Boundary Value Problems for the Willmore Functional

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This short survey aims at giving a very brief account of the author's contributions to Willmore boundary value problems. The style of this note is quite similar to a colloquium talk: Only some of the results are described and a few of the main ideas are explained. Technicalities are not provided at all, and also the bibliography is by no means exhaustive. If the reader gets interested he or she may look at the original papers which are mentioned at the beginning of each section. In these papers, also further references are given.

The author has been intensively collaborating on the Willmore topic for more than ten years with several coauthors:

- Anna Dall'Acqua (Ulm)
- Klaus Deckelnick (Magdeburg)
- Steffen Fröhlich (Mainz)
- Shinya Okabe (Sendai - Tohoku University)
- Matthias Röger (Dortmund)
- Numerics: Friedhelm Schieweck (Magdeburg)

The collaboration with Shinya Okabe was initiated during the author's visit in Japan due to the workshop in question. This is work in progress but already a couple of interesting results came out. The author is grateful to Shinya Okabe and the RIMS for the kind invitation and the inspiring atmosphere during his visit in Sendai and the workshop at the RIMS in Kyoto. The author likes also to thank Klaus Deckelnick for a careful reading of this manuscript and for helpful suggestions.

The study of boundary value problems for the Willmore functional is quite difficult since e.g. in general, no a-priori bounds are available for solutions of the corresponding Euler-Lagrange-equation and even not for minimisers / minimising sequences. For this reason the author and his collaborators have chosen to stick to somehow special situations imposing different kinds of symmetry or projectability conditions where analytic and geometric information on minimisers / minimising sequences can be obtained.

1 The Willmore functional

Definition of the Willmore functional

For sufficiently smooth (two dimensional) surfaces $S \subset \mathbb{R}^3$ (with or without boundary) we define the Willmore functional

$$W(S) := \frac{1}{4}\int_S H^2\, dS$$

with the mean curvature

$$H(x) := \kappa_1(x) + \kappa_2(x)$$

defined as the sum of the principal curvatures.

Smooth critical points of the Willmore functional are called Willmore surfaces and solve the Willmore equation

$$\Delta_S H + 2H\Big(\frac{1}{4}H^2 - K\Big) = 0 \quad \text{on } S,$$

where $K(x) := \kappa_1(x) \cdot \kappa_2(x)$ is the Gauss curvature. This equation is quasilinear, of fourth order, elliptic, but not uniformly. A strong degeneration of ellipticity takes place when large variations of the tangent planes occur.

The Willmore functional has been considered already since the 19th century in order to model thin elastic plates or biomembranes, see the detailed explanations and references in [40], cf. also [28]. Quite some interesting important differential geometric properties of Willmore surfaces were uncovered in the early 20th century, see e.g. [44]. At that time investigating the minimisation process for the Willmore functional was completely out of reach. For this reason it was only in the 1960s that Willmore renewed the interest in this functional (see e.g. [45, 46]), which for this reason is named after him.

Characteristic features: Conformal invariance & lack of comparison principles

Next to the area functional, the Willmore functional looks like the second simplest interesting differential geometric functional. And indeed, minimal surfaces ($H(x) \equiv 0$, like e.g. catenoids) are obviously Willmore surfaces. So, one may wonder:

> Is the Willmore functional a cousin of the area functional?

Indeed, some of what is explained in Section 4 is reminiscent of Giusti's (and other's) BV-approach to minimal graphs, cf. [25]. However, there are characteristic fundamental differences:

- Geometrically, the conformal invariance is possibly the most important characteristic feature of the Willmore functional and is a huge difference to the area functional: If $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ is conformal (i.e. a Möbius transformation), then for any surface $S \subset \mathbb{R}^3$ one has

  $$W(S) = W(\Phi(S)).$$

  In particular, scaling invariance holds, i.e. $\forall r > 0 : \quad W(rS) = W(S)$.

- Compact Willmore surfaces without boundary exist: an obvious example is the sphere.

- The Willmore functional involves *second* derivatives: No Stampacchia-tricks are applicable! This means that if S is locally described as the graph of a function u, one has to work in spaces like $W^{2,2}$ where passing from u to $|u|$ or u^+ is *not permitted*.

- In general, no comparison principles are available for Willmore surfaces! This is closely related to the previous item, because the availability of Stampacchia-tricks would allow for proving comparison principles. The lack of general comparison principles is a characteristic feature of fourth and higher order elliptic equations, while at least in some cases, one has remains of positivity preserving, see e.g. [24].

- Correspondingly one has no de Giorgi-Mash-Moser-theory for higher order equations and in particular not for the Willmore equation.

- In general, no a-priori-estimates are known for solutions of (the Dirichlet problem for) the Willmore equation. Geometrically one important reason for this is the conformal invariance. From the analytically technical point of view, another important reason for this is the lack of comparison principles and (somehow correspondingly) the lack of suitable testing functions.

1.1 Closed surfaces

Quite some interesting results are known about *closed* Willmore surfaces (i.e. embedded, without boundary) $S \subset \mathbb{R}^3$.

- Based upon the Gauß-Bonnet-theorem it is not too difficult to show that if $S \subset \mathbb{R}^3$ is a closed smooth surface, then $W(S) \geq 4\pi$ with equality iff S is a (round) sphere.
- The Li-Yau inequality (see [36]): If $S \subset \mathbb{R}^3$ is a closed smooth surface, $\Psi : S \to \mathbb{R}^3$ an immersion of "multiplicity" k (i.e. k distinct points of S have the same image under Ψ), then this inequality states that: $W(\Psi(S)) \geq k \cdot 4\pi$.
- Mayer-Simonett and Kuwert-Schätzle proved short-time existence for the Willmore flow (the corresponding gradient flow) for suitable "large" initial data, see [30, 39]. Moreover, Kuwert-Schätzle proved long-time existence and convergence to a round shere, if the Willmore energy of the initial surface is strictly below 8π, see [31].
- It follows from works of Simon [43] and Bauer-Kuwert [3] that for any given genus $g \in \mathbb{N}_0$ one has a Willmore minimiser in the class of compact genus-g-surfaces.
- In 2012, F. Marques and A. Neves proved the Willmore conjecture (see the survey [38] for a comprehensible outline of the history and proof of this conjecture): If the genus of S satisfies $g \geq 1$ then $W(S) \geq 2\pi^2$ with equality iff S is in the equivalence class under Möbius transformations of the stereographic projection of the Clifford torus.

For the last item it is still open whether it is essential to have codimension 1 while other results could be generalised to arbitrary codimension.

1.2 The Dirichlet problem

The goal is to find Willmore surfaces (resp. immersions) or, even better, Willmore minimisers $S \subset \mathbb{R}^3$ with boundary $\partial S \neq \emptyset$ such that

- ∂S is prescribed (fixed),
- the tangential half planes along ∂S are prescribed (fixed) as well.

Also different boundary conditions may be imposed. One may e.g. replace the second condition by prescribing the mean curvature of S on ∂S, the so called Navier problem. Compared to the theory for closed surfaces, much less is known about boundary value problems for Willmore surfaces.

As for the Dirichlet problem in this general setting, Schätzle [42] proved existence of branched Willmore minimisers in $\mathbb{S}^3 \cong \mathbb{R}^3 \cup \{\infty\}$. In general, little geometric information may be obtained for these solutions. They may not be embedded, will in general not be (global) graphs or may even contain ∞. No a-priori-bounds are obtained. The proof is heavily based on tools from geometric measure theory. Recently, a related result in a slightly more special situation and imposing an area constraint was proved by Da Lio-Palmurella-Riviére, see [9]. Their proof is based on a classical parametric approach. For an existence result for the more general Helfrich functional, one may see a recent work by Eichmann [20].

The author's and his coauthors' approach has been completely different: We have considered more special settings where we have been able to uncover more precise geometric information of solutions or minimisers respectively.

To this end we work in restricted classes of admissible (comparison) surfaces:

- Assuming invariance with respect to translations: Graphs over one-dimensional intervals.
- Assuming axial symmetry: Surfaces of revolutions. Their profile curves are graphs over one-dimensional intervals. This means that the geometry is already two-dimensional while the analysis is still one-dimensional.
- Assuming projectability: Graphs over general two-dimensional bounded smooth domains.

In these cases we have investigated the minimisation procedure of the Willmore functional. As the degree of symmetry decreases, the difficulties become more serious and, as for the third item, the results more restricted. As for Willmore graphs over general two-dimensional domains our contribution has to be considered as a first of an unknown number of future steps of unknown difficulty.

2 Assuming invariance with respect to translations: The one-dimensional case

This topic has been studied in collaboration with Klaus Deckelnick, see e.g. [16, 17].

2.1 Solutions to the symmetric Dirichlet problem

Theorem 1 ([16, Thm. 2])**.** *For any $\beta \in \mathbb{R}$ there exists a, in the class of graphs unique Willmore-energy minimising, smooth solution $u : [0,1] \to \mathbb{R}$, $u(x) = u(1-x)$, to the Dirichlet problem:*

$$\begin{cases} \dfrac{1}{\sqrt{1+u'(x)^2}} \dfrac{d}{dx} \left(\dfrac{\kappa'(x)}{\sqrt{1+u'(x)^2}} \right) + \dfrac{1}{2}\kappa^3(x) = 0, \quad x \in (0,1), \\ \kappa(x) = \dfrac{u''(x)}{(1+u'(x)^2)^{3/2}}, \\ u(0) = u(1) = 0, \quad u'(0) = -u'(1) = \beta. \end{cases} \tag{1}$$

In this special symmetric situation, the proof is based on an observation already made by Euler, see [23, pp. 233-234]: There is a constant c such that

$$\kappa(x) \cdot \left(1+u'(x)^2\right)^{1/4} = \frac{u''(x)}{(1+u'(x)^2)^{5/4}} \equiv -c.$$

One should observe that this important auxiliary function has no geometric meaning! However, this can be almost explicitly integrated in terms of the inverse of the function

$$G(t) := \int_0^t \frac{d\tau}{(1+\tau^2)^{5/4}}.$$

See Figure 1.

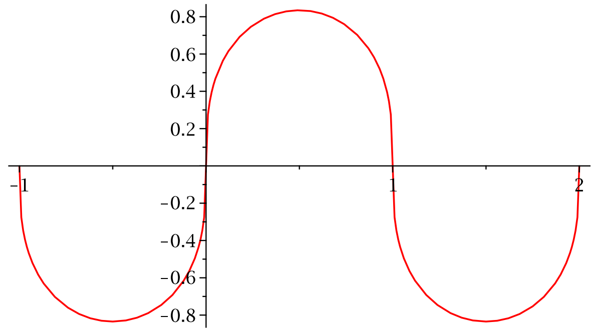


Figure 1: All solutions in Theorem 1 can be found from suitable parts of this prototype solution by suitably putting the coordinate system and rescaling. One should observe that all these solutions are uniformly bounded by $\frac{2}{B(\frac{1}{2},\frac{3}{4})} = 0.8346\ldots$.

2.2 The nonsymmetric case

If one allows beside scaling and translating also for rotating the coordinate system in Figure 1 one may deduce the following existence results for nonsymmetric boundary slopes.

Theorem 2 ([16, Thm. 4])**.** *For every $\beta_0, \beta_1 \in \mathbb{R}$, the Dirichlet boundary value problem*

$$\begin{cases} \dfrac{1}{\sqrt{1+u'(x)^2}}\dfrac{d}{dx}\left(\dfrac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right) + \dfrac{1}{2}\kappa^3(x) = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0, \qquad u'(0) = \beta_0, \quad u'(1) = -\beta_1 \end{cases} \tag{2}$$

for the Willmore equation has a smooth graph solution $u : [0,1] \to \mathbb{R}$.

Giving a close look at Figure 1 shows that prescribing different values for $u(0)$ and $u(1)$ together e.g. with $u'(0) = u'(1) = 0$ will not in any case result in a graph solution. To guarantee this one should assume that

$$|u(0) - u(1)| \le \frac{4}{B(\frac{1}{2},\frac{3}{4})} = 1.6692\ldots,$$

which is the oscillation of the prototype solution in Figure 1.

2.3 The Willmore obstacle problem in one dimension

Here we report on a first result within an ongoing project with Shinya Okabe, see [27]. The problem we are going to describe now was first studied and solved under Navier boundary conditions by Dall'Acqua-Deckelnick, see [12]. It turns out that under Dirichlet boundary conditions a rather related result can be shown: Only a small extra idea in proving a-priori-bounds for minimising sequences is needed.

Let $\psi \in H^2_0(0,1)$ be a symmetric obstacle, i.e., $\psi(x) = \psi(1-x)$ for all $x \in (0,1)$. We define a set $\mathcal{M}(\psi)$ of admissible functions as follows:

$$\mathcal{M}(\psi) := \{u \in H^2_0(0,1) \,:\, u(x) = u(1-x),\ \ u(x) \ge \psi(x),\ \text{ for any } x \in (0,1)\}.$$

This means that admissible functions have to be *above* the given obstacle.

The goal is to find minimisers of the one-dimensional Willmore functional

$$W(u) := \frac{1}{4}\int_0^1 \kappa(x)^2 \sqrt{1+u'(x)^2}\, dx$$

in $\mathcal{M}(\psi)$. This notion is consistent wirh the original one if one thinks of graphs of functions $[0,1] \times [0,1] \ni (x,y) \mapsto u(x)$, which do not depend on the y-variable (and where consequently no conditions can be imposed on $[0,1] \times \{0,1\}$).

We assume that ψ satisfies

$$\alpha := \inf_{v \in \mathcal{M}(\psi)} W(v) < c_0^2 \text{ with } c_0 = 2G(\infty) = \sqrt{\pi}\frac{\Gamma(3/4)}{\Gamma(5/4)} = 2.396280469\ldots. \tag{C}$$

In order to find nontrivial minimisers of $W(\,.\,)$ in $\mathcal{M}(\psi)$ one should always think of obstacles ψ with $\psi(x_0) > 0$ for some $x_0 \in (0,1)$.

Theorem 3. *Assume that $\psi \in H_0^2(0,1)$ satisfies Condition* (C). *Then there exists $u \in \mathcal{M}(\psi)$ such that*

$$W(u) = \inf_{v \in \mathcal{M}(\psi)} W(v).$$

In order to construct admissible obstacles, one first restricts the function in Figure 1 to $[-1/2, 3/2]$ and rescales and translates it. We so obtain the function displayed on the left in Figure 2. This has Willmore energy precisely equal to c_0^2. The following construction decreases its Willmore energy: Cut off the almost vertical parts, glue what remains together and rescale. One example is displayed on the right in Figure 2.

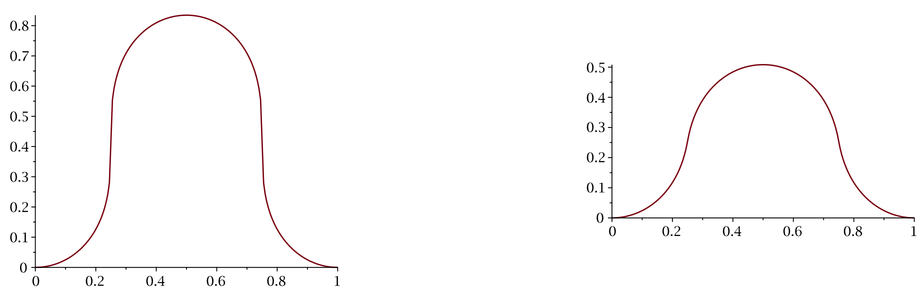


Figure 2: Left: Rescaled and translated prototype. Right: An admissible obstacle.

3 Assuming invariance with respect to rotations: Surfaces of revolution

This topic has been studied in collaboration with Anna Dall'Acqua, Klaus Deckelnick, Steffen Fröhlich and Friedhelm Schieweck, see [13, 15].

Existence, horizontal clamping

We consider surfaces of revolution S

$$(x, \varphi) \mapsto (x, u(x)\cos\varphi, u(x)\sin\varphi),\ x \in [-1,1],\ \varphi \in [0, 2\pi],$$

which arise when the sufficiently smooth profile curve $u : [-1,1] \to (0,\infty)$ is rotated around the x-axis. Here, one may calculate the mean and the Gauss curvature of S as follows:

$$H(x) = \frac{1}{u(x)\sqrt{1+(u'(x))^2}} - \frac{u''(x)}{(1+(u'(x))^2)^{3/2}},$$
$$K(x) = -\frac{u''(x)}{u(x)(1+(u'(x))^2)^2}.$$

Thus the Willmore functional is given by

$$\begin{aligned} W(u) &= \frac{1}{4}\int_S H^2\, dS \\ &= \frac{\pi}{2}\int_{-1}^{1} \left(\frac{1}{u(x)\sqrt{1+(u'(x))^2}} - \frac{u''(x)}{(1+(u'(x))^2)^{3/2}} \right)^2 u(x)\sqrt{1+(u'(x))^2}\, dx. \end{aligned} \tag{3}$$

In [13, 15] only the case of symmetric profile curves (i.e. $u(x) = u(-x)$) was studied. This case is not only technically considerably simpler than the general case but under nonsymmetric Dirichlet boundary curves it may happen that profile curves of minimisers are no longer graphs over $[-1,1]$, see [21].

The following result shows existence in the case of horizontal clamping $u'(\pm 1) = 0$.

Theorem 4 ([13, Theorem 1.1])**.** *For any $\alpha > 0$ there exists a profile curve $u \in C^\infty([-1,1],(0,\infty))$ such that:*

$$\begin{cases} \Delta_S H + 2H(\frac{1}{4}H^2 - K) = 0 & \text{in } (-1,1), \\ u(\pm 1) = \alpha, \qquad u'(\pm 1) = 0. \end{cases}$$

u is even and in this class energy minimising, $\forall x \in [0,1] : u'(x) \le 0,\ 0 \le x + u(x)u'(x)$;

$$\forall x \in [-1,1]: \quad \alpha \le u(x) \le \sqrt{\alpha^2 + 1 - x^2}, \quad |u'(x)| \le \frac{1}{\alpha}.$$

For the (symmetric) case of non-horizontal clamping $u(\pm 1) = \alpha$, $u'(\pm 1) = \mp\beta$ one may see [15, Theorem 1.1]. At a first glance this result looks like a straight generalisation of [13, Theorem 1.1]. However, technically the case $0 > \beta > -\alpha$ was quite different. We found a minimiser only in a quite restricted class while we conjecture that in this case in a larger class minimisers may be different.

3.1 Direct methods of the Calculus of Variations

Theorem 4 is proved by means of the direct methods in the calculus of variations.

For $\alpha \in (0,\infty)$ we define the following set of admissible functions:

$$N_\alpha := \{u \in H^2([-1,1]) : u \text{ even, positive}, u(1) = \alpha, u'(1) = 0\}.$$

The key step consists in finding that the bounds

$$\forall x \in [0,1] : v_k'(x) \le 0,\ 0 \le x + v_k(x)v_k'(x),$$

$$\forall x \in [-1,1]: \quad \alpha \le v_k(x) \le \sqrt{\alpha^2 + 1 - x^2}, \quad |v_k'(x)| \le \frac{1}{\alpha}$$

are obeyed by suitably modified minimising sequences $(v_k)_{k\in\mathbb{N}} \subset N_\alpha$. These bounds turn out to be strong enough to proceed with the usual local-weak-compactness and weakly-lower-semicontinuity reasoning in $H^2(-1,1)$.

Looking at the definition (3) of the Willmore functional for surfaces of revolution, there is no obvious idea at all how to modify minimising sequences in order to obey the above mentioned bounds. In particular, it is by no means obvious how to change a function, which violates $\forall x \in [-1,1]: \quad v(x) \geq \alpha$, into a function, which has smaller Willmore energy and obeys this inequality. A simple reflection trick will not work. Indeed, a number of subtle geometric constructions are involved and in this survey note, only the simplest of the required modifications is explained. Nevertheless, the key idea is the same for all these modifications. When modifying minimising sequences one has to take care to stay in H^2, i.e. not only the values of the functions but also of the derivatives have to match!

3.2 Hyperbolic geometry

That the Willmore functional for surfaces of revolution may be reformulated in the hyperbolic half plane as a simple one-dimensional curvature integral was observed e.g. by Bryant-Griffiths [8] and Pinkall, cf. [29]. This observation was heavily exploited among many others, e.g. by Langer-Singer [33, 34], where a classification of hyperbolic elastica was indicated. For comprehensible proofs, further developments, and applications of their results, one may see works by Eichmann [19] and Mandel [37].

The hyperbolic half plane is the set $\mathbb{R}^2_+ := \{(x,y) : y > 0\}$, equipped with the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

For the hyperbolic curvature of a profile curve $[-1,1] \ni x \mapsto (x, u(x)) \in \mathbb{R}^2_+$ one finds:

$$\kappa(x) = \frac{1}{\sqrt{1+u'(x)^2}} + \frac{u(x)u''(x)}{(1+u'(x)^2)^{3/2}}.$$

This looks quite similar to the mean curvature $H(x)$ of the corresponding surface of revolution: one only has to take off the factor $u(x)$ and to replace the plus by a minus sign.

Definition 1. The mapping

$$N_\alpha \ni u \mapsto \hat{W}(u) := \int_{-1}^{1} \kappa(x)^2 \, ds(x) = \int_{-1}^{1} \kappa(x)^2 \, \frac{\sqrt{1+u'(x)^2}}{u(x)} \, dx$$

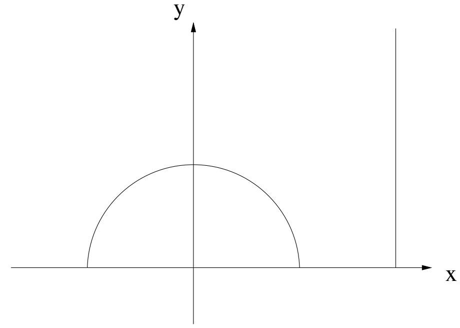
is called the **hyperbolic Willmore functional**.

One observes that $\kappa(x)^2 \frac{\sqrt{1+u'(x)^2}}{u(x)}$ and $H(x)^2 \sqrt{1+u'(x)^2}$ differ only by $4 \left(\frac{u'(x)}{\sqrt{1+u'(x)^2}} \right)'$. If $u'(\pm 1) = 0$ one therefore has:

$$\hat{W}(u) = \frac{2}{\pi} W(u).$$

Hyperbolic geodesics

Hyperbolic geodesics (which satisfy $\kappa(x) \equiv 0$) are half circles centered on the x-axis and vertical straight lines. Hence the gain of switching to hyperbolic geometry is that $\hat{W}(v)$ can be strictly decreased whenever a part of the graph of v can be replaced by a part of such a half circle.



3.3 How to suitably modify minimising sequences

As already mentioned the main step in proving Theorem 4 consists in verifying that suitable geometric modifications of minimising sequences result in sufficiently strong a-priori-bounds:

Proposition 1. *We may achieve a minimising sequence* $(v_k)_{k\in\mathbb{N}} \subset N_\alpha$ *such that:*

$$\forall x \in [0,1] : v_k'(x) \leq 0, \quad x + v_k(x)v_k'(x) \geq 0.$$

Here, we shall explain only the last step of its proof: Assume that we have already modified a suitable minimising sequence in N_α such that

$$\forall x \in [0,1] : v_k'(x) \leq 0.$$

How can we also achieve that

$$0 \leq x + v_k(x)v_k'(x) \text{ for } x \in [0,1]? \tag{4}$$

The key observation is that the Euclidean normal of the graph of v_k at the point $(x, v_k(x))$ intersects the x-axis in the point $x + v_k(x)v_k'(x)$. Certainly, (4) is satisfied for x close to 1. So, assuming that (4) is violated somewhere would yield a point $(x_0, v_k(x_0))$ with $x_0+v_k(x_0)v_k'(x_0) = 0$ and $x_0 \in (0,1)$ chosen as large as possible. The Euclidean normal of the graph of v_k at the point $(x_0, v_k(x_0))$ would intersect the x-axis at the origin. Since we are working with symmetric functions, a circular arc (i.e. a hyperbolic geodesic) would fit H^2-smoothly in and strictly decrease the hyperbolic Willmore energy. See Figure 3.

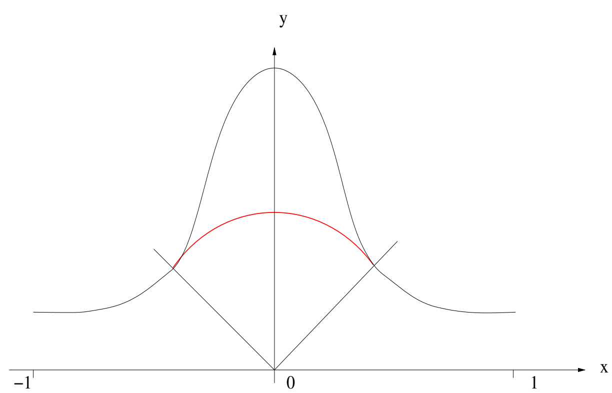


Figure 3: Replacing part of a graph by a circular arc decreases the hyperbolic Willmore energy.

Numerically calculated solutions are displayed e.g. in [15, Figure 17]. Qualitatively these look pretty much similar to the comparison function displayed in Figure 3.

The author would like to emphasise that already for surfaces of revolution, where the profile curve is not assumed to be symmetric, such replacement tricks become quite difficult to apply. In truly two-dimensional situations an application of suitable replacement tricks seems to be completely out of reach.

3.4 The Willmore obstacle problem for surfaces of revolution

Here we report on a first result within an ongoing project with Shinya Okabe, see [27].

Assume that $\psi \in H^2((-1,1),(0,\infty))$ is an even obstacle, i.e., $\psi(x) = \psi(-x)$, and satisfies

$$\psi(1) = \psi(-1) = \alpha, \quad \psi'(1) = \psi'(-1) = 0.$$

We define the admissible set $N_\alpha(\psi)$ as follows:

$$N_\alpha(\psi) := \{v \in H^2((-1,1),(0,\infty)) \,:\, v \text{ even}, \; v(1) = \alpha, \; v'(1) = 0, v(x) \le \psi(x) \text{ on } [-1,1]\}.$$

This notion is consistent with the previous sections observing that $N_\alpha = N_\alpha(\infty)$. In the following, we let $M_\alpha(\psi)$ denote the infimum of the Willmore functional in the admissible set $N_\alpha(\psi)$, i.e.,

$$M_\alpha(\psi) := \inf_{v \in N_\alpha(\psi)} W(v).$$

This means that in contrast to the one-dimensional situation, we look for minimising functions *below* the given obstacle. Typically one should think of obstacles which satisfy $\psi(x) \ge \alpha$ for $|x|$ close to 1 and $\psi(x) < \alpha$ for $|x|$ close to 0. In contrast to the situation of "free" Willmore minimisers where the profile curve is above α, we try to push these surfaces "down", i.e. towards the axis of revolution.

We call an obstacle ψ *admissible* if it satisfies

$$M_\alpha(\psi) < 4\pi. \tag{A}$$

One may observe that $\psi \le \widetilde{\psi}$ yields $N_\alpha(\psi) \subset N_\alpha(\widetilde{\psi})$ and $M_\alpha(\psi) \ge M_\alpha(\widetilde{\psi})$.

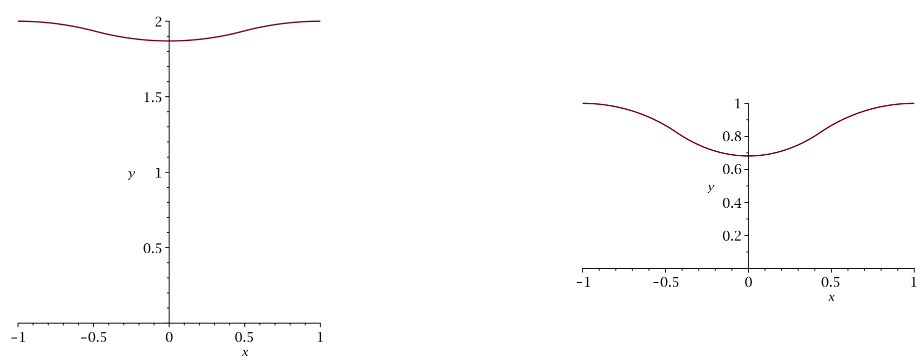


Figure 4: Admissible obstacles for $\alpha = 2$ (left) and $\alpha = 1$ (right).

Figure 5: Two admissible obstacles for $\alpha = 0.84$.

Due to the horizontal clamping in $x = \pm 1$, a vertical part in a profile curve will contribute 2π to the Willmore energy. Since we are working in classes of symmetric functions we would always have an even number of vertical parts, contributing at least 4π to the Willmore energy. This means that Condition (A) directly yields gradient bounds on minimising sequences. Bounds for the profile curve from above are then immediate. In order to show also strictly positive bounds from below one observes that approaching the x-axis by curves of bounded slope would result in infinite Willmore energy, see e.g. [15, Lemma 4.9].

Figure 6: Admissible obstacles for $\alpha = 0.7$ (left) and $\alpha = 0.5$ (right).

Theorem 5. *Assume that ψ satisfies* (A). *Then there exists $u \in N_\alpha(\psi)$ such that*

$$W(u) = M_\alpha(\psi) = \inf_{v \in N_\alpha(\psi)} W(v).$$

In order to construct admissible obstacles one may combine circles and catenoids and glue them suitably together. Some examples are displayed in Figures 4, 5 and 6. One should have in mind that any function ψ above the displayed ones is also an admissible obstacle.

4 Assuming projectability: The Willmore functional for two-dimensional graphs

This topic has been studied in collaboration with Klaus Deckelnick and Matthias Röger, see [18]. The replacement tricks, which worked so well in the previous section, cannot be applied at all in truly two-dimensional situations. One should have in mind that replacing a function on a subdomain by something better requires to match not only the functions but also their derivatives in order to stay in H^2. This fundamental difficulty is closely related to the lack of Stampacchia-tricks, the lack of a de Giorgi-Nash-Moser-technique and the lack of general comparison principles.

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and $\varphi : \overline{\Omega} \to \mathbb{R}$ a smooth boundary datum. We consider the minimisation of the Willmore functional

$$W(u) = \frac{1}{4} \int_\Omega H^2 \sqrt{1 + |\nabla u|^2}\, dx$$

for graphs

$$u : \overline{\Omega} \to \mathbb{R}$$

with mean curvature (the sum of principal curvatures, oriented with respect to the upper unit normal)

$$H = \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)$$

subject to Dirichlet boundary conditions, i.e. in the class

$$\mathcal{M} := \{u \in H^2(\Omega) : (u - \varphi) \in H^2_0(\Omega)\}.$$

In this setting to the best of my knowledge, only small data results are available, see Nitsche's survey article [40]. Schätzle's work [42] would yield solutions but they may not be graphs, i.e. they may not belong to the class $\mathcal{M}$.

4.1 Area and diameter bounds

The restriction to the class of graphs yields area and diameter bounds – this is the first main step:

Theorem 6 ([18, Theorem 2])**.** *There exists a constant $C = C(\Omega, \|\varphi\|_{W^{2,1}(\partial\Omega)})$ such that for any $u \in \mathcal{M}$ we have*

$$\sup_{x\in\Omega} |u(x)| + \int_\Omega \sqrt{1+|\nabla u(x)|^2}\,dx \le C\big(W(u)^2+1\big).$$

The key ingredients of the proof, which are explained below in some detail, are:

- A fundamental lemma by Leon Simon [43, Lemma 1.2], which estimates the diameter in terms of the boundary data and the L^1-norm (!) of the length of the second fundamental form;

- Integrating
$$\int_\Omega u\,H\,dx$$
by parts. Again, a quantity becomes crucial which has no geometric meaning.

A bound like Theorem 6 does not hold true for general surfaces. The reason is the scaling invariance (conformal invariance) of the Willmore functional: Think of an arbitrarily large ball, cut off a unit disk around the south pole and connect this to the given boundary conditions (e.g. $\varphi(x) \equiv 0$ on e.g. the disk $\Omega = B_1(0)$). This can be achieved by uniformly bounded Willmore energies, irrespective of how large the ball becomes.

In general, no *better* estimates!

Having linear Sobolev embeddings in mind one might hope that the Willmore energy could also bound stronger norms of u from above. But such an expectation does not take into account the strong degeneration of the D^2u-contribution to the Willmore energy, when ∇u becomes large. Indeed, we observe:

- In general, no $W^{1,p}$-estimates with $p > 1$ in terms of the Willmore energy are available. To see this one may think of a bowler hat, which is smooth as a surface, but which has an arbitrarily steep profile curve: $u \notin W^{1,p}(\Omega)$ but $W(u) < \infty$.

- This means that the area and diameter bounds determine $BV \cap L^\infty(\Omega)$ as solution class! $BV(\Omega)$ may be thought of as a sort of a "weak-$*$-closure" of $W^{1,1}(\Omega)$.

- Obviously, the Willmore functional is not defined there. A way out is to consider the L^1-lower semicontinuous envelope of the Willmore functional. We call this the *relaxed Willmore functional*, see Definition 2 below.

- But then one has to answer the question whether this envelope is still of relevance for the original problem. What do these functionals have to do with each other? How are the boundary conditions then encoded in $BV \cap L^\infty(\Omega)$? To answer this, finiteness of the relaxed Willmore energy is crucial.

4.2 A lower semicontinuous extension of the Willmore functional in $\mathcal{M}$

The starting point is the following compactness and L^1-lower semicontinuity result on $\mathcal{M}$, which is the second key result in [18]:

Theorem 7 ([18, Theorem 3])**.** *Let $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M} = \{u \in H^2(\Omega) : (u-\varphi) \in H_0^2(\Omega)\}$ be a given sequence that satisfies*

$$\liminf_{k\to\infty} W(u_k) < \infty.$$

Then there exists a $u \in BV(\Omega) \cap L^\infty(\Omega)$ such that up to selecting a subsequence

$$u_k \to u \text{ in } L^1(\Omega) \quad (k \to \infty).$$

If in addition $u \in H^2(\Omega)$ then

$$W(u) \le \liminf_{k\to\infty} W(u_k).$$

The proof of this result uses the area and diameter bounds from Theorem 6. In the context of currents a related lower semicontinuity result was proved by Schätzle, see [41]. The techniques are, however, completely different, and the compactness arguments from the proof of Theorem 7 are also essentially used in order to show that our minimiser attains the Dirichlet boundary conditions, at least in a suitable sense. An important step is to prove H^1-boundedness of the sequence $\left(\frac{\nabla u_k}{(1+|\nabla u_k|^2)^{5/4}}\right)_{k\in\mathbb{N}}$.

A lower semicontinuous extension of the Willmore functional

A similar approach has already been used by Ambrosio, Bellettini, Dal Maso, their collaborators and many others, see e.g. [1, 4, 5].

Recall that $\mathcal{M} = \{u \in H^2(\Omega) : (u-\varphi) \in H_0^2(\Omega)\}$.

Definition 2 (L^1-Lower semicontinuous envelope, Relaxed Willmore functional)**.**

$$\begin{aligned} \overline{W} &: L^1(\Omega) \to [0,\infty], \\ \overline{W}(u) &:= \inf\{\liminf_{k\to\infty} W(u_k) : \mathcal{M} \ni u_k \to u \text{ in } L^1(\Omega)\}. \end{aligned}$$

In the one-dimensional setting Dal Maso-Fonseca-Leoni-Morini [10] succeeded in providing a rather explicit characterisation of $\overline{W}$ which seems in general to be out of reach.

The previous lower semicontinuity result, however, implies:

Theorem 8 ([18, Theorem 4])**.** *For $u \in \mathcal{M}$ one has $\overline{W}(u) = W(u)$.*

This means: The L^1-lower semicontinuous envelope $\overline{W}$ is indeed the largest possible L^1*-lower semicontinuous extension* of W to $L^1(\Omega)$.

4.3 Existence of a minimiser

Theorem 9 ([18, Theorem 5])**.** *There exists a function $u \in BV(\Omega) \cap L^\infty(\Omega)$ such that*

$$\forall v \in L^1(\Omega): \quad \overline{W}(u) \le \overline{W}(v).$$

The Dirichlet boundary conditions are attained in the following sense:

- *$u = \varphi$ in $L^1(\partial\Omega)$ in the sense of traces.*

- *As for the derivatives, one considers the Lebesgue decomposition $\nabla u = \nabla^s u + \nabla^a u$. For this, one has $|\nabla^s u|(\partial\Omega) = 0$. Moreover, $\nabla^a u$ has an approximately continuous representative which is well defined $\mathcal{H}^1$-almost everywhere on $\partial\Omega$ and coincides there with $\nabla\varphi$.*

In general a minimiser will have a nontrivial singular part $\nabla^s u$; numerical evidence for this is given in [2, Figure 7]. However, we expect the support of $\nabla^s u$ to consist only of the jump set and the Cantor part to be absent. This conjecture seems to be rather hard to prove. A further open problem is to find explicit, geometrically motivated smallness conditions on the datum φ to ensure existence of smooth classical graphical minimisers.

4.4 An idea how to show area and diameter bounds

Here, we outline the proof of Theorem 6. We consider only $u \in C^2(\overline{\Omega})$. With $A = \frac{D^2 u}{\sqrt{1+|\nabla u|^2}}$ the second fundamental form of graph(u), we use [43, Lemma 1.2] by Leon Simon to see that:

$$\begin{aligned}\sup_{x\in\Omega}|u(x)| &\leq C\Big(\int_\Omega |A|_g\sqrt{1+|\nabla u|^2}\,dx + 1\Big)\\ &\leq C\Big(\int_\Omega |A|_g^2\sqrt{1+|\nabla u|^2}\,dx\Big)^{1/2}\cdot\Big(\int_\Omega \sqrt{1+|\nabla u|^2}\,dx\Big)^{1/2} + C.\end{aligned}$$

Using $|A|_g^2 = H^2 - 2K$ and the Gauss-Bonnet-Theorem gives

$$\sup_{x\in\Omega}|u(x)| \leq C\big(W(u)+1)\big)^{1/2}\big(\int_\Omega \sqrt{1+|\nabla u|^2}\,dx\big)^{1/2} + C. \tag{5}$$

Integration by parts (which applies to graphs only!) relates up to controllable terms the integral $\int_\Omega uH\,dx$ to the area:

$$\begin{aligned}\int_\Omega uH\,dx &= \int_\Omega u\nabla\cdot\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)\,dx\\ &= -\int_\Omega \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}}\,dx + \int_{\partial\Omega}\frac{u\frac{\partial u}{\partial\nu}}{\sqrt{1+|\nabla u|^2}}ds.\end{aligned}$$

Combining this with (5) yields:

$$\begin{aligned}&\int_\Omega \sqrt{1+|\nabla u|^2}\,dx \leq C + C\sup_{x\in\Omega}|u(x)|\big(\int_\Omega H^2\,dx\big)^{1/2}\\ &\leq\ C + C\,W(u)^{1/2} + C\,W(u)^{1/2}\cdot\Big(W(u)+1\Big)^{1/2}\Big(\int_\Omega\sqrt{1+|\nabla u|^2}\,dx\Big)^{1/2}.\end{aligned}$$

It follows that

$$\int_\Omega \sqrt{1+|\nabla u|^2}\,dx \leq C\big(W(u)^2+1\big).$$

Recalling (5) we conclude that

$$\sup_{x\in\Omega}|u(x)| \leq C\big(W(u)^2+1\big).$$

□

References

[1] L. Ambrosio, S. Masnou, A direct variational approach to a problem arising in image reconstruction, *Interfaces Free Bound.* **5**, no. 1, 63–81 (2003).

[2] J.W. Barrett, H. Garcke, R. Nürnberg, Stable variational approximations of boundary value problems for Willmore flow with Gaussian curvature, *IMA J. Numer. Anal.* **37**, no. 4, 1657–1709 (2017).

[3] M. Bauer, E. Kuwert, Existence of minimizing Willmore surfaces of prescribed genus, *Int. Math. Res. Not.* **2003**, no. 10, 553–576 (2003).

[4] G. Bellettini, G. Dal Maso, M. Paolini, Semicontinuity and relaxation properties of a curvature depending functional in 2D, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **20**, no. 2, 247–297 (1993).

[5] G. Bellettini, L. Mugnai, Characterisation and representation of the lower semicontinuous envelope of the elastica functional, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **21**, no. 6, 839–880 (2004).

[6] M. Bergner, A. Dall'Acqua, S. Fröhlich, Willmore surfaces of revolution with two prescribed boundary circles, *J. Geom. Anal.* **23**, no. 1, 283–302 (2013).

[7] M. Bergner, R. Jakob, Sufficient conditions for Willmore immersions in $\mathbb{R}^3$ to be minimal surfaces & Erratum, *Ann. Global Anal. Geom.* **45**, no. 2, 129–146 & 147–150 (2014).

[8] R. Bryant, P. Griffiths, Reduction for constrained variational problems and $\int \frac{1}{2}k^2\,ds$, *Amer. J. Math.* **108**, no. 3, 525–570 (1986).

[9] F. Da Lio, F. Palmurella, T. Rivière, A resolution of the Poisson problem for elastic plates, Preprint, arXiv:1807.09373v2.

[10] G. Dal Maso, I. Fonseca, G. Leoni, M. Morini, A higher order model for image restoration: the one-dimensional case, *SIAM J. Math. Anal.* **40**, no. 6, 2351–2391 (2009).

[11] A. Dall'Acqua, Uniqueness for the homogeneous Dirichlet Willmore boundary value problem, *Ann. Global Anal. Geom.* **42**, no. 3, 411–420, (2012).

[12] A. Dall'Acqua, K. Deckelnick, An obstacle problem for elastic graphs, *SIAM J. Math. Anal.* **50**, no. 1, 119–137 (2018).

[13] A. Dall'Acqua, K. Deckelnick, H.-Ch. Grunau, Classical solutions to the Dirichlet problem for Willmore surfaces of revolution, *Adv. Calc. Var.* **1**, no. 4, 379–397 (2008).

[14] A. Dall'Acqua, K. Deckelnick, G. Wheeler, Unstable Willmore surfaces of revolution subject to natural boundary conditions, *Calc. Var. Partial Differential Equations* **48**, no. 3-4, 293–313 (2013).

[15] A. Dall'Acqua, S. Fröhlich, H.-Ch. Grunau, F. Schieweck, Symmetric Willmore surfaces of revolution satisfying arbitrary Dirichlet boundary data, *Adv. Calc. Var.* **4**, no. 1, 1–81 (2011).

[16] K. Deckelnick, H.-Ch. Grunau, Boundary value problems for the one-dimensional Willmore equation, *Calc. Var. Partial Differential Equations* **30**, no. 3, 293–314 (2007).

[17] K. Deckelnick, H.-Ch. Grunau, Stability and symmetry in the Navier problem for the one-dimensional Willmore equation, *SIAM J. Math. Anal.* **40**, no. 5, 2055–2076 (2009).

[18] K. Deckelnick, H.-Ch. Grunau, M. Röger, Minimising a relaxed Willmore functional for graphs subject to boundary conditions, *Interfaces Free Bound.* **19**, no. 1, 109–140 (2017).

[19] S. Eichmann, Willmore surfaces of revolution satisfying Dirichlet data, Ph.D. thesis, Otto-von-Guericke-Universität Magdeburg (2017). Available online at http://edoc2.bibliothek.uni-halle.de/hs/urn/urn:nbn:de:gbv:ma9:1-9449.

[20] S. Eichmann, The Helfrich boundary value problem, *Calc. Var. Partial Differential Equations*, to appear.

[21] S. Eichmann, H.-Ch. Grunau, Existence for Willmore surfaces of revolution satisfying non-symmetric Dirichlet boundary conditions, *Adv. Calc. Var.*, to appear, available online at: https://dx.doi.org/10.1515/acv-2016-0038.

[22] S. Eichmann, A. Koeller, Symmetry for Willmore surfaces of revolution, *J. Geom. Anal.* **27** (2017), no. 1, 618–642.

[23] L. Euler, Opera Omnia, Ser. 1, **24**, Zürich: Orell Füssli (1952).

[24] F. Gazzola, H.-Ch. Grunau, G. Sweers, *Polyharmonic Boundary Value Problems*, Lecture Notes in Mathematics **1991**, Springer-Verlag, Berlin (2010).

[25] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, volume 80 of *Monographs in Mathematics*, Birkhäuser Verlag, Basel (1984).

[26] H.-Ch. Grunau, The asymptotic shape of a boundary layer of symmetric Willmore surfaces of revolution, in: Inequalities and applications 2010, *Internat. Ser. Numer. Math.* **161**, 19–29, Birkhäuser/Springer, Basel (2012).

[27] H.-Ch. Grunau, S. Okabe, Willmore obstacle problems under Dirichlet boundary conditions, in preparation.

[28] W. Helfrich, Elastic properties of lipid bilayers: Theory and possible experiments, *Z. Naturforsch. Teil C*, **28**, no. 11, 693–703 (1973).

[29] U. Hertrich-Jeromin, U. Pinkall, Ein Beweis der Willmoreschen Vermutung für Kanaltori, *J. Reine Angew. Math.* **430**, 21–34 (1992).

[30] E. Kuwert, R. Schätzle, Gradient flow for the Willmore functional, *Commun. Anal. Geom.* **10**, no. 2, 307–339 (2002).

[31] E. Kuwert, R. Schätzle, Removability of point singularities of Willmore surfaces, *Ann. of Math.* (2) **160**, no. 1, 315–357 (2004).

[32] E. Kuwert, R. Schätzle, The Willmore functional, in: *Topics in modern regularity theory*, volume 13 of *CRM Series*, pages 1–115. Ed. Norm., Pisa (2012).

[33] J. Langer, D. Singer, The total squared curvature of closed curves, *J. Differ. Geom.* **20**, no. 1, 1–22 (1984).

[34] J. Langer, D. Singer, Curves in the hyperbolic plane and mean curvature of tori in 3-space, *Bull. London Math. Soc.* **16**, no. 5, 531–534 (1984).

[35] K. Leschke, F. Pedit, U. Pinkall, Willmore tori in the 4-sphere with nontrivial normal bundle, *Math. Ann.* **332**, no. 2, 381–394 (2005).

[36] P. Li, S.-T. Yau, A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces, *Invent. Math.* **69**, no. 2, 269–291 (1982).

[37] R. Mandel, Explicit formulas, symmetry and symmetry breaking for Willmore surfaces of revolution, *Annals Global Analysis and Geometry* **54**, no. 2, 187–236 (2018).

[38] F.C. Marques and A. Neves, The Willmore Conjecture, *Jahresber. Dtsch. Math.-Ver.* **116**, no. 4, 201–222 (2014).

[39] U.F. Mayer, G. Simonett, A numerical scheme for axisymmetric solutions of curvature-driven free boundary problems, with applications to the Willmore flow, *Interfaces Free Bound.* **4**, no. 1, 89–109 (2002).

[40] J.C.C. Nitsche, Boundary value problems for variational integrals involving surface curvatures, *Quart. Appl. Math.* **51**, no. 2, 363–387 (1993).

[41] R. Schätzle, Lower semicontinuity of the Willmore functional for currents, *J. Differential Geom.* **81**, no. 2, 437–456 (2009).

[42] R. Schätzle, The Willmore boundary problem, *Calc. Var. Partial Differential Equations* **37**, no. 3-4, 275–302 (2010).

[43] L. Simon., Existence of surfaces minimizing the Willmore functional, *Comm. Anal. Geom.* **1**, no. 2, 281–326 (1993).

[44] G. Thomsen, Über konforme Geometrie I: Grundlagen der konformen Flächentheorie, *Abh. Math. Sem. Hamburg* **3**, 31–56 (1923).

[45] T.J. Willmore, Note on embedded surfaces, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Seçt. I a Mat* **11**, 493–496 (1965).

[46] T.J. Willmore, *Riemannian Geometry*, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1993).