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A CLAMPED PLATE WITH A UNIFORM WEIGHT MAY CHANGE SIGN

Hans-Christoph Grunau

Facultät für Mathematik
Otto-von-Guericke Universität
Postfach 4120
39016 Magdeburg, Germany

Guido Sweers

Mathematisches Institut
Universität zu Köln
Weyertal 86-90
50931 Köln, Germany

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Abstract. It is known that the Dirichlet bilaplace boundary value problem, which is used as a model for a clamped plate, is not sign preserving on general domains. It is also known that the corresponding first eigenfunction may change sign. In this note we will show that even a constant right hand side may result in a sign-changing solution.

1. **Introduction.**

For Ω a ball in any dimension Boggio in [3] was able to give explicit Green functions for the solution of

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial\nu} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Since his Green functions are positive, one finds a positivity preserving property: $f \geq 0$ implies $u \geq 0$. In two dimensions (1) is a model for the clamped plate (see [2]) and it seems natural that pushing that plate in the positive direction ($f \geq 0$) will imply that the plate bends in the same direction ($u \geq 0$). Boggio and Hadamard [17] were interested in the question, whether such a positivity preserving property would hold for other domains too. Hadamard in [18] even claimed to have proven such a result for all limaçons. The conjecture that such a result holds true on at least convex domains became known as the Boggio-Hadamard conjecture. Since the first counterexample by Duffin in [10] on a long, almost rectangular domain appeared, it is well known that for most domains $\Omega \subset \mathbb{R}^2$ this bilaplace boundary value problem under homogeneous Dirichlet boundary conditions is not sign preserving with the exception of domains close to the disk [15], see also [14]. Also on limaçons near the cardioid there is no positivity preserving property ([8]). For a short history of this problem see [13].

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A weaker question related to positivity is, whether or not the first eigenfunction for the Dirichlet bilaplace is of one sign. See the paper by Szegö [21]. Also this positivity does not hold in general. Coffman, Duffin and Shaffer [5, 6] were able to show that a disk with a small hole has a sign-changing first eigenfunction. Coffman [4] also considered the behaviour of eigenfunctions near corners following earlier observations of a nodal line on a square in [1]. A brief overview concerning the sign of this first eigenfunction can be found in [22]. In higher dimensions and in a more general context sign changing first eigenfunctions were studied by Kozlov, Kondrat'ev and Maz'ya in [19]. Some questions on how the sign change of both problems are related are found in [16].

An apparently still weaker question is, whether or not the solution of (1) for $f \equiv 1$ is positive. This problem was pointed out to us by Svitlana Mayboroda, for a motivation from an applied point of view see [11]. This note will show that even for that question the answer is in general negative. To obtain a suitable bounded smooth domain $\Omega \subset \mathbb{R}^2$ as a counterexample we use that any Möbius transformation h will transform the boundary value problem in (1) on Ω to a similar problem on $h(\Omega)$. Loewner pointed out in [20] that these are the only conformal transformations which enjoy nice covariance properties with respect to the bilaplacian. Moreover, any polyharmonic operator Δ^m in any dimension has related properties. We think that also Boggio was aware of this tool and had it in mind when deriving his explicit formulas on the ball and on the half space in [3].

2. **The result.** Our counterexample will yield a C^∞-domain in $\mathbb{R}^2$. For our arguments, however, it will be sufficient that $\Omega \subset \mathbb{R}^2$ is a bounded domain with $\partial\Omega \in C^{4,\gamma}$. We will assume this throughout this note.

Assumption 2.1. *Ω is a domain for which (1) is not positivity preserving in the following quantitative sense: Let G_Ω denote the Green function for (1). Then we assume that there are $x^*, y^* \in \partial\Omega$ such that*

$$\Delta_x \Delta_y G_\Omega(x^*, y^*) < 0. \tag{2}$$

According to a celebrated example of Garabedian [12] this is e.g. true for a relatively long thin ellipse (ratio of half axes $\approx 5/3$) with x^*, y^* opposite boundary points on the longer half axis.

Lemma 2.2. *If Ω satisfies Assumption 2.1, then there exist $\delta > 0$ and $C > 0$ such that for all $x \in \Omega \cap B_\delta(x^*)$ and $y \in \Omega \cap B_\delta(y^*)$:*

$$G_\Omega(x,y) \le -C \;\operatorname{dist}(x,\partial\Omega)^2 \;\operatorname{dist}(y,\partial\Omega)^2. \tag{3}$$

Proof. Since $G_\Omega \in C^{4,\gamma}\left(\bar\Omega \times \bar\Omega \setminus D_\Omega\right)$ where $D_\Omega = \{(x,x)\,; x \in \Omega\}$ one finds that $\Delta_x \Delta_y G_\Omega(x,y) < 0$ holds in a neighbourhood of (x^*, y^*). Moreover, due to the boundary conditions it holds that

$$\nabla_x^k \nabla_y^\ell G_\Omega(x,y) = 0 \text{ for } (x,y) \in \partial\Omega \times \partial\Omega,$$

whenever $|k| \le 1$ or $|\ell| \le 1$. In case that $|k| = 2$ resp. $|\ell| = 2$ at least one of the derivatives has to be taken in tangential direction. Assumption 2.1 gives then that

$$\left(\frac{\partial^2}{\partial \nu_x^2}\right)\left(\frac{\partial^2}{\partial \nu_y^2}\right) G_\Omega(x,y) < 0 \text{ for } (x,y) \in \partial\Omega \times \partial\Omega \text{ close to } (x^*, y^*)$$

and (3) follows by Taylor expansion. $\square$

We will not directly use the domain Ω for a counterexample, but will instead proceed through a Möbius-transformed Ω. For h a Möbius transformation in $\mathbb{R}^2$, see [7] or [13, p. 202], one finds

$$\left(\Delta^2 u\right)(h(x)) = J_h^{-\frac{3}{2}}(x)\ \Delta^2\left(J_h^{-\frac{1}{2}}(x)\ u(h(x))\right) \tag{4}$$

where J_h is the Jacobian of the transformation h. Next to translations, rotations and reflections in a plane, which all preserve more or less the shape of the domain, the inversion is a Möbius transformation that changes that shape. All Möbius transformations are combinations of the elementary transformations just mentioned. A sketch of an ellipse with its Möbius transformation can be found in Figure 1.

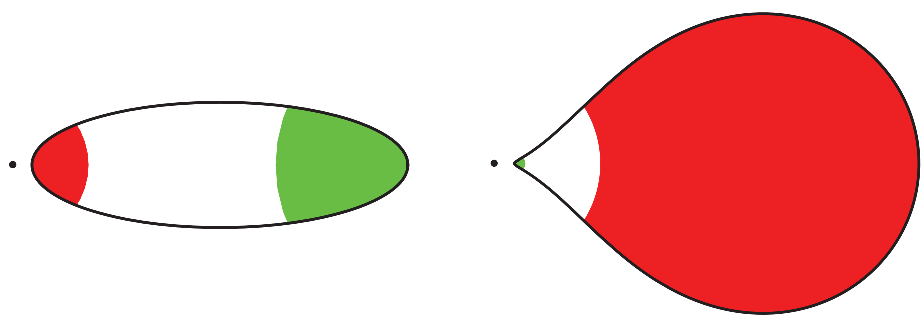

FIGURE 1. *An ellipse and its inversion with respect to the point as center. Note how both subdomains are transformed.*

Let $j : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ be the inversion: $j(x) = \frac{x}{|x|^2}$. So if we take $\tilde{x} \notin \Omega$ as the center of the inversion and set $A = j(\Omega - \tilde{x})$, that is, $h(x) = j(x - \tilde{x})$, we find that the boundary value problem

$$\begin{cases} \Delta^2 u = 1 & \text{in } A, \\ u = \frac{\partial}{\partial \nu} u = 0 & \text{on } \partial A, \end{cases} \tag{5}$$

transfers with $v(x) = J_h^{-\frac{1}{2}}(x)\ u(h(x))$ to

$$\begin{cases} \Delta^2 v = J_h^{\frac{3}{2}} & \text{in } \Omega, \\ v = \frac{\partial}{\partial \nu} v = 0 & \text{on } \partial \Omega. \end{cases} \tag{6}$$

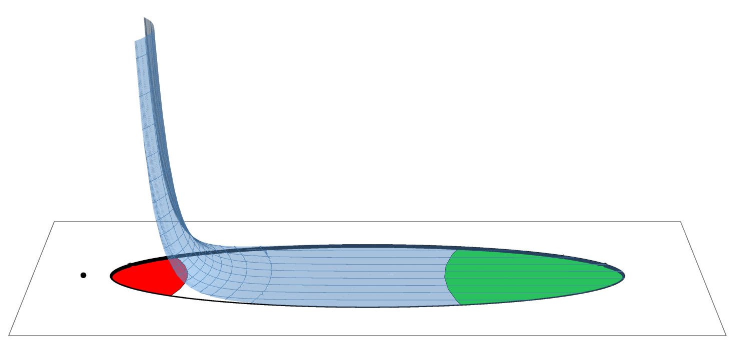

FIGURE 2. *Ω and the typical right hand side $J_h^{\frac{3}{2}}$*

If we choose $\tilde{x} \notin \Omega$ such that the solution of (6) changes sign, then so does the solution of (5). A sketch for the domain Ω with the right hand side in (6) is given in Figure 2.

Theorem 2.3. *Suppose that Ω satisfies Assumption 2.1 and let $(x^*, y^*) \in \partial\Omega \times \partial\Omega$ be as in Lemma 2.2. Then choosing $t > 0$ sufficiently small in $\tilde{x} := \tilde{x}(t) = x^* + t\,\nu^*$, where ν^* is the outside normal direction in x^*, one finds that the solution of (6) with $h(x) = j(x - \tilde{x}(t))$ changes sign.*

Proof. The Jacobian J_j that corresponds with the inversion $x \mapsto |x|^{-2}\,x$ is as follows

$$J_j(x) = \left| \det \begin{pmatrix} \frac{x_2^2 - x_1^2}{|x|^4} & \frac{-2x_1x_2}{|x|^4} \\ \frac{-2x_1x_2}{|x|^4} & \frac{x_1^2 - x_2^2}{|x|^4} \end{pmatrix} \right| = |x|^{-4}.$$

So the function v satisfies

$$\begin{cases} \Delta^2 v(x) = |x - \tilde{x}(t)|^{-6} & \text{for } x \in \Omega, \\ v = \frac{\partial}{\partial\nu} v = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Fix $\tilde{y} \in \Omega \cap B_\delta(y^*)$ with $\operatorname{dist}(\tilde{y}, \partial\Omega)^2 > \frac{1}{2}\delta$. Next we split the contribution to $v(y)$ by the right hand side into two parts:

$$\Omega_1 = \{x \in \Omega; |x - x^*| < \delta\} \text{ and } \Omega_2 = \{x \in \Omega; \delta \leq |x - x^*|\}.$$

We may assume without loss of generality, that $|x^* - y^*| > 2\delta$. For the contribution by Ω_1 we find with C as in Lemma 2.2 that

$$\begin{aligned} \int_{\Omega_1} G_\Omega(\tilde{y}, x)\,|x - \tilde{x}(t)|^{-6}\,dx &\leq -C\ \operatorname{dist}(\tilde{y}, \partial\Omega)^2 \int_{\Omega_1} \operatorname{dist}(x, \partial\Omega)^2\,|x - \tilde{x}(t)|^{-6}\,dx \\ &\leq -C_1 \int_0^\delta r^3\,(r+t)^{-6}\,r\,dr \\ &= -C_1\ t^{-2} \int_0^{\delta/t} \frac{s^3}{(1+s)^6}\,ds \\ &\leq -C_2\ t^{-2} \qquad \text{for } t > 0 \text{ small enough.} \end{aligned} \tag{8}$$

For the contribution by Ω_2 we find by the Green function estimate (see [9] or [13, Thm. 4.28])

$$|G_\Omega(x,y)| \leq c\ \operatorname{dist}(x, \partial\Omega)\ \operatorname{dist}(y, \partial\Omega)\ \min\left(1, \frac{\operatorname{dist}(x,\partial\Omega)\ \operatorname{dist}(y,\partial\Omega)}{|x-y|^2}\right) \tag{9}$$

that

$$\begin{aligned} \int_{\Omega_2} G_\Omega(\tilde{y}, x)\,|x - \tilde{x}(t)|^{-6}\,dx &\leq c\ \operatorname{dist}(\tilde{y}, \partial\Omega) \int_{\Omega_2} \operatorname{dist}(x, \partial\Omega)\,|x - \tilde{x}(t)|^{-6}\,dx \\ &\leq c_1 \int_\delta^{\operatorname{diam}(\Omega)} (r+t)^{-6}\,r\,dr \leq c_2. \end{aligned} \tag{10}$$

Letting $t \downarrow 0$ we find by (8) and (10) for the solution v of (7) that

$$v(\tilde{y}) \leq c_2 - C_2 t^{-2}.$$

So for $t > 0$ but sufficiently small, v will have a negative part. □

Corollary 2.4. *There are domains $A \subset \mathbb{R}^2$ such that the solution of*

$$\begin{cases} \Delta^2 u = 1 & \text{in } A, \\ u = \frac{\partial}{\partial\nu} u = 0 & \text{on } \partial A, \end{cases} \tag{11}$$

changes sign.

Proof. We remark that in any bounded smooth domain A, there exists $x_0 \in A$ such that $u(x_0) > 0$ for the solution u of (11). Indeed, since $\Delta u(x) \not\equiv 0$ we even have that

$$0 < \int_A (\Delta u)^2 \, dx = \int_A u \, \Delta^2 u \, dx = \int_A u \, dx.$$

We choose now Ω and h as in Theorem 2.3 and put $A = h(\Omega)$. In this case the solution u of (11) attains also strictly negative values. □

Remark 2.5. In higher dimension one may still use the inversion to transform one biharmonic Dirichlet problem to another one. Indeed the transformation rule changes into

$$\left(\Delta^2 u\right) \circ h = J_h^{-\frac{1}{2} - \frac{2}{n}} \; \Delta^2 \left(J_h^{\frac{1}{2} - \frac{2}{n}} \; u \circ h \right)$$

and the estimate in (9) and our proof could be adapted. A technical problem, however, is whether there exist domains satisfying (2). We still expect this inequality to hold for eccentric domains in higher dimensions but we have no actual reference.

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E-mail address: hans-christoph.grunau@ovgu.de
E-mail address: gsweers@math.uni-koeln.de