OPTIMAL ESTIMATES FROM BELOW FOR BIHARMONIC GREEN FUNCTIONS

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ABSTRACT. Optimal pointwise estimates are derived for the biharmonic Green function in arbitrary $C^{4,\gamma}$-smooth domains. Maximum principles do not exist for fourth order elliptic equations and the Green function may change sign. It prevents using a Harnack inequality as for second order problems and hence complicates the derivation of optimal estimates. The present estimate is obtained by an asymptotic analysis. The estimate shows that this Green function is positive near the singularity and that a possible negative part is small in the sense that it is bounded by the product of the squared distances to the boundary.

1. INTRODUCTION

It is well-known that the Green function $G(x,y)$ for second order elliptic equations on bounded domains can be estimated from above and from below by positive multiples of the same (positive) function, which is explicitly given in terms of the distances to the boundary, $d(x), d(y)$, and the distance $|x - y|$. See for example [3]. The behaviour of the biharmonic Green function for Dirichlet boundary conditions should be somehow similar but will have two crucial distinctions. The singularity of course is of lower order, namely $n - 4$ instead of $n - 2$ with n the dimension, but a more serious distinction is the fact that the biharmonic Green function is not everywhere positive for most domains. Indeed, the few known domains with a biharmonic Green function of a fixed positive sign are balls in arbitrary dimensions, small perturbations of those balls and of some limaçons in 2 dimensions. See respectively [2, 11, 10] and [6, 7]. The results in [7] extend and correct [13].

It has been observed numerically on domains with a sign changing biharmonic Green function, that the negative part is rather small and that it is also not located near the singularity. The aim of this paper is to give optimal estimates from below. Previous results concerning smallness of the negative part have been obtained in [10] for $n \geq 3$ and [4] for $n = 2$. With the estimates for the absolute value of that Green function in [5] these previously known estimates are, when $n > 4$, as follows:

$$(1) \qquad -c\, d(x)^2 d(y)^2 \leq G_\Omega(x,y) \leq c^* \, |x-y|^{4-n} \min\left(1, \frac{d(x)^2 d(y)^2}{|x-y|^4}\right)$$

for all $x, y \in \Omega \subset \mathbb{R}^n$ and where c, c^* are some positive constants only depending on the domain. The distance of x to the boundary $\partial\Omega$ is defined by $d(x) := \inf\{|x - x^*| \,;\, x^* \in \Omega\}$, and G_Ω denotes the said Green function. Let us remind the reader that the biharmonic Green function G_Ω is such that

$$u(x) = \int_\Omega G_\Omega(x,y) \ f(y) \ dy.$$

solves

$$(2) \qquad \begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = |\nabla u| = 0 & \text{on } \partial\Omega. \end{cases}$$

In (1) the dimension is restricted to $n > 4$. As has been shown in [10] for $n = 3, 4$ and in [4] for $n = 2$, the estimate from below in (1) holds in all dimensions. An estimate from above has also been proved for $n \leq 4$ but the formula for that estimate is different from (1). Those estimates can be found in [5] and contain the function H_Ω in (4).

The main result is the estimate from below in the following theorem. For the sake of completeness we include the estimate from above.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded $C^{4,\gamma}$-smooth domain. Let G_Ω denote the biharmonic Green function in Ω for (2). Then there exist constants $c_1 \geq 0$, $c_2 > 0$, depending on the domain Ω, such that we have the following Green function estimate:*

$$(3) \qquad c_2^{-1} \, H_\Omega(x, y) \leq G_\Omega(x, y) + c_1 d(x)^2 d(y)^2 \leq c_2 \, H_\Omega(x, y)$$

for all $x, y \in \Omega$, where

$$(4) \qquad H_\Omega(x, y) := \begin{cases} |x - y|^{4-n} \min\left\{1, \dfrac{d(x)^2 d(y)^2}{|x - y|^4}\right\} & \text{if } n > 4, \\[4mm] \log\left(1 + \dfrac{d(x)^2 d(y)^2}{|x - y|^4}\right) & \text{if } n = 4, \\[4mm] d(x)^{2-n/2} d(y)^{2-n/2} \min\left\{1, \dfrac{d(x)^{n/2} d(y)^{n/2}}{|x - y|^n}\right\} & \\ & \text{if } n = 2, 3. \end{cases}$$
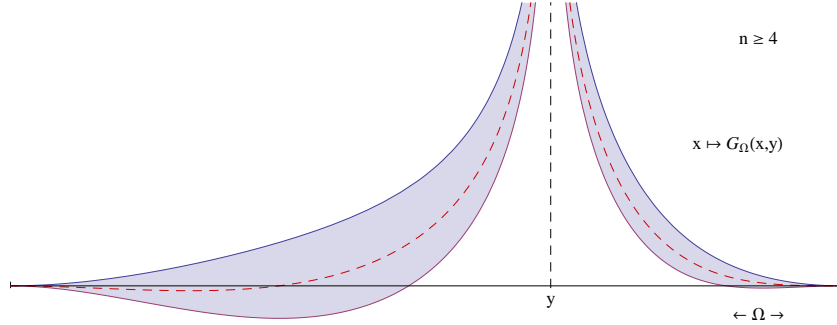

FIGURE 1. The dashed curve gives the typical behaviour of G_Ω; the shaded area describes the band given by 3.

The estimate from above follows from [5, 15, 16, 10] so that only the estimate from below has to be proved here. One should observe that $G_\Omega(x, y) + c_1 d(x)^2 d(y)^2 \geq 0$ was proved for some suitable $c_1 \geq 0$ in [10]. The preceding result may be considered as an extension of the estimates in [10] showing that close to the pole the positive singular behaviour of the fundamental solution can also be seen in the Green function.

Due to the different behaviour of the Green function we need to distinguish between $n \geq 3$ and $n = 2$ in proving Theorem 1. Finally we should remark that the lack of a maximum principle not only results in sign changing Green functions but also complicates the proof of these estimates in the fourth order case. The proof

in the second order case heavily depends on the Harnack inequality which in turn depends on the maximum principle.

An interesting consequence of Theorem 1 is a uniform local positivity result. When $n \geq 3$ this was proved in [10], while for $n = 2$ we refer to [9, Theorem 6.15]. Here the emphasis is on the interplay between Theorem 1 and the following result. Moreover, we provide a proof for $n = 2$ which is much simpler than the previous one and in the same spirit as for $n \geq 3$.

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded $C^{4,\gamma}$-smooth domain. Let G_Ω denote the biharmonic Green function in Ω for (2). Then there exist a constant $r_\Omega > 0$, such that*

$$(5) \qquad G_\Omega(x, y) > 0 \text{ for all } x, y \in \Omega \text{ with } |x - y| < r_\Omega.$$

2. Some auxiliary results for $n \geq 3$

A careful inspection of the proofs in Nehari [17] and Grunau-Sweers [12] shows the following local estimate for the biharmonic Green function from below.

Proposition 3. *Let $n \geq 3$. Then there exists constants $\delta_n > 0$ and $c_3 > 0$, which depend only on the dimension n, such that the following holds true.*

Assume $\Omega \subset \mathbb{R}^n$ to be a $C^{4,\gamma}$-smooth bounded domain and let $G_\Omega := G_{\Delta^2, \Omega}$ denote the Green function for the biharmonic operator under Dirichlet boundary conditions. If

$$(6) \qquad |x - y| \leq \delta_n \max\{d(x), d(y)\},$$

then we have

$$G_\Omega(x, y) > \begin{cases} c_3 |x - y|^{4-n} & \text{if } n > 4, \\[2mm] c_3 \log\left(1 + \dfrac{d(x)^2 d(y)^2}{|x - y|^4}\right) & \text{if } n = 4, \\[2mm] c_3 d(x)^{1/2} d(y)^{1/2} & \text{if } n = 3. \end{cases}$$

For the constant δ_n, one may achieve that $\delta_n \geq 0.5$.

In dimension $n = 2$ it seems impossible to achieve a global linear dependence of the radius of a ball of guaranteed positivity on the boundary distance, see Lemma 7 and [17]. In other words, the best result seems $\delta_2 = \delta_2(\Omega)$, which strictly depends on Ω.

Due to Proposition 3 and using the same constants $\delta_n > 0$ as there we may restrict ourselves in what follows to x, y such that

$$(7) \qquad x, y \in \Omega, x \neq y, |x - y| > \delta_n \max\{d(x), d(y)\}.$$

Lemma 4. *Suppose that $n \geq 3$ and that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{4,\gamma}$-smooth domain. Then for each $x_0 \in \overline{\Omega}$ there exists a radius $r = r_{x_0} > 0$ and a constant $C = C_{x_0} > 0$ such that for all $x, y \in \Omega_{x_0, r} := \overline{\Omega} \cap B_r(x_0)$ subject to condition (7) one has*

$$(8) \qquad G_\Omega(x, y) \geq C |x - y|^{-n} d(x)^2 d(y)^2.$$

Proof. We only need to discuss $x_0 \in \partial\Omega$ since for interior points x_0 one may choose $r = r_{x_0} > 0$ so small that condition (7) becomes void.

We assume by contradiction that there exist $x_k, y_k \in \Omega_{x_0, 1/k} = \overline{\Omega} \cap B_{1/k}(x_0)$ subject to (7) such that

$$(9) \qquad G_\Omega(x_k, y_k) < \frac{1}{k} |x_k - y_k|^{-n} d(x_k)^2 d(y_k)^2.$$

In particular we have $x_k \to x_0, y_k \to x_0$, $d(x_k), d(y_k) \to 0$, $|x_k - y_k| \to 0$. Without loss of generality we may assume that $x_0 = 0$ and that the first unit vector $\vec{e}_1$ is the exterior unit normal to $\partial\Omega$ at x_0.
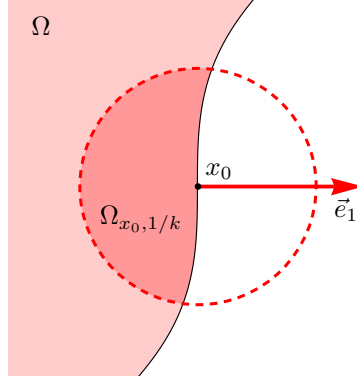

FIGURE 2. Ω and subdomain $\Omega_{x_0,1/k}$ for $x_0 \in \partial\Omega$.

For k large enough, we may define $\tilde{x}_k \in \partial\Omega$ as the closest boundary point to x_k. We introduce the rescaled biharmonic Green functions

$$G_k(\xi, \eta) := |x_k - y_k|^{n-4} G_\Omega(\tilde{x}_k + |x_k - y_k|\xi, \tilde{x}_k + |x_k - y_k|\eta)$$

for

$$\xi, \eta \in \Omega_k := \frac{1}{|x_k - y_k|}(-\tilde{x}_k + \Omega).$$

Since $\tilde{x}_k \to 0$, the exterior unit normal at $\partial\Omega$ converges to the first unit vector and so we conclude that

$$\Omega_k \to \mathcal{H} := \{x : x_1 < 0\} \text{ locally uniformly for } k \to \infty.$$

It was proved in [10, Lemma 7] that locally uniformly in $\mathcal{H}$

$$G_k(\xi, \eta) \to G_{\mathcal{H}}(\xi, \eta) = \frac{1}{4ne_n}|\xi - \eta|^{4-n} \int_1^{|\xi^* - \eta|/|\xi - \eta|} (v^2 - 1)v^{1-n} \, dv,$$

where $\xi^* = (-\xi_1, \xi_2, \ldots, \xi_n)$ and e_n is the $n-$dimensional volume of $B_1(0) \subset \mathbb{R}^n$. We remark that this step and in particular the required uniqueness proof for $G_{\mathcal{H}}$ were carried out in Grunau-Robert [10] using the assumption $n \geq 3$. The necessary modifications for $n = 2$ are emphasized in the proofs of Lemmata 7 and 6. Assumption (9) gives

$$(10) \qquad G_k(\xi_k, \eta_k) = |x_k - y_k|^{n-4}G_\Omega(x_k, y_k) < \frac{1}{k}|x_k - y_k|^{-4}d(x_k)^2 d(y_k)^2,$$

where

$$\xi_k = \frac{1}{|x_k - y_k|}(x_k - \tilde{x}_k), \quad \eta_k = \frac{1}{|x_k - y_k|}(y_k - \tilde{x}_k),$$

$$|\xi_k| = \frac{d(x_k)}{|x_k - y_k|} \leq \frac{1}{\delta_n}, \quad |\xi_k - \eta_k| = 1.$$

After passing to a further subsequence we find $\xi, \eta \in \overline{\mathcal{H}}$ with $\xi = \lim_{k\to\infty} \xi_k$, $\eta = \lim_{k\to\infty} \eta_k$. In view of the local smooth convergence of G_k to the biharmonic

Green function $G_{\mathcal{H}}$ in the half space $\mathcal{H}$, Boggio's formula and $|\xi - \eta| = 1$ there exists a positive constant $\sigma > 0$ such that for k large enough:

$$G_k(\xi_k, \eta_k) \geq \sigma d(\xi_k)^2 d(\eta_k)^2 = \sigma \left(\frac{d(x_k)}{|x_k - y_k|} \right)^2 \left(\frac{d(y_k)}{|x_k - y_k|} \right)^2$$
$$= \sigma |x_k - y_k|^{-4} d(x_k)^2 d(y_k)^2.$$

This contradicts (10) and the proof of the lemma is complete. $\qquad\square$

3. Proof of the main estimate for $n \geq 3$

Supposing that (6) holds, i.e. $|x - y| \leq \delta_n \max\{d(x), d(y)\}$, one finds that even $(1 - \delta_n)|x - y| \leq \delta_n \min\{d(x), d(y)\}$ and hence

$$\frac{d(x)d(y)}{|x - y|^2} \geq \frac{1 - \delta_n}{\delta_n^2}.$$

So in that case the estimate from below in Theorem 1 follows directly from Proposition 3. Hence, we may assume from now on, again, that x, y are subject to condition (7). Applying a compactness argument to

$$\overline{\Omega} = \bigcup_{x_0 \in \overline{\Omega}} \Omega_{x_0, r_{x_0}/2}$$

we see that there exist positive numbers $r > 0$, $c_4 > 0$, such that $|x - y| \leq r$ implies that $G_\Omega(x, y) \geq c_4 |x - y|^{-n} d(x)^2 d(y)^2$. If $|x - y| \geq r$ we take from [5], cf. also [9], that $G_\Omega(x, y) \geq -c_5 |x - y|^{-n} d(x)^2 d(y)^2$ so that

$$G_\Omega(x, y) + 2c_5 |x - y|^{-n} d(x)^2 d(y)^2 \geq c_5 |x - y|^{-n} d(x)^2 d(y)^2.$$

Since $|x - y|^{-n} \leq r^{-n}$ we end up with

$$G_\Omega(x, y) + c_6 d(x)^2 d(y)^2 \geq c_5 |x - y|^{-n} d(x)^2 d(y)^2$$

and positive constants $c_5, c_6 > 0$ in this case. The proof of Theorem 1 for $n \geq 3$ is complete. $\qquad\square$

4. Auxiliary results for $n = 2$

Lemma 5. *Let $\mathcal{H} = \{x \in \mathbb{R}^2 : x_1 < 0\}$ and let $\tilde{G} \in C^4(\overline{\mathcal{H}} \times \overline{\mathcal{H}} \setminus \{(x, x) : x \in \overline{\mathcal{H}}\})$ be a biharmonic Green function with Dirichlet boundary condition, that is*

$$\int_{\mathcal{H}} \tilde{G}(x, .)\Delta^2 \varphi \, dy = \varphi(x) + \int_{\partial \mathcal{H}} \left(\Delta\tilde{G}(x, .)\partial_\nu \varphi - \varphi \partial_\nu \Delta\tilde{G}(x, .) \right) d\sigma$$

for all $\varphi \in C_c^4(\overline{\mathcal{H}})$ and $x \in \Omega$. Moreover, we assume that $\tilde{G}(x, y) = \tilde{G}(y, x)$ for all $x \neq y$ and that a growth condition holds at infinity:

$$(11) \qquad |\tilde{G}(x, y)| \leq C \left(1 + |x|^2 + |y|^2 \right) \left(1 + (\log |x|)_+ + (\log |y|)_+ \right).$$

Then $\tilde{G}$ is uniquely determined and given by Boggio's formula [2]:

$$\tilde{G}(x, y) = G_{\mathcal{H}}(x, y) = \frac{1}{8\pi} |x - y|^2 \int_1^{|x^* - y|/|x - y|} \frac{v^2 - 1}{v} \, dv.$$

with $x^* = (-x_1, x_2)$.

Proof. We choose some arbitrary $x \in \mathcal{H}$ and keep it fixed in what follows. We write

$$\tilde{G}(x, y) = G_{\mathcal{H}}(x, y) + H(x, y),$$

where H is a regular function in $\overline{\mathcal{H}} \times \overline{\mathcal{H}}$ and $\Delta_y^2 H(x,.) \equiv 0$ in $\mathcal{H}$, $H(x,y) = \partial_{y_1} H(x,y) = 0$ for $y_1 = 0$. According to [8, 14], or by checking directly, with $y^* = (-y_1, y_2)$,

$$
H^*(x,y) := \begin{cases} H(x,y) & \text{if } y_1 \leq 0, \\ -H(x,y^*) - 2y_1 \left(\dfrac{\partial H}{\partial y_1} \right)(x,y^*) - y_1^2 \left(\Delta_y H \right)(x,y^*) & \text{if } y_1 > 0, \end{cases}
$$

satisfies $H^*(x,.) \in C^4 \left(\mathbb{R}^2 \right)$ and is biharmonic on $\mathbb{R}^2$. Since $G_{\mathcal{H}}(x,y)$ satisfies (11), so does $H(x,y)$. So we have

$$
|H(x,y)| \leq C_x \left(1 + |y|^2 \left(\log |y| \right)_+ \right).
$$

Using local elliptic estimates and their scaling behaviour for biharmonic functions satisfying Dirichlet boundary conditions on $\partial \mathcal{H}$, that is,

$$
\|D^\alpha H(x,.)\|_{L^\infty(B_R \cap \mathcal{H})} \leq \frac{C}{R^{|\alpha|}} \|H(x,.)\|_{L^\infty(B_{2R} \cap \mathcal{H})}
$$

we find that

$$
|\nabla_y H(x,y)| \leq C_x \left(1 + |y| \left(\log |y| \right)_+ \right), \qquad |\nabla_y^2 H(x,y)| \leq C_x \left(1 + \left(\log |y| \right)_+ \right).
$$

Having these estimates for $H(x,y)$ and these first two derivatives, we have an estimate for $H^*(x,y)$ and may repeat the above arguments to find similar estimates for the derivatives of $H^*(x,y)$ and even

$$
|\nabla_y^3 H^*(x,y)| \leq C_x \frac{\left(1 + \left(\log |y| \right)_+ \right)}{1 + |y|}.
$$

The maximum principle applied to the harmonic function $\nabla_y \Delta_y H^*(x,.)$ shows that

$$
\|\nabla_y \Delta_y H^*(x,.)\|_{C^0(\overline{B_R(0)})} \leq C \frac{\left(1 + |\log |R|| \right)}{1 + |R|}.
$$

Letting $R \to \infty$ yields

$$
\nabla_y \Delta_y H^*(x,.) \equiv 0, \qquad \Delta_y H^*(x,.) = a(x)
$$

with a suitable function $a(.)$. This shows that any $\nabla_y^3 H^*(x,.)$ is harmonic and, as shown above, $\nabla_y^3 H^*(x,y) \to 0$ as $y \to \infty$. Hence, any $\nabla_y^3 H^*(x,.) \equiv 0$ and by Taylor's formula and observing the boundary data we conclude that

$$
H^*(x,y) = b(x) y_1^2
$$

with a suitable function $b(.)$. By symmetry $H(x,y) = H(y,x)$ and so $H(x,y) = b(y) x_1^2 = b(x) y_1^2 = b x_1^2 y_1^2$, where $b \in \mathbb{R}$ is a suitable constant. Finally, the growth condition leads to $b = 0$, $H(x,.) \equiv 0$, and $\tilde{G} = G_{\mathcal{H}}$. $\qquad\square$

Lemma 6 (Estimates near the boundary). *Suppose that $n = 2$ and that $\Omega \subset \mathbb{R}^2$ is a bounded $C^{4,\gamma}$-smooth domain. Then for each $x_0 \in \partial \Omega$ there exists a radius $r = r_{x_0} > 0$ and a constant $C = C_{x_0} > 0$ such that for all $x,y \in \Omega_{x_0,r} := \overline{\Omega} \cap B_r(x_0)$ one has*

$$
\tag{12} G_\Omega(x,y) \geq C d(x) d(y) \min \left\{ 1, \frac{d(x) d(y)}{|x-y|^2} \right\}.
$$

Proof. We assume by contradiction that there exist $x_k, y_k \in \Omega_{x_0, 1/k} = \overline{\Omega} \cap B_{1/k}(x_0)$ such that

$$
\tag{13} G_\Omega(x_k, y_k) < \frac{1}{k} d(x_k) d(y_k) \min \left\{ 1, \frac{d(x_k) d(y_k)}{|x_k - y_k|^2} \right\}.
$$

In particular we have $x_k \to x_0, y_k \to x_0, d(x_k), d(y_k) \to 0, |x_k - y_k| \to 0$. Without loss of generality we may assume that $x_0 = 0$ and that the first unit vector $\vec{e}_1$ is

the exterior unit normal to $\partial\Omega$ at x_0. After possibly passing to a subsequence it is enough to consider one of the following two cases.

First case: $|x_k - y_k| \geq \frac{1}{2}\max\{d(x_k), d(y_k)\}$.
This proof is as above for Lemma 4; only proving the required uniform bounds for the G_k is slightly more involved. The arguments are sketched below in the second case. Thanks to Lemma 5 the convergence proof of [10, Lemma 7] can be extended to $n = 2$. One should observe that also the symmetry carries over to the limit.

Second case: $|x_k - y_k| < \frac{1}{2}\max\{d(x_k), d(y_k)\}$.
Observe that in this case

$$d(x_k) < 2d(y_k) < 4d(x_k)$$

and

$$|x_k - y_k| < \min\{d(x_k), d(y_k)\}.$$

The assumption gives that

$$(14) \qquad G_\Omega(x_k, y_k) < \frac{1}{k}d(x_k)d(y_k)$$

In this case we rescale differently, however, $\tilde{x}_k \in \partial\Omega$ denotes again the closest boundary point to x_k. We introduce the rescaled biharmonic Green functions

$$G_k(\xi, \eta) := d(x_k)^{-2}G_\Omega(\tilde{x}_k + d(x_k)\xi, \tilde{x}_k + d(x_k)\eta)$$

for

$$\xi, \eta \in \Omega_k := \frac{1}{d(x_k)}\left(-\tilde{x}_k + \Omega\right).$$

Since $\tilde{x}_k \to 0$, the exterior unit normal at $\partial\Omega$ converges to the first unit vector and so we conclude that

$$\Omega_k \to \mathcal{H} := \{x : x_1 < 0\} \text{ locally uniformly for } k \to \infty.$$

For

$$\xi_k = \frac{1}{d(x_k)}(x_k - \tilde{x_k}), \qquad \eta_k = \frac{1}{d(x_k)}(y_k - \tilde{x_k})$$

the assumption (14) transforms into

$$(15) \qquad G_k(\xi_k, \eta_k) < \frac{1}{k}d_k(\xi_k)d_k(\eta_k) < \frac{2}{k},$$

where $d_k := d(\,.\,, \partial\Omega_k)$. Since ξ_k, η_k are bounded and their boundary distances are uniformly bounded from below by $1/2$ we find after passing to a further subsequence that $\xi_k \to \xi_\infty \in \mathcal{H}, \eta_k \to \eta_\infty \in \mathcal{H}$. We claim that we have local uniform convergence in $\mathcal{H} \times \mathcal{H}$ (including the diagonal) of G_k to $G_\mathcal{H}$, since we are in dimension $n = 2$. To see this we observe first that Krasovskiĭ-type estimates (see [15, 16] and also [9, Theorem 4.20]) yield at a first instance useful information only for the third derivatives. We have

$$|\nabla^3_{\xi,\eta}G_k(\xi, \eta)| \leq \frac{C}{|\xi - \eta|} \text{ uniformly in } k.$$

Making use of

$$\forall \xi \in \partial\Omega_k, \eta \in \Omega_k : \quad \nabla_\xi\nabla_\eta G_k(\xi, \eta) = 0, \quad \nabla^2_\eta G_k(\xi, \eta) = 0$$

and of

$$\forall \xi \in \Omega_k, \eta \in \partial\Omega_k : \quad \nabla_\xi\nabla_\eta G_k(\xi, \eta) = 0, \quad \nabla^2_\xi G_k(\xi, \eta) = 0$$

we obtain upon integration that

$$|\nabla^2_{\xi,\eta}G_k(\xi, \eta)| \leq C(1 + (\log|\xi|)_+ + (\log|\eta|)_+ + |\log|\xi - \eta||) \text{ uniformly in } k$$

and further that

$$|\nabla_{\xi,\eta} G_k(\xi,\eta)| \leq C(1 + (\log|\xi|)_+ + (\log|\eta|)_+)(1 + |\xi| + |\eta|) \text{ uniformly in } k;$$

$$|G_k(\xi,\eta)| \leq C(1 + (\log|\xi|)_+ + (\log|\eta|)_+)(1 + |\xi|^2 + |\eta|^2) \text{ uniformly in } k.$$

Now, one may proceed further as in [10, Lemma 7]. So, (15) yields $G_{\mathcal{H}}(\xi_\infty, \eta_\infty) \leq 0$, while Boggio's formula shows that $G_{\mathcal{H}}(\xi,\eta) > 0$ (even if $\xi = \eta$) since $\xi, \eta \in \mathcal{H}$ are interior points. $\square$

Lemma 7 (Estimates in the interior)**.** *Suppose that $n = 2$ and that $\Omega \subset \mathbb{R}^2$ is a bounded $C^{4,\gamma}$-smooth domain. Then for each $x_0 \in \Omega$ there exists a radius $r = r_{x_0} > 0$ and a constant $C = C_{x_0} > 0$ such that for all $x, y \in \Omega_{x_0,r} := \overline{\Omega} \cap B_r(x_0)$ one has*

$$(16) \qquad G_\Omega(x,y) \geq C d(x) d(y) \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}.$$

Proof. Since $x_0 \in \Omega$, we have that $G_\Omega(x_0, x_0) > 0$ (see [17, p. 115]): since G_Ω is continuous, there exists $r, c > 0$ such that $B_r(x_0) \subset\subset \Omega$ and $G_\Omega(x,y) > c$ for all $x, y \in B_r(x_0)$. This yields (16) since Ω is bounded. $\square$

5. Proof of the main estimate for $n = 2$

Combining Lemmas 6 and 7 and applying a compactness argument to

$$\overline{\Omega} = \bigcup_{x_0 \in \overline{\Omega}} \Omega_{x_0, r_{x_0}/2}$$

we find:

Corollary 8. *Suppose that $n = 2$ and that $\Omega \subset \mathbb{R}^2$ is a bounded $C^{4,\gamma}$-smooth domain. Then there exists a radius $r > 0$ and a constant $C > 0$ such that for all $x_0 \in \overline{\Omega}$ and for all $x, y \in \Omega_{x_0,r} := \overline{\Omega} \cap B_r(x_0)$ one has*

$$(17) \qquad G_\Omega(x,y) \geq C d(x) d(y) \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}.$$

This step provides in particular a different and simpler proof for the local positivity statement from [9, Theorem 6.15] which was proved first by Dall'Acqua, Meister, and Sweers [4].

Proof of Theorem 1: If $|x - y| \geq r$ we take from [5], see also [9], that $G_\Omega(x,y) \geq -c_8|x-y|^{-2}d(x)^2 d(y)^2$ so that

$$G_\Omega(x,y) + 2c_8|x-y|^{-2}d(x)^2 d(y)^2 \geq c_8|x-y|^{-2}d(x)^2 d(y)^2$$

$$\geq c_8 d(x)d(y) \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}.$$

Since $|x - y|^{-2} \leq r^{-2}$ we end up with

$$G_\Omega(x,y) + c_9 d(x)^2 d(y)^2 \geq c_8 d(x)d(y) \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}.$$

and positive constants $c_8, c_9 > 0$ in this case. The proof of Theorem 1 is now complete also for $n = 2$ using Corollary 8. $\square$

6. Proof of Theorem 2

This theorem was proved in [10] when $n \geq 3$ and in [9, Theorem 6.15] when $n = 2$. The latter proof is quite involved and based on the extensive use of conformal maps and explicit Green functions in certain limaçons. We provide here an alternate proof which uses the same techniques for $n = 2$ as for $n \geq 3$.

Case I: $d(x)d(y) \leq |x - y|^2$. For this situation we have

$$(18) \qquad H_\Omega(x, y) = \begin{cases} |x - y|^{-n} d(x)^2 d(y)^2 & \text{if } n \neq 4, \\ \log\left(1 + \dfrac{d(x)^2 d(y)^2}{|x - y|^4}\right) & \text{if } n = 4. \end{cases}$$

Then there is $c > 0$ such that we find $c_2^{-1} H_\Omega(x, y) \geq c_1 d(x)^2 d(y)^2$ for $|x - y| < c$.

Case II: $d(x)d(y) > |x - y|^2$. Now we have

$$(19) \qquad H_\Omega(x, y) = \begin{cases} |x - y|^{4-n} & \text{if } n > 4, \\ \log\left(1 + \dfrac{d(x)^2 d(y)^2}{|x - y|^4}\right) & \text{if } n = 4, \\ d(x)^{2-n/2} d(y)^{2-n/2} & \text{if } n = 2, 3. \end{cases}$$

Since $d(x)d(y)$ is bounded on Ω one finds for $n \geq 4$ the existence of $c > 0$ such that $c_2^{-1} H_\Omega(x, y) \geq c_1 d(x)^2 d(y)^2$ for $|x - y| < c$. For dimension $n = 3$ the argument is more subtle. We fix δ_n as in Proposition 3. Taking $d(x)d(y) < \varepsilon^2 \frac{1+\delta_n}{2}$ for $\varepsilon > 0$ but sufficiently small it follows that $c_2^{-1} H_\Omega(x, y) \geq c_1 d(x)^2 d(y)^2$. It remains to consider $d(x)d(y) \geq \frac{1+\delta_n}{2}\varepsilon^2$. Assume first that $|x - y| < \frac{\delta_n}{2}\varepsilon$ and $d(x) < \frac{1}{2}\varepsilon$. Then $d(x)d(y) < \frac{1+\delta_n}{2}\varepsilon^2$ and we are in the situation just considered. So we are left with $|x - y| < \frac{\delta_n}{2}\varepsilon$ and $d(x) \geq \frac{1}{2}\varepsilon$. Then we may apply Theorem 1 of [12], see also Proposition 3, to find that for $|x - y| < \frac{1}{2}\delta_n \varepsilon \leq \delta_n \max(d(x), d(y))$ it follows that $G_\Omega(x, y) > 0$. For dimension $n = 2$ the result follows directly from Corollary 8.

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