Unexpected differences between fundamental solutions of general higher-order elliptic operators and of products of second-order operators.

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Abstract

We study fundamental solutions of elliptic operators of order $2m \geq 4$ with constant coefficients in large dimensions $n > 2m$, where their singularities become unbounded. For compositions of second-order operators these can be chosen as convolution products of positive singular functions, which are positive themselves. As soon as $n \geq 3$, the general elliptic operator $(-\Delta)^m$ may no longer serve as a prototype. We shall show that positivity of the singular fundamental solutions persists only in special dimensions, while in general the behaviour of the fundamental solutions near the unbounded singularity becomes sign changing: There are "positive" as well as "negative" directions along which the fundamental solution tends to $+\infty$ and $-\infty$ respectively, when approaching its pole.

1 Introduction & main results

General constant coefficients elliptic operators. We focus our attention to uniformly elliptic operators of order $2m$ with constant coefficients which involve only the highest order derivatives, namely

$$L = (-1)^m Q\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) = (-1)^m \sum_{i_1,\cdots,i_{2m}=1,\cdots n} A_{i_1,\cdots i_{2m}} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{2m}}}, \tag{1}$$

where the $2m$-homogeneous characteristic polynomial

$$Q(\xi) = \sum_{i_1,\cdots,i_{2m}=1,\cdots n} A_{i_1,\cdots i_{2m}} \xi_{i_1} \cdots \xi_{i_{2m}}.$$

is called (possibly up to a sign) the *symbol* of the operator.

Uniform ellipticity means then that Q is strictly positive on the unit sphere, i.e. there exists a constant $\lambda > 0$ such that

$$\forall \xi \in \mathbb{R}^n : \quad Q(\xi) \geq \lambda |\xi|^{2m}.$$

Fundamental solutions. In order to construct and to understand solutions u to the differential equation $Lu = f$ for given right-hand side f, one introduces the concept of a fundamental solution $K_L(x, \,.\,)$ for any "pole" $x \in \mathbb{R}^n$ which is defined as a solution to the equations $L^* K_L(x, \,.\,) = \delta_x$ and $L K_L(\,.\,, x) = \delta_x$ in the distributional sense where δ_x is the δ-distribution located at x. This means that for any testing function $\psi \in C_0^\infty(\mathbb{R}^n)$ one has

$$\int_{\mathbb{R}^n} K_L(x,y) L\psi(y)\, dy = \psi(x), \qquad \int_{\mathbb{R}^n} K_L(y,x) L^*\psi(y)\, dy = \psi(x)$$

with

$$L^* = (-1)^m \sum_{i_1,\cdots,i_{2m}=1,\cdots n} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{2m}}} A_{i_1,\cdots i_{2m}}$$

being the adjoint operator of L. Because L has only constant coefficients and only of the highest even order $2m$, we have that $L = L^*$. Moreover, we may achieve that

$$K_L(x,y) = K_L(0, x-y) = K_L(0, y-x) = K_L(y,x). \tag{2}$$

For given $f \in C_0^\infty(\mathbb{R}^n)$, any fundamental solution yields a solution to the differential equation $Lu = f$ in $\mathbb{R}^n$ by putting

$$u(x) := \int_{\mathbb{R}^n} K_L(x,y) f(y)\, dy.$$

One should also notice that, if the fundamental solution exists, it is not unique: one may add any smooth solution of $Lv = 0$, namely $\tilde{K}_L(x,y) = K_L(x,y) + v(x-y)$ yields another fundamental solution.

Green functions. When the problem $Lu = f$ is considered in a bounded sufficiently smooth domain, one may still obtain solution and even representation formulae by means of suitable fundamental solutions. Indeed, let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and consider the problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $f \in C^{0,\gamma}(\overline{\Omega})$ and the boundary conditions verify a complementing condition, see [ADN]. As a typical and most frequently studied prototype one may think of Dirichlet boundary conditions

$$B_D(u) := (u, \frac{\partial u}{\partial \nu}, \dots, \frac{\partial^{m-1} u}{\partial \nu^{m-1}}) = 0 \quad \text{on } \partial\Omega,$$

with ν the exterior unit normal at $\partial\Omega$. If there exists a unique solution $h_{L,\Omega,B}(x,\cdot)$ of the boundary value problem (observe that $L^* = L$)

$$\begin{cases} Lh_{L,\Omega,B}(x,\cdot) = 0 & \text{in } \Omega, \\ B(h_{L,\Omega,B}(x,\cdot)) = -B(K_L(|x-\cdot|)) & \text{on } \partial\Omega, \end{cases}$$

one can define the so called *Green function* for problem (3), given by

$$G_{L,\Omega,B}(x,y) = K_L(|x-y|) + h_{L,\Omega,B}(x,y).$$

Then the unique solution of (3) is given by

$$u(x) = \int_\Omega G_{L,\Omega,B}(x,y)\; f(y)\; dy.$$

Notice that in general is not straightforward to infer the existence of such $h_{L,\Omega,B}$. However, exploiting the general elliptic theory of Agmon, Douglis, and Nirenberg [ADN] this is always possible in our special case when the operator L has only constant coefficients of highest order, if Dirichlet boundary conditions $B = B_D$ are imposed and the domain is $C^{2m,\gamma}$-smooth.

In this case, one also infers by standard estimates that the function $h_{L,\Omega,B}$ is regular in $\overline{\Omega}$. Since in large dimensions fundamental solutions have a singularity near the pole, it becomes clear that, in order to understand $G_{L,\Omega,B}$, we need first to investigate the behaviour of fundamental solutions.

Positivity questions. Positivity properties for $G_{L,\Omega,B}$ concern the question whether a positive right-hand side yields a positive solution: if u is a solution of (3), does it hold that $f \geq 0 \Rightarrow u \geq 0$? One often expects such a behaviour for physical or geometrical reasons. However, for equations of order at least 4, such a positivity preserving property will fail in general, see [GGS] for historical remarks and detailed references. This question concerns a *nonlocal* behaviour of the full boundary value problem and often the influence of boundary conditions spoils the expected positivity. However, physically, one would hope that when applying an extremely concentrated right-hand side – a δ-Distribution – then close to this point the solution should respond in the same direction. This leads to the related but relaxed *local* question: Is a suitable fundamental solution to the differential equation positive, at least close to its pole? This question is reasonable only for large dimensions $n \geq 2m$ because only here, fundamental solutions become unbounded and they are unique only up to locally bounded regular solutions of the homogeneous equation. If $n > 2m$ one may achieve uniqueness of the fundamental solution by imposing zero (Dirichlet) boundary conditions at infinity. In this case K_L may be considered as the Green function $G_{L,\mathbb{R}^n,B_D}$ in the whole space. This means that one considers just the behaviour of the differential equation and disregards the influence of possible boundary conditions (being infinitely far apart).

Previous results. In the context of second-order equations ($m = 1$), both local and nonlocal behaviours are well established. Indeed, within the class of constant coefficients operators, the Laplacian $-\Delta$ is, up to a change of coordinates, the only such operator. Its fundamental solutions are known explicitly and in particular they are positive (if $n = 2$, at least close to the pole). Moreover, the maximum principle holds for such operators, so positive data imply positivity of solutions (see [GT]). In other words, the Green function is always positive.

When one moves to the higher-order setting ($m \geq 2$), several differences arise, even for $(-\Delta)^m$ or, equivalently, for powers of second-order operators with constant coefficients.

Indeed, if one investigates the positivity preserving property in bounded domains, then the answer is largely affected by the choice of boundary conditions. As an example, on the one hand, with Navier boundary conditions ($u = \Delta u = \cdots = \Delta^{m-1}u = 0$) one may rewrite the problem as a second-order system and thus the maximum principle implies positivity. On the other hand, this tool is in general not available when dealing with Dirichlet boundary conditions ($u = \partial_\nu u = \cdots = \partial_\nu^{m-1}u = 0$) and one cannot expect positivity, in general not even in *convex* bounded smooth domains (see [GP]). On the other hand, positivity holds in balls and their small smooth deformations (see [Bo, GR]). We refer to [GGS] for an extensive survey of the topic.

However, within that class of powers of second-order operators, if one restricts to a "local" question, meaning the positivity of Green functions under Dirichlet boundary conditions near the pole, the answer is still affirmative. Indeed, a uniform local positivity can be proved, namely the existence of a constant $r_{m,\Omega} > 0$ such that $G_{(-\Delta)^m,\Omega}(x,y) > c_{m,\Omega}^{-1} > 0$ for all $x, y \in \Omega$ with $|x-y| < r_{m,\Omega}$. This means that the negative part of the Green function is far away from the singularity.

A consequence of that result is that the size of negative part of the Green function (under Dirichlet boundary conditions), if present at all, is small compared to its positive part. Indeed, concerning Dirichlet problems, positivity for a rank-1-correction of the polyharmonic Green function is retrieved, namely

$$G_{(-\Delta)^m,\Omega}(x,y) + c_{m,\Omega} d_\Omega(x)^m d_\Omega(y)^m \geq 0,$$

where d_Ω denotes the distance to the boundary and $c_{m,\Omega}$ is a sufficiently large positive constant, see [GR, GRS].

These results have been later on extended by Pulst in his PhD-dissertation [Pu] for formally selfadjoint positive definite operators of order $2m$, such that the leading term of the operator involved is the polyharmonic $(-\Delta)^m$ (or an m-th power of an elliptic operator of order 2 with constant coefficients), provided the lower order terms can be written in divergence form and have sufficiently smooth uniformly bounded coefficients.

However, in dimensions $n > 2$ powers of second-order operators are *not* the prototype of a general operator L of order $2m$, not even in the case of constant coefficients. Moreover, it is in general not possible to rewrite L as an m-fold composition of (possibly different) second-order operators. Indeed, let us simply consider the case of a homogeneous fourth-order operator with a symbol of the kind

$$Q(x,y,z) = x^4 + y^4 + z^4 + \sum_{\substack{i+j+k=4 \\ 0 \leq i,j,k \leq 3}} c_{i,j,k} x^i y^j z^k$$

and suppose it is the composition of two second-order polynomials q_1, q_2. One may achieve that their coefficients in front of x^2 are both equal to 1 and then, they would necessarily be of the kind

$$q_1(x,y,z) = x^2 + cy^2 + dz^2 + a_1xy + a_2xz + a_3yz$$

$$q_2(x,y,z) = x^2 + \frac{1}{c}y^2 + \frac{1}{d}z^2 + b_1xy + b_2xz + b_3yz.$$

The smooth map from $\mathbb{R}^8$ into the 12-dimensional vector space of such symbols Q which maps

$$(c,d,a_1,a_2,a_3,b_1,b_2,b_3) \mapsto q_1 \cdot q_2$$

is *not surjective*. We are grateful to Guido Sweers for sharing with us this (unpublished) example.

Concerning explicit formulae and (local) positivity properties of fundamental solutions of such general elliptic operators only little is known. Existence of fundamental solution is shown in [Jo] in a very general framework, and rather involved formulae are obtained. In the particular case of a *$2m$-homogeneous higher-order uniformly elliptic operator with constant coefficients*, different implicit expressions have been found according to the parity of the dimension n. For *odd* n, the general formula for a fundamental solution [Jo, (3.44)] simplifies as

$$K_L(x,y) = -\frac{1}{4(2\pi)^{n-1}}(-\Delta_y)^{\frac{n+1-2m}{2}} \int_{|\xi|=1} \frac{|(x-y)\cdot\xi|}{Q(\xi)} d\mathcal{H}^{n-1}(\xi) \tag{4}$$

(from [Jo, (3.54)]), while for *even* n there holds (see [Jo, (3.62)])

$$K_L(x,y) = -\frac{1}{(2\pi)^{n}}(-\Delta_y)^{\frac{n-m}{2}} \int_{|\xi|=1} \frac{\log|(x-y)\cdot\xi|}{Q(\xi)}\, d\mathcal{H}^{n-1}(\xi). \tag{5}$$

We recall that Q denotes the symbol (possibly up to a sign) of the operator L. To the authors knowledge, no further results have been later on obtained in that direction.

Aim & results. The aim of this paper is to investigate the behaviour of fundamental solutions - and in particular whether they are positive or not close to the pole - for this class of uniformly elliptic operators of order $2m$ with constant coefficients. In particular, we find explicit formulae for fundamental solutions and show that for specific dimensions and specific operators one has sign change of the fundamental solution even when approaching the pole. We find this behaviour completely unexpected because this means that even when applying a right hand side, which is concentrated at some point and points into one direction, the response of any solution to the differential equation will be sign changing and so - in some regions arbitrarily close to this point - in opposite direction to the right hand side. We underline that this is in contrast to all previously known results for higher-order elliptic operators.

The first part of our work is devoted to make John's formulae above explicit. In the *odd dimensional* case, we find a nice compact formula for any dimension (Theorem 1.1). This enables us to prove that a fundamental solution is always positive only if $n = 2m+1$, while for any higher dimension (confining ourself to the simplest case $m = 2$), we can provide examples of symbols for which the fundamental solutions of the corresponding fourth-order operator are sign-changing (Theorem 1.2).

Theorem 1.1. *Let $n \geq 2m+1$ be odd. Then, the fundamental solution K_L is given by*

$$K_L(x,y) = \frac{(-1)^{\frac{n-2m-1}{2}}}{2^n\pi^{n-1}|x-y|^{n-2m}} \int_{\substack{|\xi|=1\\(x-y)\cdot\xi=0}} \nabla^{n-2m-1}\frac{1}{Q(\xi)}\left(\frac{x-y}{|x-y|}^{\otimes\, n-2m-1}\right) d\mathcal{H}^{n-2}(\xi).$$

Here, if T is a j-multilinear form and v is a vector, we use the compact tensorial notation

$$T\left(v^{\otimes j}\right) := T(\underbrace{v,v,\cdots,v}_{j-\text{times}}),$$

so, in particular,

$$\nabla^j f(\xi)\left(v^{\otimes j}\right) = \sum_{h_1,\cdots,h_j=1}^{n} \frac{\partial^j f}{\partial_{h_1},\cdots\partial_{h_j}}(\xi) v_{h_1}\cdots v_{h_j}.$$

Theorem 1.1 is proved in Section 2.2 and it follows directly from Theorem 2.3.

Let $\alpha \in \mathbb{R}$ and define the polynomial $Q_\alpha : \mathbb{R}^n = \mathbb{R}\oplus\mathbb{R}^{n-1} \to \mathbb{R}$ as

$$Q_\alpha(\xi) = \xi_1^4 - \alpha\xi_1^2|\xi'|^2 + |\xi'|^4 \tag{6}$$

where $\xi = (\xi_1, \xi') \in \mathbb{R}^n$. Notice that Q_α is a symbol of a fourth-order uniformly elliptic operator provided $\alpha^2 < 4$ and it cannot be factored as a power of second-order (smooth!) symbols as $n \geq 3$.

Theorem 1.2 (Sign-changing fundamental solutions, odd dimensional case)**.**

i) If $n = 2m + 1$, the fundamental solution K_L is always positive.

ii) Let $m = 2$. For any $n \geq 7$ odd, there exist parameters α such that the fundamental solution of the operator L associated to the elliptic symbol Q_α is sign-changing for $|x - y| \to 0$.

In the proof of Theorem 1.2 we shall distinguish several cases, according to the dimension n, the most involved one being $n \equiv 5 \pmod 8$ where one cannot use asymptotics for small α. So a detailed analysis of K_L for a specific symbol has to be performed, involving combinatorial arguments, Chebychev polynomials and other, see Section 2.3.

The situation is similar concerning *even dimensions*. However, due to the presence of the logarithm in (5), we are not able to achieve a comparable compact result, computations being much more involved. However, we show that at least for the "first cases" similar phenomena occur.

Theorem 1.3. *Let $n = 2m + 2$. Then, the fundamental solution K_L is given by*

$$\begin{aligned} K_L(x,y) = \frac{1}{2^{2m+1}\pi^{2m+2}|x-y|^2} & \left\{ \frac{n-2}{2} \int_{|\xi|=1} \frac{1}{Q(\xi)}\, d\mathcal{H}^{n-1}(\xi) \right. \\ & + \int_{\substack{|\xi|=1 \\ (x-y)\cdot\xi>0}} \log\left(\xi \cdot \frac{x-y}{|x-y|}\right) \left[(4m+4-n)\left(\xi \cdot \frac{x-y}{|x-y|}\right)\left(\nabla \frac{1}{Q(\xi)} \cdot \frac{x-y}{|x-y|}\right) \right. \\ & \left.\left. + 2m\frac{1}{Q(\xi)} + \nabla^2 \frac{1}{Q(\xi)} \left(\frac{x-y}{|x-y|}^{\otimes 2}\right) + \left(\xi \cdot \frac{x-y}{|x-y|}\right)^2 \Delta \frac{1}{Q(\xi)} \right] d\mathcal{H}^{n-1}(\xi) \right\}. \end{aligned}$$

The proof is given in Section 3.1. As before, for the fourth-order case we show examples of symbols for which the fundamental solution changes sign.

Theorem 1.4 (Examples of sign-changing fundamental solutions, even dimensional case)**.**

i) If $n = 2m$, the fundamental solution K_L is positive for $|x - y| \in B_{r_L}(0)$ with some $r_L > 0$.

ii) Let $m = 2$ and $n = 6$. There exist parameters $\alpha > 0$ such that the fundamental solution of the operator L associated to the elliptic symbol Q_α is sign-changing for $|x - y| \to 0$.

For the proof, see Section 3.2. The difference between even and odd dimensions here reminds us somehow of the same dinstinction for the wave equation. In Theorem 1.3 (even dimensional) the integration is carried out over a one-codimensional surface with a weight function, which becomes infinite at its boundary; on the other hand, in Theorem 1.1 (odd dimensional) the integration is carried out over the boundary of this surface, i.e. a 2-codimensional surface.

Notation. We denote the partial derivative as D^α or ∂_α or $\frac{\partial}{\partial\alpha}$, where α is a multiindex, with the convention that if $D^0 u = u$ for any function. Moreover, if $j \in \mathbb{N}$, $\nabla^j u$ stands for the tensor of the j-th derivatives. Finally, we denote by $\mathcal{H}^k$ the k-th dimensional Hausdorff measure.

2 Odd dimensions $n > 2m$: explicit fundamental solutions

In this section we prove Theorem 1.1 starting from John's formula (4). In other words, recalling that $K_L(x,y) = K_L(0, x - y)$, we shall compute the iterated Laplacian for

$$F(y) := \int_{\substack{|\xi|=1 \\ y\cdot\xi>0}} \frac{y\cdot\xi}{Q(\xi)} d\mathcal{H}^{n-1}(\xi). \tag{7}$$

2.1 The first iteration: ΔF

Proposition 2.1. *With the notation above, we have*

$$\nabla F(y) = \int_{\substack{|\xi|=1 \\ y\cdot\xi > 0}} \frac{\xi^T}{Q(\xi)} d\mathcal{H}^{n-1}(\xi), \tag{8}$$

$$\Delta F(y) = \frac{1}{|y|} \int_{\substack{|\xi|=1 \\ y\cdot\xi = 0}} \frac{1}{Q(\xi)} d\mathcal{H}^{n-2}(\xi). \tag{9}$$

Before going into details of the proof, let us remark an important consequence of Proposition 2.1.

Remark 1. In the case $n = 2m+1$, e.g. for a fourth-order elliptic operator in $\mathbb{R}^5$, (4) and (9) imply

$$K(x,y) = \frac{1}{32\pi^4|x-y|} \int_{\substack{|\xi|=1 \\ (x-y)\cdot\xi = 0}} \frac{1}{Q(\xi)} d\mathcal{H}^{n-2}(\xi), \tag{10}$$

which is thus a *positive* fundamental solution having the *expected order of singularity*. The only difference with the polyharmonic case is that an "angular dependent" positive factor appears. Notice that for the model polyharmonic case, as $Q_{\Delta^m}(\xi) := |\xi|^{2m}$, then such factor is identically 1 and we of course retrieve its well-known fundamental solution.

We first recall a classical result about integrations of differential forms (see for instance [Fo, Satz 3]).

Lemma 2.2. *Let M be an oriented hypersurface with exterior normal vector field $\nu(x)$, which means that for any admissible parametrisation Φ with $x = \Phi(t)$ we have* $\det\Big(\nu(x), \frac{\partial\Phi}{\partial t_1}(t), \dots, \frac{\partial\Phi}{\partial t_{n-1}}(t)\Big) > 0$. *Let further $A \subseteq M$ be a compact submanifold and $\boldsymbol{f} = (f_i)_{i=1}^n : M \to \mathbb{R}^n$ be a vector field. Then we have*

$$\int_A \boldsymbol{f}(x)\cdot\nu(x)dS(x) = \int_A \Big(\sum_{i=1}^{n-1}(-1)^{i-1} f_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\Big),$$

with the convention that $\widehat{dx_i}$ means that this factor is missing.

Proof of Proposition 1. **Step 1.** Let us first compute ∇F on points $y = (y_1, 0, \cdots, 0) \neq \mathbf{0}$. Here and everywhere in what follows we assume $y_1 > 0$. Then, by means of a rigid motion in $\mathbb{R}^n$, we will extend the result to the general case.

Writing $y = r\eta$, where $|\eta| = 1$, we have

$$F(y) = r \int_{\substack{|\xi|=1 \\ \eta\cdot\xi > 0}} \frac{\eta\cdot\xi}{Q(\xi)} d\mathcal{H}^{n-1}(\xi).$$

Noticing that the derivative in the first direction is indeed a normal derivative,

$$\frac{\partial^k}{\partial y_1^k} F(y_1, 0, \cdots, 0) = \frac{d^k}{dr^k} F(r\eta), \tag{11}$$

therefore we infer

$$\frac{\partial}{\partial y_1} F(y_1, 0, \cdots, 0) = \frac{1}{|y|} \int_{\substack{|\xi|=1 \\ y\cdot\xi > 0}} \frac{y\cdot\xi}{Q(\xi)} d\mathcal{H}^{n-1}(\xi) = \int_{\substack{|\xi|=1 \\ y\cdot\xi > 0}} \frac{\xi_1}{Q(\xi)} d\mathcal{H}^{n-1}(\xi), \tag{12}$$

$$\frac{\partial^2}{\partial y_1^2} F(y_1, 0, \cdots, 0) = 0.$$

Let us now compute all other first and second derivatives, which thus involve tangential directions. Without loss of generality, we may consider just $\partial_3 F$ and $\partial^2_{33} F$, the general case being similar. To this aim, we introduce the rotation matrix B_φ which roughly speaking exchanges the first direction with the third:

$$B_\varphi = \left[\begin{array}{ccc|c} \cos\varphi & 0 & -\sin\varphi & \\ 0 & 1 & 0 & \mathbf{0} \\ \sin\varphi & 0 & \cos\varphi & \\ \hline & \mathbf{0} & & \mathbf{I} \end{array}\right] \tag{13}$$

Moreover, define $H(\varphi) := F(B_\varphi y)$. Recalling that $y_i = 0$ for all $i > 1$ and differentiating H in $\varphi = 0$, we have

$$\begin{aligned}\frac{\partial H}{\partial \varphi}\Big|_{\varphi=0} &= \partial_1 F_{|(y_1\cos\varphi,0,y_1\sin\varphi,0,\cdots,0)}\Big|_{\varphi=0}(-\sin\varphi y_1)|_{\varphi=0} + \partial_3 F_{|(y_1\cos\varphi,0,y_1\sin\varphi,0,\cdots,0)}\Big|_{\varphi=0}(\cos\varphi y_1)|_{\varphi=0}\\ &= y_1\partial_3 F(y).\end{aligned} \tag{14}$$

On the other hand, by definition

$$H(\varphi) = \int_{\substack{|\xi|=1\\ B_\varphi y\cdot\xi>0}} \frac{B_\varphi y\cdot\xi}{Q(\xi)} d\mathcal{H}^{n-1}(\xi) \overset{\xi=B_\varphi\xi'}{=} \int_{\substack{|\xi'|=1\\ y\cdot\xi'>0}} \frac{y\cdot\xi'}{Q(B_\varphi\xi')} d\mathcal{H}^{n-1}(\xi').$$

Therefore, using Lemma 2.2,

$$\begin{aligned}\frac{\partial H}{\partial \varphi}\Big|_{\varphi=0} &= \int_{\substack{|\xi'|=1\\ y\cdot\xi'>0}} (y\cdot\xi')\frac{\partial}{\partial\varphi}\Big|_{\varphi=0}\frac{1}{Q(B_\varphi\xi')}d\mathcal{H}^{n-1}(\xi')\\ &= \int_{\substack{|\xi'|=1\\ \xi_1'>0}} (y_1\xi_1')\Big(-\xi_3'\partial_1\frac{1}{Q(\xi')} + \xi_1'\partial_3\frac{1}{Q(\xi')}\Big)d\mathcal{H}^{n-1}(\xi')\\ &= \int_{\substack{|\xi'|=1\\ \xi_1'>0}} (y_1\xi_1')\begin{pmatrix}\partial_3\frac{1}{Q(\xi')}\\ 0\\ -\partial_1\frac{1}{Q(\xi')}\\ 0\\ \cdots\\ 0\end{pmatrix}\cdot\xi' d\mathcal{H}^{n-1}(\xi')\\ &= y_1\int_{\substack{|\xi'|=1\\ \xi_1'>0}}\Big(\xi_1'\partial_3\frac{1}{Q(\xi')}\,d\xi_2'\wedge\cdots\wedge d\xi_n' \;-\; \xi_1'\partial_1\frac{1}{Q(\xi')}\,d\xi_1'\wedge d\xi_2'\wedge d\xi_4'\wedge\cdots\wedge d\xi_n'\Big).\end{aligned} \tag{15}$$

We observe that for the $(n-2)$ - form

$$\omega := -\frac{\xi_1'}{Q(\xi')}d\xi_2'\wedge d\xi_4'\wedge\cdots\wedge d\xi_n'$$

we have

$$\begin{aligned} d\omega &= -\left(d\frac{\xi_1'}{Q(\xi')}\right)\wedge d\xi_2'\wedge d\xi_4'\wedge\cdots\wedge d\xi_n'\\ &= -\partial_1\left(\frac{\xi_1'}{Q(\xi')}\right)d\xi_1'\wedge d\xi_2'\wedge d\xi_4'\wedge\cdots\wedge d\xi_n' - \partial_3\left(\frac{\xi_1'}{Q(\xi')}\right)d\xi_3'\wedge d\xi_2'\wedge d\xi_4'\wedge\cdots\wedge d\xi_n'\\ &= -\xi_1'\left(\partial_1\frac{1}{Q(\xi')}\right)d\xi_1'\wedge d\xi_2'\wedge d\xi_4'\wedge\cdots\wedge d\xi_n' \;-\; \frac{1}{Q(\xi')}d\xi_1'\wedge d\xi_2'\wedge d\xi_4'\wedge\cdots\wedge d\xi_n'\\ &+ \xi_1'\partial_3\frac{1}{Q(\xi')}d\xi_2'\wedge\cdots\wedge d\xi_n'.\end{aligned} \tag{16}$$

Hence, by (14)-(16) and Stokes' Theorem, noticing that $\omega = 0$ on $\{\xi_1' = 0\}$, we infer

$$\begin{aligned}\frac{\partial F}{\partial y_3}(y_1,0,\cdots,0) = \frac{1}{y_1}\frac{\partial H}{\partial\varphi}\Big|_{\varphi=0} &= \int_{\substack{|\xi'|=1\\ \xi_1'>0}}\frac{1}{Q(\xi')}d\xi_1'\wedge d\xi_2'\wedge d\xi_4'\wedge\cdots\wedge d\xi_n' + \int_{\substack{|\xi'|=1\\ \xi_1'>0}} d\omega\\ &= \int_{\substack{|\xi'|=1\\ \xi_1'>0}}\frac{1}{Q(\xi')}\nu_3(\xi')d\mathcal{H}^{n-1}(\xi')\\ &= \int_{\substack{|\xi'|=1\\ \xi_1'>0}}\frac{\xi_3'}{Q(\xi')}d\mathcal{H}^{n-1}(\xi').\end{aligned}$$

Analogously one may compute $\partial_k F(y_1, 0, \cdots, 0)$ for $k \neq 1$ and then obtain

$$\frac{\partial F}{\partial y_k}(y_1, 0, \cdots, 0) = \int_{\substack{|\xi'|=1 \\ y\cdot\xi' > 0}} \frac{\xi'_k}{Q(\xi')} d\mathcal{H}^{n-1}(\xi'). \tag{17}$$

Indeed, it is sufficient to consider instead of B_φ a similar matrix corresponding to a rotation in the plane $\langle y_1, y_k \rangle$. Hence, (12) and (17) yields

$$\nabla F(y_1, 0, \cdots, 0) = \int_{\substack{|\xi'|=1 \\ y\cdot\xi' > 0}} \frac{(\xi')^T}{Q(\xi')} d\mathcal{H}^{n-1}(\xi').$$

Step 2. Now we want to to extend this identity to a generic point $y \in \mathbb{R}^n \setminus \{0\}$. To this aim, let us write $y = |y| b_1$, where $|b_1| = 1$, and complete this unit vector to a matrix $B = \big(b_1 \mid \cdots \mid b_n\big) \in SO(n)$. Notice that

$$y = |y| b_1 = B \cdot \big(|y|, 0, \cdots, 0\big)^T \qquad \text{and} \qquad \big(|y|, 0, \cdots, 0\big)^T = B^T y.$$

Moreover, define $\tilde{F} := F \circ B$. From (7) we infer for all $z \in \mathbb{R}^n \setminus \{0\}$ that

$$\tilde{F}(z) = F(Bz) = \int_{\substack{|\xi|=1 \\ Bz\cdot\xi > 0}} \frac{Bz \cdot \xi}{Q(\xi)} d\mathcal{H}^{n-1}(\xi) \stackrel{\xi=B\xi'}{=} \int_{\substack{|\xi'|=1 \\ z\cdot\xi' > 0}} \frac{z \cdot \xi'}{Q(B\xi')} d\mathcal{H}^{n-1}(\xi').$$

Therefore,

$$\begin{aligned} \nabla F(y) &= \nabla F(B(|y|, 0, \cdots, 0)^T) = \nabla \tilde{F}(|y|, 0, \cdots, 0) B^T \\ &= \int_{\substack{|\xi'|=1 \\ \xi'_1 > 0}} \frac{\xi'^T}{Q(B\xi')} d\mathcal{H}^{n-1}(\xi') \cdot B^T = \int_{\substack{|\xi'|=1 \\ \xi'_1 > 0}} \frac{(B\xi')^T}{Q(B\xi')} d\mathcal{H}^{n-1}(\xi'). \end{aligned}$$

The change of variable $\xi = B\xi'$ yields finally (8).

Step 3. Now it is the turn of *second derivatives*. We compute them with the same method we applied so far, so first we consider the easier case $y = (y_1, 0, \cdots, 0)^T$ with $y_1 > 0$ and then we extend this to a general $y \in \mathbb{R}^n \setminus \{0\}$.

Once again, we may for simplicity consider just $\partial_3 F$, the other cases being similar as already mentioned. Let B_φ be as in (13) and for $y = (y_1, 0, \cdots, 0)^T$ define

$$\tilde{H}(\varphi) := \partial_3 F(B_\varphi y) = \int_{\substack{|\xi|=1 \\ B_\varphi y\cdot\xi > 0}} \frac{\xi_3}{Q(\xi)} d\mathcal{H}^{n-1}(\xi) \stackrel{\xi=B_\varphi\xi'}{=} \int_{\substack{|\xi'|=1 \\ \xi'_1 > 0}} \frac{(B_\varphi \xi')_3}{Q(B_\varphi \xi')} d\mathcal{H}^{n-1}(\xi').$$

Exactly as for F in (14), we have

$$\frac{\partial^2 F}{\partial y_3^2}(y_1, 0, \cdots, 0) = \frac{1}{y_1} \frac{\partial \tilde{H}}{\partial \varphi}\bigg|_{\varphi=0}.$$

We denote $R(z) := \frac{z_3}{Q(z)}$ for all $z \in \mathbb{R}^n$. Then we have

$$\begin{aligned}
\frac{\partial \tilde{H}}{\partial \varphi}\Big|_{\varphi=0} &= \frac{\partial}{\partial \varphi}\Big|_{\varphi=0} \int_{\substack{|\xi'|=1 \\ \xi_1' > 0}} R(B_\varphi \xi')\, d\mathcal{H}^{n-1}(\xi') \\
&= \int_{\substack{|\xi'|=1 \\ \xi_1' > 0}} \big(-\xi_3' \partial_1 R(\xi') + \xi_1' \partial_3 R(\xi') \big)\, d\mathcal{H}^{n-1}(\xi') \\
&= \int_{\substack{|\xi'|=1 \\ \xi_1' > 0}} \begin{pmatrix} \partial_3 R(\xi') \\ 0 \\ -\partial_1 R(\xi') \\ 0 \\ \cdots \\ 0 \end{pmatrix} \cdot \nu(\xi')\, d\mathcal{H}^{n-1}(\xi') \\
&= \int_{\substack{|\xi'|=1 \\ \xi_1' > 0}} \big(\partial_3 R(\xi')\, d\xi_2' \wedge \cdots \wedge d\xi_n' - \partial_1 R(\xi')\, d\xi_1' \wedge d\xi_2' \wedge d\xi_4' \wedge \cdots \wedge d\xi_n' \big) \\
&= \int_{\substack{|\xi'|=1 \\ \xi_1' > 0}} d\xi_2' \wedge \big(\partial_1 R(\xi') d\xi_1' + \partial_3 R(\xi') d\xi_3' \big) \wedge d\xi_4' \wedge \cdots \wedge d\xi_n' \\
&= -\int_{\substack{|\xi'|=1 \\ \xi_1' > 0}} d\left(R(\xi') d\xi_2' \wedge d\xi_4' \wedge \cdots \wedge d\xi_n' \right) \\
&= -\int_{\substack{|\xi'|=1 \\ \xi_1' = 0}} R(\xi')\, d\xi_2' \wedge d\xi_4' \wedge \cdots \wedge d\xi_n',
\end{aligned}$$

having applied Stokes' Theorem. Let $\tilde{\nu}$ denote the exterior normal of the half-sphere on its boundary, which gives its induced orientation. Hence, using Lemma 2.2 and the definition of R,

$$\frac{\partial \tilde{H}}{\partial \varphi}\Big|_{\varphi=0} = \int_{\substack{|\xi'|=1 \\ \xi_1' = 0}} \tilde{\nu}(\xi') \cdot \begin{pmatrix} 0 \\ R(\xi') \\ 0 \\ \cdots \\ 0 \end{pmatrix} d\mathcal{H}^{n-2}(\xi') = \int_{\substack{|\xi'|=1 \\ \xi_1' = 0}} \frac{{\xi_3'}^2}{Q(\xi')}\, d\mathcal{H}^{n-2}(\xi').$$

Therefore, we may conclude that for all $k \in \{2, \cdots, n\}$ we have

$$\frac{\partial^2 F}{\partial y_k^2}(y_1, 0, \cdots, 0) = \frac{1}{|y|} \int_{\substack{|\xi'|=1 \\ \xi_1' = 0}} \frac{{\xi_k'}^2}{Q(\xi')} d\mathcal{H}^{n-2}(\xi').$$

Recalling that $\partial_{11}^2 F(y_1, 0, \cdots, 0) = 0$, we thus infer

$$\Delta F(y_1, 0, \cdots, 0) = \frac{1}{|y|} \int_{\substack{|\xi'|=1 \\ \xi_1' = 0}} \frac{1}{Q(\xi')} d\mathcal{H}^{n-2}(\xi'). \tag{18}$$

Step 4. Let now $y \in \mathbb{R}^n \setminus \{0\}$ which we write as $y = |y| b_1$, with $|b_1| = 1$, and let us complete this unit vector to a matrix $B := \big(b_1 \mid \cdots \mid b_n\big) \in SO(n)$. Moreover, recall $\tilde{F} := F \circ B$. Then, one has

$$\Delta F(z) = Tr(\nabla^2 F(z)) = Tr(B \nabla^2 \tilde{F}_{|_{B^T z}} B^T) = Tr(\nabla^2 \tilde{F}_{|_{B^T z}}) = \Delta \tilde{F}(B^T z),$$

and therefore by (18),

$$\Delta F(y) = \Delta \tilde{F}(|y|, 0, \cdots, 0) = \frac{1}{|y|} \int_{\substack{|\xi'|=1 \\ B^T y \cdot \xi' = 0}} \frac{1}{Q(B\xi')} d\mathcal{H}^{n-2}(\xi') \overset{\xi = B\xi'}{=} \frac{1}{|y|} \int_{\substack{|\xi|=1 \\ y \cdot \xi = 0}} \frac{1}{Q(\xi)} d\mathcal{H}^{n-2}(\xi).$$

□

2.2 The k-th iteration $\Delta^k F$ and the proof of Theorem 1.1

The following theorem provides a general formula for the iterated Laplacian of F. We will see that tangential derivatives of the symbol play a fundamental role in the formula. This will have a serious impact on the analysis of the sign of fundamental solutions, see Section 2.3.

Theorem 2.3.

$$\Delta^k F(y) = \frac{1}{|y|^{2k-1}} \sum_{j=1}^{k} d_{k,j}(n,m) \int_{\substack{|\xi|=1\\ y\cdot\xi=0}} \nabla^{2(k-j)} \frac{1}{Q(\xi)} \left(\frac{y}{|y|}^{\otimes 2(k-j)} \right) d\mathcal{H}^{n-2}(\xi),^{1} \tag{19}$$

where

$$d_{k,1}(n,m) = 1$$

and for $j = 2\ \dots, k$:

$$d_{k,j}(n,m) = (-1)^{j-1} c_{k,j} \prod_{\ell=2}^{j} (n - 2m - (2k - 2\ell + 3)) \tag{20}$$

with[2]

$$c_{k,j} = \prod_{\ell=2}^{j} \frac{(k-\ell+1)(2k-2\ell+1)}{\ell-1}. \tag{21}$$

Note that Theorem 1.1 is an immediate consequence of Theorem 2.3. This follows from putting $k = \frac{n-2m+1}{2}$, where $d_{k,j} = 0$ for all $j \geq 2$.

The rest of the subsection is devoted to proving Theorem 2.3, the strategy being the following. Firstly, we show that each term of the sum, namely

$$\int_{\substack{|\xi|=1\\ y\cdot\xi=0}} \nabla^{2j} \frac{1}{Q}(\xi) \left(\frac{y}{|y|}^{\otimes 2j} \right) d\mathcal{H}^{n-2}(\xi),$$

once the Laplacian is applied, produces only terms of the same kind (so only even derivatives of $\frac{1}{Q}$ are involved) with order at most $2j+2$, each of them multiplied by the same suitable power of $\frac{1}{|y|}$. This is achieved in Proposition 2.4. As a consequence, we obtain some recurrence formulae for the coefficients $d_{k,j}$ in the proof of Theorem 2.3. These relations will be important to finally prove the theorem by induction.

Let us fix $k \in \mathbb{N}$ and $j \in \{0, \cdots, k-1\}$ and define

$$J_{k,j}(y) := \frac{1}{|y|^{2k-1}} \int_{\substack{|\xi|=1\\ y\cdot\xi=0}} \nabla^{2j} \frac{1}{Q}(\xi) \left(\frac{y}{|y|}^{\otimes 2j} \right) d\mathcal{H}^{n-2}(\xi).$$

Proposition 2.4. *There holds*

$$\begin{aligned} \nabla J_{k,j}(y) = \frac{1}{|y|^{2k}} \int_{\substack{|\xi|=1\\ y\cdot\xi=0}} \Bigg[&(2j) \nabla^{2j} \frac{1}{Q}(\xi) \left(\frac{y}{|y|}^{\otimes 2j-1}, \cdot \right) - \xi^T \cdot \nabla^{2j+1} \frac{1}{Q}(\xi) \left(\frac{y}{|y|}^{\otimes 2j+1} \right) \\ &- (2k-1+2j) \nabla^{2j} \frac{1}{Q}(\xi) \left(\frac{y}{|y|}^{\otimes 2j} \right) \frac{y^T}{|y|} \Bigg] d\mathcal{H}^{n-2}(\xi), \end{aligned} \tag{22}$$

and

$$\Delta J_{k,j}(y_1, 0, \cdots, 0) = \frac{1}{|y|^{2k+1}} \int_{\substack{|\xi'|=1\\ \xi_1'=0}} \left[c_0 \frac{\partial^{2j}}{\partial {\xi_1'}^{2j}} \frac{1}{Q}(\xi') + c_\Delta \frac{\partial^{2j-2}}{\partial {\xi_1'}^{2j-2}} \frac{1}{Q}(\xi') + \frac{\partial^{2j+2}}{\partial {\xi_1'}^{2j+2}} \frac{1}{Q}(\xi') \right] d\mathcal{H}^{n-2}(\xi'), \tag{23}$$

---

[1]To avoid redundant parenthesis we simply write $\frac{y}{|y|}^{\otimes 2j} := \left(\frac{y}{|y|} \right)^{\otimes 2j}$.

[2]Using the convention that the product is 1 whenever $j = 1$.

where

$$c_0(n,m,k,j) = 2k(2k-1) + (n-1)(1-2k-2j) + 2(4j+1)(m+j) \tag{24}$$

and

$$c_\Delta(n,m,j) = 4j(2j-1)(m+j-1)(2m+2j+1-n). \tag{25}$$

Therefore, we obtain

$$\Delta J_{k,j}(y) = c_\Delta(n,m,j)J_{k+1,j-1} + c_0(n,m,k,j)J_{k+1,j} + J_{k+1,j+1}. \tag{26}$$

Proof. In order to simplify the notation, as k, j are fixed, we write $J(y)$ instead of $J_{k,j}(y)$.
Step 1. Let $y = (y_1, 0, \cdots, 0)$, $y_1 > 0$. First of all, writing J in polar coordinates

$$J(r,\eta) = \frac{1}{r^{2k-1}} \int_{\substack{|\xi|=1\\ \eta\cdot\xi=0}} \nabla^{2j}\frac{1}{Q}(\xi)\left(\eta^{\otimes 2j}\right) d\mathcal{H}^{n-2}(\xi),$$

by (11) one infers

$$\frac{\partial}{\partial y_1} J(y_1, 0, \cdots, 0) = -\frac{2k-1}{|y|^{2k}} \int_{\substack{|\xi|=1\\ y\cdot\xi=0}} \nabla^{2j}\frac{1}{Q}(\xi)\left(\frac{y}{|y|}^{\otimes 2j}\right) d\mathcal{H}^{n-2}(\xi),$$

and

$$\frac{\partial^2}{\partial y_1^2} J(y_1, 0, \cdots, 0) = \frac{2k(2k-1)}{|y|^{2k+1}} \int_{\substack{|\xi|=1\\ y\cdot\xi=0}} \nabla^{2j}\frac{1}{Q}(\xi)\left(\frac{y}{|y|}^{\otimes 2j}\right) d\mathcal{H}^{n-2}(\xi). \tag{27}$$

As in the proof of Proposition 2.1, defining $H(\varphi) := J(B_\varphi y)$ with B_φ as in (13), we have

$$\begin{aligned} H(\varphi) &= \frac{1}{|y|^{2k-1}} \int_{\substack{|\xi|=1\\ (B_\varphi y)\cdot\xi=0}} \nabla^{2j}\frac{1}{Q}(\xi)\left(\frac{B_\varphi y}{|y|}^{\otimes 2j}\right) d\mathcal{H}^{n-2}(\xi) \\ &\overset{\xi=B_\varphi\xi'}{=} \frac{1}{|y|^{2k-1}} \int_{\substack{|\xi'|=1\\ \xi_1'=0}} \nabla^{2j}\frac{1}{Q}(B_\varphi\xi')\left(\frac{B_\varphi y}{|y|}^{\otimes 2j}\right) d\mathcal{H}^{n-2}(\xi'). \end{aligned}$$

We may rewrite the argument as

$$\nabla^{2j}\frac{1}{Q}(B_\varphi\xi')\left(\frac{B_\varphi y}{|y|}^{\otimes 2j}\right) = \sum_{h=0}^{2j}\binom{2j}{h}\partial^{2j}_{1^{2j-h}3^h}\frac{1}{Q}(B_\varphi\xi')\cos^{2j-h}\varphi\,\sin^h\varphi,$$

with the shorter notation

$$\partial^k_{i^k} := \frac{\partial^k}{\partial x_i^k}.$$

A differentiation with respect to φ yields

$$\begin{aligned} \left.\frac{\partial H}{\partial\varphi}\right|_{\varphi=0} = \frac{1}{|y|^{2k-1}} \int_{\substack{|\xi'|=1\\ \xi_1'=0}} \sum_{h=0}^{2j}\binom{2j}{h}\Bigg\{ &\left.\frac{d}{d\varphi}\right|_{\varphi=0}\left(\partial^{2j}_{1^{2j-h}3^h}\frac{1}{Q}(B_\varphi\xi')\right)\left(\cos^{2j-h}\varphi\Big|_{\varphi=0}\right)\left(\sin^h\varphi\Big|_{\varphi=0}\right) \\ &+ \partial^{2j}_{1^{2j-h}3^h}\frac{1}{Q}(\xi')\Bigg[\left(\sin^h\varphi\Big|_{\varphi=0}\right)\left(\left.\frac{d}{d\varphi}\right|_{\varphi=0}\cos^{2j-h}\varphi\right) \\ &+ \left(\cos^{2j-h}\varphi\Big|_{\varphi=0}\right)\left(\left.\frac{d}{d\varphi}\right|_{\varphi=0}\sin^h\varphi\right)\Bigg]\Bigg\} d\mathcal{H}^{n-2}(\xi'). \end{aligned}$$

Since the terms which remain must have only cosines, everything vanishes except for the term with $h = 0$ in the first sum and the one with $h = 1$ in the third. Therefore,

$$\left.\frac{\partial H}{\partial\varphi}\right|_{\varphi=0} = \frac{1}{|y|^{2k-1}} \int_{\substack{|\xi'|=1\\ \xi_1'=0}} \left\{\left.\frac{d}{d\varphi}\right|_{\varphi=0}\left(\partial^{2j}_{1^{2j}}\frac{1}{Q}(B_\varphi\xi')\right) + 2j\,\partial^{2j}_{1^{2j-1}3}\frac{1}{Q}(\xi')\right\} d\mathcal{H}^{n-2}(\xi'). \tag{28}$$

Moreover, we compute on $\{\xi'_1 = 0\}$

$$\frac{d}{d\varphi}\Big|_{\varphi=0}\left(\partial^{2j}_{1^{2j}}\frac{1}{Q}(B_\varphi\xi')\right) = \sum_{h=1}^{n}\partial^{2j+1}_{1^{2j}h}\frac{1}{Q}(\xi')\frac{\partial(B_\varphi\xi')_h}{\partial\varphi}\Big|_{\varphi=0}$$
$$= \partial^{2j+1}_{1^{2j+1}}\frac{1}{Q}(\xi')\,(-\sin\varphi\xi'_1 - \cos\varphi\xi'_3)\Big|_{\varphi=0} + \partial^{2j+1}_{1^{2j}3}\frac{1}{Q}(\xi')\,(\cos\varphi\xi'_1 - \sin\varphi\xi'_3)\Big|_{\varphi=0}$$
$$= -\xi'_3\,\partial^{2j+1}_{1^{2j+1}}\frac{1}{Q}(\xi'). \tag{29}$$

Inserting in (28), we infer

$$\frac{\partial J}{\partial y_3}\Big|_{(y_1,0,\cdots,0)} = \frac{1}{|y|}\frac{\partial H}{\partial\varphi}\Big|_{\varphi=0} = \frac{1}{|y|^{2k}}\int_{\substack{|\xi'|=1\\ \xi'_1=0}}\left(-\xi'_3\,\partial^{2j+1}_{1^{2j+1}}\frac{1}{Q}(\xi') + 2j\,\partial^{2j}_{1^{2j-1}3}\frac{1}{Q}(\xi')\right)d\mathcal{H}^{n-2}(\xi').$$

Of course, an analogous formula holds for any $h \in \{2,\cdots,n\}$, namely

$$\frac{\partial J}{\partial y_h}\Big|_{(y_1,0,\cdots,0)} = \frac{1}{|y|^{2k}}\int_{\substack{|\xi'|=1\\ \xi'_1=0}}\left(-\xi'_h\,\partial^{2j+1}_{1^{2j+1}}\frac{1}{Q}(\xi') + 2j\,\partial^{2j}_{1^{2j-1}h}\frac{1}{Q}(\xi')\right)d\mathcal{H}^{n-2}(\xi').$$

Step 2. Let us now consider $y \in \mathbb{R}^n \setminus \{0\}$, so $y = |y|b_1$, where $|b_1| = 1$, and let $B = \big(b_1 \mid \cdots \mid b_n\big) \in SO(n)$. Defining $\tilde{J} := J \circ B$, one has

$$\nabla J(y) = \nabla J(B(|y|,0,\cdots,0)^T) = \nabla\tilde{J}(|y|,0,\cdots,0)B^T$$
$$= \frac{1}{|y|^{2k}}\int_{\substack{|\xi'|=1\\ \xi'_1=0}}\begin{pmatrix} -(2k-1)\partial^{2j}_{1^{2j}}\frac{1}{Q}(B\xi') \\ 2j\,\partial^{2j}_{1^{2j-1}2}\frac{1}{Q}(B\xi') - \xi'_2\partial^{2j+1}_{1^{2j+1}}\frac{1}{Q}(B\xi') \\ \cdots \\ 2j\,\partial^{2j}_{1^{2j-1}n}\frac{1}{Q}(B\xi') - \xi'_n\partial^{2j+1}_{1^{2j+1}}\frac{1}{Q}(B\xi') \end{pmatrix}^T d\mathcal{H}^{n-2}(\xi')\cdot B^T$$
$$= \frac{1}{|y|^{2k}}\int_{\substack{|\xi'|=1\\ \xi'_1=0}} 2j\,\nabla\left(\partial^{2j-1}_{1^{2j-1}}\frac{1}{Q}(B\xi')\right)\cdot B^T - (B\xi')^T\cdot\partial^{2j+1}_{1^{2j+1}}\frac{1}{Q}(B\xi')$$
$$-(2k-1+2j)\begin{pmatrix}\partial^{2j}_{1^{2j}}\frac{1}{Q}(B\xi') \\ 0 \\ \cdots \\ 0\end{pmatrix}^T B^T d\mathcal{H}^{n-2}(\xi').$$

Returning therefore to the variable $\xi = B\xi'$, we get

$$\nabla J(y) = \frac{1}{|y|^{2k}}\Bigg[2j\int_{\substack{|\xi|=1\\ y\cdot\xi=0}}\nabla^{2j}\frac{1}{Q}(\xi)\left(\frac{y}{|y|}^{\otimes 2j-1},\cdot\right)d\mathcal{H}^{n-2}(\xi) - \int_{\substack{|\xi|=1\\ y\cdot\xi=0}}\nabla^{2j+1}\frac{1}{Q}(\xi)\left(\frac{y}{|y|}^{\otimes 2j+1}\right)\xi^T\,d\mathcal{H}^{n-2}(\xi)$$
$$-(2k-1+2j)\int_{\substack{|\xi|=1\\ y\cdot\xi=0}}\nabla^{2j}\frac{1}{Q}(\xi)\left(\frac{y}{|y|}^{\otimes 2j}\right)\frac{y^T}{|y|}d\mathcal{H}^{n-2}(\xi)\Bigg], \tag{30}$$

that is, (22).

Step 3. Let us again consider $y = (y_1,0,\cdots,0)$, $y_1 > 0$, and compute $\Delta J(y)$. Defining $\tilde{H}(\varphi) := \partial_{y_3}J(B_\varphi y)$, according to the splitting in (30), we have

$$\tilde{H}(\varphi) =: \frac{1}{|y|^{2k}}\big(2j\,\tilde{H}_0(\varphi) - \tilde{H}_1(\varphi) - (2k-1+2j)\tilde{H}_2(\varphi)\big). \tag{31}$$

Let us differentiate with respect to φ term by term. Firstly,

$$\tilde{H}_0(\varphi) = \int_{\substack{|\xi|=1 \\ (B_\varphi y)\cdot\xi=0}} \nabla^{2j}\frac{1}{Q}(\xi)\left(\frac{B_\varphi y}{|y|}^{\otimes 2j-1}, e_3\right) d\mathcal{H}^{n-2}(\xi)$$

$$\overset{\xi=B_\varphi\xi'}{=} \int_{\substack{|\xi'|=1 \\ \xi_1'=0}} \nabla^{2j}\frac{1}{Q}(B_\varphi\xi)\left(\begin{pmatrix}\cos\varphi \\ 0 \\ \sin\varphi \\ 0 \\ \dots \\ 0\end{pmatrix}^{\otimes 2j-1}, e_3\right) d\mathcal{H}^{n-2}(\xi')$$

$$= \int_{\substack{|\xi'|=1 \\ \xi_1'=0}} \sum_{h_1,\cdots,h_{2j-1}=1}^{n} \left(\nabla^{2j}\frac{1}{Q}(B_\varphi\xi)\right)_{3,h_1,\cdots,h_{2j-1}} \begin{pmatrix}\cos\varphi \\ 0 \\ \sin\varphi \\ 0 \\ \dots \\ 0\end{pmatrix}_{h_1} \cdots \begin{pmatrix}\cos\varphi \\ 0 \\ \sin\varphi \\ 0 \\ \dots \\ 0\end{pmatrix}_{h_{2j-1}} d\mathcal{H}^{n-2}(\xi')$$

$$= \int_{\substack{|\xi'|=1 \\ \xi_1'=0}} \sum_{h=0}^{2j-1}\binom{2j-1}{h}\partial^{2j}_{1^{2j-1-h}3^{h+1}}\frac{1}{Q}(B_\varphi\xi')\,\cos^{2j-1-h}\varphi\,\sin^h\varphi\, d\mathcal{H}^{n-2}(\xi').$$

Therefore there holds

$$\begin{aligned}\frac{d}{d\varphi}\bigg|_{\varphi=0}\tilde{H}_0(\varphi) = \int_{\substack{|\xi'|=1 \\ \xi_1'=0}} \sum_{h=0}^{2j-1}\binom{2j-1}{h}\Bigg\{&\frac{d}{d\varphi}\bigg|_{\varphi=0}\left(\partial^{2j}_{1^{2j-1-h}3^{h+1}}\frac{1}{Q}(B_\varphi\xi')\right)\left(\cos^{2j-1-h}\varphi\bigg|_{\varphi=0}\right)\left(\sin^h\varphi\bigg|_{\varphi=0}\right)\\ &+\partial^{2j}_{1^{2j-1-h}3^{h+1}}\frac{1}{Q}(\xi')\Bigg[\left(\sin^h\varphi\bigg|_{\varphi=0}\right)\left(\frac{d}{d\varphi}\bigg|_{\varphi=0}\cos^{2j-1-h}\varphi\right)\\ &+\left(\cos^{2j-1-h}\varphi\bigg|_{\varphi=0}\right)\left(\frac{d}{d\varphi}\bigg|_{\varphi=0}\sin^h\varphi\right)\Bigg]\Bigg\} d\mathcal{H}^{n-2}(\xi').\end{aligned}$$

As in Step 1, the terms which remain are the first with $h=0$ and the third with $h=1$, so

$$\frac{d}{d\varphi}\bigg|_{\varphi=0}\tilde{H}_0(\varphi) = \int_{\substack{|\xi'|=1 \\ \xi_1'=0}}\left[\frac{d}{d\varphi}\bigg|_{\varphi=0}\left(\partial^{2j}_{1^{2j-1}3}\frac{1}{Q}(B_\varphi\xi')\right)+(2j-1)\partial^{2j}_{1^{2j-2}3^2}\frac{1}{Q}(\xi')\right] d\mathcal{H}^{n-2}(\xi').$$

Differentiating the first term as in (29) on $\{\xi_1'=0\}$,

$$\frac{d}{d\varphi}\bigg|_{\varphi=0}\left(\partial^{2j}_{1^{2j-1}3}\frac{1}{Q}(B_\varphi\xi')\right) = \sum_{h=1}^{n}\partial^{2j+1}_{1^{2j-1}3h}\frac{1}{Q}(\xi')\left(\frac{\partial(B_\varphi\xi')_h}{\partial\varphi}\right)\bigg|_{\varphi=0} = -\xi_3'\,\partial^{2j+1}_{1^{2j}3}\frac{1}{Q}(\xi'),$$

we obtain hence

$$\frac{d}{d\varphi}\bigg|_{\varphi=0}\tilde{H}_0(\varphi) = \int_{\substack{|\xi'|=1 \\ \xi_1'=0}}\left[(2j-1)\,\partial^{2j}_{1^{2j-2}3^2}\frac{1}{Q}(\xi') - \xi_3'\,\partial^{2j+1}_{1^{2j}3}\frac{1}{Q}(\xi')\right] d\mathcal{H}^{n-2}(\xi'). \tag{32}$$

Let us now address to the second term in (31):

$$\tilde{H}_1(\varphi) = \int_{\substack{|\xi|=1 \\ (B_\varphi y)\xi=0}} \xi_3\, \nabla^{2j+1}\frac{1}{Q}(\xi) \left(\frac{B_\varphi y}{|y|}^{\otimes\, 2j+1}\right) d\mathcal{H}^{n-2}(\xi)$$

$$\overset{\xi=B_\varphi\xi'}{=} \int_{\substack{|\xi'|=1 \\ \xi_1'=0}} (B_\varphi\xi')_3\, \nabla^{2j+1}\frac{1}{Q}(B_\varphi\xi') \begin{pmatrix}\cos\varphi \\ 0 \\ \sin\varphi \\ 0 \\ \cdots \\ 0\end{pmatrix}^{\otimes\, 2j+1} d\mathcal{H}^{n-2}(\xi')$$

$$= \int_{\substack{|\xi'|=1 \\ \xi_1'=0}} (\xi_1'\sin\varphi + \xi_3'\cos\varphi) \sum_{h_1,\cdots,h_{2j+1}=1}^{n} \partial^{2j+1}_{h_1\cdots h_{2j+1}}\frac{1}{Q}(B_\varphi\xi') \begin{pmatrix}\cos\varphi \\ 0 \\ \sin\varphi \\ 0 \\ \cdots \\ 0\end{pmatrix}_{h_1} \cdots \begin{pmatrix}\cos\varphi \\ 0 \\ \sin\varphi \\ 0 \\ \cdots \\ 0\end{pmatrix}_{h_{2j+1}} d\mathcal{H}^{n-2}(\xi')$$

$$= \int_{\substack{|\xi'|=1 \\ \xi_1'=0}} (\xi_1'\sin\varphi + \xi_3'\cos\varphi) \left(\sum_{h=0}^{2j+1}\binom{2j+1}{h}\partial^{2j+1}_{1^{2j+1-h}3^h}\frac{1}{Q}(B_\varphi\xi')\cos^{2j+1-h}\varphi\,\sin^h\varphi\right) d\mathcal{H}^{n-2}(\xi').$$

Hence, differentiating with respect to φ, with similar computations as for $\tilde{H}_0$, we obtain:

$$\begin{aligned}
\frac{d}{d\varphi}\Big|_{\varphi=0}\tilde{H}_1(\varphi) &= \int_{\substack{|\xi'|=1 \\ \xi_1'=0}} (\xi_1'\cos\varphi - \xi_3'\sin\varphi)\Big|_{\varphi=0} \left(\sum_{h=0}^{2j+1}\binom{2j+1}{h}\partial^{2j+1}_{1^{2j+1-h}3^h}\frac{1}{Q}(B_\varphi\xi')\cos^{2j+1-h}\varphi\,\sin^h\varphi\right)\Big|_{\varphi=0} \\
&\quad + \xi_3' \sum_{h=0}^{2j+1}\binom{2j+1}{h}\frac{d}{d\varphi}\Big|_{\varphi=0}\left(\partial^{2j+1}_{1^{2j+1-h}3^h}\frac{1}{Q}(B_\varphi\xi')\right)\left(\cos^{2j+1-h}\varphi\Big|_{\varphi=0}\right)\left(\sin^h\varphi\Big|_{\varphi=0}\right) \\
&\quad + \xi_3' \sum_{h=0}^{2j+1}\binom{2j+1}{h}\partial^{2j+1}_{1^{2j+1-h}3^h}\frac{1}{Q}(\xi')\Big[\left(\sin^h\varphi\Big|_{\varphi=0}\right)\left(\frac{d}{d\varphi}\Big|_{\varphi=0}\cos^{2j+1-h}\varphi\right) \\
&\quad + \left(\cos^{2j+1-h}\varphi\Big|_{\varphi=0}\right)\left(\frac{d}{d\varphi}\Big|_{\varphi=0}\sin^h\varphi\right)\Big] \\
&= 0 + \int_{\substack{|\xi'|=1 \\ \xi_1'=0}} \xi_3' \left[\frac{d}{d\varphi}\Big|_{\varphi=0}\partial^{2j+1}_{1^{2j+1}}\frac{1}{Q}(B_\varphi\xi') + (2j+1)\partial^{2j+1}_{1^{2j}3}\frac{1}{Q}(\xi')\right] d\mathcal{H}^{n-2}(\xi').
\end{aligned}$$

With similar computations as in (29), we infer

$$\frac{d}{d\varphi}\Big|_{\varphi=0}\tilde{H}_1(\varphi) = \int_{\substack{|\xi'|=1 \\ \xi_1'=0}} \left[(2j+1)\,\xi_3'\,\partial^{2j+1}_{1^{2j}3}\frac{1}{Q}(\xi') - (\xi_3')^2\,\partial^{2j+2}_{1^{2j+2}}\frac{1}{Q}(\xi')\right] d\mathcal{H}^{n-2}(\xi'). \tag{33}$$

Finally, we have to consider the third term in (31):

$$\tilde{H}_2(\varphi) = \int_{\substack{|\xi|=1 \\ (B_\varphi y)\xi=0}} \nabla^{2j}\frac{1}{Q}(\xi)\left(\frac{B_\varphi y}{|y|}^{\otimes 2j}\right)\left(\frac{B_\varphi y}{|y|}\right)_3 d\mathcal{H}^{n-2}(\xi)$$

$$\overset{\xi=B_\varphi\xi'}{=} \int_{\substack{|\xi'|=1 \\ \xi_1'=0}} \sum_{h_1,\cdots,h_{2j}=1}^{n} \partial^{2j}_{h_1\cdots h_{2j}}\frac{1}{Q}(B_\varphi\xi') \begin{pmatrix}\cos\varphi \\ 0 \\ \sin\varphi \\ 0 \\ \cdots \\ 0\end{pmatrix}_{h_1} \cdots \begin{pmatrix}\cos\varphi \\ 0 \\ \sin\varphi \\ 0 \\ \cdots \\ 0\end{pmatrix}_{h_{2j}} \sin\varphi\, d\mathcal{H}^{n-2}(\xi')$$

$$= \int_{\substack{|\xi'|=1 \\ \xi_1'=0}} \sum_{h=0}^{2j}\binom{2j}{h}\partial^{2j}_{1^{2j-h}3^h}\frac{1}{Q}(B_\varphi\xi')\cos^{2j-h}\varphi\,\sin^{h+1}\varphi\; d\mathcal{H}^{n-2}(\xi').$$

Hence,

$$\frac{d}{d\varphi}\bigg|_{\varphi=0}\tilde{H}_2(\varphi) = \int_{\substack{|\xi'|=1\\ \xi_1'=0}} \sum_{h=0}^{2j}\binom{2j}{h}\bigg[\frac{d}{d\varphi}\bigg|_{\varphi=0}\left(\partial^{2j}_{1^{2j-h}3^h}\frac{1}{Q}(B_\varphi\xi')\right)\left(\cos^{2j-h}\varphi\bigg|_{\varphi=0}\right)\left(\sin^{h+1}\varphi\bigg|_{\varphi=0}\right)$$
$$+\,\partial^{2j}_{1^{2j-h}3^h}\frac{1}{Q}(\xi')\Bigg(\left(\sin^{h+1}\varphi\bigg|_{\varphi=0}\right)\left(\frac{d}{d\varphi}\bigg|_{\varphi=0}\cos^{2j-h}\varphi\right)$$
$$+\left(\cos^{2j-h}\varphi\bigg|_{\varphi=0}\right)\left(\frac{d}{d\varphi}\bigg|_{\varphi=0}\sin^{h+1}\varphi\right)\Bigg)\bigg]d\mathcal{H}^{n-2}(\xi').$$

The first two terms vanish for any choice of h, while the last one remains only for $h=0$, so

$$\frac{d}{d\varphi}\bigg|_{\varphi=0}\tilde{H}_2(\varphi) = \int_{\substack{|\xi'|=1\\ \xi_1'=0}} \partial^{2j}_{1^{2j}}\frac{1}{Q}(\xi')\,d\mathcal{H}^{n-2}(\xi'). \tag{34}$$

Hence, recalling the splitting (31), by (32)-(34) we obtain (omitting from now on in each integral its differential $d\mathcal{H}^{n-2}(\xi')$):

$$\frac{\partial^2 J}{\partial y_3^2}\bigg|_{(y_1,0,\cdots,0)} = \frac{1}{|y|}\frac{d}{d\varphi}\bigg|_{\varphi=0}\tilde{H}(\varphi)$$
$$= \frac{1}{|y|^{2k+1}}\left[2j\,\frac{d}{d\varphi}\bigg|_{\varphi=0}\tilde{H}_0(\varphi) - \frac{d}{d\varphi}\bigg|_{\varphi=0}\tilde{H}_1(\varphi) - (2k-1+2j)\,\frac{d}{d\varphi}\bigg|_{\varphi=0}\tilde{H}_2(\varphi)\right]$$
$$= \frac{1}{|y|^{2k+1}}\bigg[-(2k-1+2j)\int_{\substack{|\xi'|=1\\ \xi_1'=0}}\partial^{2j}_{1^{2j}}\frac{1}{Q}(\xi')\;+\;2j(2j-1)\int_{\substack{|\xi'|=1\\ \xi_1'=0}}\partial^{2j}_{1^{2j-2}3^2}\frac{1}{Q}(\xi')$$
$$-(4j+1)\int_{\substack{|\xi'|=1\\ \xi_1'=0}}\xi_3'\;\partial^{2j+1}_{1^{2j}3}\frac{1}{Q}(\xi')\;+\;\int_{\substack{|\xi'|=1\\ \xi_1'=0}}(\xi_3')^2\;\partial^{2j+2}_{1^{2j+2}}\frac{1}{Q}(\xi')\bigg].$$

Therefore, the same being valid for any variable y_h with $h\in\{2,\cdots,n\}$, and recalling (27) if $h=1$, we may compute the Laplacian of J:

$$\Delta J\bigg|_{(y_1,0,\cdots,0)} = \frac{\partial^2 J}{\partial y_1^2}\bigg|_{(y_1,0,\cdots,0)} + \sum_{h=2}^{n}\frac{\partial^2 J}{\partial y_h^2}\bigg|_{(y_1,0,\cdots,0)}$$
$$= \frac{1}{|y|^{2k+1}}\Bigg\{2k(2k-1)\int_{\substack{|\xi'|=1\\ \xi_1'=0}}\partial^{2j}_{1^{2j}}\frac{1}{Q}(\xi') + (n-1)(1-2k-2j)\int_{\substack{|\xi'|=1\\ \xi_1'=0}}\partial^{2j}_{1^{2j}}\frac{1}{Q}(\xi')$$
$$+\;2j(2j-1)\int_{\substack{|\xi'|=1\\ \xi_1'=0}}\sum_{h=2}^{n}\partial^{2j}_{1^{2j-2}h^2}\frac{1}{Q}(\xi') - (4j+1)\int_{\substack{|\xi'|=1\\ \xi_1'=0}}\sum_{h=1}^{n}\xi_h'\,\partial_h\left(\partial^{2j}_{1^{2j}}\frac{1}{Q}\right)(\xi')$$
$$+\int_{\substack{|\xi'|=1\\ \xi_1'=0}}\underbrace{\left(\sum_{h=1}^{n}(\xi_h')^2\right)}_{|\xi'|^2=1}\partial^{2j+2}_{1^{2j+2}}\frac{1}{Q}(\xi')\Bigg\}$$
$$= \frac{1}{|y|^{2k+1}}\Bigg\{\big[2k(2k-1)+(n-1)(1-2k-2j)\big]\int_{\substack{|\xi'|=1\\ \xi_1'=0}}\partial^{2j}_{1^{2j}}\frac{1}{Q}(\xi')$$
$$+\,2j(2j-1)\int_{\substack{|\xi'|=1\\ \xi_1'=0}}\Delta'\left(\partial^{2j-2}_{1^{2j-2}}\frac{1}{Q}\right)(\xi') - (4j+1)\int_{\substack{|\xi'|=1\\ \xi_1'=0}}\partial_\nu\left(\partial^{2j}_{1^{2j}}\frac{1}{Q}\right)(\xi')$$
$$+\int_{\substack{|\xi'|=1\\ \xi_1'=0}}\partial^{2j+2}_{1^{2j+2}}\frac{1}{Q}(\xi')\Bigg\}. \tag{35}$$

Here, we denote

$$\Delta' = \partial^2_{22}+\ldots+\partial^2_{nn}.$$

By homogeneity of the symbol, one has (see Lemma 2.5 below)

$$\partial_\nu \left(\partial^{2j}_{1^{2j}} \frac{1}{Q}\right) = -2(m+j) \left(\partial^{2j}_{1^{2j}} \frac{1}{Q}\right). \tag{36}$$

Moreover, in order to handle the term $\Delta\partial^{2j-2}_{1^{2j-2}} \frac{1}{Q}$, we may apply the following well-known identity

$$\Delta' u = \Delta_S u + H_S \partial_\nu u + \partial^2_{\nu\nu} u \tag{37}$$

with $u = \partial^{2j-2}_{1^{2j-2}} \frac{1}{Q}$ and $S := \{|\xi'| = 1\,,\, \xi'_1 = 0\}$ the manifold on which we are integrating, and where H_S stands for the mean curvature of S, so we have $H_S = (n-2)$. Noticing that the normal derivatives in (37) may be handled as in (36), it remains the term with the tangential part of the Laplacian. However, it vanishes when integrated on S. Hence,

$$\begin{aligned}\int_{\substack{|\xi'|=1\\ \xi'_1=0}} \Delta' \left(\partial^{2j-2}_{1^{2j-2}} \frac{1}{Q}\right)(\xi') &= (n-2) \int_{\substack{|\xi'|=1\\ \xi'_1=0}} \partial_\nu \left(\partial^{2j-2}_{1^{2j-2}} \frac{1}{Q}\right) + \int_{\substack{|\xi'|=1\\ \xi'_1=0}} \partial^2_{\nu\nu} \left(\partial^{2j-2}_{1^{2j-2}} \frac{1}{Q}\right) \\ &\overset{(36)}{=} \big[(n-2)\big(-2(m+j-1)\big) + 2(m+j-1)\big(2m+2j-1\big)\big] \int_{\substack{|\xi'|=1\\ \xi'_1=0}} \partial^{2j-2}_{1^{2j-2}} \frac{1}{Q}(\xi') \\ &= 2(m+j-1)(2m+2j+1-n) \int_{\substack{|\xi'|=1\\ \xi'_1=0}} \partial^{2j-2}_{1^{2j-2}} \frac{1}{Q}(\xi')\end{aligned} \tag{38}$$

Inserting (36) and (38) in (35) and summing the costants, we finally end up with (23) and thus with our formula (26). □

Lemma 2.5. *Let $Q(\cdot)$ be positive and p-homogeneous. Then, one has for any multi-index $\alpha \in \mathbb{N}_0^n$ and any $x \in \mathbb{R}^n \setminus \{0\}$:*

$$x \cdot \nabla \left(D^\alpha \frac{1}{Q}\right)(x) = -(p + |\alpha|) \left(D^\alpha \frac{1}{Q}\right)(x).$$

Proof. By assumption we have for $r > 0$ that

$$\frac{1}{Q}(rx) = r^{-p} \frac{1}{Q}(x).$$

Differentiation with respect to x yields:

$$r^{|\alpha|} \left(D^\alpha \frac{1}{Q}\right)(rx) = r^{-p} \left(D^\alpha \frac{1}{Q}\right)(x) \quad \Rightarrow \quad \left(D^\alpha \frac{1}{Q}\right)(rx) = r^{-p-|\alpha|} \left(D^\alpha \frac{1}{Q}\right)(x).$$

Differentiating now with respect to r gives:

$$x \cdot \nabla \left(D^\alpha \frac{1}{Q}\right)(rx) = (-p - |\alpha|) r^{-p-|\alpha|-1} \left(D^\alpha \frac{1}{Q}\right)(x).$$

The claim follows by putting $r = 1$. □

Proof of Theorem 2.3. Notice that for $k = 1$, ΔF is already in the form (19) by Proposition 2.1. We thus proceed by induction and let us suppose that $\Delta^k F$ has the form (19) for some $k \in \mathbb{N}$, $k > 1$, with coefficients as in (20)-(21), namely

$$\Delta^k F = \sum_{j=1}^{k} d_{k,j}(n,m) J_{k,k-j}.$$

Applying the recursive formula (26), we thus have

$$\Delta^{k+1} F = \sum_{j=1}^{k+1} d_{k+1,j}(n,m) J_{k+1,k+1-j}$$

with

$$d_{k+1,j}(n,m) = d_{k,j}(n,m) + d_{k,j-1}(n,m)c_0(n,m,k,k-j+1) + d_{k,j-2}(n,m)c_\Delta(n,m,k-j+2),$$

for $j = 3,\ldots,k+1$. We have $d_{k,k+1}$ according to (21) and put $d_{k,0} := d_{k,-1} := 0$. Hence, for $j = 1$ and $j = 2$ we have the recurrence relations

$$d_{k+1,2} = d_{k,2} + d_{k,1}\, c_0(n,m,k,k-1),$$

$$d_{k+1,1} = d_{k,1},$$

and for those the formulae (20),(21) are easily checked.

We show the shape of $d_{k+1,j}(n,m)$ for $j = 3,\ldots,k+1$, the cases $j = 1,2$ being analogous but simpler. In order to prove that $d_{k+1,j}(n,m)$ has the shape as in (19) with $k+1$, which will prove Theorem 2.3, we show that the following term is equal to 0:

$$\begin{aligned}
&d_{k,j}(n,m) + d_{k,j-1}(n,m)c_0(n,m,k,k-j+1) + d_{k,j-2}(n,m)c_\Delta(n,m,k-j+2)\\
&\qquad -(-1)^{j-1}c_{k+1,j}\prod_{\ell=2}^{j}(n-2m-2k+2\ell-5)\\
&= \quad (-1)^{j-1}\prod_{\ell=2}^{j-2}(n-2m-2k+2\ell-3)\\
&\qquad \cdot\Big\{ c_{k,j}(n-2m-2k+2j-5)(n-2m-2k+2j-3)\\
&\qquad\quad -c_{k,j-1}(n-2m-2k+2j-5)\\
&\qquad\qquad \cdot\Big(2k(2k-1)+(n-1)(2j-4k-1)+2(4k-4j+5)(m+k+1-j)\Big)\\
&\qquad\quad +4c_{k,j-2}(k+2-j)(2k+3-2j)(m+k+1-j)(-n+2m+2k-2j+5)\\
&\qquad\quad -c_{k+1,j}(n-2m-2k-1)(n-2m-2k+2j-5)\Big\}\\
&= \quad (-1)^{j-1}\prod_{\ell=2}^{j-1}(n-2m-2k+2\ell-3)\\
&\qquad \cdot\Big\{ c_{k,j}(n-2m-2k+2j-3)\\
&\qquad\quad -c_{k,j-1}\cdot\Big(2k(2k-1)+(n-1)(2j-4k-1)+2(4k-4j+5)(m+k+1-j)\Big)\\
&\qquad\quad -4c_{k,j-2}(k+2-j)(2k+3-2j)(m+k+1-j)\\
&\qquad\quad -c_{k+1,j}(n-2m-2k-1)\Big\}\\
&= \quad (-1)^{j-1}\left(\prod_{\ell=2}^{j-1}(n-2m-2k+2\ell-3)\right)\left(\prod_{\ell=2}^{j}\frac{1}{\ell-1}\right)\left(\prod_{\ell=2}^{j-2}(k-\ell+1)(2k-2\ell+1)\right)\\
&\qquad \cdot\Big\{(k-j+2)(2k-2j+3)(k-j+1)(2k-2j+1)(n-2m-2k+2j-3)\\
&\qquad\quad -(j-1)(k-j+2)(2k-2j+3)\\
&\qquad\qquad \cdot\Big(2k(2k-1)+(n-1)(2j-4k-1)+2(4k-4j+5)(m+k+1-j)\Big)\\
&\qquad\quad -4(j-1)(j-2)(k+2-j)(2k+3-2j)(m+k+1-j)\\
&\qquad\quad -k(2k-1)(k-j+2)(2k-2j+3)(n-2m-2k-1)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{j-1}\left(\prod_{\ell=2}^{j-1}(n-2m-2k+2\ell-3)\right)\left(\prod_{\ell=2}^{j}\frac{1}{\ell-1}\right)\left(\prod_{\ell=2}^{j-1}(k-\ell+1)(2k-2\ell+1)\right)\\
&\quad\cdot\Big\{(k-j+1)(2k-2j+1)(n-2m-2k+2j-3)\\
&\quad -(j-1)\cdot\Big(2k(2k-1)+(n-1)(2j-4k-1)+2(4k-4j+5)(m+k+1-j)\Big)\\
&\quad -4(j-1)(j-2)(m+k+1-j)-k(2k-1)(n-2m-2k-1)\Big\}\\
&= (-1)^{j-1}\left(\prod_{\ell=2}^{j-1}(n-2m-2k+2\ell-3)\right)\left(\prod_{\ell=2}^{j}\frac{1}{\ell-1}\right)\left(\prod_{\ell=2}^{j-1}(k-\ell+1)(2k-2\ell+1)\right)\\
&\quad\cdot\Big\{(k-j+1)(2k-2j+1)(n-2m-2k+2j-3)\\
&\quad -k(2k-1)(n-2m-2k+2j-3)-(n-1)(2j-4k-1)(j-1)\\
&\quad -2(j-1)(m+k-j+1)(4k-2j+1)\Big\}\\
&= (-1)^{j-1}\left(\prod_{\ell=2}^{j}(n-2m-2k+2\ell-3)\right)\left(\prod_{\ell=2}^{j}\frac{1}{\ell-1}\right)\left(\prod_{\ell=2}^{j-1}(k-\ell+1)(2k-2\ell+1)\right)\\
&\quad\cdot\Big\{(k-j+1)(2k-2j+1)-k(2k-1)+(j-1)(4k-2j+1)\Big\}\\
&= 0.
\end{aligned}
$$

This concludes the proof of Theorem 2.3. □

2.3 Sign-changing fundamental solutions near the pole in large odd dimensions

As a consequence of the first iteration of the Laplacian for the function F (Proposition 2.1), we pointed out in Remark 1 that if $n = 2m + 1$ then the fundamental solutions for any symbol are of one sign near the pole. Indeed, the explicit formula (10) involves just the symbol itself. In particular, this holds for all homogeneous fourth-order operators with constant coefficients in dimension 5. However, the formula for fundamental solutions found in Theorem 1.1 shows that in higher dimensions tangential derivatives of the symbol appear, whose sign is not a-priori predictable for all symbols.

The aim of this section is thus to prove Theorem 1.2, namely that for any odd dimension $n > 2m + 1$ we can provide examples of elliptic operators for which the corresponding fundamental solutions are *not* of one sign in any arbitrary small neighbourhood of the pole.

For sake of simplicity, we restrict our analysis for the fourth-order setting, i.e. $m = 2$, as already here the proof is quite technical and all the features are already present. As a particular case of Theorem 1.1 we have then

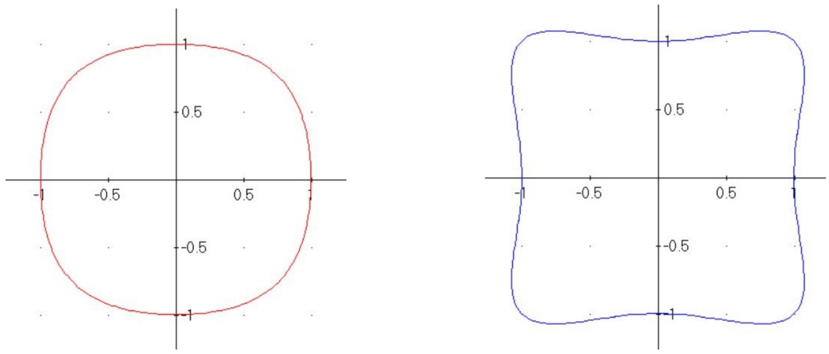
$$
K(0,y) = \frac{(-1)^{\frac{n-1}{2}}}{2(2\pi)^{n-1}|y|^{n-4}}\int_{\substack{|\xi|=1\\ y\cdot\xi=0}} \nabla^{n-5}\frac{1}{Q(\xi)}\left(\frac{y}{|y|}^{\otimes(n-5)}\right) d\mathcal{H}^{n-2}(\xi).
$$

Writing $\mathbb{R}^n \ni \xi = (\xi_1, \xi')$, we choose our symbols among polynomials of the kind

$$
Q_\alpha(\xi) = \xi_1^4 - \alpha\xi_1^2|\xi'|^2 + |\xi'|^4,
$$

where we recall that Q_α is a symbol of a fourth-order uniformly elliptic operator provided $\alpha^2 < 4$. Moreover, Q_α cannot be factored as a power of a second-order (smooth) symbol. Notice also that if $\alpha < 0$, then the sublevels of Q_α are strictly convex, while they are not convex when $\alpha > 0$, see Figure 1.

By (2), without loss of generality, we may consider 0 as the pole of K_L and in what follows we prove that one can find two directions where K_L has different signs. We start with proving the existence of "positive" directions (observe our sign convention for ellipticity) for the fundamental solutions which is somehow the simpler case and which one expects from the notion of ellipticity.



(a) $\alpha = -1$ (b) $\alpha = 1$

Figure 1: The sublevels of Q_α for $\alpha < 0$ (left) and $\alpha > 0$ (right).

Theorem 2.6. *Assume that $n > 2m$, L is a uniformly elliptic operator with constant coefficients of order $2m$ as introduced in (1) and consider the $2m$-homogeneous fundamental solution K_L according to (4) and (5), respectively. Then there exists $y \in \mathbb{R}^n \setminus \{0\}$ such that*

$$K_L(0, y) > 0.$$

Proof. We assume by contradiction that

$$\forall y \in \mathbb{R}^n \setminus \{0\} : \quad K_L(0, y) \leq 0.$$

Certainly, $K_L(0, y) \not\equiv 0$. By continuity there exists a nonempty open set $\Omega \subset \mathbb{S}^{n-1}$ such that we have

$$\forall y \in \mathcal{C}_\Omega : \quad K_L(0, y) < 0$$

on the corresponding cone

$$\mathcal{C}_\Omega := \{r\eta : \ \eta \in \Omega, \ r \in \mathbb{R} \setminus \{0\}\}.$$

We consider a fixed radially symmetric $\varphi \in C_0^\infty(\mathbb{R}^n)$ with

$$0 \leq \varphi(x) \leq 1; \qquad \varphi(x) > 0 \Leftrightarrow |x| \leq \frac{1}{2}.$$

We introduce a corresponding solution (defined in the whole space) of the differential equation by

$$U(x) := \int_{\mathbb{R}^n} K_L(x, y)\varphi(y)\, dy = \int_{\mathbb{R}^n} K_L(0, y)\varphi(x - y)\, dy.$$

Since for $x \in \overline{B_{1/2}(0)}$ the intersection of $(x + \mathcal{C}_\Omega) \cap B_{1/2}(0)$ is nonempty, U is strictly negative there. By compactness we find a constant $C_0 > 0$ such that

$$\forall x \in \overline{B_{1/2}(0)} : \quad U(x) \leq -C_0 < 0.$$

Next, we introduce a scaling parameter

$$\sigma \in (0, 1]$$

and consider for

$$\varphi_\sigma(x) := \varphi(x/\sigma)$$

the solution of the corresponding Dirichlet problem in $\overline{B_1(0)}$

$$\overline{B_1(0)} \ni x \mapsto u_\sigma(x) := \int_{B_1(0)} G(x, y)\varphi_\sigma(y)\, dy.$$

Here,

$$G(x, y) := G_{L, B_1(0)}(x, y) =: K_L(x, y) + h(x, y) = K_L(x, y) + h_{L, B_1(0)}(x, y)$$

denotes the corresponding Green function and its decomposition into fundamental solution and regular part. By continuous dependence on parameters and general elliptic theory (see [ADN]) we find that

$$\forall (x,y) \in \overline{B_{1/2}(0) \times B_{1/2}(0)} : \quad |h(x,y)| \le C_1$$

with a suitable constant C_1. In what follows we consider only $x \in \overline{B_{1/2}(0)}$. By $(2m-n)$-homogeneity of the fundamental solution we obtain:

$$\begin{aligned} u_\sigma(\sigma x) &= \sigma^n \int_{B_1(0)} \left(K_L(\sigma x, \sigma y) + h(\sigma x, \sigma y)\right) \varphi(y)\, dy \\ &= \sigma^{2m} U(x) + \sigma^n \int_{B_{1/2}(0)} h(\sigma x, \sigma y) \varphi(y)\, dy \\ &\le -C_0 \sigma^{2m} + \sigma^n C_1 |B_{1/2}(0)| \\ &\le -\frac{C_0}{2} \sigma^{2m}, \end{aligned}$$

provided that $\sigma \in (0,1]$ is chosen small enough. We fix such a suitable parameter and keep the corresponding u_σ and φ_σ fixed. We recall that we have shown:

$$\varphi_\sigma(x) > 0 \text{ in } B_{\sigma/2}(0), \quad \varphi_\sigma(x) = 0 \text{ outside } B_{\sigma/2}(0), \quad u_\sigma(x) < 0 \text{ in } B_{\sigma/2}(0).$$

This yields (we recall that λ denotes the ellipticity constant of L)

$$\begin{aligned} 0 &> \int_{B_{\sigma/2}(0)} u_\sigma(x) \varphi_\sigma(x)\, dx = \int_{B_1(0)} u_\sigma(x) \varphi_\sigma(x)\, dx = \int_{B_1(0)} u_\sigma(x) \left(L u_\sigma(x)\right)\, dx \\ &\ge \lambda \|u_\sigma\|^2_{H_0^m(B_1(0))} > 0, \end{aligned}$$

a contradiction. In the last step we used the elementary form of Gårding's inequality (see [GL]) for operators, which have only constant coefficients and only of highest order, which follows from the ellipticity condition by employing the Fourier transform. □

It remains to find a "negative" direction which is the more difficult and more interesting case, as such behaviour cannot be observed in the second-order setting. By the particular form of our symbol Q_α, we will produce it by considering $y = e_1$. In that case,

$$K(0, e_1) = \frac{(-1)^{\frac{n-1}{2}}}{2(2\pi)^{n-1}} \int_{\substack{|\xi|=1 \\ \xi_1 = 0}} \partial_1^{n-5} \frac{1}{Q_\alpha(\xi)}\, d\mathcal{H}^{n-2}(\xi).$$

Firstly, we compute the derivatives in the direction e_1 of the symbol. From now on we will refer to it simply as Q to shorten the notation.

$$\partial_1 Q(\xi) = 4\xi_1^3 - 2\alpha \xi_1 |\xi'|^2, \qquad \partial_1^2 Q(\xi) = 12\xi_1^2 - 2\alpha |\xi'|^2, \qquad \partial_1^3 Q(\xi) = 24\xi_1, \qquad \partial_1^4 Q(\xi) = 24,$$

so, on $M := \{\xi \in \mathbb{R}^n \mid \xi_1 = 0, |\xi'| = 1\}$ we have

$$Q(\xi) = 1, \qquad \partial_1 Q(\xi) = 0, \qquad \partial_1^2 Q(\xi) = -2\alpha, \qquad \partial_1^3 Q(\xi) = 0, \qquad \partial_1^4 Q(\xi) = 24. \tag{39}$$

We have now to distinguish 4 cases according to the dimension n.

Case 1: $n \equiv 1$ (mod 8). Let $n = 8k+1$, $k \ge 1$. As $(-1)^{\frac{n-1}{2}} = 1$, we are aiming to prove

$$\partial_1^{4(2k-1)} \frac{1}{Q}\bigg|_M < 0 \tag{40}$$

We notice that when we compute derivatives of the reciprocal Q, thanks to (39), the only terms which remain once we evaluate on M are the ones involving just the second and the fourth derivative of Q. Consequently, on M there holds

$$
\begin{aligned}
\partial_1^{4(2k-1)} \frac{1}{Q} &= \sum_{j=1}^{2k} c_j \frac{(\partial_1^4 Q)^{2k-j} (\partial_1^2 Q)^{2(j-1)}}{Q^{2k+j-1}} \\
&= \sum_{j=1}^{2k} c_j 24^{2k-j} (-2\alpha)^{2(j-1)} \\
&= 24^{2k-1} \sum_{j=1}^{2k} c_j \left(\frac{4\alpha^2}{24}\right)^{j-1} \\
&= 24^{2k-1} \left(c_1 + \sum_{h=1}^{2k-1} c_{h+1} \left(\frac{\alpha^2}{6}\right)^h \right).
\end{aligned}
$$

Noticing that $(-1)^j c_j > 0$, we deduce $c_1 < 0$. Therefore, for values of α so that α^2 is small, then (40) holds. For instance, $\alpha = 0$ is a good choice. Notice that here we are not prescribing the sign of α.

Case 2: $n \equiv 3$ (mod 8). Let $n = 8k + 3$, $k \geq 1$. As $(-1)^{\frac{n-1}{2}} = -1$, we are looking for

$$
\partial_1^{2(4k-1)} \frac{1}{Q} \bigg|_M > 0. \tag{41}
$$

Using (39) as in case 1, on M there holds

$$
\begin{aligned}
\partial_1^{2(4k-1)} \frac{1}{Q} &= \sum_{j=1}^{2k} c_j \frac{(\partial_1^4 Q)^{2k-j} (\partial_1^2 Q)^{2j-1}}{Q^{2k+j}} \\
&= \sum_{j=1}^{2k} (-c_j) 24^{2k-j} (2\alpha)^{2j-1} \\
&= 24^{2k-1} \sum_{j=1}^{2k} (-2\alpha c_j) \left(\frac{4\alpha^2}{24}\right)^{j-1} \\
&= 24^{2k-1} \left(-2\alpha c_1 + \sum_{h=1}^{2k-1} (-2\alpha\, c_{h+1}) \left(\frac{\alpha^2}{6}\right)^h \right).
\end{aligned}
$$

In this case $(-1)^{j-1} c_j > 0$, so in particular $c_1 > 0$. Therefore (41) is obtained provided α^2 is small *and* $\alpha < 0$.

Case 3: $n \equiv 7$ (mod 8). Let $n = 8k + 7$, $k \geq 0$. As $(-1)^{\frac{n-1}{2}} = -1$, we are aiming to prove

$$
\partial_1^{2(4k+1)} \frac{1}{Q} \bigg|_M > 0. \tag{42}
$$

Similarly as in case 2, on M there holds

$$
\begin{aligned}
\partial_1^{2(4k+1)} \frac{1}{Q} &= \sum_{j=1}^{2k+1} c_j \frac{(\partial_1^4 Q)^{2k+1-j} (\partial_1^2 Q)^{2j-1}}{Q^{2k+j+1}} \\
&= \sum_{j=1}^{2k+1} (-2\alpha c_j) 24^{2k+1-j} (-2\alpha)^{2(j-1)} \\
&= 24^{2k} \sum_{j=1}^{2k+1} (-2\alpha c_j) \left(\frac{\alpha^2}{6}\right)^{j-1} \\
&= 24^{2k} \left(-2\alpha c_1 + \sum_{h=1}^{2k} (-2\alpha\, c_{h+1}) \left(\frac{\alpha^2}{6}\right)^h \right).
\end{aligned}
$$

In this case $(-1)^j c_j > 0$, so in particular $c_1 < 0$. Therefore (42) is obtained provided α^2 is small *and* $\alpha > 0$.

Case 4: $n \equiv 5 \pmod 8$ This case turns out to be the most difficult case as, unlike as in the previous ones, one cannot choose small values of α, but has to prescribe them in a much more accurate way. Indeed this time, as $n = 8k + 5$, $k \geq 1$, then $(-1)^{\frac{n-1}{2}} = 1$, so we are aiming to prove

$$\partial_1^{8k} \frac{1}{Q}\bigg|_M < 0,$$

and on M one has

$$\begin{aligned}
\partial_1^{8k} \frac{1}{Q} &= \sum_{j=1}^{2k+1} c_j \frac{(\partial_1^4 Q)^{2k+1-j} (\partial_1^2 Q)^{2(j-1)}}{Q^{2k+j}} \\
&= \sum_{j=1}^{2k+1} c_j 24^{2k+1-j} (-2\alpha)^{2(j-1)} \\
&= 24^{2k} \sum_{j=1}^{2k+1} c_j \left(\frac{\alpha^2}{6}\right)^{j-1}.
\end{aligned} \tag{43}$$

However, here $(-1)^{j-1} c_j > 0$, so in particular $c_1 > 0$. Therefore, in order to end up with a negative sign, we cannot just consider the first term of the sum, being positive independently on α. Hence, we really need not only the sign of the coefficients c_j but also their precise value. Tanking into account all differentiations occurring in the computations for $\partial_1^{8k} \frac{1}{Q}$, one ends up with

$$\begin{aligned}
c_j &= \frac{(2k+j-1)!(-1)^{j-1} \binom{8k}{8k-4}\binom{8k-4}{8k-8} \cdots \binom{4j}{4j-4} \cdot \binom{4j-4}{4j-6} \cdots \binom{4}{2}}{(2k+1-j)!(2j-2)!} \\
&= (-1)^{j-1} \binom{2k+j-1}{2j-2} \frac{(8k)!}{(24)^{2k+1-j}\, 2^{2j-2}}
\end{aligned}$$

Indeed, the first product of binomial coefficients comes from the differentiations with respect to 4-tuples of indexes of $8k$, while the second product from the differentiations with respect to couples of indexes of $8k$; but as the terms of kind $\partial_1^4 Q$ or $\partial_1^2 Q$ cannot be distinguished within the same class, one should also divide by the respective number of permutations.

Therefore, from (43) we get

$$\begin{aligned}
\partial_1^{8k} \frac{1}{Q} &= 24^{2k} (8k)! \sum_{j=1}^{2k+1} (-1)^{j-1} \binom{2k+j-1}{2j-2} \frac{\left(\frac{\alpha^2}{6}\right)^{j-1}}{(24)^{2k} \left(\frac{4}{24}\right)^{j-1}} \\
&= (8k)! \sum_{j=1}^{2k+1} (-1)^{j-1} \binom{2k+j-1}{2j-2} \alpha^{2j-2} \\
&= (8k)! \sum_{j=0}^{2k} (-1)^j \binom{2k+j}{2j} \alpha^{2j}.
\end{aligned} \tag{44}$$

Now we need the following Lemma.

Lemma 2.7. *For any $k \in \mathbb{N}$ and $|y| < 1$ there holds*

$$\sum_{j=0}^{2k} (-1)^j \binom{2k+j}{2j} (2y)^{2j} = \frac{\cos\big((4k+1)\arcsin(y)\big)}{\sqrt{1-y^2}}.$$

Proof. Proving the statement is equivalent to prove the identity between the antiderivative of both sides, namely:

$$\sum_{j=0}^{2k} (-1)^j \binom{2k+j}{2j} 4^j \frac{y^{2j+1}}{2j+1} = \frac{\sin\big((4k+1)\arcsin(y)\big)}{4k+1}. \tag{45}$$

Now, recall the following facts.

i) For any $n \equiv 1 \pmod 4$ we have $\sin(n \arcsin(t)) = T_n(t)$, where T_n is the n-th Chebychev polynomial of the first kind. Indeed,

$$\begin{aligned} T_n(t) &:= \cos(n \arccos(t)) = \cos\left(\frac{n\pi}{2} - n \arcsin(t)\right) = \sin\left(\frac{(n+1)\pi}{2} - n\arcsin(t)\right) \\ &= \sin(n \arcsin(t)) \;\Leftrightarrow\; \frac{n+1}{2} \equiv 1 \,(\mathrm{mod}\ 2) \;\Leftrightarrow\; n \equiv 1 \,(\mathrm{mod}\ 4). \end{aligned}$$

ii) There holds (see [Kr, Chapter 6.4]):

$$T_n(t) = \frac{n}{2}\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{(n-j-1)!}{j!(n-2j)!}(2t)^{n-2j}.$$

Therefore, we have:

$$\begin{aligned} \frac{\sin\big((4k+1)\arcsin(y)\big)}{4k+1} &= \frac{T_{4k+1}(y)}{4k+1} = \frac{1}{2}\sum_{j=0}^{2k}(-1)^j \frac{(4k-j)!}{j!(4k+1-2j)!}(2y)^{4k+1-2j} \\ &= y\sum_{j=0}^{2k}(-1)^j \frac{(4k-j)!}{j!(4k+1-2j)!}(2y)^{4k-2j}. \end{aligned} \tag{46}$$

Set now $h := 2k - j$. Then $h \in \{0, \cdots, 2k\}$ and we have can rewrite (46) as

$$\begin{aligned} \frac{\sin\big((4k+1)\arcsin(y)\big)}{4k+1} &= y\sum_{h=0}^{2k}(-1)^h \frac{(2k+h)!}{(2k-h)!(2h+1)!}(2y)^{2h} \\ &= y\sum_{h=0}^{2k}(-1)^h \frac{(2k+h)!}{(2k-h)!(2h)!}4^h \frac{y^{2h}}{2h+1} \\ &= \sum_{h=0}^{2k}(-1)^h \binom{2k+h}{2h}\frac{y^{2h+1}}{2h+1}, \end{aligned}$$

which is exactly (45). By differentiating both sides, the proof is complete. □

Let us now conclude the proof of the case $n \equiv 5 \pmod 8$. From (44) and applying Lemma 2.7 we get

$$\partial_1^{8k}\frac{1}{Q} = 2(8k)!\frac{\cos\big((4k+1)\arcsin\left(\frac{\alpha}{2}\right)\big)}{\sqrt{4-\alpha^2}}$$

As all terms are positive but the cosine, we need to choose α so that

$$\sin\left(\frac{\pi}{8k+2}\right) < \frac{\alpha}{2} < \sin\left(\frac{3\pi}{8k+2}\right)$$

Notice that this condition automatically implies either that our symbol is positive and that the denominator in (44) is well-defined. This choice concludes case 4 and, recalling Theorem 2.6, Theorem 1.2 is finally proved.

3 Even dimensions $n \geq 2m$

Here the starting point is John's formula (5), according to which we have

$$\begin{aligned} K(0,y) &= -\frac{1}{(2\pi)^n}(-\Delta_y)^{(n-2m)/2}\int_{|\xi|=1}\frac{\log|y\cdot\xi|}{Q(\xi)}\,d\mathcal{H}^{n-1}(\xi) \\ &= -\frac{2}{(2\pi)^n}(-\Delta_y)^{(n-2m)/2}\int_{\substack{|\xi|=1 \\ y\cdot\xi>0}}\frac{\log(y\cdot\xi)}{Q(\xi)}\,d\mathcal{H}^{n-1}(\xi). \end{aligned} \tag{47}$$

For $n = 2m$ we immediately see that the fundamental solution of L satisfies

$$K(0, y) \to +\infty \text{ as } y \to 0,$$

which is parallel to the case $n = 2m + 1$ above.

Hence, in order to obtain an explicit expression for K_L, we thus have to compute the Laplacian of

$$G(y) := \int_{\substack{|\xi|=1 \\ y\cdot\xi>0}} \frac{\log(y \cdot \xi)}{Q(\xi)} d\mathcal{H}^{n-1}(\xi). \tag{48}$$

Due to the logarithmic term, the calculations for even dimensions cannot be simplified similarly to the previous section. An application of Stokes' theorem would change $\log(\xi_1)$ into $\frac{1}{\xi_1}$, a non-integrable singularity. For this reason we restrict ourselves to the case $n = 2m + 2$.

3.1 Proof of Theorem 1.3

The strategy is similar to the one applied in the proofs of Propositions 2.1 and 2.4.
Step 1. Let $y = (y_1, 0, \cdots, 0)$ with $y_1 > 0$. Splitting G as

$$\begin{aligned} G(y) &= \int_{\substack{|\xi|=1 \\ y\cdot\xi>0}} \frac{\log(y_1\xi_1)}{Q(\xi)} d\mathcal{H}^{n-1}(\xi) \\ &= \log(y_1) \int_{\substack{|\xi|=1 \\ y\cdot\xi>0}} \frac{1}{Q(\xi)} d\mathcal{H}^{n-1}(\xi) + \int_{\substack{|\xi|=1 \\ y\cdot\xi>0}} \frac{\log(\xi_1)}{Q(\xi)} d\mathcal{H}^{n-1}(\xi), \end{aligned}$$

we first infer

$$\frac{\partial G}{\partial y_1}(y) = \frac{1}{y_1} \int_{\substack{|\xi|=1 \\ \xi_1>0}} \frac{1}{Q(\xi)} d\mathcal{H}^{n-1}(\xi) = \frac{1}{2y_1} \int_{|\xi|=1} \frac{1}{Q(\xi)} d\mathcal{H}^{n-1}(\xi) \tag{49}$$

as $Q(\xi) = Q(-\xi)$ by homogeneity of the symbol Q. Moreover,

$$\frac{\partial^2 G}{\partial y_1^2}(y) = -\frac{1}{2y_1^2} \int_{|\xi|=1} \frac{1}{Q(\xi)} d\mathcal{H}^{n-1}(\xi). \tag{50}$$

In order to compute the other first derivatives of G, let us split it as

$$\begin{aligned} G(y) &= \log|y| \int_{\substack{|\xi|=1 \\ y\cdot\xi>0}} \frac{1}{Q(\xi)} d\mathcal{H}^{n-1}(\xi) + \int_{\substack{|\xi|=1 \\ y\cdot\xi>0}} \frac{\log\left(\frac{y}{|y|} \cdot \xi\right)}{Q(\xi)} d\mathcal{H}^{n-1}(\xi) \\ &=: \log|y|\, \tilde{G}_1(y) + \tilde{G}_2(y). \end{aligned}$$

Hence,

$$\frac{\partial G}{\partial y_3}(y) = \frac{y_3}{|y|^2} \tilde{G}_1(y) + \log|y| \frac{\partial \tilde{G}_1}{\partial y_3}(y) + \frac{\partial \tilde{G}_2}{\partial y_3}(y). \tag{51}$$

Introduce the matrix B_φ defined in (13) and define

$$\tilde{H}_1(\varphi) = \tilde{G}_1(B_\varphi y) \qquad \text{and} \qquad \tilde{H}_2(\varphi) = \tilde{G}_2(B_\varphi y).$$

Concerning the first term, noticing that $Q(\xi) = Q(-\xi)$, for any y we have

$$\tilde{G}_1(y) = \frac{1}{2} \int_{|\xi|=1} \frac{1}{Q(\xi)}\, d\mathcal{H}^{n-1}(\xi), \qquad \Rightarrow \qquad \frac{\partial \tilde{G}_1}{\partial y_3}(y) = 0 \quad \text{for any } \; y. \tag{52}$$

Concerning the second, reasoning as in (14), we get

$$\begin{aligned} y_1 \frac{\partial \tilde{G}_2}{\partial y_3}(y) &= \frac{\partial \tilde{H}_2}{\partial \varphi}(\varphi)\Big|_{\varphi=0} = \frac{\partial}{\partial \varphi}\Big|_{\varphi=0} \int_{\substack{|\xi|=1 \\ \xi_1>0}} \frac{\log\left(\frac{y}{|y|} \cdot \xi\right)}{Q(B_\varphi \xi)} d\mathcal{H}^{n-1}(\xi) \\ &= \int_{\substack{|\xi|=1 \\ \xi_1>0}} \log(\xi_1) \Big(-\xi_3 \partial_1 \frac{1}{Q}(\xi) + \xi_1 \partial_3 \frac{1}{Q}(\xi) \Big) d\mathcal{H}^{n-1}(\xi) \end{aligned} \tag{53}$$

Therefore, by (51)-(53) we conclude

$$\frac{\partial G}{\partial y_3}(y) = \frac{y_3}{|y|^2}\int_{\substack{|\xi|=1\\ y\cdot\xi>0}} \frac{1}{Q(\xi)} d\mathcal{H}^{n-1}(\xi) + \frac{1}{|y|}\int_{\substack{|\xi|=1\\ \xi_1>0}} \log(\xi_1)\Big(-\xi_3\partial_1\frac{1}{Q}(\xi) + \xi_1\partial_3\frac{1}{Q}(\xi)\Big) d\mathcal{H}^{n-1}(\xi)$$

and, analogously, for any $k \in \{2, \cdots, n\}$, one has

$$\begin{aligned}\frac{\partial G}{\partial y_k}(y_1, 0\cdots, 0) = &\frac{y_k}{2|y|^2}\int_{|\xi|=1} \frac{1}{Q(\xi)} d\mathcal{H}^{n-1}(\xi)\\ &+ \frac{1}{|y|}\int_{\substack{|\xi|=1\\ \xi_1>0}} \log\left(\xi\cdot\frac{y}{|y|}\right)\Big(-\xi_k\Big(\nabla\frac{1}{Q}(\xi)\cdot\frac{y}{|y|}\Big) + \Big(\xi\cdot\frac{y}{|y|}\Big)\partial_k\frac{1}{Q}(\xi)\Big) d\mathcal{H}^{n-1}(\xi).\end{aligned}$$

As this formula above is coherent with (49), we can simply write it in the more compact way as

$$\begin{aligned}\nabla G(y) = &\frac{y^T}{2|y|^2}\int_{|\xi|=1} \frac{1}{Q(\xi)} d\mathcal{H}^{n-1}(\xi)\\ &+ \frac{1}{|y|}\int_{\substack{|\xi|=1\\ y\cdot\xi>0}} \log\left(\xi\cdot\frac{y}{|y|}\right)\Big(-\xi^T\Big(\nabla\frac{1}{Q}(\xi)\cdot\frac{y}{|y|}\Big) + \Big(\xi\cdot\frac{y}{|y|}\Big)\nabla\frac{1}{Q}(\xi)\Big) d\mathcal{H}^{n-1}(\xi).\end{aligned} \tag{54}$$

Step 2. Let now $y \in \mathbb{R}^n \setminus \{0\}$ so $y = |y|b_1$, with $|b_1| = 1$, and let $B = \big(b_1 \mid \cdots \mid b_n\big) \in SO(n)$. Defining

$$\tilde{G}(y) := (G\circ B)(y) = \int_{\substack{|\xi|=1\\ y\cdot\xi>0}} \frac{\log(y\cdot\xi)}{Q(B(\xi))} d\mathcal{H}^{n-1}(\xi),$$

we have

$$\nabla G(y) = \nabla G\big(B(|y|, 0, \cdots, 0)^T\big) = \nabla\tilde{G}\big((|y|, 0, \cdots, 0)^T\big)B^T.$$

Therefore,

$$\begin{aligned}\nabla\tilde{G}\big((|y|, 0, \cdots, 0)^T\big) = &\frac{1}{2|y|}e_1^T\int_{|\xi|=1} \frac{1}{Q(B(\xi))} d\mathcal{H}^{n-1}(\xi)\\ &+ \frac{1}{|y|}\int_{\substack{|\xi|=1\\ y\cdot\xi>0}} \log(\xi_1)\Big(-\xi^T\big(\nabla\frac{1}{Q}(B(\xi))\cdot Be_1\big) + \xi_1\,\nabla\frac{1}{Q}(B(\xi))\cdot B\Big) d\mathcal{H}^{n-1}(\xi)\\ = &\frac{1}{2|y|}e_1^T\int_{|\xi|=1} \frac{1}{Q(\xi)} d\mathcal{H}^{n-1}(\xi)\\ &+ \frac{1}{|y|}\int_{\substack{|\xi|=1\\ (B^T\xi)_1>0}} \log((B^T\xi)_1)\Big(-\xi^T B\big(\nabla\frac{1}{Q}(\xi)\cdot\frac{y}{|y|}\big) + (B^T\xi)_1\,\nabla\frac{1}{Q}(\xi)\cdot B\Big) d\mathcal{H}^{n-1}(\xi).\end{aligned}$$

Observing that $(B^T\xi)_1 = \frac{y\cdot\xi}{|y|}$ and multiplying by B^T shows that (54) holds also for any $y \in \mathbb{R}^n \setminus \{0\}$.

Step 3. Let us consider again $y = (y_1, 0, \cdots, 0)$ with $y_1 > 0$ and let us compute $\Delta G(y)$. Similarly as in Step 1, we get

$$\begin{aligned}\partial_3^2\, G(y_1, 0, \cdots, 0) = &\frac{1}{2|y|^2}\int_{|\xi|=1} \frac{1}{Q(\xi)} d\mathcal{H}^{n-1}(\xi) + \frac{1}{|y|}\frac{\partial}{\partial\varphi}\Big|_{\varphi=0}\frac{1}{|B_\varphi y|}\int_{\substack{|\xi|=1\\ B_\varphi y\cdot\xi>0}} \log\left(\frac{B_\varphi y\cdot\xi}{|B_\varphi y|}\right)\\ &\cdot\Big(-\xi_3\Big(\nabla\frac{1}{Q}(\xi)\cdot\frac{B_\varphi y}{|B_\varphi y|}\Big) + \Big(\frac{\xi\cdot B_\varphi y}{|B_\varphi y|}\,\partial_3\frac{1}{Q}(\xi)\Big)\Big) d\mathcal{H}^{n-1}(\xi).\end{aligned}$$

A change of variables and (13) imply

$$\partial_3^2 G(y_1,0,\cdots,0) - \frac{1}{2|y|^2}\int_{|\xi|=1}\frac{1}{Q(\xi)}d\mathcal{H}^{n-1}(\xi)$$

$$= \frac{1}{y_1}\frac{1}{|y|}\frac{\partial}{\partial\varphi}\Big|_{\varphi=0}\int_{\substack{|\xi|=1\\ \xi_1>0}}\log(\xi_1)\Big(-(\xi_1\sin\varphi+\xi_3\cos\varphi)\nabla\frac{1}{Q(B_\varphi\xi)}\cdot\begin{pmatrix}\cos\varphi\\ 0\\ \sin\varphi\\ 0\\ \cdots\\ 0\end{pmatrix}+\xi_1\,\partial_3\frac{1}{Q(B_\varphi\xi)}\Big)d\mathcal{H}^{n-1}(\xi)$$

$$= \frac{1}{y_1^2}\int_{\substack{|\xi|=1\\ \xi_1>0}}\log(\xi_1)\Big(-\xi_1\,\partial_1\frac{1}{Q}-\xi_3\,\partial_3\frac{1}{Q}+\xi_3^2\,\partial_1^2\frac{1}{Q}+\xi_1^2\,\partial_3^2\frac{1}{Q}-2\xi_1\xi_3\,\partial_{13}^2\frac{1}{Q}\Big)d\mathcal{H}^{n-1}(\xi).$$

As the same holds for any $k\in\{2,\cdots,n\}$, we infer

$$\partial_3^2 G(y_1,0,\cdots,0) = \frac{1}{2|y|^2}\int_{|\xi|=1}\frac{1}{Q(\xi)}d\mathcal{H}^{n-1}(\xi)+\frac{1}{|y|^2}\int_{\substack{|\xi|=1\\ \xi_1>0}}\log\left(\xi\cdot\frac{y}{|y|}\right)$$
$$\cdot\Big(-\xi_1\,\partial_1\frac{1}{Q}-\xi_k\,\partial_k\frac{1}{Q}+\xi_k^2\,\partial_1^2\frac{1}{Q}+\xi_1^2\,\partial_k^2\frac{1}{Q}-2\xi_1\xi_k\,\partial_{1k}^2\frac{1}{Q}\Big)d\mathcal{H}^{n-1}(\xi).$$

This, together with (50) yields (omitting from now on the differentials)

$$\begin{aligned}\Delta G(y_1,0,\cdots,0) &= \frac{n-2}{2|y|^2}\int_{|\xi|=1}\frac{1}{Q}+\frac{1}{|y|^2}\int_{\substack{|\xi|=1\\ \xi_1>0}}\log\left(\xi\cdot\frac{y}{|y|}\right)\Big[-(n-1)\xi_1\,\partial_1\frac{1}{Q}\\ &-\sum_{k=2}^n\xi_k\partial_k\frac{1}{Q}+\Big(\sum_{k=2}^n\xi_k^2\Big)\partial_1^2\frac{1}{Q}+\xi_1^2\Big(\sum_{k=2}^n\partial_k^2\frac{1}{Q}\Big)-2\xi_1\Big(\sum_{k=2}^n\xi_k\partial_k\Big(\partial_1\frac{1}{Q}\Big)\Big)\Big]\\ &= \frac{n-2}{2|y|^2}\int_{|\xi|=1}\frac{1}{Q}+\frac{1}{|y|^2}\int_{\substack{|\xi|=1\\ y\cdot\xi>0}}\log\left(\xi\cdot\frac{y}{|y|}\right)\Big[-(n-2)\xi_1\,\partial_1\frac{1}{Q}-\xi\cdot\nabla\frac{1}{Q}\\ &+\partial_1^2\frac{1}{Q}+\xi_1^2\Delta\frac{1}{Q}-2\xi_1\,\xi\cdot\nabla\Big(\partial_1\frac{1}{Q}\Big)\Big].\end{aligned}$$

Step 4. Let now $y\in\mathbb{R}^n\setminus\{0\}$ and reasoning as in Step 2 and with the same notations as therein we obtain

$$\begin{aligned}\Delta G(y) &= \Delta\tilde{G}(B^Ty)=\Delta\tilde{G}((|y|,0,\cdots,0)^T)=\frac{n-2}{2|y|^2}\int_{|\xi|=1}\frac{1}{Q}+\frac{1}{|y|^2}\int_{\substack{|\xi|=1\\ \xi_1>0}}\log(\xi\cdot e_1)\\ &\cdot\Big[-(n-2)(\xi\cdot e_1)\Big(e_1\cdot\Big(\nabla\frac{1}{Q}(B\xi)\cdot B\Big)-\xi\cdot\Big(\nabla\frac{1}{Q}(B\xi)\cdot B\Big)+e_1^T\cdot B^TD\nabla^2\frac{1}{Q}(B\xi)B\cdot e_1\\ &+(\xi\cdot e_1)^2\Delta\frac{1}{Q}(B\xi)-2(\xi\cdot e_1)\Big(\xi^T\cdot B^T\nabla^2\frac{1}{Q}(B\xi)B\cdot e_1\Big)\Big].\end{aligned} \tag{55}$$

Recalling that $B\cdot e_1=\frac{y}{|y|}$, (55) implies

$$\begin{aligned}\Delta G(y) &= \frac{n-2}{2}\frac{1}{|y|^2}\int_{|\xi|=1}\frac{1}{Q(\xi)}\,d\mathcal{H}^{n-1}(\xi)+\frac{1}{|y|^2}\int_{\substack{|\xi|=1\\ y\cdot\xi>0}}\log\left(\xi\cdot\frac{y}{|y|}\right)\\ &\cdot\Big[-(n-2)\Big(\xi\cdot\frac{y}{|y|}\Big)\Big(\nabla\frac{1}{Q(\xi)}\cdot\frac{y}{|y|}\Big)-\xi\cdot\nabla\frac{1}{Q(\xi)}+\frac{y^T}{|y|}\cdot\nabla^2\frac{1}{Q(\xi)}\cdot\frac{y}{|y|}\\ &+\Big(\xi\cdot\frac{y}{|y|}\Big)^2\Delta\frac{1}{Q(\xi)}-2\Big(\xi\cdot\frac{y}{|y|}\Big)\Big(\xi^T\cdot\nabla^2\frac{1}{Q(\xi)}\cdot\frac{y}{|y|}\Big)\Big]\,d\mathcal{H}^{n-1}(\xi).\end{aligned}$$

Finally, due to the $2m$-homogeneity of Q by means of Lemma 2.5 we have

$$\forall\xi\in\mathbb{S}^{n-1}:\quad \xi\cdot\nabla\frac{1}{Q(\xi)}=-2m\frac{1}{Q(\xi)},\quad \xi^T\cdot\nabla^2\frac{1}{Q(\xi)}\cdot\frac{y}{|y|}=-(2m+1)\left(\nabla\frac{1}{Q(\xi)}\cdot\frac{y}{|y|}\right).$$

Hence the previous formula simplifies to

$$\begin{aligned}\Delta G(y) &= \frac{n-2}{2}\frac{1}{|y|^2}\int_{|\xi|=1}\frac{1}{Q(\xi)}\,d\mathcal{H}^{n-1}(\xi)+\frac{1}{|y|^2}\int_{\substack{|\xi|=1\\ y\cdot\xi>0}}\log\left(\xi\cdot\frac{y}{|y|}\right)\\ &\quad\cdot\left((4m+4-n)\left(\xi\cdot\frac{y}{|y|}\right)\left(\nabla\frac{1}{Q(\xi)}\cdot\frac{y}{|y|}\right)+2m\frac{1}{Q(\xi)}\right.\\ &\quad\left.+\frac{y^T}{|y|}\cdot\nabla^2\frac{1}{Q(\xi)}\cdot\frac{y}{|y|}+\left(\xi\cdot\frac{y}{|y|}\right)^2\Delta\frac{1}{Q(\xi)}\right)\,d\mathcal{H}^{n-1}(\xi)\end{aligned}\tag{56}$$

and, recalling (2), the proof is concluded.

3.2 Sign-changing fundamental solutions near the pole in large even dimensions

The aim of this section is to show that in even dimensions a similar behaviour of fundamental solutions can be observed as in odd dimensions. In particular, as we remarked at the beginning of Section 3, if $n=2m$ from John's formula (5) it is clear that fundamental solutions are of one sign near the pole. Hence, $n=2m$ can be seen as corresponding to the case $n=2m+1$. Using now the explicit expression found in Theorem 1.3 and restricting again our analysis to the fourth-order setting, we prove that the behaviour of fundamental solutions for $n=6$ is similar to the one appearing in Section 2.3 when $n=7$. In particular, our examples will be found in the same class of symbols.

Let thus $m=2$, $n=6$ and consider again the symbol $Q_\alpha=\xi_1^4-\alpha\xi_1^2|\xi'|^2+|\xi'|^4$ defined in (6). Recall that Q_α defines an elliptic operator provided $\alpha^2<4$. Similarly as for $n=7$, we shall find the desired examples for values of $\alpha>0$, namely when the sublevels of Q_α are non-convex (see Section 2.3). Notice that Theorem 2.6 already guarantees the existence of a positive direction for the fundamental solution, so here it is enough to find a negative direction.

From (47) we immediately see that $\mathrm{sgn}(K(0,y))=\mathrm{sgn}(\Delta G(y))$ for any $y\in\mathbb{R}^n$, where G has the form (48). By the peculiar form of Q_α, we choose as in Section 2.3 $y=e_1$. Hence, by (56), recalling the notation $\mathbb{R}^n\ni\xi=(\xi_1,\xi')$ and that all integrals are on $S:=\mathbb{S}^{n-1}$, so $\xi_1=\sqrt{1-|\xi'|^2}$, we get

$$\begin{aligned}\Delta G(e_1)&=4\int_{|\xi'|<1}\frac{1}{Q(\sqrt{1-|\xi'|^2},\xi')}\frac{d\mathcal{H}^{n-1}(\xi')}{\sqrt{1-|\xi'|^2}}+\frac{1}{2}\int_{|\xi'|<1}\log\left(1-|\xi'|^2\right)\\ &\quad\cdot\left(6\sqrt{1-|\xi'|^2}\,\partial_1\frac{1}{Q}(\sqrt{1-|\xi'|^2},\xi')+\frac{4}{Q(\sqrt{1-|\xi'|^2},\xi')}\right.\\ &\quad\left.+(2-|\xi'|^2)\partial_1^2\frac{1}{Q}(\sqrt{1-|\xi'|^2},\xi')+\left(1-|\xi'|^2\right)\Delta'\frac{1}{Q(\sqrt{1-|\xi'|^2},\xi')}\right)\frac{d\mathcal{H}^{n-1}(\xi')}{\sqrt{1-|\xi'|^2}},\end{aligned}\tag{57}$$

where $\Delta':=\sum_{k=2}^6\partial_k^2$. Exploiting the form of Q and writing $Q(\xi'):=Q(\sqrt{1-|\xi'|^2},\xi')$, one has

$$\partial_1\frac{1}{Q}\bigg|_S=-\frac{\partial_1 Q}{Q^2}\bigg|_S=-\frac{4\xi_1^3-2\alpha\xi_1|\xi'|^2}{Q(\xi)^2}\bigg|_S=\frac{-4(1-|\xi'|^2)^{3/2}+2\alpha|\xi'|^2(1-|\xi'|^2)^{1/2}}{Q(\xi')^2},\tag{58}$$

$$\begin{aligned}\partial_{11}^2\frac{1}{Q}\bigg|_S&=\left(-\frac{\partial_{11}^2Q}{Q^2}+\frac{2(\partial_1Q)^2}{Q^3}\right)\bigg|_S=-\frac{12\xi_1^2-2\alpha|\xi'|^2}{Q(\xi)^2}\bigg|_S+\frac{2(4\xi_1^3-2\alpha\xi_1|\xi'|^2)^2}{Q(\xi)^3}\bigg|_S\\ &=-\frac{12(1-|\xi'|^2)-2\alpha|\xi'|^2}{Q(\xi')^2}+2\frac{16(1-|\xi'|^2)^3+4\alpha^2(1-|\xi'|^2)^2|\xi'|^4-16\alpha|\xi'|^2(1-|\xi'|^2)^2}{Q(\xi')^3}.\end{aligned}\tag{59}$$

Analogously, we find

$$\partial_k\frac{1}{Q}\bigg|_S=-\frac{\partial_kQ}{Q^2}\bigg|_S=-\frac{-2\alpha\xi_1^2\xi_k+4\xi_k|\xi'|^2}{Q(\xi')^2}\bigg|_S=\frac{2\alpha(1-|\xi'|^2)\xi_k-4\xi_k|\xi'|^2}{Q(\xi')^2},$$

$$\begin{aligned}\partial_{kk}^2\frac{1}{Q}\bigg|_S&=\left(-\frac{\partial_{kk}^2Q}{Q^2}+\frac{2(\partial_kQ)^2}{Q^3}\right)\bigg|_S=\frac{2\alpha\xi_1^2-4|\xi'|^2-8\xi_k^2}{Q(\xi')^2}\bigg|_S+\frac{2(4\alpha^2\xi_1^4\xi_k^2+16\xi_k^2|\xi'|^4-16\alpha\xi_k^2\xi_1^2|\xi'|^2)}{Q(\xi')^3}\bigg|_S\\ &=\frac{2\alpha(1-|\xi'|^2)-4|\xi'|^2-8\xi_k^2}{Q(\xi')^2}+2\frac{4\alpha^2(1-|\xi'|^2)^2\xi_k^2+16\xi_k^2|\xi'|^4-16\alpha\xi_k^2(1-|\xi'|^2)|\xi'|^2}{Q(\xi')^3}.\end{aligned}$$

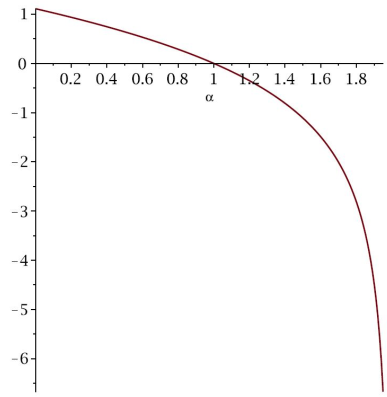


Figure 2: The graph of $\Delta G(e_1)$, with $\alpha \in [0, 1.95]$.

Therefore,

$$\Delta' \frac{1}{Q}\Big|_S = \sum_{k=2}^{6} \partial_k^2 \frac{1}{Q}\Big|_S = \frac{10\alpha(1-|\xi'|^2) - 28|\xi'|^2}{Q(\xi')^2} + \frac{2}{Q(\xi')^3}\big(4\alpha^2(1-|\xi'|^2)^2|\xi'|^2 + 16|\xi'|^6 - 16\alpha|\xi'|^4(1-|\xi'|^2)\big). \tag{60}$$

Hence, we insert (58)-(60) in (57) and write in polar coordinates, denoting the 5-dimensional volume of the unit ball by e_5. We obtain

$$\begin{aligned}\frac{1}{5e_5}\Delta G(e_1) = &\int_0^1 \frac{4r^4}{\sqrt{1-r^2}Q(r)}dr + \int_0^1 \frac{r^4\log(1-r^2)}{\sqrt{1-r^2}}\Bigg[6(1-r^2)\left(\frac{-2(1-r^2)+\alpha r^2}{Q(r)^2}\right)\\ &+ \frac{2}{Q(r)} - (2-r^2)\frac{6(1-r^2)-\alpha r^2}{Q(r)^2} + 4(2-r^2)\frac{4(1-r^2)^3 + \alpha^2(1-r^2)r^4 - 4\alpha r^2(1-r^2)^2}{Q(r)^3}\\ &+ \frac{4(1-r^2)r^2}{Q(r)^3}\big(\alpha^2(1-r^2)^2 + 4r^4 - 4\alpha r^2(1-r^2)\big) + (1-r^2)\frac{5\alpha(1-r^2)-14r^2}{Q(r)^2}\Bigg]dr.\end{aligned}$$

Using maple™ gives evidence that for any value of $\alpha < 2$ but *close* to 2 we find $K(0, e_1) < 0$. In Figure 2 the graph of $\Delta G(e_1)$ is displayed and this behaviour is here well observable.

Recalling Theorem 2.6, the proof of Theorem 1.4 is complete. □

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