Uniform Estimates and Convexity in Capillary Surfaces*

Hans-Christoph Grunau
Fakultät für Mathematik
Otto-von-Guericke-Universität
Postfach 4120
39016 Magdeburg, Germany

Stephan Lenor
Stockheimer Straße 5
70435 Stuttgart, Germany

October 14, 2013

Abstract

We prove uniform convexity of solutions to the capillarity boundary value problem for fixed boundary angle in $(0, \pi/2)$ and strictly positive capillarity constant provided that the base domain $\Omega \subset \mathbb{R}^2$ is sufficiently close to a disk in a suitable C^4-sense.

1 Introduction

The goal of the present note is to investigate convexity properties of classical solutions $u : \overline{\Omega} \to \mathbb{R}$ to the capillarity equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \kappa u, \quad \text{in } \Omega, \tag{1}$$

together with the (fully) nonlinear boundary condition

$$\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nu = \cos(\gamma), \quad \text{on } \partial\Omega. \tag{2}$$

Here, $\Omega \subset \mathbb{R}^2$ denotes a bounded smooth domain with exterior unit normal $\nu : \partial\Omega \to \mathbb{S}^1$, $\kappa > 0$ and $\gamma \in [0, \pi]$ are physical constants. A profound explanation of the physical background together with a number of historical remarks can be found in Finn's monograph [4, Chapter 1]. The boundary value problem (1, 2) goes back to Laplace, Young and Gauß and models the following physical situation, see also Figure 1. A vertical capillary tube with horizontal cross section Ω is put into a huge container filled with liquid up to the zero-reference-level. Above the liquid, the space is filled with a specific gas. The solution $u : \overline{\Omega} \to \mathbb{R}$ describes in how far the capillary surface deviates inside the capillary tube from the zero reference level, which is also the equilibrium shape of the surface in case that the tube is not present. The capillarity constant $\kappa > 0$ depends on the interplay of surface tension (i.e. the interaction between liquid and gas) and gravity while the constant angle $\gamma \in [0, \pi]$ between the lower normal of the capillary surface and the exterior normal of the capillary tube depends on the interaction between the liquid, gas and the material of the tube. We observe that when u is a solution to (1, 2), then $-u$ solves the same boundary value problem with γ replaced by $\pi - \gamma$. For this reason we may confine ourselves in what follows to $\gamma \in [0, \pi/2]$.

Since the differential equation (1) is not uniformly elliptic and the boundary condition (2) is (fully) nonlinear developing existence theories was somehow involved.

On the one hand one may first minimise the corresponding functional to obtain BV-solutions which under suitable conditions on Ω turn out to be smooth. In this context one has to mention

*This note is an abridged version of the second author's diploma thesis [13] written under the guidance of the first author.

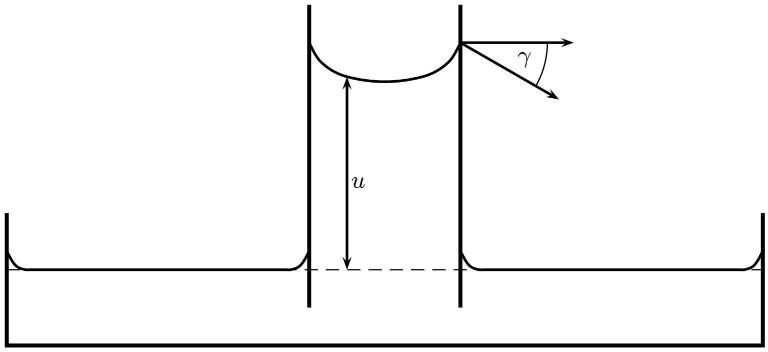

Figure 1: Capillary surface

authors like Emmer, Finn, Gerhardt, Giusti, Miranda (see e.g. [3, 5, 6, 7, 9, 16]), for further references and a comprehensive exposition one may also see Finn's monograph [4, Chapter 7].

Here, however, we shall use the theory based on a priori estimates in classical Hölder spaces and fixed point arguments. According to works of Simon, Spruck, and Liebermann [14, 17, 19], in bounded C^4-domains Ω one has a unique solution $u \in C^{3,\alpha}(\overline{\Omega}) \cap C^\infty(\Omega)$ for some suitable $\alpha \in (0,1)$ in the case $\gamma \in (0, \pi/2]$. If $\gamma = 0$ one has a solution $u \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ in the sense that u is the uniform limit of smooth solutions u_γ for $\gamma \searrow 0$. The seemingly unnatural strong regularity requirements on the domain Ω are due to the fully nonlinear character of the boundary conditions. Uniqueness of these solutions is immediate thanks to a suitable comparison principle, see [4, Chapter 5]. One should observe that in the case $\gamma = \pi/2$ one has the trivial solution $u(x) \equiv 0$ so that this case need not be considered in what follows. Moreover, as discussed in [4, Chapter 5], those comparison principles allow for a number of interesting results on geometric, qualitative and quantitative properties of solutions.

In this respect, a particularly interesting and still widely open question concerns the possible convexity of solutions. Brulois [2] considered the case of Ω a disk and proved that for any $\gamma \in [0, \pi/2)$ these (due to uniqueness) radially symmetric solutions are uniformly convex, see Lemma 11 below. Korevaar [10] proved convexity of solutions for strictly convex C^1-smooth domains under the extreme boundary condition $\gamma = 0$, i.e. infinite slope at the boundary $\partial\Omega$. On the other hand, if $\gamma \in (0, \pi/2)$, there exist smooth convex domains Ω such that $\partial\Omega$ contains an edge with a sufficiently small angle, which in a suitable way is smoothly rounded over, where the corresponding solution is no longer convex. See [10, Theorem 2.4] and also [4], Theorem 5.5 and the remark at the end of Chapter 5.5. One would expect convexity of solutions if, depending on the boundary angle $\gamma \in (0, \pi/2)$, the boundary curvature of the uniformly convex domain does not deviate too much from a constant. Our main result is a first step in this direction. It shows that the convexity property of the radial solution in any disk persists under sufficiently small C^4-domain perturbations.

Theorem 1. *Suppose that $\kappa > 0$ and $\gamma \in (0, \pi/2)$. We consider a disk $B_R(x_0)$ and for some small $\mu > 0$, a neighbourhood $\mathcal{U} := B_{R+\mu}(x_0)$. Then there exists an $\varepsilon = \varepsilon(\mu, R, \kappa, \gamma) > 0$ such that the following holds true:*

Consider any C^4-diffeomorphism $\Phi : \mathcal{U} \to \Phi(\mathcal{U})$ and $\Omega_\Phi := \Phi(B_R(x_0)) \subset \mathbb{R}^2$ with exterior unit normal $\nu_\Phi : \partial\Omega_\Phi \to \mathbb{S}^1$ such that $\|\Phi - \mathrm{Id}\|_{C^4(\overline{\Omega})} < \varepsilon$. Then the solution $u_\Phi \in C^3(\overline{\Omega}_\Phi)$ of the capillarity boundary value problem

$$\operatorname{div}\left(\frac{\nabla u_\Phi}{\sqrt{1+|\nabla u_\Phi|^2}}\right) = \kappa u_\Phi, \quad \text{in } \Omega_\Phi, \qquad \frac{\nabla u_\Phi \cdot \nu_\Phi}{\sqrt{1+|\nabla u_\Phi|^2}} = \cos(\gamma), \quad \text{on } \partial\Omega_\Phi,$$

is uniformly convex.

The first key observation in proving this result is that the classical $C^{1,\alpha}$-estimates (see Spruck's work [19] and references therein) are uniform for families of simply connected C^4-smooth domains as long as they obey the same quantitative bounds specified in Proposition 8 below. It is explained in Section 2 that all the underlying computations are in principle constructive and the estimation constants are in principle explicit.

Then, using an argument of Ladyzhenskaya and Uraltseva (see [11, Chapter 10, pp. 463–465]) we show by means of a proof by contradiction a $C^{2,\beta}$-stability result for the solutions with respect to small C^4-domain perturbations. See Theorem 9 below. In view of the uniform convexity of radial solutions in balls due to Brulois (see [2] and also Lemma 11) this yields the proof of Theorem 1 which is given in Section 4.

Most of our notation is quite standard and mostly adopted from [8].

2 Uniform $C^{1,\alpha}$-estimates

$C^{1,\alpha}$-estimates for solutions of (1, 2) are well known. The goal of this section is to recall them step by step, to provide references and to explain the reason that these are uniform for families of domains as considered in Theorem 1.

2.1 Maximum estimates

Since we may restrict ourselves to $\gamma \in [0, \pi/2]$ we observe first that solutions of (1, 2) are positive.

Lemma 2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded C^1-smooth domain, $\kappa > 0$, $\gamma \in [0, \pi/2]$. Then any solution $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ of the capillarity problem (1, 2) satisfies either $u > 0$ in Ω or $u \equiv 0$ in Ω.*

Proof. This is a direct consequence of the comparison principle [4, Theorem 5.1]. □

For the notion of interior and exterior sphere condition, which will be used frequently in what follows, see e.g. [8, p. 27].

Lemma 3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded C^1-smooth domain, $\kappa > 0$, $\gamma \in [0, \pi/2]$ and assume that $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ solves the capillarity problem (1, 2).*

(a) On $B_r(x_M) \subset \Omega$, one has

$$\sup_{x \in B_r(x_M)} |u(x)| \le \frac{2}{\kappa r} + r.$$

(b) If Ω satisfies an interior sphere condition with radius r then it holds:

$$\sup_{x \in \Omega} |u(x)| \le \frac{2}{\kappa r} + r.$$

(c) If $\partial\Omega$ is C^2-smooth there exists a constant $C = C(\Omega, \kappa)$, independent of γ, such that

$$\sup_{x \in \Omega} |u(x)| \le C.$$

Proof. See [4, Theorem 5.2]. One should always have in mind that C^2-smooth domains satisfy interior and exterior sphere conditions. The radius may be chosen even uniformly for C^2-smooth families of domains, where this notion has to be understood in the spirit of Theorem 1. □

2.2 A gradient maximum principle

Lemma 4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded C^1-smooth domain, $\kappa > 0$ and $u \in C^1(\overline{\Omega}) \cap C^3(\Omega)$ be a solution of the capillarity equation*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = \kappa u, \quad \text{in } \Omega.$$

then one has

$$\sup_{\Omega} |\nabla u| = \sup_{\partial\Omega} |\nabla u|.$$

Proof. See [8, Theorem 15.1]. □

2.3 Interior gradient bounds

In view of the preceding gradient maximum principle one may think that in order to achieve global gradient bounds it is enough to deduce boundary gradient estimates. The derivation of these (see Proposition 6 below), however, makes somehow unexpectedly use of the following interior gradient estimates.

Lemma 5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $\Omega' \Subset \Omega$ be any subdomain, $\kappa > 0$, $M > 0$. Then there exists a constant $C_I = C_I(\operatorname{dist}(\Omega', \partial\Omega), M, \kappa)$ such that for any solution $u \in C^2(\overline{\Omega})$ of the capillarity equation*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = \kappa u, \quad \text{in } \Omega$$

obeying $\sup_\Omega |u| \le M \in \mathbb{R}$ one has the following

$$\sup_{\Omega'} |\nabla u| \le C_I.$$

Proof. See [12, Theorem 4]. □

2.4 Global gradient estimates

In view of the gradient maximum principle above it suffices to deduce boundary gradient estimates. We begin with recalling estimates for the tangential derivative.

From now on we assume for simplicity the bounded domains $\Omega \subset \mathbb{R}^2$ to be always simply connected, i.e. that every closed curve in Ω is there null-homotopic. As a consequence, (see [18]) we have that by duality the boundary $\partial\Omega$ is connected.

Proposition 6. *There exists a universal combinatorial constant $C_{univ} > 0$, which can be calculated explicitly such that the following holds true.*

Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded C^4-smooth domain, $\kappa > 0$, $\gamma \in (0, \pi/2]$. We assume that $c \in C^4(\mathbb{R}; \partial\Omega)$ is a periodic parametrisation by arclength of the boundary $\partial\Omega$ which is positively oriented such that $(-\dot{c}^2, \dot{c}^1)$ gives the interior unit normal. Let $\mathcal{K} \in C^2(\mathbb{R})$ denote the boundary curvature which is defined by

$$\begin{pmatrix} \ddot{c}^1 \\ \ddot{c}^2 \end{pmatrix} = \mathcal{K} \begin{pmatrix} -\dot{c}^2 \\ \dot{c}^1 \end{pmatrix}.$$

Take $\varepsilon > 0$ small enough such that

$$2\varepsilon \sup_{\mathbb{R}} |\mathcal{K}| \le 1 \quad \text{and} \quad \forall (r, s) \in [0, \varepsilon] \times \mathbb{R} : \begin{pmatrix} c^1(s) - r\dot{c}^2(s) \\ c^2(s) + r\dot{c}^1(s) \end{pmatrix} \in \overline{\Omega} \tag{3}$$

yields a smooth regular and (up to periodicity with respect to s) smoothly invertible parametrisation of the ε-strip along $\partial\Omega$. We choose further $\varepsilon_I \in (0, \varepsilon)$ and consider the compactly contained subset

$$\Omega' := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon_I\} \Subset \Omega.$$

For any numbers $M > 0$, $C_I > 0$ consider a solution $u \in C^3(\overline{\Omega})$ of the capillarity problem (1, 2) obeying

$$\sup_{\Omega} |u| \le M \quad \text{and} \quad \sup_{\Omega'} |\nabla u| \le C_I.$$

Then the tangential derivatives of u along $\partial\Omega$ satisfy the following estimate:

$$\sup_{s \in \mathbb{R}} |\nabla u(c(s)) \cdot \dot{c}(s)| \le C_{univ} \max \Big\{ \varepsilon \sup_{\mathbb{R}} |\mathcal{K}'| M, \kappa^{-1} \sup_{\mathbb{R}} |\mathcal{K}'|, \kappa^{-1}\varepsilon^2 \sup_{\mathbb{R}} |\mathcal{K}'|^2, \kappa^{-1}\varepsilon \sup_{\mathbb{R}} |\mathcal{K}''|, C_I \Big\}.$$

Proof. The proof of this result is based on a closer analysis of the C^1-bounds from Spruck's paper [19]. There the bounds on the tangential derivatives are not stated in this explicit way. We will briefly outline the strategy of proof here.

Within the ε-strip along $\partial\Omega$, by means of the parametrisation

$$\phi : [0,\varepsilon] \times \mathbb{R} \to \mathbb{R}^2, \quad (r,s) \mapsto \begin{pmatrix} c^1(s) - r\dot{c}^2(s) \\ c^2(s) + r\dot{c}^1(s) \end{pmatrix},$$

the capillarity problem (1, 2) is transformed to the following problem for $\overline{u} := u \circ \phi$:

$$\kappa \overline{u} = \frac{1}{1 - r\mathcal{K}} \left(\frac{\partial}{\partial r} \left[(1 - r\mathcal{K}) \frac{\overline{u}_r}{W} \right] + \frac{\partial}{\partial s} \left[(1 - r\mathcal{K})^{-1} \frac{\overline{u}_s}{W} \right] \right), \quad \text{in } [0,\varepsilon] \times \mathbb{R}, \tag{4}$$

$$\cos(\gamma) = -\frac{\overline{u}_r}{\sqrt{1 + \overline{u}_r^2 + \overline{u}_s^2}}, \quad \text{on } \{0\} \times \mathbb{R}. \tag{5}$$

The modulus of the "tangential" derivative $|\overline{u}_s|$ attains its maximum in some $(r_0, s_0) \in [0,\varepsilon] \times \mathbb{R}$. According to [19] one has to distinguish the three cases $r_0 = \varepsilon$, $0 < r_0 < \varepsilon$, and $r_0 = 0$.

In the case $r_0 = \varepsilon$ we use the assumption on the gradient bound in Ω'. This allows for the following estimate

$$|\overline{u}_s(r_0, s_0)| = (1 - r_0\mathcal{K}(s_0))|\dot{c}(s_0) \cdot \nabla \overline{u}(\phi(r_0, s_0))| \le \frac{3}{2} |\nabla \overline{u}(\phi(r_0, s_0))| \le \frac{3}{2} C_I(\varepsilon_I, M, \kappa).$$

For the case $0 < r_0 < \varepsilon$ the transformed capillarity equation (4) has to be differentiated with respect to s. It turns out that from the term $\kappa\overline{u}_s$ of the differentiated equation, by estimating the right-hand side, one can obtain a bound on $|\overline{u}_s|$ in terms of ε, M, κ^{-1}, $\sup_{\mathbb{R}} |\mathcal{K}'|$ and $\sup_{\mathbb{R}} |\mathcal{K}''|$. In the case $r_0 = 0$ one has to first differentiate the boundary condition (5) with respect to s to find that also in this case (r_0, s_0) is a critical point of $\overline{u}_s$. Then one proceeds similarly as in the case $0 < r_0 < \varepsilon$ with some obvious simplifications thanks to $r_0 = 0$.

Since the bound on $|\overline{u}_s|$ depends among others on $\mathcal{K}''$, it seems to be unavoidable to formulate our results in the restricted class of C^4-smooth domains and C^4-perturbations.

For an extensive version of these calculations one may see also the second author's thesis [13, pp. 32–39]. One observes that for C^4-smooth families of domains in the sense of Theorem 1, in view of Lemmas 3 and 5, one may choose the same M and C_I for the whole family. □

From the previous estimate and the boundary condition (2), bounds for the normal derivative and hence for the full gradient on $\partial\Omega$ are immediate. We combine all the previous results of this section into the following corollary.

Corollary 7. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected C^4-smooth domain, $\kappa > 0$ and $\gamma \in (0, \pi/2]$. Then, there exists a constant $C = C(\Omega, \kappa, \gamma)$ such that for each solution $u \in C^3(\overline{\Omega})$ of the capillarity problem (1, 2) one has*

$$\sup_{\overline{\Omega}} |\nabla u| \le C.$$

For C^4-smooth families of domains, close to Ω in the sense of Theorem 1, one may choose the same uniform constant.

2.5 $C^{1,\alpha}$-bounds

Proposition 8. *We suppose that $\Omega \subset \mathbb{R}^2$ is a simply connected bounded C^4-smooth domain, let $R > 0$ be such that Ω satisfies an exterior as well as an interior sphere condition with radius R. Let $\kappa > 0$, $\gamma \in (0, \pi/2]$ and $u \in C^3(\overline{\Omega})$ be a solution of the capillarity problem (1, 2). Let the boundary $\partial\Omega$ be parametrised by arclength and let $\mathcal{K}$ be the corresponding curvature.*

We assume that we have bounds $S_{\mathcal{K}} \ge 0$, $S_{\mathcal{K}'} \ge 0$, $C_{C^0} \ge 0$ and $C_{C^1} \ge 0$ for $\mathcal{K}$ and u, respectively:

$$\sup_{\mathbb{R}} |\mathcal{K}| \le S_{\mathcal{K}}, \quad \sup_{\mathbb{R}} |\mathcal{K}'| \le S_{\mathcal{K}'}, \quad \sup_{\overline{\Omega}} |u| \le C_{C^0}, \quad \sup_{\overline{\Omega}} |\nabla u| \le C_{C^1}.$$

Then there exist constants $\alpha \in (0,1)$ and $C_{C^{1,\alpha}} > 0$ which depend explicitly and only on κ, γ, $S_{\mathcal{K}}$, $S_{\mathcal{K}'}$, C_{C^0}, C_{C^1} and R such that u obeys the bound

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le C_{C^{1,\alpha}}.$$

Proof. It is well known (see e.g. [8, Chapter 15.1]) that the derivatives of u also solve a differential equation closely related to (1). The coefficients depend on ∇u, but the C^1-estimates of Corollary 7 ensure that these enjoy quantitative properties which are sufficient for what follows. As for the derivation of $C^{0,\alpha}$-bounds for ∇u close to $\partial\Omega$ we may adapt the reasoning from [11, pp. 466–468]. For interior $C^{0,\alpha}$-bounds for ∇u we refer to [8, Theorem 8.24]. For an extensive exposition one may see [13, pp. 42–55]. □

3 A $C^{2,\beta}$-stability result

Theorem 9. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded C^4-smooth domain with exterior unit normal $\nu : \partial\Omega \to \mathbb{S}^1$, $\mathcal{U} = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \Omega) < \mu\}$ for some small enough $\mu > 0$ a neighbourhood. Suppose that $\kappa > 0$, $\gamma \in (0, \pi/2]$ und $\alpha \in (0, 1)$. We consider a sequence $(\Phi_k)_{k\in\mathbb{N}}$ of C^4-diffeomorphisms $\Phi_k : \mathcal{U} \to \Phi_k(\mathcal{U})$ obeying*

$$\|\Phi_k - \operatorname{Id}\|_{C^4(\overline{\Omega})} \to 0, \quad (k \to \infty). \tag{6}$$

Let $\Omega_k := \Phi_k(\Omega) \subset \mathbb{R}^2$ denote the C^4-smooth image domains and $\nu_k : \partial\Omega_k \to \mathbb{S}^1$ their exterior unit normals. We consider further boundary angles $(\gamma_k)_{k\in\mathbb{N}} \subset (0, \pi/2]$ converging to γ:

$$\gamma_k \to \gamma, \quad k \to \infty. \tag{7}$$

Let $u_k \in C^3(\overline{\Omega}_k)$ be the uniquely determined solutions of the capillarity problems

$$\operatorname{div}\left(\frac{\nabla u_k}{\sqrt{1 + |\nabla u_k|^2}}\right) = \kappa u_k, \quad \text{in } \Omega_k, \qquad \frac{\nabla u_k \cdot \nu_k}{\sqrt{1 + |\nabla u_k|^2}} = \cos(\gamma_k), \quad \text{on } \partial\Omega_k. \tag{8}$$

We assume that there exists a uniform bound $C_{C^{1,\alpha}} > 0$ such that

$$\forall k \in \mathbb{N} : \quad \|u_k\|_{C^{1,\alpha}(\overline{\Omega}_k)} \le C_{C^{1,\alpha}}. \tag{9}$$

Then for any $\beta \in (0, \alpha)$, the sequence $\hat{u}_k := u_k \circ \Phi_k \in C^3(\overline{\Omega})$ converges in $C^{2,\beta}(\overline{\Omega})$ to the solution $u \in C^3(\overline{\Omega})$ of the boundary value problem

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = \kappa u, \quad \text{in } \Omega, \qquad \frac{\nabla u \cdot \nu}{\sqrt{1 + |\nabla u|^2}} = \cos(\gamma), \quad \text{on } \partial\Omega. \tag{10}$$

Proof. The proof is given by contradiction following ideas and arguments by Ladyzhenskaya and Ural'tseva, see [11, Chapter 10, pp. 463–465]. We choose some $\tilde{\beta} \in (\beta, \alpha)$ and keep this fixed in what follows.

We shall first prove by contradiction boundedness of $\|\hat{u}_k\|_{C^{2,\tilde{\beta}}(\overline{\Omega})}$. We shall select a convergent subsequence in $C^{1,\tilde{\beta}}(\overline{\Omega})$ and deduce a linear elliptic differential equation satisfied by differences of elements of this sequence. Linear elliptic estimates will then allow for deducing a contradiction. Once boundedness in $C^{2,\tilde{\beta}}(\overline{\Omega})$ is shown the stated convergence follows with the help of a compactness argument.

In what follows we use the following notation

$$a_{ij} : \mathbb{R}^2 \to \mathbb{R}, \quad p \mapsto \frac{\delta_{i,j}}{\sqrt{1 + |p|^2}} - \frac{p^i p^j}{(1 + |p|^2)^{\frac{3}{2}}}, \quad i, j \in \{1, 2\},$$
$$a : \mathbb{R}^2 \to \mathbb{R}^2, \quad p \mapsto \frac{p}{\sqrt{1 + |p|^2}}.$$

Since the sequence $(\Phi_k)_{k\in\mathbb{N}}$ is uniformly bounded in $C^4(\overline{\Omega})$ we obtain from (9) also a uniform $C^{1,\alpha}(\overline{\Omega})$-bound $\tilde{C}_{C^{1,\alpha}} = \tilde{C}_{C^{1,\alpha}}(C_{C^{1,\alpha}}, (\Phi_k)_{k\in\mathbb{N}}, \Omega)$ for the transformed sequence $(\hat{u}_k)_{k\in\mathbb{N}}$. We choose $\tilde{C}_{C^{1,\alpha}}$ such that

$$\forall k \in \mathbb{N} : \quad \|u_k\|_{C^{1,\alpha}(\overline{\Omega}_k)} + \|\hat{u}_k\|_{C^{1,\alpha}(\overline{\Omega})} \le \tilde{C}_{C^{1,\alpha}}. \tag{11}$$

Moreover, since also the Φ_k^{-1} enjoy uniform bounds we find a further constant $\tilde{C} = \tilde{C}((\Phi_k)_{k\in\mathbb{N}}, \mu)$ such that for $k \in \mathbb{N}$ and $\delta \in \{\beta, \tilde{\beta}, \alpha\}$ we have

$$\sum_{i,j=1}^{2} \left\| u_{k,x^ix^j} \circ \Phi \right\|_{C^{0,\delta}(\overline{\Omega})} + \sum_{i,j=1}^{2} \left\| \frac{\partial}{\partial x^i} \left(u_{k,x^j} \circ \Phi \right) \right\|_{C^{0,\delta}(\overline{\Omega})} \leq \tilde{C} \|\hat{u}_k\|_{C^{2,\delta}(\overline{\Omega})}. \tag{12}$$

By smoothness $a_{ij} \in C^\infty(\mathbb{R}^2)$ and $a \in C^\infty(\mathbb{R}^2;\mathbb{R}^2)$, we find a bound $\tilde{C}_a$ for the compact set $K := \overline{B_{\tilde{C}_{C^{1,\alpha}}}(0)} \subset \mathbb{R}^2$ such that

$$\|a\|_{C^3(K)} + \sum_{i,j=1}^{2} \|a_{ij}\|_{C^3(K)} \leq \tilde{C}_a. \tag{13}$$

Thanks to (11) and the compactness of the embedding $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^{1,\tilde{\beta}}(\overline{\Omega})$ (see e.g. [1, Theorem 1.31]) we have, after selecting a subsequence, convergence of $(\hat{u}_k)_{k\in\mathbb{N}}$ in $C^{1,\tilde{\beta}}(\overline{\Omega})$.

For arbitrary $\ell, m \in \mathbb{N}$ we consider $v := \hat{u}_\ell - \hat{u}_m$ and a corresponding linear boundary value problem the coefficients and right-hand sides of which depend on $\hat{u}_\ell$ and $\hat{u}_m$. Using the Φ_k we pull back the boundary value problems (8) to $\overline{\Omega}$. For any $k \in \mathbb{N}$ we have

$$\begin{aligned}
\kappa \hat{u}_k &= \sum_{i,j=1}^{2} a_{ij}\left(\nabla u_k \circ \Phi_k\right) u_{k,x^ix^j} \circ \Phi_k \qquad (14)\\
&= \sum_{i,j=1}^{2} \left(a_{ij}\left(\nabla \hat{u}_k\right) + \left[a_{ij}\left(\nabla u_k \circ \Phi_k\right) - a_{ij}\left(\nabla \hat{u}_k\right)\right]\right) \left(\hat{u}_{k,x^ix^j} + \left[u_{k,x^ix^j} \circ \Phi_k - \hat{u}_{k,x^ix^j}\right]\right),\\
\cos(\gamma_k) &= a\left(\nabla u_k \circ \Phi_k\right) \cdot \left(\nu_k \circ \Phi_k\right) \qquad (15)\\
&= \left(a\left(\nabla \hat{u}_k\right) + \left[a\left(\nabla u_k \circ \Phi_k\right) - a\left(\nabla \hat{u}_k\right)\right]\right) \cdot \left(\nu + \left[\nu_k \circ \Phi_k - \nu\right]\right).
\end{aligned}$$

This gives for v the following differential equation

$$\begin{aligned}
Lv \;&:=\; \sum_{i,j=1}^{2} \left[a_{ij}(\nabla \hat{u}_\ell)\right] v_{x^ix^j} + \sum_{k=1}^{2} \left[\sum_{i,j=1}^{2} \hat{u}_{m,x^ix^j} \int_0^1 \frac{\partial a_{ij}}{\partial p^k}(\nabla \hat{u}_m + t\left(\nabla \hat{u}_\ell - \nabla \hat{u}_m\right))dt\right] v_{x^k} - \kappa v\\
&=\; \sum_{i,j=1}^{2} a_{ij}(\nabla \hat{u}_\ell)\left(\hat{u}_{\ell,x^ix^j} - \hat{u}_{m,x^ix^j}\right) + \sum_{i,j=1}^{2} \left[a_{ij}(\nabla \hat{u}_\ell) - a_{ij}(\nabla \hat{u}_m)\right] \hat{u}_{m,x^ix^j} - \kappa(\hat{u}_\ell - \hat{u}_m) \qquad (16)\\
&=\; -\sum_{i,j=1}^{2} \Big(a_{ij}\left(\nabla \hat{u}_\ell\right)\left[u_{\ell,x^ix^j} \circ \Phi_\ell - \hat{u}_{\ell,x^ix^j}\right] + \left[a_{ij}\left(\nabla u_\ell \circ \Phi_\ell\right) - a_{ij}\left(\nabla \hat{u}_\ell\right)\right] u_{\ell,x^ix^j} \circ \Phi_\ell\Big)\\
&\quad + \sum_{i,j=1}^{2} \Big(a_{ij}\left(\nabla \hat{u}_m\right)\left[u_{m,x^ix^j} \circ \Phi_m - \hat{u}_{m,x^ix^j}\right] + \left[a_{ij}\left(\nabla u_m \circ \Phi_m\right) - a_{ij}\left(\nabla \hat{u}_m\right)\right] u_{m,x^ix^j} \circ \Phi_m\Big)
\end{aligned}$$

together with the boundary condition

$$\begin{aligned}
Bv \;&:=\; \sum_{k=1}^{2} \left[\int_0^1 \frac{\partial a}{\partial p^k}(\nabla \hat{u}_m + t\left(\nabla \hat{u}_\ell - \nabla \hat{u}_m\right))dt \cdot \nu\right] v_{x^k}\\
&=\; \left(a(\nabla \hat{u}_\ell) - a(\nabla \hat{u}_m)\right) \cdot \nu \qquad (17)\\
&=\; \cos(\gamma_\ell) - a\left(\nabla \hat{u}_\ell\right) \cdot \left[\nu_\ell \circ \Phi_\ell - \nu\right] - \left[a\left(\nabla u_\ell \circ \Phi_\ell\right) - a\left(\nabla \hat{u}_\ell\right)\right] \cdot \left(\nu_\ell \circ \Phi_\ell\right)\\
&\quad - \cos(\gamma_m) + a\left(\nabla \hat{u}_m\right) \cdot \left[\nu_m \circ \Phi_m - \nu\right] + \left[a\left(\nabla u_m \circ \Phi_m\right) - a\left(\nabla \hat{u}_m\right)\right] \cdot \left(\nu_m \circ \Phi_m\right).
\end{aligned}$$

Defining the constants

$$c_L := c_B := \frac{1}{\left(1 + \tilde{C}^2_{C^{1,\alpha}}\right)^{\frac{3}{2}}} > 0$$

we see that Condition [11, Chapter 3, (3.3)] for all $\xi \in \mathbb{R}^2$

$$\sum_{i,j=1}^{2} a_{ij}(\nabla\hat{u}_\ell)\xi^i\xi^j = \frac{|\xi|^2 + |\xi|^2\,|\nabla\hat{u}_\ell|^2 - (\xi\cdot\nabla\hat{u}_\ell)^2}{\left(1+|\nabla\hat{u}_\ell|^2\right)^{\frac{3}{2}}} \geq \frac{|\xi|^2}{\left(1+|\nabla\hat{u}_\ell|^2\right)^{\frac{3}{2}}} \geq c_L|\xi|^2 \tag{18}$$

is satisfied as well as Condition [11, Chapter 3, (3.4)]: Putting $q(t) := \nabla\hat{u}_m + t\,(\nabla\hat{u}_\ell - \nabla\hat{u}_m)$ we have

$$\begin{aligned}\sum_{i=1}^{2}\left[\int_0^1 \frac{\partial a}{\partial p^i}(q(t))dt\cdot\nu\right]\nu^i &= \sum_{i,j=1}^{2}\int_0^1 \frac{\partial a^j}{\partial p^i}(q(t))dt\,\nu^i\nu^j \\ &= \int_0^1\sum_{i,j=1}^{2} a_{ij}(q(t))\nu^i\nu^j dt \geq c_L|\nu|^2 = c_B.\end{aligned} \tag{19}$$

In view of (11) and $\tilde{\beta} < \alpha$ we have a further constant $c_1 = c_1(\tilde{C}_{C^{1,\alpha}}, \kappa)$ such that, independently of ℓ und m, the coefficients of L and B obey for all $i,j \in \{1,2\}$ the following estimates:

$$\begin{aligned}&\|a_{ij}(\nabla\hat{u}_\ell)\|_{C^{0,\tilde{\beta}}(\overline{\Omega})} + \|-\kappa\|_{C^{0,\tilde{\beta}}(\overline{\Omega})} \\ &\quad + \left\|\int_0^1 \frac{\partial a}{\partial p^i}(\nabla\hat{u}_m + t\,(\nabla\hat{u}_\ell - \nabla\hat{u}_m))dt\cdot\nu\right\|_{C^{0,\tilde{\beta}}(\partial\Omega)} \leq c_1 \\ &\left\|\sum_{\tilde{n},\tilde{m}=1}^{2}\hat{u}_{m,x^{\tilde{n}}x^{\tilde{m}}}\int_0^1\frac{\partial a_{\tilde{n}\tilde{m}}}{\partial p^i}(\nabla\hat{u}_m + t\,(\nabla\hat{u}_\ell - \nabla\hat{u}_m))dt\right\|_{C^0(\overline{\Omega})} \\ &\quad + \left\|\int_0^1 \frac{\partial a}{\partial p^i}(\nabla\hat{u}_m + t\,(\nabla\hat{u}_\ell - \nabla\hat{u}_m))dt\cdot\nu\right\|_{C^1(\partial\Omega)} \\ &\leq c_1\left(1 + \|\hat{u}_\ell\|_{C^2(\overline{\Omega})} + \|\hat{u}_m\|_{C^2(\overline{\Omega})}\right), \\ &\left\|\sum_{\tilde{n},\tilde{m}=1}^{2}\int_0^1\hat{u}_{m,x^{\tilde{n}}x^{\tilde{m}}}\frac{\partial a_{\tilde{n}\tilde{m}}}{\partial p^i}(\nabla\hat{u}_m + t\,(\nabla\hat{u}_\ell - \nabla\hat{u}_m))dt\right\|_{C^{0,\tilde{\beta}}(\overline{\Omega})} \\ &\quad + \left\|\int_0^1 \frac{\partial a}{\partial p^i}(\nabla\hat{u}_m + t\,(\nabla\hat{u}_\ell - \nabla\hat{u}_m))dt\cdot\nu\right\|_{C^{1,\tilde{\beta}}(\partial\Omega)} \\ &\leq c_1\left(1 + \|\hat{u}_\ell\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} + \|\hat{u}_m\|_{C^{2,\tilde{\beta}}(\overline{\Omega})}\right).\end{aligned} \tag{20}$$

Finally we need to estimate the right-hand sides in (16) and (17). We find for $k \in \mathbb{N}$ and $i,j \in \{1,2\}$

$$\begin{aligned}\hat{u}_{k,x^i} &= \frac{\partial}{\partial x^i}\left(u_k\circ\Phi_k\right) = \sum_{\tilde{n}=1}^{2}\Phi^{\tilde{n}}_{k,x^i}u_{k,x^{\tilde{n}}}\circ\Phi_k, \\ \hat{u}_{k,x^ix^j} &= \frac{\partial^2}{\partial x^i\partial x^j}\left(u_k\circ\Phi_k\right) = \sum_{\tilde{n},\tilde{m}=1}^{2}\Phi^{\tilde{n}}_{k,x^i}\Phi^{\tilde{m}}_{k,x^j}u_{k,x^{\tilde{n}}x^{\tilde{m}}}\circ\Phi_k + \sum_{\tilde{n}=1}^{2}\Phi^{\tilde{n}}_{k,x^ix^j}u_{k,x^{\tilde{n}}}\circ\Phi_k.\end{aligned} \tag{21}$$

Making use of (12) and of the convergence in (6) we obtain

$$\begin{aligned}\left\|\hat{u}_{k,x^ix^j} - u_{k,x^ix^j}\circ\Phi_k\right\|_{C^0(\overline{\Omega})} &= \Bigg\|\left(\Phi^i_{k,x^i}\Phi^j_{k,x^j} - 1\right)u_{k,x^ix^j}\circ\Phi_k + \Phi^{3-i}_{k,x^i}\Phi^j_{k,x^j}u_{k,x^{3-i}x^j}\circ\Phi_k \\ &\quad + \Phi^i_{k,x^i}\Phi^{3-j}_{k,x^j}u_{k,x^ix^{3-j}}\circ\Phi_k + \Phi^{3-i}_{k,x^i}\Phi^{3-j}_{k,x^j}u_{k,x^{3-i}x^{3-j}}\circ\Phi_k \\ &\quad + \sum_{\tilde{n}=1}^{2}\Phi^{\tilde{n}}_{k,x^ix^j}u_{k,x^{\tilde{n}}}\circ\Phi_k\Bigg\|_{C^0(\overline{\Omega})} \\ &\leq o(1)\left(\sum_{\tilde{n},\tilde{m}=1}^{2}\left\|u_{k,x^{\tilde{n}}x^{\tilde{m}}}\circ\Phi_k\right\|_{C^0(\overline{\Omega})} + \sum_{\tilde{n}=1}^{2}\left\|u_{k,x^{\tilde{n}}}\circ\Phi_k\right\|_{C^0(\overline{\Omega})}\right) \\ &\leq o(1)\left(\|\hat{u}_k\|_{C^2(\overline{\Omega})} + \tilde{C}_{C^{1,\alpha}}\right),\end{aligned} \tag{22}$$

and similarly

$$\left\|\hat{u}_{k,x^ix^j} - u_{k,x^ix^j} \circ \Phi_k\right\|_{C^{0,\tilde{\beta}}(\overline{\Omega})} \leq o(1)\left(\|\hat{u}_k\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} + \tilde{C}_{C^{1,\alpha}}\right). \tag{23}$$

By virtue of (13) and product and chain rule we have for $i, j \in \{1, 2\}$ and $k \in \mathbb{N}$:

$$\begin{aligned}
\|a_{ij}(\nabla u_k \circ \Phi_k) - a_{ij}(\nabla \hat{u}_k)\|_{C^0(\overline{\Omega})} &\leq o(1)\tilde{C}^{1,\alpha}, \\
\|a_{ij}(\nabla u_k \circ \Phi_k) - a_{ij}(\nabla \hat{u}_k)\|_{C^{0,\tilde{\beta}}(\overline{\Omega})} &\leq o(1)\tilde{C}^{1,\alpha}, \\
\|a(\nabla u_k \circ \Phi_k) - a(\nabla \hat{u}_k)\|_{C^1(\partial\Omega)} &\leq o(1)\left(\tilde{C}^{1,\alpha} + \|\hat{u}_k\|_{C^2(\overline{\Omega})}\right), \\
\|a(\nabla u_k \circ \Phi_k) - a(\nabla \hat{u}_k)\|_{C^{1,\tilde{\beta}}(\partial\Omega)} &\leq o(1)\left(\tilde{C}^{1,\alpha} + \|\hat{u}_k\|_{C^{2,\tilde{\beta}}(\overline{\Omega})}\right).
\end{aligned} \tag{24}$$

Thanks to Assumption (6) concerning domain convergence we have

$$\|\nu_k \circ \Phi_k - \nu\|_{C^{1,\tilde{\beta}}(\partial\Omega)} = o(1). \tag{25}$$

We combine (7), (22), (23), (24), and (25) and see that the right-hand sides in (16) and (17) can be estimated as follows:

$$\begin{aligned}
\|Lv\|_{C^0(\overline{\Omega})} &\leq o(1)\left(1 + \|\hat{u}_\ell\|_{C^2(\overline{\Omega})} + \|\hat{u}_m\|_{C^2(\overline{\Omega})}\right), \\
\|Lv\|_{C^{0,\tilde{\beta}}(\overline{\Omega})} &\leq o(1)\left(1 + \|\hat{u}_\ell\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} + \|\hat{u}_m\|_{C^{2,\tilde{\beta}}(\overline{\Omega})}\right), \\
\|Bv\|_{C^1(\partial\Omega)} &\leq o(1)\left(1 + \|\hat{u}_\ell\|_{C^2(\overline{\Omega})} + \|\hat{u}_m\|_{C^2(\overline{\Omega})}\right), \\
\|Bv\|_{C^{1,\tilde{\beta}}(\partial\Omega)} &\leq o(1)\left(1 + \|\hat{u}_\ell\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} + \|\hat{u}_m\|_{C^{2,\tilde{\beta}}(\overline{\Omega})}\right).
\end{aligned} \tag{26}$$

Since the conditions (18), (19) are satisfied, we may apply [11, Chapter 3, Theorem 3.1] and use the estimate [11, Chapter 3, (3.5)]. Making further use of (20) and (26) we find a constant $c_2 = c_2(\tilde{C}_{C^{1,\alpha}}, c_L, c_B)$, which is independent of ℓ and m, and the following estimate:

$$\begin{aligned}
\|v\|_{2,\tilde{\beta}(\overline{\Omega})} \leq c_2\Bigg(& o(1)\left(1 + \|\hat{u}_\ell\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} + \|\hat{u}_m\|_{C^{2,\tilde{\beta}}(\overline{\Omega})}\right) + \|v\|_{C^0(\overline{\Omega})} \\
&+ \|v\|_{C^1(\overline{\Omega})}\left(1 + \|\hat{u}_\ell\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} + \|\hat{u}_m\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} + \left(\|\hat{u}_\ell\|_{C^2(\overline{\Omega})} + \|\hat{u}_m\|_{C^2(\overline{\Omega})}\right)^{1+\tilde{\beta}}\right)\Bigg).
\end{aligned}$$

Applying the interpolation inequality in Hölder spaces

$$\forall w \in C^{2,\tilde{\beta}}(\overline{\Omega}): \quad \|w\|_{C^2(\overline{\Omega})} \leq c_3 \|w\|_{C^{2,\tilde{\beta}}(\overline{\Omega})}^{\frac{1}{1+\tilde{\beta}}} \|w\|_{C^1(\overline{\Omega})}^{\frac{\tilde{\beta}}{1+\tilde{\beta}}} \tag{27}$$

(see e.g. [11, Chapter 3, (2.1)] or [15]) we end up with the following crucial estimate

$$\|v\|_{2,\tilde{\beta}(\overline{\Omega})} \leq c\left(o(1) + \|v\|_{C^1(\overline{\Omega})}\right)\left(1 + \|\hat{u}_\ell\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} + \|\hat{u}_m\|_{C^{2,\tilde{\beta}}(\overline{\Omega})}\right). \tag{28}$$

Here, $c = c(\tilde{C}_{C_{1,\alpha}}, c_3, c_L, c_B)$ is a constant which is independent of ℓ and m.

We assume now by contradiction that the sequence $(\hat{u}_k)_{k\in\mathbb{N}}$ is unbounded $C^{2,\tilde{\beta}}(\overline{\Omega})$. Then one could select a subsequence $(\hat{u}_{k_i})_{i\in\mathbb{N}}$ such that

$$\|\hat{u}_{k_{i+1}}\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} \geq 2\|\hat{u}_{k_i}\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} \quad \text{and} \quad \|\hat{u}_{k_i}\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} \geq 1$$

holds true for all $i \in \mathbb{N}$. We introduce $\ell(i) = k_{i+1}$, $m(i) = k_i$ and $v_i = \hat{u}_{\ell(i)} - \hat{u}_{k(i)}$. Inequality (28) implies then that

$$\begin{aligned}
\|\hat{u}_{k_{i+1}}\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} &= 2\|\hat{u}_{k_i} + \hat{u}_{k_{i+1}} - \hat{u}_{k_i}\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} - \|\hat{u}_{k_{i+1}}\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} \\
&\leq 2\|\hat{u}_{k_i}\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} + 2\|v_i\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} - \|\hat{u}_{k_{i+1}}\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} \leq 2\|v_i\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} \\
&\leq 2c\left(o(1) + \|v_i\|_{C^1(\overline{\Omega})}\right)\left(1 + \|\hat{u}_{k_i}\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} + \|\hat{u}_{k_{i+1}}\|_{C^{2,\tilde{\beta}}(\overline{\Omega})}\right)
\end{aligned}$$

$$\leq 5c\left(o(1) + \|v_i\|_{C^1(\overline{\Omega})}\right) \|\hat{u}_{k_{i+1}}\|_{C^{2,\tilde{\beta}}(\overline{\Omega})}.$$

Since the $C^{1,\tilde{\beta}}(\overline{\Omega})$-convergence of $(\hat{u}_k)_{k\in\mathbb{N}}$ implies $\|v_i\|_{C^1(\overline{\Omega})} \to 0$ for $i \to \infty$, we conclude that $\|\hat{u}_{k_{i+1}}\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} \leq \|\hat{u}_{k_{i+1}}\|_{C^{2,\tilde{\beta}}(\overline{\Omega})}/2$ for sufficiently large i. In view of $\|\hat{u}_{k_{i+1}}\|_{C^{2,\tilde{\beta}}(\overline{\Omega})} \geq 1 > 0$ we achieve a contradiction.

To conclude: We have shown so far that $(\hat{u}_k)_{k\in\mathbb{N}}$ is bounded in $C^{2,\tilde{\beta}}(\overline{\Omega})$. Due to $\beta < \tilde{\beta}$, $C^{2,\tilde{\beta}}(\overline{\Omega})$ is compactly embedded in $C^{2,\beta}(\overline{\Omega})$. Hence we find a $\widetilde{u} \in C^{2,\beta}(\overline{\Omega})$ such that after selecting a subsequence, $(\hat{u}_k)_{k\in\mathbb{N}}$ converges in $C^{2,\beta}(\overline{\Omega})$ to $\widetilde{u} \in C^{2,\beta}(\overline{\Omega})$. Employing the convergence properties (7), (22), (24) and (25) we conclude from the fact that the $\hat{u}_k$ solve the boundary value problems (14, 15) that

$$\kappa\widetilde{u} = \lim_{k\to\infty} \sum_{i,j=1}^{2} \Big(\underbrace{a_{ij}(\nabla\hat{u}_k)}_{\to a_{ij}(\nabla\widetilde{u})} + \underbrace{[a_{ij}(\nabla u_k \circ \Phi_k) - a_{ij}(\nabla\hat{u}_k)]}_{\to 0} \Big)\Big(\underbrace{\hat{u}_{k,x^ix^j}}_{\to \widetilde{u}_{x^ix^j}} + \underbrace{[u_{k,x^ix^j} \circ \Phi_k - \hat{u}_{k,x^ix^j}]}_{\to 0} \Big)$$

$$= \sum_{i,j=1}^{2} a_{ij}(\nabla\widetilde{u})\,\widetilde{u}_{x^ix^j},$$

$$\cos(\gamma) = \lim_{k\to\infty} \Big(\underbrace{a(\nabla\hat{u}_k)}_{\to a(\nabla\widetilde{u})} + \underbrace{[a(\nabla u_k \circ \Phi_k) - a(\nabla\hat{u}_k)]}_{\to 0} \Big) \cdot \Big(\nu + \underbrace{[\nu_k \circ \Phi_k - \nu]}_{\to 0} \Big) = a(\nabla\widetilde{u}) \cdot \nu.$$

Hence, $\widetilde{u}$ solves the capillarity problem (10) and by uniqueness, $\widetilde{u} = u$ follows.

Since the previous reasoning applies also to any subsequence of $(\hat{u}_k)_{k\in\mathbb{N}}$, a standard reasoning by contradiction shows that the whole sequence converges to u. ☐

Since in Section 2 we have deduced uniform $C^{1,\alpha}$-bounds the preceding theorem yields the following stability result.

Corollary 10. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded C^4-smooth domain with exterior unit normal $\nu : \partial\Omega \to \mathbb{S}^1$, $\mathcal{U} = \{x \in \mathbb{R}^2 : \operatorname{dist}(x,\Omega) < \mu\}$ for some small enough $\mu > 0$ a neighbourhood. Suppose that $\kappa > 0$ and $\gamma \in (0, \pi/2]$. We consider a sequence $(\Phi_k)_{k\in\mathbb{N}}$ of C^4-diffeomorphisms $\Phi_k : \mathcal{U} \to \Phi_k(\mathcal{U})$ satisfying*

$$\|\Phi_k - \operatorname{Id}\|_{C^4(\overline{\Omega})} \to 0, \quad (k \to \infty).$$

Let $\Omega_k := \Phi_k(\Omega) \subset \mathbb{R}^2$ denote the C^4-smooth image domains and $\nu_k : \partial\Omega_k \to \mathbb{S}^1$ their exterior unit normals. Let $u_k \in C^3(\overline{\Omega}_k)$ be the uniquely determined solutions of the capillarity problems

$$\operatorname{div}\left(\frac{\nabla u_k}{\sqrt{1 + |\nabla u_k|^2}}\right) = \kappa u_k, \quad \text{in } \Omega_k, \qquad \frac{\nabla u_k \cdot \nu_k}{\sqrt{1 + |\nabla u_k|^2}} = \cos(\gamma), \quad \text{on } \partial\Omega_k.$$

Then there exists a Hölder exponent $\beta \in (0,1)$ such that we have the following convergence of the perturbed solutions to the original solution:

$$\|u_k \circ \Phi_k - u\|_{C^{2,\beta}(\overline{\Omega})} \to 0, \quad k \to \infty.$$

Proof. We combine Corollary 7 and Proposition 8 to conclude from the C^4-convergence of Φ_k to Id that there exists a Hölder exponent $\alpha = \alpha(\Omega, (\Phi_k)_{k\in\mathbb{N}}, \kappa, \gamma, \mu)$ and a constant $C_{C^{1,\alpha}} = C_{C^{1,\alpha}}(\Omega, \kappa, \gamma, \mu)$ such that we have for all $k \in \mathbb{N}$ the following uniform $C^{1,\alpha}$-estimate

$$\|u_k\|_{C^{1,\alpha}(\overline{\Omega}_k)} < C_{C^{1,\alpha}}.$$

The claim follows now from Theorem 9. ☐

4 Convexity of capillary surfaces in domains close to a disk

4.1 Radially symmetric solutions

In case that Ω is a disk the solution to the capillarity problem is radially symmetric and hence, this problem becomes much simpler. Indeed, in this situation we have the following convexity result by Brulois [2] from 1982. For the reader's convenience we also recall its proof.

Lemma 11. *Consider arbitrary $R > 0$, $x_0 \in \mathbb{R}^2$, $\kappa > 0$ and $\gamma \in [0, \pi/2)$ and let $\Omega := B_R(x_0)$. Then the solution u of (1, 2) is uniformly convex in the sense that the eigenvalues of* $\operatorname{Hess} u(x)$ *are uniformly bounded from below by a strictly positive number.*

Proof. We may assume $x_0 = 0$. Rotational symmetry of the solution is immediate due to uniqueness. Writing $w(|x|) = u(x)$, (1, 2) is transformed into the following ODE-problem:

$$\frac{d}{dr}\left[\frac{rw'(r)}{\sqrt{1+w'(r)^2}}\right] = \kappa r w(r) \text{ for } r \in [0,R], \quad w'(0) = 0, \quad w'(R) = \cot(\gamma). \tag{29}$$

Integration yields

$$r\frac{w'(r)}{\sqrt{1+w'(r)^2}} = \kappa \int_0^r \tau w(\tau) d\tau. \tag{30}$$

By assumption, $\gamma \in [0, \pi/2)$ implying that $w \not\equiv 0$. Lemma 2 yields that $w > 0$ on $[0, R]$. (30) shows then that $w' > 0$ for $r \in (0, R]$, so that $w(r) > w(0) > 0$ on $(0, R]$. Integrating (30) by parts gives

$$\frac{w'(r)}{\sqrt{1+w'(r)^2}} = \frac{1}{r}\kappa \int_0^r \tau w(\tau) d\tau = \frac{1}{2}\kappa r w(r) - \frac{1}{r}\kappa \int_0^r \frac{\tau^2}{2} w'(\tau) d\tau < \frac{1}{2}\kappa r w(r). \tag{31}$$

We combine (29) and (31) and find for $r \in (0, R]$

$$\kappa r w(r) = \frac{r}{(1+w'(r)^2)^{\frac{3}{2}}} w''(r) + \frac{w'(r)}{\sqrt{1+w'(r)^2}} < \frac{r}{(1+w'(r)^2)^{\frac{3}{2}}} w''(r) + \frac{1}{2}\kappa r w(r).$$

Hence, we have for $r \in [0, R]$

$$w''(r) \geq \frac{1}{2}\kappa w(r)\left(1+w'(r)^2\right)^{\frac{3}{2}} \geq \frac{1}{2}\kappa w(0) > 0.$$

This means that for $u(x) = w(|\cdot|)$, $\overline{\Omega} \ni x \neq 0$ and $\xi \in \mathbb{R}^2$ one has

$$\begin{aligned}\sum_{i,j=1}^{2} u_{x^i x^j}(x)\xi^i \xi^j &= \left[\frac{1}{|x|}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \cdot \xi\right]^2 \underbrace{w''(|x|)}_{>0} + \left[\frac{1}{|x|}\begin{pmatrix} -x^2 \\ x^1 \end{pmatrix} \cdot \xi\right]^2 \underbrace{\frac{w'(|x|)}{|x|}}_{>0} \\ &\geq \min\left\{w''(|x|), \frac{w'(|x|)}{|x|}\right\} |\xi|^2.\end{aligned}$$

Since $\lim_{r \searrow 0} \frac{w'(r)}{r} = w''(0) > 0$ there is a positive number $\lambda_0 > 0$ such that $\min\left\{w''(|x|), \frac{w'(|x|)}{|x|}\right\} \geq \lambda_0$. This finally shows the claimed uniform convexity:

$$\forall x \in \overline{\Omega}, \quad \forall \xi \in \mathbb{R}^2 : \qquad \sum_{i,j=1}^{2} u_{x^i x^j}(x)\xi^i \xi^j \geq \lambda_0 |\xi|^2.$$

□

4.2 Close to a disk

In view of the stability result from Corollary 10 it is now evident that convexity of radial solutions over disks as just proved in Lemma 11 remains under small domain perturbations. This means that we may now give the proof of our main result.

Proof of Theorem 1. According to Lemma 11, $u := u_{\mathrm{Id}}$ is uniformly convex meaning that there exists a positive constant $\lambda_0 > 0$ such that the eigenvalues $\lambda_1, \lambda_2 : \overline{\Omega} \to \mathbb{R}$ of $\operatorname{Hess} u$ are bounded from below by λ_0. We always choose $\lambda_1(x)$ to be the smaller eigenvalue.

We assume by contradiction that there is no $\varepsilon > 0$ as in Theorem 1. Then one would find a sequence of diffeomorphisms $\Phi_k \to \mathrm{Id}$ in $C^4(\overline{\Omega})$ such that $u_k := u_{\Phi_k}$ is not uniformly convex. We find then a sequence $x_k \in \overline{\Omega}_k$, such that $\operatorname{Hess} u_k(x_k)$ is not (strictly) positive definit. This means that $\lambda_1(u_k, x_k) \leq 0$. By compactness we may assume, after possibly passing to a subsequence, that $(\hat{x}_k := \Phi_k^{-1}(x_k))_{k\in\mathbb{N}} \in \overline{\Omega}$ converges to some $x_0 \in \overline{\Omega}$.

According to Corollary 10 the sequence $(\hat{u}_k := u_k \circ \Phi_k)_{k\in\mathbb{N}}$ converges in $C^2(\overline{\Omega})$ to u. In particular, $\|\hat{u}_k\|_{C^2(\overline{\Omega})}$ is uniformly bounded and making use of (22) we find for $i, j \in \mathbb{N}$

$$\left\|\hat{u}_{k,x^ix^j} - u_{k,x^ix^j} \circ \Phi_k\right\|_{C^0(\overline{\Omega})} \to 0, \quad k \to \infty.$$

Since $\Phi_k \to \mathrm{Id}$ in $C^4(\overline{\Omega})$, convergence of $\operatorname{Hess} u_k(x_k)$ to $\operatorname{Hess} u(x_0)$ follows; we have for $i, j \in \{1, 2\}$:

$$\begin{aligned} \left|u_{k,x^ix^j}(x_k) - u_{x^ix^j}(x_0)\right| \leq{}& \underbrace{\left|u_{k,x^ix^j}(x_k) - u_{k,x^ix^j} \circ \Phi_k(\hat{x}_k)\right|}_{=0} + \underbrace{\left|u_{k,x^ix^j} \circ \Phi_k(\hat{x}_k) - \hat{u}_{k,x^ix^j}(\hat{x}_k)\right|}_{\to 0} \\ &+ \underbrace{\left|\hat{u}_{k,x^ix^j}(\hat{x}_k) - u_{x^ix^j}(\hat{x}_k)\right|}_{\to 0} + \underbrace{\left|u_{x^ix^j}(\hat{x}_k) - u_{x^ix^j}(x_0)\right|}_{\to 0} \to 0. \end{aligned}$$

Since eingenvalues of matrices depend continuously on the matrix entries we would end up with

$$0 \geq \lim_{k\to\infty} \lambda_1(u_k, x_k) = \lambda_1(u, x_0) \geq \lambda_0 > 0,$$

a contradiction. □

References

[1] Robert A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics **65**, Academic Press, New York etc., 1975.

[2] Frédéric P. Brulois, Asymptotic expansions and estimates for the capillary problem, in: *Proceedings of the Second International Colloquium on Drops and Bubbles*, Jet Propulsion Laboratory, 344–349, 1982.

[3] Michele Emmer, Esistenza, unicità e regolarità nelle superfici di equilibrio nei capillari, *Ann. Univ. Ferrara Sez. VII (N.S.)* **18**, 79–94, 1973.

[4] Robert Finn, *Equilibrium Capillary Surfaces*, Grundlehren der Mathematischen Wissenschaften **284**, Springer-Verlag, New York etc., 1986.

[5] Robert Finn and Claus Gerhardt, The internal sphere condition and the capillary problem, *Ann. Mat. Pura Appl. (4)* **112**, 13–31, 1977.

[6] Claus Gerhardt, Existence and regularity of capillary surfaces, *Boll. Un. Mat. Ital. (4)* **10**, 317–335, 1974.

[7] Claus Gerhardt, Global regularity of the solutions to the capillarity problem, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **3**, 157–175, 1976.

[8] David Gilbarg and Neil S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Grundlehren der Mathematischen Wissenschaften **224**, second edition, Springer-Verlag, Berlin etc., 1983.

[9] Enrico Giusti, Generalized solutions for the mean curvature equation, *Pacific J. Math.* **88**, 297–321, 1980.

[10] Nicholas Korevaar, Capillary surface convexity above convex domains, *Indiana Univ. Math. J.* **32**, 73–81, 1983.

[11] Olga A. Ladyzhenskaya and Nina N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York etc., 1968.

[12] Olga A. Ladyzhenskaya and Nina N. Ural'tseva, Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations, *Commun. Pure Appl. Math.* **23**, 677–703, 1970.

[13] Stephan Lenor, *Konvexität und gleichmäßige Abschätzungen bei der Kapillaritätsgleichung*, Diplomarbeit, Otto-von-Guericke-Universität Magdeburg, 2012, available online at http://www-ian.math.uni-magdeburg.de/home/grunau/papers/LenorDiplomarbeit.pdf

[14] Gary M. Lieberman, Solvability of quasilinear elliptic equations with nonlinear boundary conditions, *Trans. Amer. Math. Soc.* **273**, 753–765, 1982.

[15] Carlo Miranda, *Equazioni alle Derivate Parziali di Tipo Ellittico*, Springer-Verlag, Berlin etc., 1955.

[16] Mario Miranda, Superficie minime illimitate, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **4**, 313–322, 1977.

[17] Leon M. Simon and Joel Spruck, Existence and regularity of a capillary surface with prescribed contact angle, *Arch. Rational Mech. Anal.* **61**, 19–34, 1976.

[18] E. D. Solomentsev, Simply-connected domain, *Encyclopedia of Mathematics*, http://www.encyclopediaofmath.org/index.php?title=Simply-connected_domain&oldid=14141

[19] Joel Spruck, On the existence of a capillary surface with prescribed contact angle, *Commun. Pure Appl. Math.* **28**, 189–200, 1975.