by Nakai and Sario in [16].

The precise statement of our main result is as follows.

Theorem 1 *For any integer $n \geq 2$, there are bounded smooth domains $\Omega \subset \mathbb{R}^n$ such that the solution u of (2) changes sign.*

2 An inductive procedure

In [13, Theorem 2.4] one finds the following result:

- *There are bounded C^∞-smooth domains $\Omega \subset \mathbb{R}^2$ such that the solution of (2) changes sign.*

The proof is based on the fact that a solution u of (2) composed with an inversion $h(x) = |x|^{-2} x$ with $0 \notin \Omega$ satisfies

$$\Delta^2 (u \circ h(x)) = |x|^{-6}. \tag{3}$$

One takes a domain Ω for which the Green function changes sign near opposite boundary points and moves Ω such that the center of inversion 0 is located outside of Ω but near one such a boundary point. The final step consists of showing that the singularity in (3) is sufficient close to a δ-distribution near the first boundary point in order to keep the negative sign near the opposite boundary point. See [13] for details.

With [13, Theorem 2.4] it suffices to show the following.

Theorem 2 *Let $n \geq 2$. Assume that there is a bounded smooth domain $A \subset \mathbb{R}^n$ for which the solution of (2) with $\Omega = A$ is sign-changing. Then there exists a bounded smooth domain $A^* \subset \mathbb{R}^{n+1}$ for which the solution of (2) with $\Omega = A^*$ is sign-changing.*

In order to prove this result we pick a dimension $n \geq 2$ and a bounded smooth domain $A \subset \mathbb{R}^n$ and assume that the corresponding smooth solution $u : \bar{A} \to \mathbb{R}$ of (2) is sign changing. Writing $x = (x', x_{n+1}) \in \mathbb{R}^{n+1}$ and putting

$$A^*_\infty := A \times \mathbb{R}, \quad u_\infty(x', x_{n+1}) := u(x'), \tag{4}$$

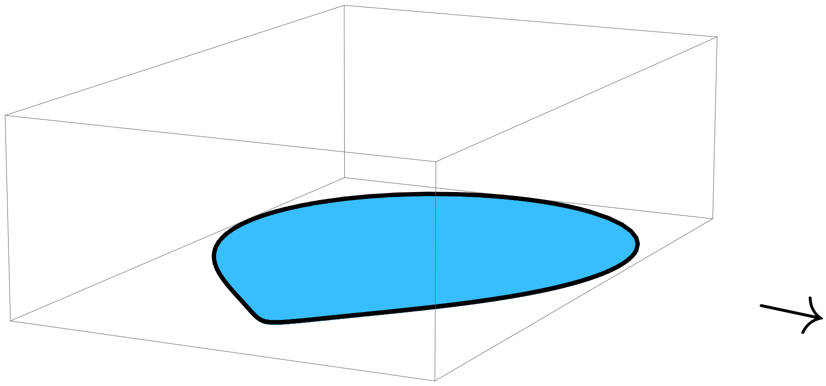
we immediately get a sign changing solution of (2) in the unbounded cylindrical domain $A^*_\infty \subset \mathbb{R}^{n+1}$. The idea is to suitably cap off A^*_∞ to a bounded smooth domain A^*_h. See Figure 1. We solve (2) for these bounded domains and will show that the corresponding solution is still sign changing when h is large enough.

We start with a technical result.

Lemma 3 *Let $A \subset \mathbb{R}^n$ be a smooth and bounded domain. Then there exists a function $g_A \in C^0(\bar{A}, [0,1]) \cap C^\infty(A, [0,1])$ such that for any $h > 0$ the domains $A^*_h \subset \mathbb{R}^{n+1}$, defined by*

$$A^*_h := \left\{ \left(x', x_{n+1}\right) : x \in A,\ -h - g_A(x') < x_{n+1} < h + g_A(x') \right\} \tag{5}$$

are smooth.



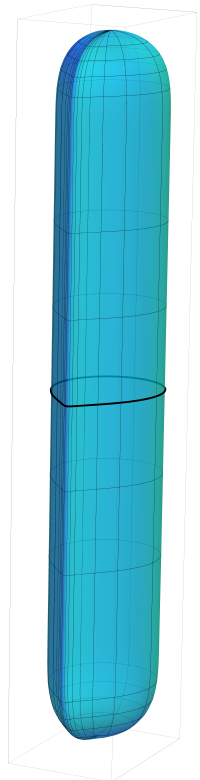


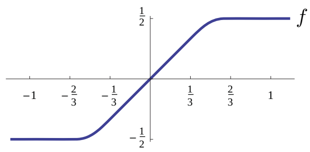
Figure 1: $A \subset \mathbb{R}^2$ and $A_h^* \subset \mathbb{R}^3$

Proof. The signed distance $d(\partial A, \cdot) : \mathbb{R}^n \to \mathbb{R}$ to the boundary of A is defined by

$$d\left(\partial A, x'\right) = \begin{cases} \inf\left\{|x' - \tilde{x}|\,; \tilde{x} \in \partial A\right\} & \text{for } x' \in \bar{A}, \\ -\inf\left\{|x' - \tilde{x}|\,; \tilde{x} \in \partial A\right\} & \text{for } x' \notin \bar{A}. \end{cases}$$

Since ∂A is smooth and bounded, there exists $r_A \in (0,1)$, such that A satisfies a uniform interior sphere condition as well as a uniform exterior sphere condition both with spheres of radius r_A. Moreover, the function $d(\partial A, \cdot)$ is smooth on $\partial A + B_{r_A}(0)$. See [9]. Let $f \in C^\infty(\mathbb{R})$ be nondecreasing such that

$$f(s) = \begin{cases} -\frac{1}{2} & \text{for} \quad s < -\frac{2}{3}, \\ s & \text{for} \quad s \in [-\frac{1}{3}, \frac{1}{3}], \\ \frac{1}{2} & \text{for} \quad s > \frac{2}{3}, \end{cases}$$

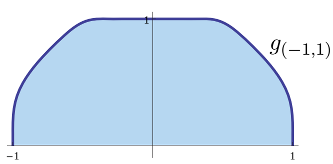


and we find a function $\tilde{d}_A \in C^\infty\left(\mathbb{R}^n\right)$, which coincides with $d(\partial A, \cdot)$ on $\partial A + B_{\frac{1}{3}r_A}(0)$, through

$$\tilde{d}_A(x') := r_A\, f\left(\frac{d\left(\partial A, x'\right)}{r_A}\right).$$

The function $g_A : \bar{A} \to [0,1]$, defined by

$$g_A\left(x'\right) = \frac{\log\left(\frac{1}{10}\right)}{\log\left(\frac{1}{5}\tilde{d}_A\left(x'\right)\right)} \text{ for } x' \in A$$

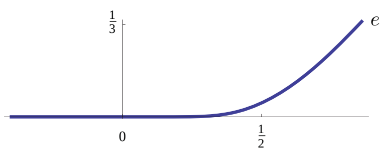


and extended by 0 for $x' \in \partial A$, is a function that satisfies the properties in the lemma:

$$\left(x' \mapsto (x', h + g_A(x'))\right) \in C^0(\bar{A}; \partial A_h^*) \cap C^\infty(A; \partial A_h^*) \tag{6}$$

parametrises the cap. In order to show that we indeed find a smooth parametrisation also where the cap is glued to the cylinder, we define

$$e(t) = \begin{cases} 5\exp\left(\frac{-\log 10}{t}\right) & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t \le 0. \end{cases}$$



One may see that, if

$$(z \mapsto \xi) \in C^\infty(U; \partial A)$$

for some open $U \subset \mathbb{R}^{n-1}$ is locally a smooth parametrisation of ∂A, then for $t_0 > 0$ but small enough and with $\nu(\xi)$ the outside unit normal of A at ξ,

$$((z,t) \mapsto (\xi - e(t)\nu(\xi), h + t)) \in C^\infty(U \times (-t_0, t_0); \partial A_h^*) \tag{7}$$

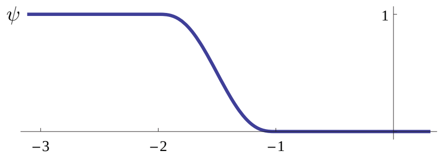
is locally a smooth parametrisation of a neighbourhood of where the cap is glued to the cylinder. For $t \in [0, t_0)$ the parametrisation in (7) describes locally the same set as the parametrisation in (6) for $x' \in \bar{A} \cap (\partial A + B_r(0))$ with $r = e(t_0)$. ∎

We prove Theorem 2 by showing the following proposition.

Proposition 4 *Let $u_h : \overline{A_h^*} \to \mathbb{R}$ denote the solutions to (2) for $\Omega = A_h^*$ and let u_∞ as in (4). On any compact subset of $\overline{A_\infty^*}$ we have smooth convergence of u_h to u_∞, as $h \to \infty$.*

Proof. We assume throughout the proof that $h \gg 2$ and introduce a cut-off function $\psi \in C^\infty(\mathbb{R}, [0,1])$ with

$$\psi(s) = \begin{cases} 1 & \text{for } s < -2, \\ 0 & \text{for } s > -1. \end{cases}$$



We consider modified (close to the caps of A_h^*) differences between u_h and u_∞:

$$v_h(x', x_{n+1}) := u_\infty(x', x_{n+1}) \cdot \psi(x_{n+1} - h) \cdot \psi(-x_{n+1} - h) - u_h(x', x_{n+1}).$$

These functions solve the following boundary value problems

$$\begin{cases} \Delta^2 v_h = f_h & \text{in } A_h^*, \\ v_h = \frac{\partial}{\partial \nu} v_h = 0 & \text{on } \partial A_h^*, \end{cases} \tag{8}$$

with suitable right hand sides f_h, that depend on u_∞, ψ and h but not on u_h, which satisfy

$$f_h(x', x_{n+1}) \equiv 0 \quad \text{for} \quad -h + 2 \le x_{n+1} \le h - 2$$

and such that for any integer $k \in \mathbb{N}_0$ there exists $C(k)$ with

$$\|f_h\|_{C^k(\overline{A_h^*})} \le C(k).$$

We first deduce uniform H^2-estimates for v_h. Note that for $u \in H_0^2(\Omega)$ one has

$$\int_\Omega |D^2 u|^2 \, dx = \int_\Omega |\Delta u|^2 \, dx. \tag{9}$$

Moreover, there exists a uniform $c > 0$ such that for all $u \in H_0^2(\Omega)$ with $\Omega \subset (-\ell_1, \ell_1) \times \mathbb{R}^n$

$$\int_\Omega |u^2| \, dx \le c \, \ell_1^2 \int_\Omega \left| \tfrac{\partial^2}{\partial x_1 \partial x_1} u^2 \right| \, dx. \tag{10}$$

Hence we find uniform Poincaré-Friedrichs-inequalities on A_h^* by using integration by parts only with respect to the x'-directions:

$$\int_{A_h^*} |D^2 v_h|^2 \, d(x', x_{n+1}) = \int_{A_h^*} f_h v_h \, d(x', x_{n+1}) = \int_{A_h^* \cap \{h-2 \le |x_{n+1}| \le h\}} f_h v_h \, d(x', x_{n+1})$$
$$\le C_1 \left(\int_{A_h^* \cap \{h-2 \le |x_{n+1}| \le h\}} v_h^2 \, d(x', x_{n+1}) \right)^{1/2}$$
$$\le C_1 \left(\int_{A_h^*} v_h^2 \, d(x', x_{n+1}) \right)^{1/2} \le C_2 \left(\int_{A_h^*} |D^2 v_h|^2 \, d(x', x_{n+1}) \right)^{1/2}.$$

We emphasise that all constants are independent of h. Hence, for $v_h \in H_0^2(A_h^*) \hookrightarrow H_0^2(A_\infty^*)$ we have the following uniform estimates:

$$\|v_h\|_{H^2(A_h^*)} \le C_3. \tag{11}$$

Applying local elliptic estimates [1] on $\overline{A_h^*} \cap \{|x_{n+1} - a| \le 2\}$, where a varies in $[-h, h]$, yields

$$\|v_h\|_{C^{4,\gamma}(\overline{A_h^*})} \le C_4,$$

where again the constant is independent of h. By local compactness and in view of (11) we find $v_\infty \in C^{4,\gamma}(\overline{A_\infty^*}) \cap H_0^2(A_\infty^*)$ such that after selecting a subsequence we have as $h \to \infty$

$$v_h \to v_\infty \text{ in } C^4 \text{ on compact subsets of } \overline{A_\infty^*}, \qquad v_h \rightharpoonup v_\infty \text{ in } H_0^2(A_\infty^*).$$

We conclude from (8) that

$$\begin{cases} \Delta^2 v_\infty = 0 & \text{in } A_\infty^*, \\ v_\infty = \frac{\partial}{\partial \nu} v_\infty = 0 & \text{on } \partial A_\infty^*. \end{cases} \tag{12}$$

Writing (8) in a weak form we find that also (12) holds in the following weak form. We have

$$\int_{A_\infty^*} \sum_{i,j=1,\dots,n+1} (\partial_i \partial_j v_\infty) \cdot (\partial_i \partial_j \varphi) \, dx' dx_{n+1} = 0$$

for all $\varphi \in C_0^\infty(A_\infty^*)$ and by density also for all $\varphi \in H_0^2(A_\infty^*)$. Hence, we may choose $\varphi = v_\infty$ and see that all $\partial_i \partial_j v_\infty = 0$. This immediately yields $v_\infty = 0$. Since the previous reasoning applies to any subsequence of $(v_h)_{h>0}$ the claim is proved. ■

A Green function estimates in infinite cylinders

In the proof of Proposition 4 we might have profited from a fact like $G_{A_h^*}(\cdot, y)$ converges to $G_{A_\infty^*}(\cdot, y)$ for $h \to \infty$. But to do so one should first clarify that there is decent notion of Green function on an unbounded domain. For an unbounded domain the equations

$$\begin{cases} LG_\Omega(\cdot, y) = \delta_y(\cdot) \text{ in } \Omega, \\ BG_\Omega(\cdot, y) = 0 \text{ on } \partial\Omega, \end{cases} \tag{13}$$

are in general not sufficient for having a unique solution not even when similar conditions for the adjoint are prescribed. Some conditions at ∞ have to be added but usually not that the function goes to 0 at ∞.

Example A.1 *For the half plane* $\mathbb{R}^+ \times \mathbb{R}$ *Boggios formula for the Dirichlet biharmonic gives us*

$$G(x,y) = \frac{1}{16\pi}\left(4x_1y_1 - |x-y|^2 \log\left(1 + \frac{4x_1y_1}{|x-y|^2}\right)\right).$$

This function is not bounded for x *or* y *going to* ∞ *but in some sense it is the solution with the lowest growth rate.*

Without some restriction at ∞ *one could add the function* $u(x;y) = x_1^2y_1^2$ *which is biharmonic in* x *and satisfies the boundary conditions. It is even symmetric in* x, y.

For a cylindrical domain the situation is better but even there some kind of boundary condition at infinity is necessary for uniqueness. Let us recall that the cylindrical domains we consider are defined as follows:

$$\Omega = \left\{(x', x_{n+1}); x' \in A \text{ and } x_{n+1} \in \mathbb{R}\right\} \text{ with } A \subset \mathbb{R}^n \text{ a bounded domain.}$$

Here, i.e. $\Omega = A^*_\infty = A \times \mathbb{R}$, there is a unique symmetric function $G_\Omega(\cdot,\cdot) : \bar{\Omega} \times \bar{\Omega} \mapsto \mathbb{R} \cup \{\infty\}$, which lies in $C^4\left(\left\{(x,y) \in \bar{\Omega} \times \bar{\Omega};\ x \neq y\right\}\right)$, that satisfies

$$\begin{cases} \Delta^2 G_\Omega(\cdot, y) = \delta_y(\cdot) \text{ in } \Omega, \\ G_\Omega(\cdot, y) = \frac{\partial}{\partial\nu} G_\Omega(\cdot,y) = 0 \text{ on } \partial\Omega, \\ \lim\limits_{\Omega \ni x \to \infty} G_\Omega(x,y) = 0. \end{cases} \tag{14}$$

A theory for elliptic boundary value problems in cylinders by Kondratiev was developed by Kozlov, Maz'ya and Rossmann in [15, Chapter 5]. In order to verify the applicability of this theory to our present case, one has to replace ∂_{n+1} in the biharmonic operator Δ^2 on A^*_∞ by a spectral parameter. In other words, consider for $\lambda \in \mathbb{C}$ the boundary value problems

$$\begin{cases} \left(\Delta^2 + 2\lambda^2\Delta + \lambda^4\right)\varphi = 0 & \text{in } A, \\ \varphi = \frac{\partial}{\partial\nu}\varphi = 0 & \text{on } \partial A, \end{cases} \tag{15}$$

that have a nontrivial solution. Theorem 5.2.1 of [15] states that there at most countably many such λ, which are isolated, and that these λ are located in double sector around the real axis. For $\lambda = 0$ one finds only the trivial solution for (15) and hence one may assume that there exists $\lambda_A > 0$ such that all λ with a nontrivial solution satisfy $|\mathrm{Re}\lambda| \geq \lambda_A$. With these preliminaries one may find that for each $f \in C_0^\infty(A^*_\infty)$ there is a solution $u \in C^\infty(\overline{A^*_\infty})$

$$\begin{cases} \Delta^2 u = f & \text{in } A^*_\infty, \\ u = \frac{\partial}{\partial\nu} u = 0 & \text{on } \partial A^*_\infty, \\ \lim\limits_{A^*_\infty \ni x \to \infty} u(x) = 0. \end{cases} \tag{16}$$

and moreover the following estimates hold.

Theorem A.2 *Suppose that* $f \in C_0^\infty(A^*_\infty)$ *with* support$(f) \subset \bar{A} \times (-1,1)$. *Then there is* $u \in C^\infty(\overline{A^*_\infty})$ *which satisfies (16). Moreover, for* $|\gamma| < \lambda_A$ *and* $k \in \mathbb{Z}$ *there exists a constant* $C = C(A,k,\gamma)$ *such that the following estimate holds for all* $|m| > 3$:

$$\|u\|_{W^k(A \times (m-1,m+1))} \leq Ce^{-\gamma|m|} \|f\|_{W^{k-4}(A \times (1,1))}. \tag{17}$$

Proof. Let us recall the spaces $\mathcal{W}^k_{2,\gamma}(A^*_\infty)$ and the corresponding norms from [15]. These are defined in [15, Equation (5.2.1)] for $k \in \mathbb{N}$ by

$$w \in \mathcal{W}^k_{2,\gamma}(A^*_\infty) \Leftrightarrow e^{\gamma x_{n+1}} w \in H^k(A^*_\infty).$$

Through duality ([15, Equation (5.3.1)]) one may also give a meaning to $\mathcal{W}^k_{2,\gamma}(A^*_\infty)$ with a negative coefficient k. We may use Theorem 5.2.2 of [15] with the operator C equal to 0 and a homogeneous boundary condition $\underline{g}= 0$. Then for any $k \geq 4$ problem (16) has a unique solution $u \in \mathcal{W}^k_{2,\gamma}(A^*_\infty)$ for each $f \in \mathcal{W}^{k-4}_{2,\gamma}(A^*_\infty)$, which a priori depends on γ, and moreover there exists a constant $C_{A,k,\gamma} > 0$ such that

$$\|u\|_{\mathcal{W}^k_{2,\gamma}(A^*_\infty)} \leq C_{A,k,\gamma} \ \|f\|_{\mathcal{W}^{k-4}_{2,\gamma}(A^*_\infty)}, \tag{18}$$

which implies (17) for $k \geq 4$. Since (16) is selfadjoint, by duality, see [15, Theorem 5,3,2], one finds the similar result as in (18) for $k < 4$. By [15, Corollary 5.4.2] the solutions in $\mathcal{W}^k_{2,\gamma_1}(A^*_\infty)$ and in $\mathcal{W}^k_{2,\gamma_2}(A^*_\infty)$ coincide if $\gamma_1, \gamma_2 \in (-\lambda_A, \lambda_A)$ and lie in any $\mathcal{W}^k_{2,\gamma_1}(A^*_\infty)$ with $k \in \mathbb{Z}$ for the present f and A with C^∞-boundary. ■

As a consequence one finds the existence and uniqueness of a Green function on A^*_∞ with a corresponding estimate.

Corollary A.3 *There exists a unique Dirichlet biharmonic Green function on A^*_∞, that is, a function $G_{A^*_\infty} : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}^+ \cup \{\infty\}$ which satisfies (14).Moreover, there exists a constant C such that for all $\alpha \in \mathbb{N}^{n+1}$, $x' \in A$ and $|x_{n+1}| > 2$*

$$\left|D^\alpha G_{A^*_\infty}\left((x', x_{n+1}), (y', 0)\right)\right| \leq C e^{-\gamma|x_{n+1}|}, \tag{19}$$

and for any $k, \ell \in \mathbb{N}$ one finds for $h \to \infty$ that

$$G_{A^*_h}\left(\cdot, (y', 0)\right) \to G_{A^*_\infty}\left(\cdot, (y', 0)\right) \ \text{in } C^k\left(\bar{A} \times [-\ell, \ell]\right).$$

Proof. Starting with a weighted Sobolev space of suitable negative order, by means of local elliptic estimates as in [1], one finds the desired estimates for the Green function. ■

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