WILLMORE OBSTACLE PROBLEMS UNDER DIRICHLET BOUNDARY CONDITIONS

HANS-CHRISTOPH GRUNAU AND SHINYA OKABE

Abstract. We consider obstacle problems for the Willmore functional in the class of graphs of functions and surfaces of revolution with Dirichlet boundary conditions. We prove the existence of minimisers of the obstacle problems under the assumption that the Willmore energy with the unilateral constraint is below a universal bound. We address the question whether such bounds are necessary in order to ensure the solvability of the obstacle problems. Moreover, we give several instructive examples of obstacles such that minimisers exist.

1. Introduction

The present paper is concerned with obstacle problems for the Willmore functional with Dirichlet boundary conditions. More precisely, we consider minimisation problems for the Willmore functional among curves or surfaces of revolution with Dirichlet boundary conditions under a unilateral constraint. Recently the minimisation problem for the one-dimensional Willmore or elastica functional among graphs of functions with Navier boundary conditions under a unilateral constraint has been intensively studied ([1, 9, 10, 16]): For a given obstacle function $\psi : [0,1] \to \mathbb{R}$, find a function $u : [0,1] \to \mathbb{R}$ such that u attains

$$\inf_{v \in K(\psi)} W(v), \tag{1.1}$$

where

$$W(v) := \int_0^1 \kappa(x)^2 \sqrt{1+v'(x)^2}\, dx = \int_0^1 \left(\frac{v''(x)}{(1+v'(x)^2)^{3/2}} \right)^2 \sqrt{1+v'(x)^2}\, dx = \int_0^1 \frac{v''(x)^2}{(1+v'(x)^2)^{5/2}}\, dx,$$

is the one-dimensional Willmore or elastica functional and

$$K(\psi) := \{ v \in H^2(0,1) \mid v(0) = v(1) = 0,\ v \ge \psi \text{ in } [0,1] \}$$

is the class of admissible functions. Problem (1.1) was firstly studied by Dall'Acqua–Deckelnick [1] and they proved that W has a minimiser in $K(\psi)$ under an explicit smallness condition on ψ. In view of this result it is an obvious question whether there is a threshold for the obstacles beyond which they no longer permit a solution of the minimisation problem for W in $K(\psi)$. For symmetric cone obstacles ψ, i.e., $\psi(x) = \psi(1-x)$ for all $x \in [0,1]$ and ψ is affine on $(0, 1/2)$, such that $\psi(0) = \psi(1) < 0$ and $\psi(1/2) > 0$, the question was completely solved as follows: (i) if $\psi(1/2) < 2/c_0$, then there exists a unique minimiser of W in $K_{sym}(\psi)$; (ii) if

Date: March 17, 2021.

$\psi(1/2) \geq 2/c_0$, then there is no minimiser of W not only in $K_{sym}(\psi)$ but also in $K(\psi)$, where

$$K_{sym}(\psi) := \{v \in K(\psi) \mid v(x) = v(1-x) \text{ for all } x \in [0,1]\},$$

and

$$c_0 := \int_{\mathbb{R}} \frac{d\tau}{(1+\tau^2)^{5/4}} = \mathcal{B}\left(\frac{1}{2}, \frac{3}{4}\right) = \sqrt{\pi}\frac{\Gamma(3/4)}{\Gamma(5/4)} = 2.396280469\ldots.$$

We note that the existence of a minimiser in the first case follows from Dall'Acqua–Deckelnick [1]. Its uniqueness was independently proved by Miura [9] and Yoshizawa [16]. The non-existence of minimisers in the case $\psi(1/2) > 2/c_0$ was proved by Müller [10] and in the critical case $\psi(1/2) = 2/c_0$ by Yoshizawa [16].

One of the purposes of this paper is to extend the studies of problem (1.1) to graphs of functions and surfaces of revolution with *Dirichlet* boundary conditions. In the one-dimensional setting, special emphasis is laid on studying the necessity of smallness conditions for general symmetric obstacles.

We describe now our main results in some more detail. First we consider the obstacle problem for $W(\cdot)$ among symmetric graphs. Namely, we only consider symmetric obstacles ψ which are subject to the following basic condition:

$$\text{(A)} \qquad \begin{aligned} &\psi \in C^0([0,1]), \quad \psi(x) = \psi(1-x) \text{ for all } x \in (0,1), \\ &\exists \sigma > 0: \quad \psi < 0 \text{ on } [0,\sigma) \cup (1-\sigma, 1], \quad \exists x_0 \in (0,1): \psi(x_0) > 0. \end{aligned}$$

The latter condition is imposed to avoid $u(x) \equiv 0$ as a possible trivial minimiser. The negativity condition is needed to ensure regularity of minimisers up to the boundary $\{0,1\}$. For such ψ we consider the minimisation problem

$$\inf_{v \in \mathcal{M}(\psi)} W(v) \tag{1.2}$$

with

$$\mathcal{M}(\psi) := \{u \in H_0^2(0,1) \mid u(x) = u(1-x), \ \ u(x) \geq \psi(x), \ \text{ for all } x \in (0,1)\}.$$

As for the existence of minimisers we have:

Theorem 1.1. *Assume that ψ satisfies conditions* (A) *and*

$$\text{(B)} \qquad \inf_{v \in \mathcal{M}(\psi)} W(v) < 4c_0^2.$$

Then there exists $u \in \mathcal{M}(\psi)$ such that

$$W(u) = \inf_{v \in \mathcal{M}(\psi)} W(v).$$

For the regularity of minimisers u of W in $\mathcal{M}(\psi)$, we obtain as in [1] that $u \in W^{3,\infty}$ and $u''' \in BV$, see Proposition 2.9. Moreover, Theorem 1.1 can be extended to the non-symmetric case (see Theorem 2.18), but then a more restrictive smallness condition has to be imposed.

In Remark 2.4 we explain how typical obstacles look like such that condition (B) is satisfied. While we have to leave open whether condition (B) is optimal, Theorem 1.2 below shows that indeed some kind of smallness condition is necessary in order to have existence of minimisers of W in $\mathcal{M}(\psi)$. It is natural to ask whether there exists a specific universal bound such that problem (1.2) has no solution, if the obstacle violates this bound. For this question we obtain an affirmative answer:

Theorem 1.2. *We assume that the obstacle ψ satisfies condition* (A) *and that there exists a minimiser $u \in \mathcal{M}(\psi)$ of $W(\cdot)$. Then the obstacle has to obey the following bound*:

$$\forall x \in [0,1]: \quad \psi(x) \leq \frac{1}{2} \max_{y\in[0,\infty]} \frac{2+2(1+y^2)^{-1/4}}{c_0 - G(y)} = 1.1890464540\ldots. \tag{1.3}$$

Here the function G, which appears in (1.3) and plays a crucial role in the one-dimensional elastica equation, is defined by

$$G:\mathbb{R} \to \left(-\frac{c_0}{2}, \frac{c_0}{2}\right), \quad G(t) := \int_0^t \frac{d\tau}{(1+\tau^2)^{5/4}}. \tag{1.4}$$

Theorem 1.2 says that for any obstacle ψ violating (1.3) the minimisation problem has no solution in the class $\mathcal{M}(\psi)$.

We derive (1.3) as a universal bound for all sufficiently smooth supersolutions of the elastica equation under Dirichlet boundary conditions. Although its optimality for "admissible" obstacles ψ (which permit a minimiser) is not proved here, we are confident that the universal bound cannot be improved for sufficiently smooth supersolutions, see Remark 2.15. Moreover, the proof of Theorem 1.2 can be adapted to the Navier boundary value problem, where one can obtain the universal bound $2/c_0$. This means that the proof of Theorem 1.2 yields a significant generalisation of [1, Lemma 4.3] and [10, Theorem 1.1].

We adapt Theorem 1.1 to the minimisation problem for the Willmore functional for surfaces of revolution under Dirichlet boundary conditions and a unilateral constraint. Let $u : [-1,1] \to (0,\infty)$ be a profile curve with Dirichlet boundary conditions

$$u(1) = u(-1) = \alpha > 0, \quad u'(1) = u'(-1) = 0. \tag{1.5}$$

Then the Willmore functional of the corresponding surface of revolution $\mathcal{R}(u)$: $(x,\theta) \mapsto (x, u(x)\cos(\theta), u(x)\sin(\theta))$ with mean curvature H (defined as the mean value of the principal curvatures) is given by

$$\begin{aligned} W(u) &= \int_{\mathcal{R}(u)} H^2\, dS \\ &= \frac{\pi}{2}\int_{-1}^{1} \left(\frac{1}{u(x)\sqrt{1+(u'(x))^2}} - \frac{u''(x)}{(1+(u'(x))^2)^{3/2}} \right)^2 u(x)\sqrt{1+(u'(x))^2}\, dx. \end{aligned}$$

For obstacles ψ satisfying
(C)
$\psi \in C^0([-1,1];(0,\infty)), \quad \psi(1) = \psi(-1) > \alpha, \quad \psi(x) = \psi(-x)$ for all $x \in (-1,1)$,

we consider the minimisation problem

$$\inf_{v \in N_\alpha(\psi)} W(v), \tag{1.6}$$

with

$$\begin{aligned} N_\alpha &:= \{v \in H^2((-1,1);(0,\infty)) \mid v \text{ satisfies (1.5)}, \\ &\qquad\qquad v(x) = v(-x) \text{ for all } x \in [-1,1]\}, \\ N_\alpha(\psi) &:= \{v \in N_\alpha \mid v(x) \leq \psi(x) \text{ for all } x \in [-1,1]\}. \end{aligned}$$

This means that in contrast to the one-dimensional situation, we look for minimising functions *below* the given obstacle. Typically one should think of obstacles which satisfy $\psi(x) < \alpha$ for $|x|$ close to 0. Typical "admissible" obstacles (where a minimiser exists) are constructed in Remark 3.3.

We think that this is here the more interesting setting because solutions of the obstacle problem look then completely different from minimisers of the free problem (without obstacle), which are all strictly above α. The arguments for minimisation in classes of functions above a given obstacle, however, will be similar.

First we have similarly as in Theorem 1.1:

Theorem 1.3. *Assume that ψ satisfies conditions* (C) *and*

$$\inf_{v \in N_\alpha(\psi)} W(v) < 4\pi. \tag{D}$$

Then there exists $u \in N_\alpha(\psi)$ such that

$$W(u) = \inf_{v \in N_\alpha(\psi)} W(v).$$

We obtain the same regularity result of minimisers u of W in $N_\alpha(\psi)$ as above: $u \in W^{3,\infty}$, $u''' \in BV$, see Proposition 3.10.

We show in Proposition 3.4 that for $\alpha \to \infty$, only almost constant functions $v(x) \approx \alpha$ satisfy condition (D). This means that Theorem 1.3 is interesting (only) for "small" α, while for $\alpha \to \infty$ we should seek a condition different from (and weaker than) (D). Indeed, we have:

Theorem 1.4. *Assume that ψ satisfies condition* (C) *and is such that*

$$\inf_{v \in N_\alpha(\psi)} W(v) < \pi \max_{S \in [0,\alpha]} g_\alpha(S) \text{ with } g_\alpha(S) := (\alpha - S)G(S)^2. \tag{1.7}$$

Then there exists $u \in N_\alpha(\psi)$ such that

$$W(u) = \inf_{v \in N_\alpha(\psi)} W(v).$$

We discuss in Remark 3.5 that condition (1.7) is actually weaker than condition (D) for large α (beyond ≈ 6.1). The occurrence of the function G indicates that in this case one-dimensional arguments come into play again.

We construct many examples belonging to the admissible sets $\mathcal{M}(\psi)$ and $N_\alpha(\psi)$ by employing particular solutions of the elastica equation and prototype Willmore surfaces of revolution (spheres and catenoids). These examples illustrate how obstacles may look like in order to obey condition (B), (D) or (1.7), respectively.

The gradient flow for W defined on graphs with a unilateral constraint and satisfying homogeneous Navier boudary conditions has also recently been studied in [11, 12, 13, 16]. It would be also interesting to investigate gradient flows corresponding to problems (1.2) and (1.6).

This paper is organised as follows: Section 2 is devoted to the obstacle problem for the one-dimentional Willmore functional with Dirichlet boundary conditions. In Section 2.1 we introduce some notations and collect fundamental facts about the elastica equation. We prove Theorem 1.1 in Section 2.2 and the regularity of minimisers in Section 2.3. We prove Theorem 1.2 in Section 2.4. In Section 2.5 we briefly study the non-symmetric case.

Section 3 is concerned with the obstacle problem for the Willmore functional defined on surfaces of revolution, again with Dirichlet boundary conditions. In Section 3.1 we recall the basic existence and symmetry result for "free" minimisers (with no obstacle). In Section 3.2 we prove Theorems 1.3 and 1.4 and present many instructive examples. Finally we prove the regularity of minimisers in Section 3.3.

2. The one-dimensional Dirichlet obstacle problem

2.1. **Explicit solutions for the one-dimensional Willmore equation.** We first collect some facts concerning the one-dimensional Willmore or elastica equation without obstacle which were already known to Euler [6, pp. 231–297] (see in particular pp. 233–234) and which will be relevant also in order to understand the shape of admissible obstacles in what follows. For a more convenient reference one may also see [4].

For $u \in H_0^2(0,1)$, i.e. a sufficiently smooth function subject to homogeneous Dirichlet boundary conditions (horizontal clamping), we define the one-dimensional Willmore or elastica functional:

$$W(u) := \int_0^1 \kappa(x)^2 \sqrt{1+u'(x)^2}\,dx = \int_0^1 \left(\frac{u''(x)}{(1+u'(x)^2)^{3/2}}\right)^2 \sqrt{1+u'(x)^2}\,dx$$
$$= \int_0^1 \frac{u''(x)^2}{(1+u'(x)^2)^{5/2}}\,dx.$$

For the graph of $u : [0,1] \to \mathbb{R}$, its curvature is given by

$$\kappa(x) := \kappa_u(x) := \frac{u''(x)}{(1+u'(x)^2)^{3/2}}.$$

Critical points of the one-dimensional Willmore or elastica functional satisfy the one-dimensional Willmore or elastica equation:

$$\frac{1}{\sqrt{1+u'(x)^2}}\frac{d}{dx}\left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right) + \frac{1}{2}\kappa(x)^3 = 0, \quad x \in (0,1). \tag{2.1}$$

Symmetric solutions to this equation are known explicitly. Here the function G defined in (1.4) plays an important role. This is smooth and strictly increasing and so is $G^{-1} : (-c_0/2, c_0/2) \to \mathbb{R}$.

Lemma 2.1. ([4, Lemma 4]) *Let $u \in C^4([0,1])$ be a function symmetric around $x = 1/2$. Then u solves the Willmore equation (2.1) iff there exists $c \in (-c_0, c_0)$ such that*

$$u'(x) = G^{-1}\left(\frac{c}{2} - cx\right) \quad in \quad [0,1].$$

For the curvature, one has that

$$\kappa(x) = -\frac{c}{\sqrt[4]{1+G^{-1}\left(\frac{c}{2}-cx\right)^2}}.$$

Moreover, if we additionally assume that $u(0) = u(1) = 0$, then one has

$$u(x) = \frac{2}{c\sqrt[4]{1+G^{-1}\left(\frac{c}{2}-cx\right)^2}} - \frac{2}{c\sqrt[4]{1+G^{-1}\left(\frac{c}{2}\right)^2}}, \quad c \neq 0.$$

Finally, if one additionally assumes also that $u'(0) = u'(1) = 0$, then one has

$$u(x) \equiv 0.$$

One should observe that this result holds true *only in the class of graphs of smooth functions*. The graph of the function $\hat{u}_{c_0}$ as displayed in Figure 1 is a non-trivial solution to the Dirichlet problem for the one-dimensional Willmore (elastica) equation under homogeneous boundary conditions, but:

- although the curve is smooth and a graph, due to $\hat{u}_{c_0}'(1/4) = -\hat{u}_{c_0}'(3/4) = \infty$, it is *not a smooth graph solution*,
- it is not minimising the Willmore energy.

2.2. **The obstacle problem in the symmetric case.** In this section we only consider symmetric obstacles ψ satisfying condition (A). We define a set $\mathcal{M}(\psi)$ of admissible functions as follows:

$$\mathcal{M}(\psi) := \{u \in H_0^2(0,1) \mid u(x) = u(1-x), \;\; u(x) \geq \psi(x), \;\; \text{for any } \; x \in (0,1)\}.$$

One should observe that unlike Section 3 we consider functions *above* the given obstacle. Thanks to condition (A) this set is not empty. We assume further that ψ satisfies condition (B), i.e.,

$$\alpha := \inf_{v \in \mathcal{M}(\psi)} W(v) < 4c_0^2.$$

In what follows, c_1 denotes a constant such that $c_1^2 \in (\alpha/4, c_0^2)$, i.e., $c_1 \in (\sqrt{\alpha}/2, c_0)$.

Lemma 2.2. *Assume that $v \in \mathcal{M}(\psi)$ satisfies $W(v) \leq 4c_1^2$. Then*

$$\max_{x \in [0,1]} |v'(x)| \leq G^{-1}\left(\frac{c_1}{2}\right) < \infty.$$

Proof. Let $x_{\max} \in (0,1)$ be such that

$$v'(x_{\max}) = \max_{x \in [0,1]} |v'(x)|.$$

Since $v \in \mathcal{M}(\psi)$ is symmetric, we have $v'(x_{\max}) = -v'(1-x_{\max})$. To begin with, we consider the case where $x_{\max} \in (0, 1/2)$. By Hölder's inequality, we have

$$\begin{aligned}
&4G(v'(x_{\max}))\\
&= G(v'(x_{\max})) + \{G(v'(x_{\max})) - G(v'(1-x_{\max}))\} - G(v'(1-x_{\max}))\\
&= \int_0^{v'(x_{\max})} \frac{d\tau}{(1+\tau^2)^{\frac{5}{4}}} + \int_{v'(1-x_{\max})}^{v'(x_{\max})} \frac{d\tau}{(1+\tau^2)^{\frac{5}{4}}} + \int_{v'(1-x_{\max})}^{0} \frac{d\tau}{(1+\tau^2)^{\frac{5}{4}}}\\
&= \int_0^{x_{\max}} \frac{v''(x)}{(1+v'(x)^2)^{\frac{5}{4}}}\,dx + \int_{x_{\max}}^{1-x_{\max}} \frac{-v''(x)}{(1+v'(x)^2)^{\frac{5}{4}}}\,dx + \int_{1-x_{\max}}^{1} \frac{v''(x)}{(1+v'(x)^2)^{\frac{5}{4}}}\,dx\\
&\leq \int_0^1 |\kappa_v(x)|(1+v'(x)^2)^{1/4}\,dx \leq W(v)^{1/2} \leq 2c_1,
\end{aligned}$$

i.e.,

$$G(v'(x_{\max})) \leq \frac{c_1}{2}.$$

Then it follows from the monotonicity of G that

$$v'(x_{\max}) \leq G^{-1}\left(\frac{c_1}{2}\right) < \infty. \tag{2.2}$$

We turn to the case where $x_{\max} \in (1/2, 1)$. Considering $-v$ instead of v and $1-x_{\max}$ instead of $x_{\max}$, we observe as in the first case that (2.2) holds. The proof is complete. ◻

The preceding estimate is the key to prove our first existence result.

Proof of Theorem 1.1. Let $\{u_j\} \subset \mathcal{M}(\psi)$ be a minimising sequence, i.e.,

$$\lim_{j\to\infty} W(u_j) = \alpha := \inf_{v\in\mathcal{M}(\psi)} W(v).$$

Then, for $c_1 \in (\sqrt{\alpha}, c_0)$, extracting a subsequence, we have

$$W(u_j) \le c_1 \quad \text{for any} \quad j \in \mathbb{N}, \tag{2.3}$$

where for brevity $\{u_j\}$ denotes also the subsequence. Then, thanks to Lemma 2.2, we obtain

$$|u_j(x)| + |u_j'(x)| \le \int_0^{\frac{1}{2}} |u_j'(\xi)|\, d\xi + |u_j'(x)| \le \frac{3}{2} G^{-1}\left(\frac{c_1}{2}\right)$$

for any $x \in [0,1]$. Thus $\{u_j\}$ is bounded in $C^1(0,1)$. This together with (2.3) implies that $\{u_j\}$ is also bounded in $H^2(0,1)$. Indeed, we have

$$c_1 \ge W(u_j) \ge \frac{1}{(1+G^{-1}(c_1/2)^2)^{5/2}} \int_0^1 u''(x)^2\, dx.$$

Thus we find $u \in H_0^2(0,1)$ such that

$$u_j \rightharpoonup u \quad \text{weakly in} \quad H_0^2(0,1), \tag{2.4}$$

up to a subsequence. Since the embedding $H_0^2(0,1) \subset C^{1,\gamma}(0,1)$ is compact for any $\gamma \in (0,1/2]$, we have in particular that

$$u_j \to u \quad \text{in} \quad C^1(0,1). \tag{2.5}$$

Then it follows from (2.4) and (2.5) that

$$\liminf_{j\to\infty} W(u_j) \ge W(u).$$

Recalling that (2.5) yields $u \in \mathcal{M}(\psi)$, we see that u is a minimiser of W. The proof is complete. $\square$

Remark 2.3. Since $\mathcal{M}(\psi) \subset H_0^2(0,1)$ is convex, one finds as usual that any minimiser $u \in \mathcal{M}(\psi)$ of $W(\cdot)$ satisfies the variational inequality:

$$W'(u)(v-u) \ge 0 \quad \text{for all} \quad v \in \mathcal{M}(\psi).$$

We find from [4, Lemma 2] that

$$W'(u)(\varphi) = 2\int_0^1 \frac{\kappa(x)}{1+u'(x)^2}\varphi''(x)\, dx - 5\int_0^1 \frac{\kappa(x)^2 u'(x)}{\sqrt{1+u'(x)^2}}\varphi'(x)\, dx \tag{2.6}$$

for all $\varphi \in H_0^2(0,1)$.

In order to construct admissible obstacles we first recall the scaling behaviour of the one-dimensional Willmore functional. For $v : [0,1] \to \mathbb{R}$ and $\rho > 0$, we consider $v_\rho : [0,\rho] \to \mathbb{R}$ defined by $v_\rho(x) := \rho v(x/\rho)$. Then it holds that

$$v_\rho'(x) = v'\left(\frac{x}{\rho}\right), \quad v_\rho''(x) = \frac{1}{\rho} v''\left(\frac{x}{\rho}\right),$$

and then

$$\kappa_\rho(x) := \frac{v_\rho''(x)}{(1+v_\rho'(x)^2)^{3/2}} = \frac{1}{\rho}\frac{v''(\frac{x}{\rho})}{(1+v'(\frac{x}{\rho})^2)^{3/2}} =: \frac{1}{\rho}\kappa\left(\frac{x}{\rho}\right).$$

Hence we have

$$W_{[0,\rho]}(v_\rho) := \int_0^\rho \kappa_\rho(x)^2 \sqrt{1+v_\rho'(x)^2}\, dx$$
$$= \frac{1}{\rho^2}\int_0^\rho \kappa\left(\frac{x}{\rho}\right)^2 \sqrt{1+v'\left(\frac{x}{\rho}\right)^2}\, dx = \frac{1}{\rho}W(v).$$

The idea is now in order to find admissible obstacles to glue different pieces of explicit solutions together and to rescale them.

Remark 2.4. For each $c \in (0, c_0)$, we let $u_c : [0,1] \to \mathbb{R}$ denote the solution of (2.1) obtained by Lemma 2.1. Thanks to Lemma 2.1, we see that $W(u_c) = c^2$. Here we oddly extend u_c to $U_c \in H^2([-1/2, 3/2], \mathbb{R})$ as follows:

$$U_c(x) := \begin{cases} -u_c(-x) & x \in [-\frac{1}{2}, 0], \\ u_c(x), & x \in [0,1], \\ -u_c(2-x) & x \in [1, \frac{3}{2}]. \end{cases}$$

One should observe that only for $c = c_0$ this extension yields a solution to (2.1) also for x around 0 and 1.

Since u_c is oddly extended to U_c, it is clear that $W(U_c) = 2c^2$. By means of a scaling and translation, we define $\hat{u}_c : [0,1] \to \mathbb{R}$ as follows:

$$\hat{u}_c(x) := \frac{1}{2}U_c(-\tfrac{1}{2} + 2x) - \frac{1}{2}U_c(-\tfrac{1}{2}).$$

Recalling the scaling property of W, we observe that

$$W(\hat{u}_c) = 2W(U_c) = 4c^2.$$

Thus, for any $c \in (0, c_0)$, we find that $W(\hat{u}_c) < 4c_0^2$. This implies that for any sufficiently small $\varepsilon > 0$, any $\psi \in C^0([0,1])$ with $\psi \le \hat{u}_c - \varepsilon$ obeys conditions (A) and (B). In particular, any $\hat{u}_c - \varepsilon$ with $c \in (0, c_0)$ is itself an admissible obstacle.

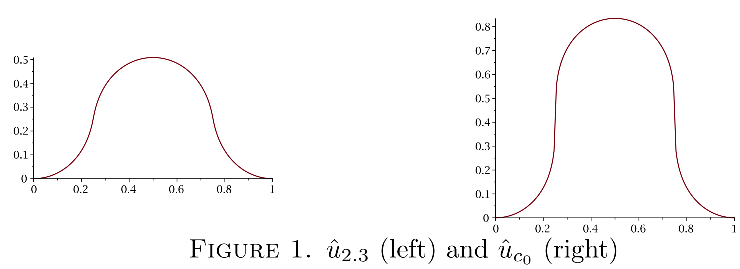

FIGURE 1. $\hat{u}_{2.3}$ (left) and $\hat{u}_{c_0}$ (right)

One may observe that

$$\max_{x\in[0,1]} \hat{u}_{c_0} = \hat{u}_{c_0}(1/2) = \frac{2}{c_0} = 0.8346268418\ldots.$$

We consider only $c > 0$ in order to obtain obstacles with $\psi(x_0) > 0$, i.e. satisfying condition (A).

2.3. **Regularity of symmetric minimisers.** In order to show regularity we follow the reasoning of Dall'Acqua and Deckelnick, one may see [1, Proof of Theorem 5.1].

Let $u \in \mathcal{M}(\psi)$ be a minimiser of $W(\cdot)$. Recalling Remark 2.3 we notice first that u is a weak supersolution of the elastica equation.

Corollary 2.5. *For all $\varphi \in H_0^2(0,1)$ with $\varphi \geq 0$ we have that*

$$W'(u)(\varphi) \geq 0.$$

Proof. Take $u + \varphi$ as comparison function in Remark 2.3. $\square$

This corollary shows that $W'(u)$ is a nonnegative distribution on $C_c^\infty(0,1)$. This nonnegativity yields that $W'(u)$ is even a distribution on $C_c^0(0,1)$. Thus, by the Riesz representation theorem we find a nonnegative Radon measure μ such that

$$W'(u)(\varphi) = \int_0^1 \varphi \, d\mu \tag{2.7}$$

for all $\varphi \in C_c^\infty(0,1)$. See [15, Lemma 37.2]. We define $\mathcal{N} \subset (0,1)$ by

$$\mathcal{N} := \{x \in (0,1) \mid u(x) > \psi(x)\}.$$

Since u and ψ are continuous in $[0,1]$, we see that the set $\mathcal{N}$ is an open, and

$$\mu(\mathcal{N}) = 0. \tag{2.8}$$

In other words, the restriction $u|_{\mathcal{N}}$ is even a *solution* of the elastica equation.

Indeed, for all $\varphi \in C_c^\infty(\mathcal{N})$, we have $u + \varepsilon\varphi \geq \psi$ in $[0,1]$ for $\varepsilon > 0$ small enough, and then $W'(u)(\varphi) = 0$. This implies that $u|_{\mathcal{N}}$ is even a weak *solution* of the elastica equation.

Lemma 2.6. *Assume that ψ satisfies condition* (A). *Suppose that there exists a minimiser $u \in \mathcal{M}(\psi)$ of $W(\cdot)$. Then there exist $a \in (0,1/2)$ such that $(0,a) \cup (1-a,1) \subset \mathcal{N}$.*

Proof. Since ψ is continuous in $[0,1]$, there exists $\delta > 0$ such that

$$\psi(x) < \frac{3}{4}\psi(0) \quad \text{for all} \quad x \in [0,\delta).$$

By Sobolev's embedding $H^2(0,1) \subset W^{1,\infty}(0,1)$ and since $\mathcal{M}(\psi) \subset H^2(0,1)$, we find a constant $C_1 > 0$ such that

$$\|u\|_{W^{1,\infty}(0,1)} \leq C\|u\|_{H^2(0,1)} \leq C_1.$$

This together with $u(0) = 0$ implies that

$$|u(x)| \leq C_1 x \quad \text{for all} \quad x \in [0,1].$$

Thus, taking $\delta' > 0$ such that $\delta' < -\psi(0)/4C_1$, we see that

$$u(x) \geq \frac{1}{4}\psi(0) \quad \text{for all} \quad x \in [0,\delta'].$$

Setting $a := \min\{\delta,\delta'\}$, we deduce that $(0,a) \subset \mathcal{N}$. The symmetry of u yields that also $(1-a,1) \subset \mathcal{N}$. The proof of Lemma 2.6 is complete. $\square$

We next show that the nonnegative Radon measure μ is finite.

Lemma 2.7. *Assume that ψ satisfies condition* (A)*. Suppose that there exists a minimiser $u \in \mathcal{M}(\psi)$ of $W(\cdot)$. Then*

$$\mu(0,1) < \infty.$$

Proof. Thanks to Lemma 2.6 and (2.8), we find $a \in (0,1/2)$ such that $\mu((0,1)) = \mu([a,1-a])$. Fix $\eta \in C_c^\infty(0,1)$ with $\eta \equiv 1$ in $[a,1-a]$ and $0 \le \eta \le 1$ in $(0,1)$. Since by assumption $u \in \mathcal{M}(\psi) \subset H^2(0,1) \hookrightarrow C^1([0,1])$ we conclude from (2.6):

$$\begin{aligned}\mu((0,1)) = \mu([a,1-a]) &\le \int_0^1 \eta(x)\,d\mu = W'(u)(\eta)\\ &\le C(\|u''\|_{L^2(0,1)}\|\eta''\|_{L^2(0,1)} + \|u''\|^2_{L^2(0,1)}\|\eta'\|_{L^\infty(0,1)})\\ &\le C(\|\eta''\|_{L^2(0,1)} + \|\eta'\|_{L^\infty(0,1)}).\end{aligned}$$

This proves the claim. □

In order to study the regularity of minimisers, we employ ideas used in [1, Proposition 3.2] and [2, Theorem 3.9].

Lemma 2.8. *Fix $\eta \in C_c^\infty(0,1)$ and set*

$$\varphi_1(x) := \int_0^x \int_0^y \eta(s)\,dsdy + \alpha x^2 + \beta x^3,$$

$$\varphi_2(x) := \int_0^x \eta(y)\,dy + (-3x^2 + 2x^3)\int_0^1 \eta(y)\,dy,$$

for $x \in [0,1]$, where

$$\alpha := \int_0^1 \eta(y)\,dy - 3\int_0^1\int_0^y \eta(s)\,dsdy, \qquad \beta := -\alpha - \int_0^1\int_0^y \eta(s)\,dsdy.$$

Then, $\varphi_1, \varphi_2 \in H_0^2(0,1)$ and there exists $C > 0$ such that

$$\|\varphi_1\|_{C^1(0,1)}, |\alpha|, |\beta| \le C\|\eta\|_{L^1(0,1)},$$

$$\|\varphi_2\|_{L^\infty(0,1)} \le C\|\eta\|_{L^1(0,1)}, \qquad \|\varphi_2'\|_{L^p(0,1)} \le C\|\eta\|_{L^p(0,1)} \quad \text{for} \quad p \in [1,\infty).$$

Proposition 2.9. *Assume that ψ satisfies condition* (A)*. Suppose that there exists a minimiser $u \in \mathcal{M}(\psi)$ of $W(\cdot)$. Then $u \in C^2([0,1])$, u'' is weakly differentiable and $u''' \in BV(0,1)$.*

Proof. We define a function $m : (0,1) \to \mathbb{R}$ by

$$m(x) = \mu(0,x) \quad \text{for} \quad x \in (0,1).$$

Then m is increasing and bounded on $(0,1)$, and the Lebesgue–Stieltjes integral induced by m is well-defined. Using integration by parts for Lebesgue–Stieltjes integrals ([14, Chapter III, Theorem 14.1]), we obtain

$$\int_0^1 \varphi\,d\mu(x) = -\int_0^1 m(x)\varphi'(x)\,dx \tag{2.9}$$

for all $\varphi \in C_c^\infty(0,1)$. It follows from (2.6), (2.7) and (2.9) that

$$2\int_0^1 \frac{\kappa(x)}{1+u'(x)^2}\varphi''(x)\,dx = 5\int_0^1 \frac{\kappa(x)^2 u'(x)}{\sqrt{1+u'(x)^2}}\varphi'(x)\,dx - \int_0^1 m(x)\varphi'(x)\,dx \tag{2.10}$$

for all $\varphi \in C_c^\infty(0,1)$. By a density argument, we see that (2.10) also holds for all $\varphi \in H_0^2(0,1)$.

Fix $\eta \in C^\infty_{\mathrm{c}}(0,1)$ arbitrarily. Taking φ_1 as φ in (2.10), where φ_1 is defined in Lemma 2.8, we have

$$
\begin{aligned}
2\int_0^1 \frac{u''(x)}{(1+u'(x)^2)^{5/2}}\eta(x)\,dx = &-4\int_0^1 \frac{u''(x)}{(1+u'(x)^2)^{5/2}}(\alpha+3\beta x)\,dx \\
&+5\int_0^1 \frac{|u''(x)|^2u'(x)}{(1+u'(x)^2)^{7/2}}\varphi_1'(x)\,dx \\
&-\int_0^1 m(x)\varphi_1'(x)\,dx =: I_1+I_2+I_3.
\end{aligned} \tag{2.11}
$$

Since by assumption $u \in \mathcal{M}(\psi) \subset H^2(0,1) \hookrightarrow C^1([0,1])$, we observe from Lemmas 2.7 and 2.8 that

$$
\begin{aligned}
&|I_1| \le (4|\alpha|+12|\beta|)\|u''\|_{L^1(0,1)} \le C\|\eta\|_{L^1(0,1)},\\
&|I_2| \le 5\|u''\|^2_{L^2(0,1)}\|\varphi_1\|_{C^1([0,1])} \le C\|\eta\|_{L^1(0,1)},\\
&|I_3| \le \sup_{x\in[0,1]} m(x)\|\varphi_1\|_{C^1([0,1])} \le C\mu(0,1)\|\eta\|_{L^1(0,1)} \le C\|\eta\|_{L^1(0,1)}.
\end{aligned}
$$

This together with (2.11) implies that

$$
\|u''(x)(1+u'(x)^2)^{-5/2}\|_{L^\infty(0,1)} \le C. \tag{2.12}
$$

In view of $u \in C^1([0,1])$ we obtain from (2.12):

$$
\|u''\|_{L^\infty(0,1)} \le C. \tag{2.13}
$$

Fix $\eta \in C^\infty_{\mathrm{c}}(0,1)$ arbitrarily. Taking φ_2 as φ in (2.10), where φ_2 is defined by Lemma 2.8, we have

$$
\begin{aligned}
\int_0^1 \frac{u''(x)}{(1+u'(x)^2)^{5/2}}\eta'(x)\,dx = 6&\left(\int_0^1 (1-2x)\frac{u''(x)}{(1+u'(x)^2)^{5/2}}\,dx\right)\left(\int_0^1 \eta(x)\,dx\right)\\
&+\frac{5}{2}\int_0^1 \frac{u''(x)^2u'(x)\varphi_2'(x)}{(1+u'(x)^2)^{7/2}}\,dx\\
&-\frac{1}{2}\int_0^1 m(x)\varphi_2'(x)\,dx =: I_1'+I_2'+I_3'.
\end{aligned}
$$

We deduce from (2.13) and $u \in H^2(0,1) \hookrightarrow C^1([0,1])$ that

$$
|I_2'| \le C\|u''\|^2_{L^\infty(0,1)}\|\varphi_2'\|_{L^1(0,1)} \le C\|\eta\|_{L^1(0,1)}.
$$

Along the same lines as above, we obtain

$$
\begin{aligned}
&|I_1'| \le 6\|u''\|_{L^\infty(0,1)}\|\eta\|_{L^1(0,1)} \le C\|\eta\|_{L^1(0,1)},\\
&|I_3'| \le \frac{1}{2}\Big(\sup_{x\in[0,1]} m(x)\Big)\|\varphi_2'\|_{L^1(0,1)} \le C\mu(0,1)\|\eta\|_{L^1(0,1)} \le C\|\eta\|_{L^1(0,1)}.
\end{aligned}
$$

Thus we observe that

$$
\Big|\int_0^1 \frac{u''(x)}{(1+u'(x)^2)^{5/2}}\eta'(x)\,dx\Big| \le C\|\eta\|_{L^1(0,1)}
$$

for all $\eta \in C^\infty_{\mathrm{c}}(0,1)$, and then

$$
\|(u''(1+(u')^2)^{-5/2})'\|_{L^\infty(0,1)} \le C.
$$

This together with $u \in C^1([0,1])$ and (2.13) implies that

$$
\|u'''\|_{L^\infty(0,1)} < \infty. \tag{2.14}
$$

We observe from (2.10) and (2.14) that for all $\varphi \in C_c^\infty(0,1)$ we have

$$\int_0^1 \Big[-2\Big(\frac{u''(x)}{(1+u'(x)^2)^{5/2}}\Big)' - 5\frac{u''(x)^2 u'(x)}{(1+u'(x)^2)^{7/2}} + m(x)\Big]\varphi'(x)\,dx = 0.$$

Thus there exists a constant $c \in \mathbb{R}$ such that

$$-2\Big(\frac{u''(x)}{(1+u'(x)^2)^{5/2}}\Big)' - 5\frac{u''(x)^2 u'(x)}{(1+u'(x)^2)^{7/2}} + m(x) = c \quad \text{for all} \quad x \in (0,1).$$

Recalling that $m(\cdot)$ is of bounded variation, we see that $u''' \in BV(0,1)$. The proof of Proposition 2.9 is complete. $\square$

2.4. **Further properties of minimisers in the symmetric case.** We deduce some properties of solutions of the obstacle problem as constructed in the previous section. That means that in what follows we consider

$$u \in \mathcal{M}(\psi): \quad W(u) = \inf_{v \in \mathcal{M}(\psi)} W(v).$$

We recall that according to condition (A) we always have that

$$\exists x_0 \in (0,1): \quad \psi(x_0) > 0.$$

So, for sure, the solution u is strictly positive somewhere. In what follows we shall find out much more. Among others we prove Theorem 1.2 which implies that for obstacles exceeding a specific universal bound our minimisation problem has *no solution.*

One should always have in mind that the following results are valid due to the fact that we are looking for minimisers in the class of "relatively smooth" symmetric *graphs.* In this class existence results are more restrictive while we have more and stronger results concerning qualitative properties of minimisers. We expect that this situation will change fundamentally when admitting all "sufficiently smooth" *curves.* We emphasise that smooth curves, which are graphs, need not be smooth graphs. As for this one should always have the function $\hat{u}_{c_0}$ in mind, which is defined in Remark 2.4.

We recall from Corollary 2.5 that the minimiser u under consideration is a weak supersolution of the elastica equation, i.e. for all $\varphi \in H_0^2(0,1)$ with $\varphi \geq 0$ we have that $W'(u)(\varphi) \geq 0$. Beside the regularity result in Proposition 2.9 only this is exploited in what follows.

We associate with u the following corresponding auxiliary function:

$$V(x) := V_u(x) := \kappa_u(x)(1+u'(x)^2)^{1/4} = \frac{u''(x)}{(1+u'(x)^2)^{5/4}}.$$

This turned to out to be extremely useful in [4], see e.g. Lemma 3 therein. Obviously, already Euler [6] was aware of the importance of this function. Similarly as in [4] we see that this function is a weak subsolution of a second order differential equation in divergence form:

Proposition 2.10. *We have for all $\varphi \in H_0^2(0,1)$ with $\varphi \geq 0$ that*

$$0 \geq \int_0^1 \varphi'(x) \left(\frac{V'(x)}{(1+u'(x)^2)^{5/4}}\right) dx. \tag{2.15}$$

Moreover, one has

$$0 \geq \int_0^1 (\varphi(x)u'(x))' \left(\frac{u'(x)V'(x)}{(1+u'(x)^2)^{5/4}} - \frac{1}{2}V(x)^2 \right) dx. \tag{2.16}$$

That the right hand side of (2.15) is the weak form of the elastica operator was already observed by Müller [10, Proposition 2.3].

Proof. We conclude from Remark 2.3, Corollary 2.5, and Proposition 2.9 that the following holds for all such φ:

$$0 \geq \int_0^1 \varphi'(x) \left(2\left(\frac{\kappa(x)}{1+u'(x)^2} \right)' + 5\frac{\kappa(x)^2 u'(x)}{\sqrt{1+u'(x)^2}} \right) dx.$$

Observing that

$$\begin{aligned} 2\left(\frac{\kappa(x)}{1+u'(x)^2} \right)' &= 2\left(\frac{V(x)}{(1+u'(x)^2)^{5/4}} \right)' = 2\frac{V'(x)}{(1+u'(x)^2)^{5/4}} - 5\frac{V(x)u'(x)u''(x)}{(1+u'(x)^2)^{9/4}} \\ &= 2\frac{V'(x)}{(1+u'(x)^2)^{5/4}} - 5\frac{V(x)^2 u'(x)}{1+u'(x)^2} \end{aligned}$$

we find

$$0 \geq 2\int_0^1 \varphi'(x) \frac{V'(x)}{(1+u'(x)^2)^{5/4}}\, dx.$$

This shows the first claim. In order to see the second one we use $\varphi(u')^2$ as a test function and observe that $\left(\varphi(u')^2\right)' = u'\,(\varphi u')' + \varphi\, u'\, u''$:

$$\begin{aligned} 0 &\geq \int_0^1 \left(\varphi(u')^2\right)'(x) \left(\frac{V'(x)}{(1+u'(x)^2)^{5/4}} \right) dx \\ &= \int_0^1 (\varphi u')'(x) \left(\frac{u'(x)V'(x)}{(1+u'(x)^2)^{5/4}} \right) dx + \int_0^1 \varphi(x)u'(x)V(x)V'(x)\, dx \\ &= \int_0^1 (\varphi u')'(x) \left(\frac{u'(x)V'(x)}{(1+u'(x)^2)^{5/4}} \right) dx + \int_0^1 \varphi(x)u'(x) \left(\frac{V^2}{2} \right)'(x)\, dx, \end{aligned}$$

and also the second claim (2.16) follows. $\square$

The previous proposition reflects on the one hand the well known fact that the elastica operator as a whole is of divergence form. On the other hand this shows that V is a subsolution of a second order differential inequality without zeroth-order term and hence obeys a strong (!) maximum principle. Observing further that by assumption u must be strictly positive somewhere, this implies in view of the homogeneous Dirichlet boundary conditions that u must be strictly concave somewhere in the interior of $(0,1)$ and strictly convex as well left as also right from this. Combining this with the strong Hopf-type maximum principle for V and making use of the symmetric setting we immediately obtain the following result.

Proposition 2.11. *The auxiliary function V satisfies $V'(x) < 0$ on $[0,1/2)$ and $V'(x) > 0$ on $(1/2,1]$ and obeys*

$$V(0) = V(1) > 0, \qquad V(\frac{1}{2}) < 0.$$

There exists a point $a \in (0, 1/2)$ such that $V > 0$ (and hence u is strictly convex) on $[0, a) \cup (1-a, 1]$ and $V < 0$ (and hence u is strictly concave) on $[a, 1-a]$. Moreover we have

$$\forall x \in (0, \frac{1}{2}) : \quad u'(x) > 0, \qquad \forall x \in (\frac{1}{2}, 1) : \quad u'(x) < 0;$$

$$\forall x \in [0, \frac{1}{2}) : \quad \kappa'(x) < 0, \qquad \forall x \in (\frac{1}{2}, 1] : \quad \kappa'(x) > 0. \tag{2.17}$$

Finally we have

$$\forall x \in (\frac{1}{2}, 1) : \quad V(1/2)^2 \leq \left(V(x)^2 - 2\frac{u'(x)V'(x)}{(1+u'(x)^2)^{5/4}} \right) \leq V(1)^2.$$

In particular

$$0 < -\kappa(1/2) \leq \kappa(1).$$

Proof. The last result follows from (2.16) and it remains only to show (2.17). Since κ is symmetric around $1/2$ it suffices to consider $x \in (1/2, 1]$ where $u'(x) < 0$. Here we find from the strict monotonicity of V for $x \in (1/2, 1]$:

$$\begin{aligned} 0 < V'(x) &= \left(\kappa(x)(1+u'(x)^2)^{1/4} \right)' \\ &= \kappa'(x)(1+u'(x)^2)^{1/4} + \frac{1}{2}\kappa(x)(1+u'(x)^2)^{-3/4}u'(x)u''(x) \\ &= \kappa'(x)(1+u'(x)^2)^{1/4} + \frac{1}{2}\kappa(x)^2(1+u'(x)^2)^{3/4}u'(x) \\ &\leq \kappa'(x)(1+u'(x)^2)^{1/4}, \end{aligned}$$

and (2.17) follows. □

Although higher order equations do in general not obey any kind of comparison principle, here we can show that symmetric supersolutions lie above symmetric solutions which obey the same Dirichlet boundary conditions.

Lemma 2.12. *Assume that $u \in C^2([0,1]) \cap W^{3,\infty}(0,1)$ is symmetric around $1/2$, satisfies for some $\beta \in \mathbb{R}$ Dirichlet boundary conditions*

$$u(0) = u(1) = 0, \qquad u'(0) = -u'(1) = \beta$$

and is a supersolution of the elastica equation in the sense that we have for all $\varphi \in H_0^2(0,1)$ with $\varphi \geq 0$ that

$$0 \geq \int_0^1 \varphi'(x) \left(\frac{V'(x)}{(1+u'(x)^2)^{5/4}} \right) dx.$$

Let $w \in C^4([0,1])$ be a solution of the elastica equation

$$\frac{1}{\sqrt{1+w'(x)^2}} \frac{d}{dx} \left(\frac{\kappa_w'(x)}{\sqrt{1+w'(x)^2}} \right) + \frac{1}{2}\kappa_w(x)^3 = 0, \quad x \in (0,1).$$

which is symmetric around $1/2$ and satisfies the same Dirichlet boundary conditions

$$w(0) = w(1) = 0, \qquad w'(0) = -w'(1) = \beta.$$

Then the following comparison statement holds:

$$\forall x \in (0,1) : \qquad u(x) \geq w(x).$$

Moreover we either have equality or strict inequality everywhere.

Proof. We define

$$c := -2G(\beta).$$

According to [4, Theorem 2] the solution w is unique in the class of symmetric smooth graphs and satisfies

$$w'(x) = G^{-1}\left(c(x-1/2)\right),$$

cf. also Lemma 2.1. We let $V = V_u = \frac{u''}{(1+(u')^2)^{5/4}}$ denote the usual auxiliary function and show first that

$$V(0) \geq c \tag{2.18}$$

and assume by contradiction that

$$V(0) < c.$$

We only used that u is a sufficiently smooth supersolution to show Proposition 2.10. Hence the strong maximum principle applies to V and yields that

$$\forall x \in [0,1] : V(x) < c.$$

Observe that the symmetry of u yields that $u'(1/2) = 0$. Integration yields

$$\begin{aligned} c &> 2\int_0^{1/2} \frac{u''(\xi)}{(1+u'(\xi)^2)^{5/4}}\, d\xi = 2\int_{u'(0)=\beta}^{u'(1/2)=0} \frac{1}{(1+\tau^2)^{5/4}}\, d\tau \\ &= 2G(0) - 2G(\beta) = c, \end{aligned}$$

a contradiction. This proves (2.18).

In case that $V(x) \equiv c$ integration as in [4, Theorem 2] shows that $u(x) \equiv w(x)$. Otherwise we have thanks to the strict maximum principle that either

$$\forall x \in (0,1) \setminus \{1/2\} : \qquad V(x) > c.$$

or

$$\exists x_0 \in [0,1/2) : \quad \forall x \in (0,x_0) \cup (1-x_0) : \quad V(x) > c$$

and

$$\forall x \in (x_0, 1-x_0) : \quad V(x) < c.$$

In view of the first part of the proof it follows that $x_0 > 0$. In the first situation we conclude for all $x \in (0,1/2]$ that

$$\begin{aligned} cx &< \int_0^x \frac{u''(\xi)}{(1+u'(\xi)^2)^{5/4}}\, d\xi = \int_\beta^{u'(x)} \frac{1}{(1+\tau^2)^{5/4}}\, d\tau = G(u'(x)) + \frac{c}{2}, \\ &\Rightarrow w'(x) < u'(x). \end{aligned}$$

In particular, $0 = w'(1/2) < u'(1/2) = 0$, which is impossible. So we are left with the second situation where we obtain in the same way as before that

$$\forall x \in (0,x_0) : \quad w'(x) < u'(x).$$

For $x \in (x_0, 1/2)$ we integrate instead on $(x, 1/2)$ and find thanks to the symmetry of u:

$$\begin{aligned} c\,(1/2-x) &> \int_x^{1/2} \frac{u''(\xi)}{(1+u'(\xi)^2)^{5/4}}\, d\xi = \int_{u'(x)}^{u'(1/2)=0} \frac{1}{(1+\tau^2)^{5/4}}\, d\tau = -G(u'(x)), \\ &\Rightarrow w'(x) < u'(x). \end{aligned}$$

So we conclude that

$$\forall x \in (0,1/2) \setminus \{x_0\} : \quad w'(x) < u'(x).$$

Hence for any $x \in (0,1)$ it holds that $u(x) > w(x)$ except when $V(x) \equiv c$ and hence $u(x) \equiv w(x)$. □

Remark 2.13. Since β was arbitrary in the previous lemma and multiplication by -1 changes a subsolution into a supersolution, an analogous statement holds also for subsolutions.

For what follows we fix the following notation: $u \in C^2([0,1]) \cap W^{3,\infty}(0,1)$ always denotes a supersolution (which is not a solution) of the elastica equation which is symmetric around $1/2$ and satisfies homogeneous Dirichlet boundary conditions

$$u(0) = u(1) = 0, \qquad u'(0) = -u'(1) = 0.$$

We call the reparameterisation by arclength of the graph of u

$$y : [-L, L] \to [0,1] \times \mathbb{R}, \quad L = \frac{1}{2} \int_0^1 \sqrt{1 + u'(\xi)^2}\, d\xi$$

so that

$$\forall x \in [0,1] : \quad y(-L + \int_0^x \sqrt{1 + u'(\xi)^2}\, d\xi) = (x, u(x)).$$

We let $\kappa : [-L, L] \to \mathbb{R}$ denote its curvature function in arclength parameterisation,

$$\tau, n : [-L, L] \to \mathbb{R}^2, \quad \tau(s) := y_s(s), \quad n(s) := \begin{pmatrix} -\tau^2 \\ \tau^1 \end{pmatrix}$$

its unit tangent and normal, respectively. Being a supersolution of the elastica equation means that

$$\kappa_{ss} + \frac{1}{2}\kappa^3 \geq 0.$$

According to Proposition 2.11 we know that

$$\forall s \in (0, L] : \quad \kappa_s(s) > 0. \tag{2.19}$$

We shall compare u with a suitable solution w of the elastica equation, which is symmetric around $1/2$ and which will be specified later. One should have in mind that such a solution is smooth as a curve, it is a graph, but it may have infinite slope and is hence there not a smooth graph. At points, where w has infinite slope, the corresponding auxiliary function V_w may have even jump discontinuouties.

In any case we shall assume in $x = 1/2$, that w has its maximum and its curvature its minimum there. Let Y denote the arclength parameterisation of its graph (which is smooth) with $Y(0) = (1/2, w(1/2))$ and k, T, N denote its curvature function, unit tangent and normal field, respectively.

Lemma 2.14. *Assume that*

$$0 > \kappa(0) = k(0).$$

Then

$$\forall s \in [0, L] : \quad \kappa(s) \geq k(s).$$

Proof. We assume first that

$$\kappa_{ss} + \frac{1}{2}\kappa^3 > 0$$

and prove that then even

$$\forall s \in (0, L]: \quad \kappa(s) > k(s). \tag{2.20}$$

We first see that $\kappa_{ss}(0) > -\frac{1}{2}\kappa(0)^3 = -\frac{1}{2}k(0)^3 = k_{ss}(0)$ so that (2.20) is satisfied for small positive s. We assume by contradiction that this is not the case on the whole interval and choose $s_0 \in (0, L]$ minimal with

$$\kappa(s_0) = k(s_0), \text{ so that } \forall s \in (0, s_0): \quad \kappa(s) > k(s).$$

This implies that

$$0 < \kappa_s(s_0) \leq k_s(s_0). \tag{2.21}$$

We obtain from the differential inequality and equation, respectively, and from (2.19) that for all $s \in (0, L]$:

$$\frac{d}{ds}\left(4\kappa_s^2 + \kappa^4\right) = 8\kappa_s\left(\kappa_{ss} + \frac{1}{2}\kappa^3\right) > 0 = \frac{d}{ds}\left(4k_s^2 + k^4\right).$$

Integrating over $[0, s_0]$ yields

$$\left[4\kappa_s^2 + \kappa^4\right]_0^{s_0} > \left[4k_s^2 + k^4\right]_0^{s_0}.$$

Making use of $\kappa_s(0) = k_s(0) = 0$, of $\kappa(s_0) = k(s_0)$, of $\kappa(0) = k(0)$ and of (2.21) we find

$$4\kappa_s(s_0)^2 > 4k_s(s_0)^2 \geq 4\kappa_s(s_0)^2,$$

a contradiction.

In the general case that we apply first (2.20) to $\kappa + \varepsilon s^2$ and let then $\varepsilon \searrow 0$. $\square$

Proof of Theorem 1.2. We introduce angle functions

$$\alpha : [-L, L] \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad A : [-L, L] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

such that

$$\tau(s) = \begin{pmatrix}\cos(\alpha(s))\\ \sin(\alpha(s))\end{pmatrix}, \qquad T(s) = \begin{pmatrix}\cos(A(s))\\ \sin(A(s))\end{pmatrix}.$$

We take from planar differential geometry that

$$\forall s \in [-L, L]: \quad \kappa(s) = \dot{\alpha}(s), \quad k(s) = \dot{A}(s).$$

Since $\alpha(0) = A(0) = 0$, Lemma 2.14 yields that

$$\forall s \in [0, L]: \quad \alpha(s) \geq A(s), \quad \alpha(s) \in \Big(-\frac{\pi}{2}, 0\Big].$$

The last claim follows from Proposition 2.11. This means that both angle functions are in the interval where cos as well as sin are monotonically increasing so that we have componentwise ordering of the tangent vectors:

$$\forall s \in [0, L]: \quad \tau(s) \geq T(s), \text{ meaning that for } j = 1, 2: \quad \tau^j(s) \geq T^j(s).$$

We specify the solution w such that $w(1/2) = u(1/2)$ and that in arclength reparameterisation $k(s = 0) = \kappa(s = 0)$ so that in particular $Y(0) = y(0)$. We come up with

$$\forall s \in [0, L]: \quad y(s) \geq Y(s)$$

to be interpreted compenentwise as before. This means that the supersolution lies above and right from the corresponding solution with $k(0) = \kappa(0)$ and $Y(0) = y(0)$.

This means that the average slope of the supersolution is less than the average slope of the solution, but restricted to $[0, L]$. In order to find a bound for the maximal average slope of symmetric solutions (which is scaling invariant) one has to maximise (cf. Lemma 2.1)

$$[0, 1/2] \ni x \mapsto \frac{\dfrac{2}{c_0 \sqrt[4]{1 + G^{-1}\left(\frac{c_0}{2} - c_0 x\right)^2}} + \dfrac{2}{c_0}}{x + \frac{1}{2}}.$$

By substituting $y := G^{-1}\left(\frac{c_0}{2} - c_0 x\right)$ this is equivalent to maximising

$$[0, \infty] \ni y \mapsto \frac{2 + 2(1 + y^2)^{-1/4}}{c_0 - G(y)}.$$

This maximum is given by the unique solution $y_0 \in (0, \infty)$ of the equation

$$y_0(c_0 - G(y_0)) = 2 + 2(1 + y_0^2)^{-1/4}.$$

We think that this equation cannot be solved explicitly. With the help of maple™ or mathematica™ one finds that $y_0 = 2.3780929080\ldots$ is the solution and so, a bound for this function. Hence, *any supersolution* of the elastica equation as considered in this section obeys the *universal bound*:

$$\forall x \in [0, 1] : \quad 0 \le u(x) \le u\left(\frac{1}{2}\right) \le \frac{y_0}{2} = 1.1890464540\ldots. \tag{2.22}$$

In view of Proposition 2.9 and Corollary 2.5 this yields the proof of Theorem 1.2. $\square$

In view of [10, Theorem 1.1] and Remark 2.16 below one may have expected that the optimal bound in (1.3) in Theorem 1.2 would be $2/c_0 = 0.8346268418\ldots$. However, the following remark shows that the bound (2.22) can presumably not be improved, at least not for supersolutions in the class of "sufficiently smooth" graphs.

Remark 2.15. We consider a solution $Y(\cdot)$ as before, parameterised by arclength on $[-s_0, s_0]$. The point is s_0 chosen such that $Y(0) - Y(s_0)$ has in modulus the maximal slope $-2.3780929080\ldots$ mentioned before. Beyond s_0 we extend this curve by the solution of the initial value problem

$$\begin{cases} \tilde{k}_{ss} + \frac{1}{2}\tilde{k} = 0, \quad s \ge s_0 \\ \tilde{Y}(s_0) = Y(s_0), \quad \tilde{Y}_s(s_0) = Y_s(s_0), \quad \tilde{k}(s_0) = k(s_0), \quad \tilde{k}_s(s_0) >> k_s(s_0). \end{cases}$$

Choosing $\tilde{k}_s(s_0)$ arbitrarily large yields an extension $\tilde{Y}$ which becomes horizontal in $s_1 > s_0$ arbitrarily close to s_0. We consider this composed symmetric curve on $[-s_1, s_1]$ and rescale and translate it such that it satisfies Dirichlet conditions over $[0, 1]$. Since the derivative of its curvature jumps upwards in $\{\pm s_0\}$ we find a smooth enough supersolution of the elastica equation. As a curve it has the regularity as in Proposition 2.9 and it is a graph. However, one should observe that it is not even C^1 as a graph.

So, formally this example does not show the optimality of the bound (2.22) for "smooth" graphical supersolutions. However, we are confident that approximating the delta distributions on the right hand side of the elastica equation by smooth functions would show the optimality also rigorously.

Moreover, we believe that one may find obstacles such that these supersolutions solve the corresponding obstacle problems. If this expectation is correct the bound (1.3) would also be optimal for admissible obstacles.

Remark 2.16. The same reasoning leading to (2.22) could also be applied to the Navier problem where in arclength parameterisation $\kappa(L) = 0$. Using a comparison solution as before with $k(0) = \kappa(0)$, $y(0) = Y(0)$, $\tau(0) = T(0)$ would lead to $k(L) \leq 0$, i.e. there one is still in the concave regime, where the average slope is $\leq 4/c_0$. As before, y is right and above Y so that the average slope of u is at most $4/c_0$. This shows that any sufficiently smooth supersolution over $[0,1]$, which is symmetric around $1/2$ and satisfies Navier boundary conditions, obeys the universal bound

$$\forall x \in [0,1] : \quad 0 \leq u(x) \leq \frac{2}{c_0}.$$

This is a significant generalisation of [1, Lemma 4.3] and [10, Theorem 1.1].

2.5. **The obstacle problem in the non-symmetric case.** Let ψ be an obstacle function, which needs no longer to be symmetric and is always subject to the following condition:

(A_*)

$$\psi \in C^0([0,1]), \quad \exists \sigma > 0 : \quad \psi < 0 \text{ on } [0,\sigma) \cup (1-\sigma, 1], \quad \exists x_0 \in (0,1) : \psi(x_0) > 0.$$

We define the set of admissible functions as follows:

$$\mathcal{N}(\psi) := \{v \in H_0^2(0,1) \mid v(x) \geq \psi(x) \text{ for all } x \in [0,1]\}.$$

Assume that ψ satisfies

$$\alpha := \inf_{v \in N(\psi)} W(v) < c_0^2. \tag{B$_*$}$$

From now on, we let $c_2 \in (\sqrt{\alpha}/2, c_0)$.

Lemma 2.17. *Assume that $v \in \mathcal{N}(\psi)$ satisfies $W(v) \leq c_2^2$. Then*

$$\max_{x \in [0,1]} |v'(x)| \leq G^{-1}\left(\frac{c_2}{2}\right) < \infty.$$

Proof. Let $x_{\max} \in (0,1)$ be such that $|v'(x_{\max})| = \max_{x \in [0,1]} |v'(x)|$. First we consider the case where $v'(x_{\max}) > 0$. Then there exists $x_0 \in (x_{\max}, 1)$ where v attains its maximum which means that

$$v'(x_0) = 0, \quad v''(x_0) \leq 0.$$

Considering $v|_{[0,x_0]}$, we can adopt the argument from the proof of Lemma 2.2. Then we have

$$\begin{aligned}
2G(v'(x_{\max})) &= \int_0^{v'(x_{\max})} \frac{d\tau}{(1+\tau^2)^{5/4}} + \int_0^{v'(x_{\max})} \frac{d\tau}{(1+\tau^2)^{5/4}} \\
&= \int_0^{x_{\max}} \frac{v''(x)}{(1+v'(x)^2)^{5/4}}\,dx + \int_{x_{\max}}^{x_0} \frac{-v''(x)}{(1+v'(x)^2)^{5/4}}\,dx \\
&\leq W(v)^{1/2} \leq c_2.
\end{aligned}$$

Thanks to the monotonicity of of G, we obtain

$$v'(x_{\max}) \leq G^{-1}\left(\frac{c_2}{2}\right).$$

For the case where $v'(x_{\max}) < 0$, considering $-v$ instead of v and using the same argument as in the first case, we obtain the required conclusion. The proof is complete. $\square$

Combining Lemma 2.17 with the same argument as in the proof of Theorem 1.1 and Proposition 2.9, we obtain the following:

Theorem 2.18. *Assume that $\psi \in C^0([0,1])$ satisfies conditions* (A_*) *and* (B_*). *Then there exists $u \in \mathcal{N}(\psi)$ such that*

$$W(u) = \inf_{v\in\mathcal{N}(\psi)} W(v).$$

Moreover, $u \in C^2([0,1])$, u'' is weakly differentiable and $u''' \in BV(0,1)$.

Remark 2.19. Analogues of Propositions 2.10 and 2.11 can be proved also in the nonsymmetric case.

Remark 2.20. We use the same notation as in Remark 2.4 and consider the functions $\hat{u}_c : [0,1] \to \mathbb{R}$ as there. We recall that

$$W(\hat{u}_c) = 2W(U_c) = 4c^2.$$

In order to satisfy condition (B_*) we have now to assume that

$$c \in (0, c_0/2), \qquad \frac{c_0}{2} = \sqrt{\pi}\frac{\Gamma(3/4)}{2\Gamma(5/4)} = 1.198140234\ldots,$$

here we have that $W(\hat{u}_c) < c_0^2$. This implies that for any sufficiently small $\varepsilon > 0$, any $\psi \in C^0([0,1])$ with $\psi \le \hat{u}_c - \varepsilon$ obeys conditions (A_*) and (B_*). In particular, any $\hat{u}_c - \varepsilon$ with $c \in (0, c_0/2)$ is itself an admissible obstacle.

FIGURE 2. $\hat{u}_{1.0}$ (left) and $\hat{u}_{c_0/2}$ (right)

One may observe that

$$\max_{x\in[0,1]} \hat{u}_{c_0/2} = \hat{u}_{c_0/2}(1/2) = \frac{4}{c_0}\left(1 - \frac{1}{\sqrt[4]{1 + G^{-1}\left(\frac{c_0}{4}\right)^2}}\right) = 0.1628208198\ldots.$$

3. An obstacle problem for surfaces of revolution

Let $u : [-1,1] \to (0,\infty)$ be a profile curve with Dirichlet boundary conditions

$$u(1) = u(-1) = \alpha > 0, \quad u'(1) = u'(-1) = 0. \tag{3.1}$$

Then the mean curvature H and Gauss curvature K of the surface of revolution which are obtained by rotating the graph of u around the x-axis

$$[-1,1] \times [0, 2\pi] \ni (x,\varphi) \mapsto (x, u(x)\cos\varphi, u(x)\sin\varphi)$$

are given as follows:

$$H := \frac{1}{u(x)\sqrt{1+(u'(x))^2}} - \frac{u''(x)}{(1+(u'(x))^2)^{3/2}},$$
$$K := -\frac{u''(x)}{u(x)(1+(u'(x))^2)^2}.$$

Thus the Willmore functional is given by
(3.2)

$$\begin{aligned}
W(u) &= \frac{\pi}{2}\int_{-1}^{1}\Big(\frac{1}{u(x)\sqrt{1+(u'(x))^2}} - \frac{u''(x)}{(1+(u'(x))^2)^{\frac{3}{2}}}\Big)^2 u(x)\sqrt{1+(u'(x))^2}\,dx\\
&= \frac{\pi}{2}\int_{-1}^{1}\frac{u''(x)^2u(x)}{(1+(u'(x))^2)^{\frac{5}{2}}}\,dx + \frac{\pi}{2}\int_{-1}^{1}\frac{1}{u(x)\sqrt{1+(u'(x))^2}}\,dx\\
&\quad - \pi\,\Big[\,\frac{u'(x)}{\sqrt{1+u'(x)^2}}\,\Big]_{-1}^{1}\\
&= \frac{\pi}{2}\int_{-1}^{1}\frac{u''(x)^2u(x)}{(1+(u'(x))^2)^{\frac{5}{2}}}\,dx + \frac{\pi}{2}\int_{-1}^{1}\frac{1}{u(x)\sqrt{1+(u'(x))^2}}\,dx.
\end{aligned}$$

Here we reformulate the curvature κ and the Willmore functional $W(u)$ in the metric $ds^2 = \frac{1}{y^2}(dx^2+dy^2)$ of the hyperbolic half plane as follows:

$$\kappa_h(x) := \frac{u(x)u''(x)}{(1+(u'(x))^2)^{3/2}} + \frac{1}{\sqrt{1+(u'(x))^2}},$$
$$W^h(u) := \int_{-1}^{1}\kappa_h(x)^2\frac{\sqrt{1+(u'(x))^2}}{u(x)}\,dx$$

Concerning the original and the hyperbolic Willmore functional, it follows from a simple calculation that

$$W_{[a,b]}(u) = \frac{\pi}{2}W^h_{[a,b]}(u) - 2\pi\,\Big[\,\frac{u'(x)}{\sqrt{1+u'(x)^2}}\,\Big]_b^a, \tag{3.3}$$

see, e.g., [2, p. 384] or [7, p. 378].

3.1. **Willmore surfaces of revolution under Dirichlet boundary conditions.** As in the one-dimensional case it is good to have results about the existence, the regularity and the shape of minimisers (without obstacle) under Dirichlet boundary conditions (3.1) in mind.

Combining [2, Theorem 1.1], [3, Lemma 3.20], [5, Theorem 1.1], one has the following result:

Theorem 3.1. *For each $\alpha > 0$ and there exists a function $u \in C^\infty([-1,1],(0,\infty))$ which minimises the Willmore energy $W(\,.\,)$ in the class*

$$\{v \in H^2((-1,1);(0,\infty))\,|\,v(1) = v(-1) = \alpha, \quad v'(1) = v'(-1) = 0\}.$$

The corresponding surface of revolution $\Gamma \subset \mathbb{R}^3$ solves the Dirichlet problem

$$\begin{cases} \triangle_\Gamma H + 2H(H^2-K) = 0 \quad \text{in } (-1,1),\\ u(-1) = u(+1) = \alpha, \qquad u'(-1) = -u'(+1) = 0. \end{cases}$$

Any such minimiser is symmetric (i.e. $u(x) = u(-x)$) and has the following properties:

$$-\frac{x}{\alpha} < u'(x) < 0, \quad x + u(x)u'(x) > 0, \quad \textit{in} \quad (0,1),$$

and

$$\alpha < u(x) < \sqrt{1+\alpha^2 - x^2} \quad \textit{in} \quad (-1,1).$$

For plots of numerically calculated solutions one may see [3, Figure 17]. For $\alpha \searrow 0$ one finds boundary layers close to ± 1 where any minimiser approaches after rescaling a catenoid, while in any compact subset of $(-1,1)$ any minimiser approaches the upper unit half circle centered at $(0,0)$, see [8].

3.2. **The obstacle problem.** Assume that $\psi \in C^0([-1,1];(0,\infty))$ satisfies condition (C). We define the admissible set $N_\alpha(\psi)$ as in Section 1. In the following, $M_\alpha(\psi)$ denotes the infimum of the Willmore functional $W(v)$ in the admissible set $N_\alpha(\psi)$, i.e.,

$$M_\alpha(\psi) := \inf_{v \in N_\alpha(\psi)} W(v).$$

Lemma 3.2. *Assume that $v \in N_\alpha(\psi)$. If*

$$W(v) < 4\pi \tag{3.4}$$

is satisfied then it holds for all $x \in [-1,1]$ that

$$|v'(x)| \le K := \frac{1}{\sqrt{\left(\frac{4\pi}{W(v)}\right)^2 - 1}}\ . \tag{3.5}$$

If for some K, $|v'(x)| \le K$ is satisfied on $[-1,1]$, we have

$$\alpha + K \ge v(x) \ge M := \frac{K}{\exp(\frac{2K}{\pi}\sqrt{1+K^2}\ W(v)) - 1} \tag{3.6}$$

for all $x \in [-1,1]$.

Proof. To begin with, we prove (3.5). Let $x_{\max} \in (0,1)$ be such that

$$|v'(x_{\max})| = \max_{x \in [-1,1]} |v'(x)| =: K_1.$$

First we consider the case where $v'(x_{\max}) < 0$. Since $v \in N_\alpha(\psi)$, we find

$$v'(-x_{\max}) = -v'(x_{\max}) > 0. \tag{3.7}$$

It follows from (3.3) and (3.7) that

$$\begin{aligned} W(v) &= W_{[-1,-x_{\max}]}(v) + W_{[-x_{\max},x_{\max}]}(v) + W_{[x_{\max},1]}(v) \\ &\ge \frac{\pi}{2} W^h_{[-x_{\max},x_{\max}]}(v) - 2\pi \left[\frac{v'(x)}{\sqrt{1+v'(x)^2}} \right]_{-x_{\max}}^{x_{\max}} \ge 4\pi \frac{K_1}{\sqrt{1+K_1^2}}, \end{aligned}$$

and then we have

$$\left(\frac{W(v)}{4\pi}\right)^2 \ge 1 - \frac{1}{1+K_1^2}. \tag{3.8}$$

By (3.4) we deduce from (3.8) that $K_1 \le K$. We turn to the case where $v'(x_{\max}) > 0$. In this case, we have

$$v'(-x_{\max}) = -v'(x_{\max}) < 0. \tag{3.9}$$

Using (3.3) again, we obtain

$$\begin{aligned}
W(v) &= W_{[-1,-x_{\max}]}(v) + W_{[-x_{\max},x_{\max}]}(v) + W_{[x_{\max},1]}(v) \\
&\geq W_{[-1,-x_{\max}]}(v) + W_{[x_{\max},1]}(v) \\
&\geq \frac{\pi}{2} W^h_{[-1,-x_{\max}]}(v) - 2\pi \left[\frac{v'(x)}{\sqrt{1+v'(x)^2}} \right]_{-1}^{-x_{\max}} \\
&\quad + \frac{\pi}{2} W^h_{[x_{\max},1]}(v) - 2\pi \left[\frac{v'(x)}{\sqrt{1+v'(x)^2}} \right]_{x_{\max}}^{1} \\
&\geq 2\pi \left(\frac{v'(x_{\max})}{\sqrt{1+v'(x_{\max})^2}} - \frac{v'(-x_{\max})}{\sqrt{1+v'(-x_{\max})^2}} \right),
\end{aligned}$$

This together with (3.9) implies that

$$W(v) \geq 4\pi \frac{K_1}{\sqrt{1+K_1^2}}.$$

Then we find $K_1 \leq K$, as in the first case.

We prove (3.6). Thanks to (3.5), we observe that for $x \in [-1, 0]$

$$v(x) = \alpha + \int_{-1}^{x} v'(\xi)\, d\xi \leq \alpha + \int_{-1}^{0} |v'(\xi)|\, d\xi \leq \alpha + K.$$

Since v is even, the estimate from above follows.

For the estimate from below, we let $x_{\min} \in (0, 1)$ be such that

$$v(x_{\min}) = \min_{x \in [-1,1]} v(x) =: v_{\min} > 0.$$

Then, for any $x \in [-1, x_{\min}]$, we have

$$v(x) = v_{\min} - \int_{x}^{x_{\min}} v'(\xi)\, d\xi \leq v_{\min} + K(x_{\min} - x) \tag{3.10}$$

Thanks to $v'(-1) = v'(1) = 0$, we obtain

$$\begin{aligned}
W(v) &= \frac{\pi}{2} W^h(v) \\
&= \frac{\pi}{2} \int_{-1}^{1} \frac{v(v'')^2}{(1+(v')^2)^{5/2}}\, dx + \pi \int_{-1}^{1} \frac{v''}{(1+(v')^2)^{3/2}}\, dx \\
&\qquad + \frac{\pi}{2} \int_{-1}^{1} \frac{1}{v\sqrt{1+(v')^2}}\, dx \\
&\geq \pi \left[\frac{v'(x)}{\sqrt{1+v'(x)^2}} \right]_{-1}^{1} + \frac{\pi}{2} \int_{-1}^{1} \frac{1}{v\sqrt{1+(v')^2}}\, dx \\
&= \frac{\pi}{2} \int_{-1}^{1} \frac{1}{v\sqrt{1+(v')^2}}\, dx.
\end{aligned} \tag{3.11}$$

Combining (3.11) with (3.5) and (3.10), we deduce that

$$
\begin{aligned}
W(v) &\geq \frac{\pi}{2}\frac{1}{\sqrt{1+K^2}}\int_{x_{\min}-1}^{x_{\min}} \frac{1}{v}\,dx \\
&\geq \frac{\pi}{2}\frac{1}{\sqrt{1+K^2}}\int_{x_{\min}-1}^{x_{\min}} \frac{1}{v_{\min}+K(x_{\min}-x)}\,dx \\
&\geq \frac{\pi}{2}\frac{1}{\sqrt{1+K^2}}\int_0^1 \frac{1}{v_{\min}+K\xi}\,d\xi = \frac{\pi}{2K\sqrt{1+K^2}}\log\frac{v_{\min}+K}{v_{\min}}.
\end{aligned}
\tag{3.12}
$$

By a direct calculation, we observe from (3.12) that

$$v_{\min} \geq M.$$

The proof is complete. □

We may now prove our first existence result for surfaces of revolution.

Proof of Theorem 1.3. Let

$$\tilde{M}_\alpha(\psi) := \frac{1}{2}(4\pi + M_\alpha(\psi)) \in (M_\alpha(\psi), 4\pi),$$

and consider a minimising sequence $\{v_k\}_{k\in\mathbb{N}} \subset N_\alpha(\psi)$. We may assume that

$$M_\alpha(\psi) \leq W(v_k) \leq \tilde{M}_\alpha(\psi). \tag{3.13}$$

By Lemma 3.2 we see that $\{v_k\}_{k\in\mathbb{N}}$ is uniformly C^1-bounded, to be more precisely,

$$|v_k'(x)| \leq \tilde{K} := \frac{\tilde{M}_\alpha(\psi)}{4\pi}\frac{1}{\sqrt{1-\left(\frac{\tilde{M}_\alpha(\psi)}{4\pi}\right)^2}},$$

$$\alpha + \tilde{K} \geq v_k(x) \geq \tilde{M} := \frac{\tilde{K}}{\exp(\frac{2\tilde{K}}{\pi}\sqrt{1+\tilde{K}^2}\tilde{M}_\alpha(\psi)) - 1},$$

for all $x \in [-1,1]$ and $k \in \mathbb{N}$. This together with (3.13) implies that $\{v_k\}_{k\in\mathbb{N}}$ is uniformly bounded in H^2. Then, analogously to the proof of Theorem 1.1, we find $u \in N_\alpha(\psi)$ as the limit of $\{v_k\}_{k\in\mathbb{N}}$ such that

$$W(u) = \min_{v\in N_\alpha(\psi)} W(v).$$

In particular, the minimiser u satisfies $u(x) \geq \tilde{M} > 0$ for all $x \in [-1,1]$. The proof is complete. □

Remark 3.3. For which α do we easily obtain interesting and admissible obstacles? We construct a function $v_\alpha \in N_\alpha$ with $W(v_\alpha) < 4\pi$. Any $\psi \in C^0([-1,1];(0,\infty))$ with $\psi(\pm 1) > \alpha$ and $\psi \geq v_\alpha$ would then be an admissible obstacle and $v_\alpha \in N_\alpha(\psi)$.

To this end, let $b_0 > 0$ denote the solution of the equation $b_0 \cdot \tanh(b_0) = 1$. One sees that the space above and including the graph of $x \mapsto \left|\frac{\cosh(b_0)}{b_0}\cdot x\right|$ coincides with the set $\{(x, \cosh(bx)/b) \,|\, x \in \mathbb{R},\, b > 0\} \cap \{(0,0)\}$. The smallest circle around $(1,0)$ which intersects this set has the radius

$$\alpha_0 := \sqrt{1 - \frac{1}{1+\left(\frac{\cosh(b_0)}{b_0}\right)^2}} = 0.8335565596\ldots, \quad \text{where } \frac{\cosh(b_0)}{b_0} = 1.508879561\ldots.$$

In what follows we assume first that $\alpha \geq \alpha_0$. For $\alpha \geq 1$ and $\alpha = \alpha_0$ one finds one catenoid which touches the circles around $(1, 0)$ and $(-1, 0)$ with radii α. For $\alpha \in (\alpha_0, 1)$ one has even two such catenoids.

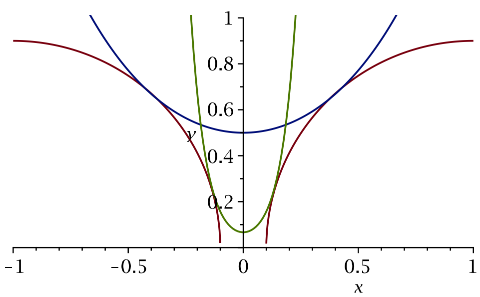

FIGURE 3. Any smooth function built from the brown circular arcs and a part either of the green or of the blue cosh and slightly enlarged near ± 1 is an admissible obstacle for $\alpha = 0.9$.

To be more explicit, we put

$$v_\alpha(x) := \begin{cases} \sqrt{\alpha^2 - (1+x)^2} & \text{for } -1 \leq x \leq -x_b, \\ \frac{1}{b}\cosh(bx) & \text{for } -x_b \leq x \leq x_b, \\ \sqrt{\alpha^2 - (1-x)^2} & \text{for } x_b \leq x \leq 1, \end{cases}$$

where b and x_b are chosen such that $v_\alpha \in H^2((-1,1);(0,\infty))$, i.e.:

$$\begin{cases} \dfrac{1}{b}\cosh(bx_b) & = \sqrt{\alpha^2 - (1-x_b)^2} \\ \quad \sinh(bx_b) & = \dfrac{1-x_b}{\sqrt{\alpha^2 - (1-x_b)^2}}. \end{cases}$$

After some elementary calculations one finds that this is equivalent to solving first

$$\cosh(b - \sqrt{(\alpha b)^2 - (\alpha b)}) = \sqrt{\alpha b}$$

for b and putting then:

$$x_b := 1 - \sqrt{\alpha^2 - \frac{\alpha}{b}}.$$

As in the proof of Lemma 3.2 one has that

$$W(v_\alpha) = 4\pi \tanh(bx_b) < 4\pi. \tag{3.14}$$

Some numerically calculated examples:

α	b	x_b	$v_\alpha(0) = 1/b$
5	$0.2020339962\ldots$	$0.4983130059\ldots$	$4.949662031\ldots$
2	$0.5349618217\ldots$	$0.4887119667\ldots$	$1.869292274\ldots$
1	$1.467396505\ldots$	$0.4356234114\ldots$	$0.6814790662\ldots$
0.99	$1.502200407\ldots$	$0.4333724672\ldots$	$0.6656901405\ldots$
0.99	$304.6450597\ldots$	$0.0116426170\ldots$	$0.003282508507\ldots$
0.9	$1.986626006\ldots$	$0.4025298387\ldots$	$0.5033660070\ldots$
0.9	$14.46598282\ldots$	$0.1352543279\ldots$	$0.06912769166\ldots$
0.84	$3.077899286\ldots$	$0.3422108342\ldots$	$0.3248969206\ldots$
0.84	$5.266858981\ldots$	$0.2610059859\ldots$	$0.1898664847\ldots$

As mentioned before, for $\alpha \in (\alpha_0, 1)$ we find two catenoids which touch the circles around $(1, 0)$ and $(-1, 0)$ with radii α. For $\alpha = \alpha_0$ and $\alpha \geq 1$ exactly one catenoid fits in like this. For $\alpha \searrow \alpha_0$ the two constructed branches of "admissible obstacles" (up to being slightly enlarged near ± 1) converge to the same one while for $\alpha \nearrow 1$ one of these branches persists (and becomes less and less interesting for increasing α and the other one becomes singular (convergence to two quarter circles). See Figures 4–6.

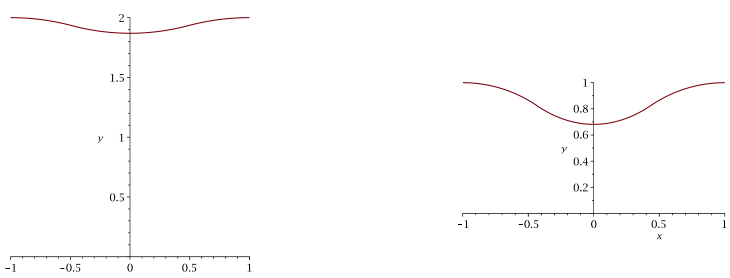

FIGURE 4. Admissible obstacles (up to enlarging near ± 1) for $\alpha = 2$ (left) and $\alpha = 1$ (right).

FIGURE 5. Two admissible obstacles (up to enlarging near ± 1) for $\alpha = 0.99$.

FIGURE 6. Two admissible obstacles (up to enlarging near ± 1) for $\alpha = 0.84$.

We consider now the case $\alpha \in (0, \alpha_0)$. Here, the previous construction has to be modified as follows: One *chooses* suitably (in a sense which will be explained below) $x_b \in (1-\alpha, 1)$ such that the catenoids

$$x \mapsto \frac{\cosh(b(|x| - x_0))}{b}$$

are in $\pm x_b$ tangential to the circles

$$x \mapsto \sqrt{\alpha^2 - (1-|x|)^2}.$$

The parameters b and x_0 are given by the equations

$$b = \frac{\sqrt{1+\beta^2}}{\gamma}, \quad \sinh(b(x_b - x_0)) = \beta,$$

where

$$\gamma := \sqrt{\alpha^2 - (1-x_b)^2}, \quad \beta := \frac{1-x_b}{\sqrt{\alpha^2 - (1-x_b)^2}}.$$

One puts then a circle around the origin which touches these catenoids. Its radius r is given by

$$r := \sqrt{x_{\min}^2 + \frac{\cosh(b(x_0 - x_{\min}))^2}{b^2}}$$

where $x_{\min} \in (0, x_0)$ is defined as the solution of the equation

$$b x_{\min} = \cosh(b(x_0 - x_{\min})) \sinh(b(x_0 - x_{\min})).$$

One then defines

$$v_\alpha(x) := \begin{cases} \sqrt{r^2 - x^2} & \text{for } |x| \le x_{\min}, \\ \frac{1}{b}\cosh(b(|x| - x_0)) & \text{for } x_{\min} \le |x| \le x_b, \\ \sqrt{\alpha^2 - (1-|x|)^2} & \text{for } x_b \le |x| \le 1, \end{cases}$$

Finally, in order that v_α becomes an admissible obstacle (up to enlarging near ± 1), x_b has to be chosen such that

$$W(v_\alpha) = 4\pi \left(\tanh(b(x_b - x_0)) + \tanh(b(x_0 - x_{\min}))\right) \overset{!}{<} 4\pi.$$

This is certainly possible because for $x_b \nearrow 1$ the function v_α converges to the prototype function used in [3]. There, for any $\alpha > 0$ it was shown in Proposition 6.6 that this function has Willmore energy strictly below 4π.

On the other hand, for $\alpha \nearrow \alpha_0$, x_b may be chosen such that x_0 is still very close to 0 and the admissible obstacles (up to enlarging ...) still resemble those from the case $\alpha > \alpha > 0$.

Some numerically calculated parameters for admissible obstacles (up to enlarging ...), see also Figure 7:

α	x_b	b	x_0	$x_{\min}$	r
0.7	0.5	0.4898979486...	0.1928412335...	0.09881364931...	0.3692969430...
0.5	0.79	2.428363283...	0.6056404249...	0.3419392371...	0.6050456522...
0.1	0.995	10.02506266...	0.9900083375...	0.8159959886...	0.8673937881...

FIGURE 7. Admissible obstacles (up to enlarging ...) for $\alpha = 0.7$ (left) and $\alpha = 0.5$ (right).

The previous examples are in our opinion quite interesting for α in a neighbourhood of 1. But giving a look at Figure 4 and the table just below (3.14) shows that already for $\alpha = 2$ the constructed "admissible obstacle" resembles somehow a straight line. Indeed, it is not difficult to see that $\lim_{\alpha \nearrow \infty} \|\alpha - v_\alpha\|_{C^0([-1,1])} = 0$ and $\lim_{\alpha \nearrow \infty} W(v_\alpha) = 0$.

The following result shows that the Willmore energy of "interesting" examples increases of the order α when $\alpha \nearrow \infty$. On the other hand it can be interpreted in that way that one has suitable a-priori-bounds in $C^1([-1,1],(0,\infty))$ for functions whose energy is below α times a suitable factor. The underlying observation is that in this regime the second term in $W(\,\cdot\,)$ in (3.2) becomes small compared to the first one and may be neglected. Then, up to a factor we are back in the one-dimensional situation. In what follows, the function G defined in (1.4) will play again an important role.

Proposition 3.4. *Let $S \in (0, \alpha)$. Assume that $u \in N_\alpha$ is such that*

$$\max_{x \in [-1,1]} |u'(x)| \geq S. \tag{3.15}$$

Then

$$W(u) > \pi(\alpha - S) G(S)^2. \tag{3.16}$$

Equivalently we may conclude for $u \in N_\alpha$

$$W(u) \leq \pi(\alpha - S) G(S)^2 \quad \Rightarrow \quad \max_{x \in [-1,1]} |u'(x)| < S. \tag{3.17}$$

Proof. We consider $u \in N_\alpha$ which satisfies (3.15). Choose $x_0 \in (-1, 0)$ minimal such that $|u'(x_0)| = S$, i.e. $\forall x \in [-1, x_0) : \quad |u'(x)| < S$ and $\forall x \in [-1, x_0] : u(x) > \alpha - S > 0$. Neglecting the second term in $W(\,\cdot\,)$ we find

$$\begin{aligned}\frac{2}{\pi} W(u) &\geq \int_{-1}^{1} \frac{u(x) u''(x)^2}{(1+u'(x)^2)^{5/2}}\, dx > 2(\alpha - S) \int_{-1}^{x_0} \frac{u''(x)^2}{(1+u'(x)^2)^{5/2}}\, dx \\ &\geq 2(\alpha - S) \underbrace{\frac{1}{1+x_0}}_{\geq 1} \left(\int_{-1}^{x_0} \frac{u''(x)}{(1+u'(x)^2)^{5/4}}\, dx \right)^2 \\ &\geq 2(\alpha - S) \left(\int_{u'(-1)=0}^{u'(x_0)} \frac{1}{(1+\tau^2)^{5/4}}\, d\tau \right)^2 = 2(\alpha - S) G(S)^2\end{aligned}$$

and (3.16) follows. □

This derivative estimate yields our second existence result for surfaces of revolution.

Proof of Theorem 1.4. Minimising sequences obey the bound in (3.17) for some suitable $S \in (0, \alpha)$ and hence also in (3.6) with S instead of K. This shows that minimising sequences satisfy sufficiently strong C^1-a-priori-estimates. Proceeding similarly as in the proof of Theorem 1.3, we obtain the assertion in Theorem 1.4. □

Remark 3.5. In order to interpret condition (1.7) one may observe that already $G(2.1) > 1$ and that $\sup_{S \in (0,\infty)} G(S)^2 = (c_0/2)^2 = 1.43\ldots$. This means that for $\alpha > 2.1$ the right hand side of (1.7) is estimated from below by $(\alpha - 2.1)\pi$, i.e. for $\alpha > 6.1$, condition (1.7) is weaker than condition (D). For $\alpha \nearrow \infty$ one has asymptotically that $\max_{S \in [0,\alpha]} g_\alpha(S) = (c_0/2)^2 \alpha = 1.43\ldots \cdot \alpha$.

The optimal S for $g_\alpha(\,\cdot\,)$ cannot be calculated explicitly. However, analysing also $g'_\alpha(\,\cdot\,)$ shows that this is increasing in α and becomes unbounded for $\alpha \nearrow \infty$. See Figure 8 for plots of $g_\alpha(\,\cdot\,)$ for $\alpha = 6.1,\ 10,\ 50$.

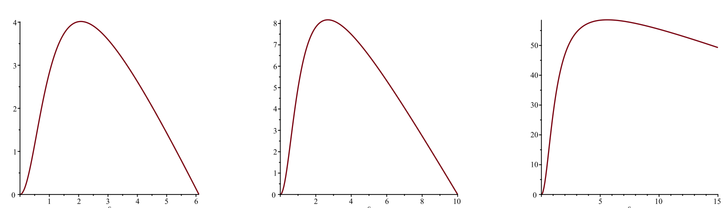

FIGURE 8. Plots of $g_\alpha(\,\cdot\,)$ for $\alpha = 6.1,\ 10,\ 50$.

Remark 3.6. How do obstacles look like that satisfy the assumptions of Theorem 1.4? Since this result is of interest only for (relatively) large α, we shall consider this case here only. Then, as mentioned above, (3.2) shows that the Willmore functional resembles, up to the factor $u(x)$, the one-dimensional functional. For this reason we recall the functions $U_c : [-\frac{1}{2}, \frac{3}{2}] \to [-2/c_0, 2/c_0]$ from Remark 2.4 and define for $\alpha >> 2$ and $0 < c < c_0$:

$$u_{\alpha,c} : [-1, 1] \to [\alpha - 4/c_0, \alpha], \quad u_{\alpha,c}(x) := \alpha - U_c(x + 1/2) + U_c(-1/2).$$

According to Remark 2.4 the one-dimensional Willmore energy of U_c over $[-\frac{1}{2}, \frac{3}{2}]$ is $2c^2$. In view of this we obtain from (3.2):

$$
\begin{aligned}
\frac{2}{\pi} W(u_{\alpha,c}) &= \int_{-1}^{1} \frac{u''_{\alpha,c}(x)^2 u_{\alpha,c}(x)}{(1+(u'_{\alpha,c}(x))^2)^{5/2}}\,dx + \int_{-1}^{1} \frac{1}{u_{\alpha,c}(x)\sqrt{1+(u'_{\alpha,c}(x))^2}}\,dx \\
&< \alpha \underbrace{\int_{-1}^{1} \frac{u''_{\alpha,c}(x)^2}{(1+(u'_{\alpha,c}(x))^2)^{5/2}}\,dx}_{\text{the one-dimensional functional}} + 2\frac{1}{\alpha - \frac{4}{c_0}} \\
W(u_{\alpha,c}) &< \pi\alpha c^2 + \frac{\pi}{\alpha - 2}.
\end{aligned}
$$

In order that $u_{\alpha,c}$ obeys the condition in Theorem 1.4 we need to choose $c \in (0, c_0)$ such that

$$
\alpha c^2 + \frac{1}{\alpha - 2} \overset{!}{\leq} \max_{S\in[0,\alpha]} \left((\alpha - S) G(S)^2 \right) \tag{3.18}
$$

is satisfied. This condition cannot be resolved explicitly. For $\alpha \approx \infty$ the left hand side of (3.18) behaves asymptotically like αc^2 and the right hand side like $\alpha(c_0/2)^2$. Hence, for $\alpha \approx \infty$, condition (3.18) is satisfied if $0 < c < c_0/2 \approx 1.198$. The following table displays numerically calculated threshold values c_{thre} such that for $0 < c < c_{\text{thre}}$, slightly enlarged $u_{\alpha,c}$ yield admissible obstacles.

α	$\max_{S\in[0,\alpha]} g_\alpha(S)$	c_{thre}
10	$8.170\ldots$	$0.896\ldots$
25	$26.231\ldots$	$1.023\ldots$
50	$58.583\ldots$	$1.082\ldots$
100	$125.756\ldots$	$1.121\ldots$

In Figure 9 we display how much the straight line $x \mapsto \alpha$ may be pushed down, when $c = 1$ and $c = c_0/2$ are admissible, respectively. One may observe that for $\alpha > 25$ the admissible profiles change only a little; in particular $u_{\alpha,1}$ is always admissible.

FIGURE 9. Plots of $x \mapsto -U_c(x+1/2) + U_c(-1/2)$ for $c = 1$ (left) and $c = c_0/2$ (right).

3.3. **Regularity of minimisers.** In order to show regularity we follow the strategy in Section 2.1. The formula for the first variation of $W^h(u)$ is derived in [2, formula (A.1)] as follows:

$$
\begin{aligned}
\left(W^h\right)'(u)(\varphi) = & 2\int_{-1}^{1} \frac{\kappa_h(x)}{1+u'(x)^2}\varphi''(x)\,dx + \int_{-1}^{1} \kappa_h(x)^2 \frac{\sqrt{1+u'(x)^2}}{u(x)^2}\varphi(x)\,dx \\
& - 5\int_{-1}^{1} \kappa_h(x)^2 \frac{u'(x)}{u(x)\sqrt{1+u'(x)^2}}\varphi'(x)\,dx - 2\int_{-1}^{1} \frac{\kappa_h(x)}{u(x)^2}\varphi(x)\,dx \\
& + 4\int_{-1}^{1} \kappa_h(x) \frac{u'(x)}{u(x)(1+u'(x)^2)}\varphi'(x)\,dx
\end{aligned} \tag{3.19}
$$

for all $\varphi \in H_0^2(0,1)$. Let u be a minimiser of $W^h(\cdot)$ in the admissible set $N_\alpha(\psi)$. Then u satisfies the variational inequality:

$$\forall v \in N_\alpha(\psi): \quad \left(W^h\right)'(u)(v-u) \geq 0.$$

One may in particular choose $v = u - \varphi$ with $\varphi \in H_0^2(0,1)$ and $\varphi \geq 0$ small enough because we push any admissible function *below* the obstacle. This means that the minimiser u is here a *subsolution* of the Willmore equation, i.e.

$$\forall \varphi \in H_0^2(0,1): \quad \varphi \geq 0 \quad \Rightarrow \quad \left(W^h\right)'(u)(\varphi) \leq 0. \tag{3.20}$$

This shows that $-\left(W^h\right)'(u)$ is a nonnegative distribution on $C_c^\infty(-1,1)$ and hence even on $C_c^0(-1,1)$, cf. Section 2.3 or [15, Lemma 37.2]. Hence, combining (3.20) with the Riesz representation theorem, we find a nonnegative Radon measure μ such that

$$\left(W^h\right)'(u)(\varphi) = -\int_{-1}^{1} \varphi\,d\mu \tag{3.21}$$

for all $\varphi \in C_c^\infty(-1,1)$. We define $\mathcal{N} \subset (-1,1)$ by

$$\mathcal{N} := \{x \in (-1,1) \mid u(x) < \psi(x)\}.$$

Since u and ψ are continuous on $[-1,1]$, the set $\mathcal{N}$ is open, and we have:

Lemma 3.7. *Assume that $\psi \in C^0([-1,1])$ satisfies condition* (C). *Suppose that there exists a minimiser $u \in N_\alpha(\psi)$ of $W^h(\cdot)$. Then there exist $a \in (0,1)$ such that $(-1,-a) \cup (a,1) \subset \mathcal{N}$.*

Lemma 3.8. *Assume that $\psi \in C^0([-1,1])$ satisfies condition* (C). *Suppose that there exists a minimiser $u \in N_\alpha(\psi)$ of $W^h(\cdot)$. Then*

$$\mu(-1,1) < \infty.$$

Proof. As in Section 2.3 one finds first that

$$\mu(\mathcal{N}) = 0.$$

Hence, by Lemma 3.7 we find $a \in (0,1)$ such that $\mu(-1,1) = \mu([-a,a])$. Fix $\eta \in C_c^\infty(-1,1)$ with $\eta \equiv 1$ in $[-a,a]$ and $0 \leq \eta \leq 1$ in $(-1,1)$. It follows from $u \in N_\alpha(\psi) \subset H^2((-1,1);(0,\infty)) \hookrightarrow C^1([-1,1],(0,\infty))$ that

$$|\kappa_h(x)| \leq C(|u''(x)| + 1),$$

and further that

$$\mu(-1,1) = \mu([-a,a]) \leq \int_{-1}^{1} \eta(x)d\mu = -\left(W^h\right)'(u)(\eta)$$

$$
\begin{aligned}
&\le C(\|u''\|_{L^2(-1,1)}\|\eta\|_{H^2(-1,1)} + \|\eta\|_{W^{2,1}(-1,1)} \\
&\qquad\qquad + \|u''\|^2_{L^2(-1,1)}\|\eta\|_{W^{1,\infty}(-1,1)}) \\
&\le C\|\eta\|_{H^2(-1,1)}.
\end{aligned}
$$

Therefore Lemma 3.8 follows. □

In order to study the regularity of minimisers, we employ the same ideas already used in [2, Proof of Theorem 3.9, Step 2], see also Section 2.3.

Lemma 3.9. *Fix $\eta \in C^\infty_{\mathrm{c}}(-1,1)$ and set*

$$
\begin{aligned}
\varphi_1(x) &:= \int_{-1}^{x}\int_{-1}^{y} \eta(s)\, ds dy + \alpha(x+1)^2 + \beta(x+1)^3, \\
\varphi_2(x) &:= \int_{-1}^{x} \eta(y)\, dy + \frac{1}{4}(-3(x+1)^2 + (x+1)^3)\int_{-1}^{1}\eta(y)\, dy,
\end{aligned}
$$

for $x \in [-1,1]$, where

$$
\alpha := \frac{1}{2}\int_{-1}^{1}\eta(y)\, dy - \frac{3}{4}\int_{-1}^{1}\int_{-1}^{y}\eta(s)\, dsdy, \qquad \beta := -\frac{1}{2}\alpha - \frac{1}{8}\int_{-1}^{1}\int_{-1}^{y}\eta(s)\, dsdy.
$$

Then, $\varphi_1, \varphi_2 \in H^2_0(-1,1)$ and there exists $C > 0$ such that

$$
\|\varphi_1\|_{C^1(-1,1)}, |\alpha|, |\beta| \le C\|\eta\|_{L^1(-1,1)},
$$
$$
\|\varphi_2\|_{L^\infty(-1,1)} \le C\|\eta\|_{L^1(-1,1)}, \qquad \|\varphi_2'\|_{L^p(-1,1)} \le C\|\eta\|_{L^p(-1,1)} \quad \text{for} \quad p \in [1,\infty).
$$

Proposition 3.10. *Assume that $\psi \in C^0([-1,1])$ satisfies condition* (C). *Suppose that there exists a minimiser $u \in N_\alpha(\psi)$ of $W^h(\cdot)$. Then $u \in C^2([-1,1])$, u'' is weakly differentiable and $u''' \in BV(-1,1)$.*

Proof. We define a nonnegative bounded increasing function $m : (-1,1) \to \mathbb{R}$ by

$$
m(x) = \mu(-1,x) \quad \text{for} \quad x \in (-1,1).
$$

Then, along the same lines as in the proof of Proposition 2.9, we obtain

$$
\int_{-1}^{1} \varphi\, d\mu(x) = -\int_{-1}^{1} m(x)\varphi'(x)\, dx \tag{3.22}
$$

for all $\varphi \in C^\infty_{\mathrm{c}}(-1,1)$. It follows from (3.19), (3.21) and (3.22) that

$$
\begin{aligned}
&2\int_{-1}^{1}\frac{\kappa_h(x)}{1+u'(x)^2}\varphi''(x)\, dx \\
&\quad = -\int_{-1}^{1}\frac{\kappa_h(x)^2\sqrt{1+u'(x)^2}}{u(x)^2}\varphi(x)\, dx \\
&\qquad + 5\int_{-1}^{1}\frac{\kappa_h(x)^2u'(x)}{u(x)\sqrt{1+u'(x)^2}}\varphi'(x)\, dx + 2\int_{-1}^{1}\frac{\kappa_h(x)}{u(x)^2}\varphi(x)\, dx \\
&\qquad - 4\int_{-1}^{1}\frac{\kappa_h(x)u'(x)}{u(x)(1+u'(x)^2)}\varphi'(x)\, dx + \int_{-1}^{1} m(x)\varphi'(x)\, dx
\end{aligned} \tag{3.23}
$$

for all $\varphi \in C^\infty_{\mathrm{c}}(-1,1)$. By a density argument, we see that (3.23) also holds for all $\varphi \in H^2_0(-1,1)$.

Fix $\eta \in C_{\mathrm{c}}^{\infty}(-1,1)$ arbitrarily. Taking φ_1 as φ in (3.23), where φ_1 is defined in Lemma 3.9, we have

$$
\begin{aligned}
&2\int_{-1}^{1} \frac{\kappa_h(x)}{1+u'(x)^2}\eta(x)\,dx \\
&\quad= -4\int_{-1}^{1} \frac{\kappa_h(x)}{1+u'(x)^2}(\alpha+3\beta x+3\beta)\,dx - \int_{-1}^{1} \frac{\kappa_h(x)^2\sqrt{1+u'(x)^2}}{u(x)^2}\varphi_1(x)\,dx \\
&\qquad + 5\int_{-1}^{1} \frac{\kappa_h(x)^2 u'(x)}{u(x)\sqrt{1+u'(x)^2}}\varphi_1'(x)\,dx + 2\int_{-1}^{1}\frac{\kappa_h(x)}{u(x)^2}\varphi_1(x)\,dx \\
&\qquad - 4\int_{-1}^{1} \frac{\kappa_h(x)u'(x)}{u(x)(1+u'(x)^2)}\varphi_1'(x)\,dx + \int_{-1}^{1} m(x)\varphi_1'(x)\,dx \\
&\quad=: J_1+J_2+J_3+J_4+J_5+J_6.
\end{aligned} \tag{3.24}
$$

Because any minimiser belongs to $N_\alpha(\psi) \subset C^1([-1,1],(0,\infty))$, we observe from Lemmas 3.8 and 3.9 that

$$
\begin{aligned}
|J_1| &\le C(|\alpha|+|\beta|)(\|u''\|_{L^1(-1,1)}+1) \le C\|\eta\|_{L^1(-1,1)}, \\
|J_2|+|J_3| &\le C(\|u''\|_{L^2(-1,1)}^2+1)\|\varphi_1\|_{C^1([-1,1])} \le C\|\eta\|_{L^1(-1,1)}, \\
|J_4| &\le C(\|u''\|_{L^1(-1,1)}+1)\|\varphi_1\|_{C^0([-1,1])} \le C\|\eta\|_{L^1(-1,1)}, \\
|J_5| &\le C(\|u''\|_{L^1(-1,1)}+1)\|\varphi_1\|_{C^1([-1,1])} \le C\|\eta\|_{L^1(-1,1)}, \\
|J_6| &\le \Big(\sup_{x\in[-1,1]} m(x)\Big)\|\varphi_1\|_{C^1([-1,1])} \le C\mu(-1,1)\|\eta\|_{L^1(-1,1)} \le C\|\eta\|_{L^1(-1,1)}.
\end{aligned}
$$

This together with (3.24) implies that

$$
\|\kappa_h(x)(1+u'(x)^2)^{-1}\|_{L^\infty(-1,1)} \le C. \tag{3.25}
$$

Combining (3.25) with $u \in C^1([-1,1],(0,\infty))$, we obtain

$$
\|u''\|_{L^\infty(-1,1)} \le C. \tag{3.26}
$$

This shows that we already have $u \in W^{2,\infty}((-1,1);(0,\infty))$.

Fix $\eta \in C_{\mathrm{c}}^{\infty}(-1,1)$ arbitrarily. Taking φ_2 as φ in (3.23), where φ_2 is defined by Lemma 3.9, we have

$$
\begin{aligned}
2\int_{-1}^{1} \frac{\kappa_h(x)}{1+u'(x)^2}\eta'(x)\,dx &= -3\left(\int_{-1}^{1} x\frac{\kappa_h(x)}{1+u'(x)^2}\,dx\right)\left(\int_{-1}^{1}\eta(x)\,dx\right) \\
&\quad - \int_{-1}^{1} \frac{\kappa_h(x)^2\sqrt{1+u'(x)^2}}{u(x)^2}\varphi_2(x)\,dx \\
&\quad + 5\int_{-1}^{1} \frac{\kappa_h(x)^2 u'(x)}{u(x)\sqrt{1+u'(x)^2}}\varphi_2'(x)\,dx + 2\int_{-1}^{1}\frac{\kappa_h(x)}{u(x)^2}\varphi_2(x)\,dx \\
&\quad - 4\int_{-1}^{1} \frac{\kappa_h(x)u'(x)}{u(x)(1+u'(x)^2)}\varphi_2'(x)\,dx + \int_{-1}^{1} m(x)\varphi_2'(x)\,dx \\
&=: J_1'+J_2'+J_3'+J_4'+J_5'+J_6'.
\end{aligned}
$$

We deduce from (3.26) and $u \in H^2((-1,1);(0,\infty)) \hookrightarrow C^1([-1,1],(0,\infty))$ that

$$
|J_2'|+|J_3'| \le C(\|u''\|_{L^\infty(-1,1)}^2+1)\|\varphi_2\|_{W^{1,1}(-1,1)} \le C\|\eta\|_{L^1(-1,1)}.
$$

Similarly as above, we obtain

$$
\begin{aligned}
&|J_1'| \le C(\|u''\|_{L^\infty(-1,1)} + 1)\|\eta\|_{L^1(-1,1)} \le C\|\eta\|_{L^1(-1,1)},\\
&|J_4'| \le C(\|u''\|_{L^\infty(-1,1)} + 1)\|\varphi_2\|_{L^1(-1,1)} \le C\|\eta\|_{L^1(-1,1)},\\
&|J_5'| \le C(\|u''\|^2_{L^\infty(-1,1)} + 1)\|\varphi_2'\|_{L^1(-1,1)} \le C\|\eta\|_{L^1(-1,1)},\\
&|J_6'| \le \Big(\sup_{x\in[-1,1]} m(x)\Big)\|\varphi_2'\|_{L^1(-1,1)} \le C\mu(-1,1)\|\eta\|_{L^1(-1,1)} \le C\|\eta\|_{L^1(-1,1)}.
\end{aligned}
$$

Thus we observe that

$$\Big|\int_{-1}^1 \frac{\kappa_h(x)}{1+u'(x)^2}\eta'(x)\,dx\Big| \le C\|\eta\|_{L^1(-1,1)}$$

for all $\eta \in C_c^\infty(-1,1)$, and then

$$\|(\kappa_h(1+(u')^2)^{-1})'\|_{L^\infty(-1,1)} \le C.$$

This together with $u \in W^{2,\infty}((-1,1);(0,\infty))$ (see (3.26)) implies that

$$\|u'''\|_{L^\infty(-1,1)} < \infty.$$

Finally, with the help of the absolutely continuous function

$$f(x) := \int_{-1}^x \frac{\kappa_h(\xi)^2\sqrt{1+u'(\xi)^2}}{u(\xi)^2}\,d\xi - 2\int_{-1}^x \frac{\kappa_h(\xi)}{u(\xi)^2}\,d\xi,$$

(3.23) may be written in the following form

$$
\begin{aligned}
2\int_{-1}^1 &\left(\frac{\kappa_h(x)}{1+u'(x)^2}\right)'\varphi'(x)\,dx + \int_{-1}^1 f(x)\varphi'(x)\,dx\\
&+5\int_{-1}^1 \frac{\kappa_h(x)^2u'(x)}{u(x)\sqrt{1+u'(x)^2}}\varphi'(x)\,dx - 4\int_{-1}^1 \frac{\kappa_h(x)u'(x)}{u(x)(1+u'(x)^2)}\varphi'(x)\,dx\\
&+\int_{-1}^1 m(x)\varphi'(x)\,dx = 0
\end{aligned}
$$

for all $\varphi \in C_c^\infty(-1,1)$. This shows that there exists a constant $c \in \mathbb{R}$ such that

$$2\left(\frac{\kappa_h(x)}{1+u'(x)^2}\right)' + f(x) + 5\frac{\kappa_h(x)^2u'(x)}{u(x)\sqrt{1+u'(x)^2}} - 4\frac{\kappa_h(x)u'(x)}{u(x)(1+u'(x)^2)} + m(x) = c.$$

Since m is increasing and bounded and we already know that $u \in W^{3,\infty}(-1,1)$ we conclude that κ_h' and also u''' is of bounded variation. Proposition 3.10 follows. ☐

Acknowledgements. This work was initiated during the first author's visit at Tohoku University (before the Corona pandemic). The first author is very grateful to second author for his warm hospitality and the inspiring working atmosphere. The second author was supported in part by JSPS KAKENHI Grant Numbers JP19H05599 and JP20KK0057.

References

[1] A. Dall'Acqua and K. Deckelnick, *An obstacle problem for elastic graphs*, SIAM J. Math. Anal. **50** (2018), no. 1, 119–137.

[2] A. Dall'Acqua, K. Deckelnick, H.-Ch. Grunau, *Classical solutions to the Dirichlet problem for Willmore surfaces of revolution*, Adv. Calc. Var. **1** (2008), no. 4, 379–397.

[3] A. Dall'Acqua, S. Fröhlich, H.-Ch. Grunau, F. Schieweck, *Symmetric Willmore surfaces of revolution satisfying arbitrary Dirichlet boundary data*, Adv. Calc. Var. **4** (2011), no. 1, 1–81.

[4] K. Deckelnick and H.-Ch. Grunau, *Boundary value problems for the one-dimensional Willmore equation*, Calc. Var. Partial Differential Equations **30** (2007), no. 3, 293–314.
[5] S. Eichmann, A. Koeller, *Symmetry for Willmore surfaces of revolution*, J. Geom. Anal. **27** (2017), no. 1, 618–642.
[6] L. Euler, *Opera Omnia*, Ser. 1, **24**, Zürich: Orell Füssli (1952).
[7] F. Gazzola, H.-Ch. Grunau, G. Sweers, *Polyharmonic Boundary Value Problems*, Lecture Notes in Mathematics **1991**, Springer-Verlag, Berlin, 2010.
[8] H.-Ch. Grunau, *The asymptotic shape of a boundary layer of symmetric Willmore surfaces of revolution*, in: Inequalities and applications 2010, Internat. Ser. Numer. Math. **161**, 19–29, Birkhäuser/Springer, Basel, 2012.
[9] T. Miura, *Polar tangential angles and free elasticae*, Math. Eng. **3** (2021), no. 4, Paper No. 034, 12pp.
[10] M. Müller, *An obstacle problem for elastic curves: existence results*, Interfaces Free Bound. **21** (2019), no. 1, 87–129.
[11] M. Müller, *On gradient flows with obstacle and Euler's elstica*, Nonlinear Anal. **192** (2020), Paper No. 111676, 48pp.
[12] M. Müller, *The elastic flow with obstacle: small obstacle results*, arXiv:2010.14112.
[13] S. Okabe and K. Yoshizawa, *A dynamical approach to the variational inequality on modified elastic graphs*, Geom. Flows, **5** (2020), no. 1, 78–101.
[14] S. Saks, *Theory of the integral*, Second revised edition, English translated by L. C. Young, With two additional notes by Stefan Banach, Dover Publications, Inc., New York, 1964.
[15] L. Tartar, *An introduction to Sobolev spaces and interpolation spaces*, Lecture Notes of the Unione Matematica Italiana **3**, Springer-Verlag Berlin, UMI Bologna, 2007.
[16] K. Yoshizawa, *A remark on elastic graphs with the symmetric cone obstacle*, SIAM J. Math. Anal., to appear.

(H.-Ch. Grunau) Fakultät für Mathematik, Otto-von-Guericke-Universität, Postfach 4120,, 39016 Magdeburg, Germany
Email address: `hans-christoph.grunau@ovgu.de`

(S. Okabe) Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan
Email address: `shinya.okabe@tohoku.ac.jp`