STABILITY OF THE POSITIVITY OF BIHARMONIC GREEN'S FUNCTIONS UNDER PERTURBATIONS OF THE DOMAIN

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Dedicated to Prof. Wolf von Wahl on the occasion of his 65th birthday

ABSTRACT. In general, higher order elliptic equations and boundary value problems like the biharmonic equation or the linear clamped plate boundary value problem do not enjoy neither a maximum principle nor a comparison principle or – equivalently – a positivity preserving property. The problem is rather involved since the clamped boundary conditions prevent the boundary value problem from being written as a system of second order boundary value problems.

On the other hand, the biharmonic Green's function in balls $B \subset \mathbb{R}^n$ under Dirichlet (i.e. clamped) boundary conditions is known explicitly and is positive. Previously it was shown that this property also remains under small regular perturbations of the domain, if $n = 2$. In the present paper, such a stability result is proved for $n \geq 3$.

1. INTRODUCTION

Although simple examples show that strong maximum principles as satisfied e.g. by harmonic functions cannot hold true for solutions of higher order elliptic equations, it is reasonable to ask whether higher order *boundary value problems* may possibly enjoy a *positivity preserving property*. To be specific, we consider the clamped plate boundary value problem:

$$\begin{cases} \Delta^2 u = f \text{ in } \Omega, \\ u_{|\partial\Omega} = \frac{\partial}{\partial\nu} u_{|\partial\Omega} = 0. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain with exterior unit normal ν at $\partial\Omega$, and f is a sufficiently smooth datum. If $n = 2$, the unknown u models the vertical deflection of a horizontally clamped thin elastic plate from the horizontal equilibrium shape under the vertical load f. The boundary conditions $u_{|\partial\Omega} = \frac{\partial}{\partial\nu} u_{|\partial\Omega} = 0$ model the horizontal clamping and are called *Dirichlet boundary conditions*. Throughout the present paper, always these boundary conditions will be considered. We say that the clamped plate boundary value problem enjoys a *positivity preserving property* in Ω, if $f \geq 0$ always implies that also the solution in nonnegative, i.e. $u \geq 0$. Equivalently one may ask whether the corresponding Green's function is nonnegative or even strictly positive.

Boggio [2] (1901) and Hadamard [13] (1908) conjectured that in arbitrary reasonable (two dimensional) domains Ω, the positivity preserving property should hold true. Boggio [3] could show with the help of a beautiful explicit formula that this is indeed the case for balls in $\mathbb{R}^n$, even for the Dirichlet problem for *polyharmonic operators*. In 1909 Hadamard [14] already knew, that the positivity conjecture is false in annuli with small inner radius.

Starting about 40 years later, numerous counterexamples disproved the Boggio-Hadamard conjecture, see e.g. [6, 8, 24]. For a more extensive survey and further references we refer to [10]. Although no positivity preserving property holds true even in general arbitrarily smooth uniformly convex domains, one may ask whether there are at least families of domains enjoying a positivity preserving property.

For two dimensional domains, this question was addressed in [10]. There, it was shown that in domains $\Omega \subset \mathbb{R}^2$ being sufficiently close in C^4-sense to the (unit) disk $B \subset \mathbb{R}^2$, the biharmonic Green's function (under Dirichlet boundary conditions) is positive. Recently, Sassone [20] could relax the assumption on the domains to be close to B in $C^{2,\alpha}$-sense. The authors could take advantage of conformal maps and the Riemann mapping theorem, pulling back the clamped plate boundary value problem to Dirichlet problems in the unit disk with the biharmonic operator as principal part and only with (small) *lower order perturbations*. The latter were treated in $B \subset \mathbb{R}^n$ (n arbitrary) in [11]. The methods of [10], however, do not carry over to higher dimensions due to a lack of sufficiently many conformal maps. So, the question, whether the positivity of the biharmonic Green's function in the unit ball $B \subset \mathbb{R}^n$ is stable under *domain perturbations*, was left open.

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This is precisely the question which we address in the present paper. Assuming $n > 2$, we show that in domains $\Omega \subset \mathbb{R}^n$, which are sufficiently close to the unit ball in a suitable $C^{4,\alpha}$-sense, the biharmonic Green's function under Dirichlet boundary conditions is indeed positive. More precisely, we prove the following theorem:

Theorem 1. *Let B be a unit ball of $\mathbb{R}^n$, $n \geq 3$. Then, there exists $\varepsilon_0 = \varepsilon_0(n) > 0$ such that the following holds true:*

We assume that $\Omega \subset \mathbb{R}^n$ is a $C^{4,\alpha}$-smooth domain, which for some suitable $\varepsilon \in [0, \varepsilon_0]$ is ε-close to the ball B in the $C^{4,\alpha}$-sense, i.e.:

There exists a surjective $C^{4,\alpha}$-map $\psi : \overline{B} \to \overline{\Omega}$ such that $\|Id - \psi\|_{C^{4,\alpha}(\overline{B})} \leq \varepsilon$.

Then, the Green's function H_Ω for Δ^2 in Ω under Dirichlet boundary conditions is strictly positive:

$$\forall x, y \in \Omega, \quad x \neq y : \qquad H_\Omega(x, y) > 0.$$

Assuming ε_0 small enough, this notion of closeness implies that there is a fixed neighbourhood U of $\overline{B}$, $C^{4,\alpha}$-smooth injective extensions $\psi : \overline{U} \to \mathbb{R}^n$, $\|Id - \psi\|_{C^{4,\alpha}(\overline{U})} = O(\varepsilon)$ and $C^{4,\alpha}$-smooth inverse maps $\phi = \psi^{-1} : \psi(\overline{U}) \to \overline{U}$ such that $\psi\left(\overline{B}\right) = \overline{\Omega}$, $\psi(B) = \Omega$.

For $n = 2$, a direct and explicit proof based on perturbation series, Green's function estimates and conformal maps was given in [10, 11]. This means that there, in principle, ε_0 may be calculated explicitly. Moreover, in the case $n = 2$, closeness has to be assumed only in a weaker norm, see [20].

Here, the proof is somehow more indirect since a number of proofs by contradiction are involved so that it will be very difficult to calculate ε_0 from our proofs. Furthermore, we have to make extensive use of general elliptic theory as provided by Agmon, Douglis and Nirenberg [1]. We emphasize that Theorem 1 is by no means just a continuous dependence on data result, since the involved Green's "functions" are not simply functions but families of functions depending on the position of the singularity. The problem consists in gaining uniformity with respect to the position of the singularity, in particular when it approaches the boundary.

We are confident that using precisely the same techniques will allow for proving a result like Theorem 1 also for polyharmonic operators $(-\Delta)^m$ under boundary conditions such that the corresponding Green's function in the ball is positive and does not display any kind of degeneracies (additional zeros) on the boundary. In particular, we expect Dirichlet boundary conditions to be covered. The proof, however, will become more and more technical; in particular the dimensions $2 < n \leq 2m$ will require a separate discussion.

We are optimistic that Theorem 1 may help in order to show that in general domains, where the Green's function is sign changing, the negative part, however, will turn out to be "small" when compared with the positive part. First steps in this direction have been done in [4, 5, 12, 18]. In [5] a family of nonconvex smooth domains with positive Green's function was studied, while in [4] it was shown, that in smooth two dimensional domains, one has a more restrictive bound for the (negative part of the) Green's function from below than (for the positive part) from above. In [12, 18] it was shown in a quantitative way, that in general domains $\Omega \subset \mathbb{R}^n$, $n \geq 3$, the biharmonic Green's function is positive *around the singularity.*

2. A more general result

In order to prove Theorem 1, below in Theorem 2 we describe the possible situations how transition from positivity to sign change may occur within a smooth family of domains. It is then easy to see that none of these situations occurs in the (unit) ball in $\mathbb{R}^n$, $n > 2$.

To provide a more flexible result in Theorem 2, we will also include lower order perturbations. The formulation is somehow technical and requires in particular the notion of smooth domain perturbations, which we make precise in the following definition.

Definition 1. *Let Ω, $(\Omega_k)_{k\in\mathbb{N}}$ be domains of $\mathbb{R}^n$. We say that $(\Omega_k)_{k\in\mathbb{N}}$ is a* $C^{4,\alpha}$-smooth perturbation *of the bounded $C^{4,\alpha}$-smooth domain Ω, and we write*

$$\lim_{k\to+\infty} \Omega_k = \Omega$$

if the following facts are satisfied:

(i) *There exist $N \in \mathbb{N}$, $p_1, \dots, p_N \in \partial\Omega$, $\delta > 0$ and open subsets $\omega \subset\subset \Omega$, $\omega \subset\subset \omega_0 \subset\subset \Omega_k$ such that*

$$\Omega \subset \omega \cup \bigcup_{i=1}^N B_\delta(p_i); \qquad \Omega_k \subset \omega \cup \bigcup_{i=1}^N B_\delta(p_i);$$

(ii) *for any $i \in \{1, \dots, N\}$, there exists an open subset of $U_i \subset \mathbb{R}^n$ such that $0 \in U_i$, and a $C^{4,\alpha}$-smooth diffeomorphism $\Phi_i : U_i \to B_{2\delta}(p_i)$ such that $\Phi_i(0) = p_i$ and*

$$\Phi_i(U_i \cap \{x_1 < 0\}) = \Phi_i(U_i) \cap \Omega\ ,\ \Phi_i(U_i \cap \{x_1 = 0\}) = \Phi_i(U_i) \cap \partial\Omega;$$

(iii) *for any* $i \in \{1, \dots, N\}$ *and* $k \in \mathbb{N}$, *there exists* $\Phi_{k,i} : U_i \to B_{2\delta}(p_i)$ *such that* $\Phi_{k,i}(U_i)$ *is an open subset,* $B_{3\delta/2}(p_i) \subset \Phi_{k,i}(U_i)$ *and* $\Phi_{k,i} : U_i \to \Phi_{k,i}(U_i)$ *is a* $C^{4,\alpha}$*-smooth diffeomorphism and*

$$\Phi_{k,i}(U_i \cap \{x_1 < 0\}) = \Phi_{k,i}(U_i) \cap \Omega_k \ , \ \Phi_{k,i}(U_i \cap \{x_1 = 0\}) = \Phi_{k,i}(U_i) \cap \partial\Omega_k;$$

(iv) *for any* $i \in \{1, \dots, N\}$, $\lim_{k\to+\infty} \Phi_{k,i} = \Phi_i$ *in* $C^{4,\alpha}_{loc}(U_i)$.

This definition implies that we have a well defined smooth exterior normal vector field so that Ω, Ω_k, $\partial\Omega$ and $\partial\Omega_k$ carry a canonical orientation. In what follows, the local charts will be chosen such that this orientation is observed, i.e. such that Jac $\Phi_i \circ \Phi_j^{-1} > 0$, Jac $\Phi_{k,i} \circ \Phi_{k,j}^{-1} > 0$.

This definition covers in particular the following more special situation of smooth domain perturbation, which we make use of in proving Theorem 1: Let a sequence of mappings $(\psi_k)_{k\in\mathbb{N}}$ be such that there exists an open subset of $U \subset \mathbb{R}^n$ and $\psi_k : U \to \mathbb{R}^n$ for all $k \in \mathbb{N}$. We assume that $\lim_{k\to+\infty} \psi_k = Id$ in $C^{4,\alpha}_{loc}(U)$. Let $\Omega \subset\subset U$ be a $C^{4,\alpha}$-smooth bounded subset of $\mathbb{R}^n$ and let $\Omega_k := \psi_k(\Omega)$ for all $k \in \mathbb{N}$. Then the sequence $(\Omega_k)_{k\in\mathbb{N}}$ is a smooth perturbation of Ω.

Basing upon the notion of smooth domain perturbation we are now able to formulate our main result (where Theorem 1 is a direct consequence of):

Theorem 2. *Let* $n \geq 3$, *and* $(\Omega_k)_{k\in\mathbb{N}}$ *be a* $C^{4,\alpha}$*-smooth perturbation of the bounded* $C^{4,\alpha}$*-smooth domain* Ω *in the sense of Definition 1. We consider a sequence* $(a_k)_{k\in\mathbb{N}} \in C^{0,\alpha}(U_0)$, *where* $\Omega \subset\subset U_0$ *and assume that there exists* $a_\infty \in C^{0,\alpha}(U_0)$ *such that*

$$\lim_{k\to+\infty} a_k = a_\infty \ \text{in} \ C^{0,\alpha}_{loc}(U_0).$$

We assume further that there exists $\lambda > 0$ *such that*

$$\int_{\Omega_k} \left((\Delta\varphi)^2 + a_k\varphi^2\right) dx \geq \lambda \int_{\Omega_k} \varphi^2 \, dx$$

for all $\varphi \in C^\infty_c(\Omega_k)$ *and all* $k \in \mathbb{N}$. *Let* G_k *be the Green's function of* $\Delta^2 + a_k$ *on* Ω_k, *and* G *be the Green's function of* $\Delta^2 + a_\infty$ *on* Ω, *all with Dirichlet boundary conditions.*

Finally, we suppose that there exist two sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}$ *such that* $x_k, y_k \in \Omega_k$ *and*

$$G_k(x_k, y_k) = 0 \ \text{for all} \ k \in \mathbb{N}.$$

Up to a subsequence, let $x_\infty := \lim_{k\to+\infty} x_k$ *and* $y_\infty := \lim_{k\to+\infty} y_k$. *Then* $x_\infty, y_\infty \in \overline{\Omega}$, $x_\infty \neq y_\infty$ *and we are in one of the following situations:*

(i) *if* $x_\infty, y_\infty \in \Omega$, *then* $G(x_\infty, y_\infty) = 0$;
(ii) *if* $x_\infty \in \Omega$ *and* $y_\infty \in \partial\Omega$, *then* $\Delta_y G(x_\infty, y_\infty) = 0$;
(iii) *if* $x_\infty \in \partial\Omega$ *and* $y_\infty \in \Omega$, *then* $\Delta_x G(x_\infty, y_\infty) = 0$;
(iv) *if* $x_\infty, y_\infty \in \partial\Omega$, *then* $\Delta_x\Delta_y G(x_\infty, y_\infty) = 0$.

In the above statement, $\Delta_x G$ denotes the Laplacian with respect to the first variables, and $\Delta_y G$ denotes the Laplacian with respect to the second variables.

More general lower order "self adjoint" perturbations of the biharmonic operator may be covered by precisely the same techniques. However, here we prefer to stick to a relatively simple situation in order to avoid too many technical details.

In the one dimensional context (clamped bars), related and quite concrete results were obtained by Schröder [21, 22, 23].

Throughout the paper we assume that

$$n \geq 3.$$

A first essential step in proving Theorem 2 consists in providing uniform bounds (in k) for the Green's functions like

$$|G_k(x,y)| \leq C \cdot \begin{cases} |x-y|^{4-n}, & \text{if } n > 4, \\ (1 + |\log|x-y||), & \text{if } n = 4, \\ 1, & \text{if } n = 3. \end{cases} \tag{2.1}$$

Moreover, if $n = 3, 4$, the somehow irregular estimates for G_k require to focus first on the gradients, where estimates like

$$|\nabla G_k(x,y)| \leq C \cdot \begin{cases} |x-y|^{-1}, & \text{if } n = 4, \\ 1, & \text{if } n = 3, \end{cases} \tag{2.2}$$

are available, which are compatible with the scaling arguments performed below. In this respect, the proof is more difficult in dimensions $n = 3$ and in particular $n = 4$. With $C = C(\Omega_k)$, the estimates (2.1) and (2.2) are due to Krasovskiĭ [17], while we prove uniformity in Ω_k in Theorem 3 in Section 4 below. Preliminary properties of the Green's functions are shown in Section 3, while Section 5 is devoted to convergence properties of families of Green's functions in $(\Omega_k)_{k\in\mathbb{N}}$. The proofs of Theorems 2 and 1 are finally given in Section 6.

Notation. Straightening the boundary requires to work in $\mathbb{R}^n_- := \{x \in \mathbb{R}^n : x_1 < 0\}$, where we write $\mathbb{R}^n \ni x = (x_1, \bar{x})$. e_n denotes the n-dimensional volume of $B_1(0) \subset \mathbb{R}^n$.

3. The Green's function G for the perturbed biharmonic operator

In the first part of this section, we consider a fixed operator $\Delta^2 + a$ in a fixed smooth domain and construct and investigate the corresponding Green's function.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{4,\alpha}$-smooth domain and $a \in C^{0,\alpha}\left(\overline{\Omega}\right)$. We assume that $\Delta^2 + a$ is coercive. Then, for every $x \in \Omega$, there exists a unique Green's function $G_x \in L^1(\Omega) \cap C^{4,\alpha}\left(\overline{\Omega} \setminus \{x\}\right)$ such that $G_x|_{\partial\Omega} = \frac{\partial G_x}{\partial\nu}|_{\partial\Omega} = 0$ and that for all $\varphi \in C^4\left(\overline{\Omega}\right)$ with $\varphi|_{\partial\Omega} = \frac{\partial\varphi}{\partial\nu}|_{\partial\Omega} = 0$ one has the following representation formula:*

$$\varphi(x) = \int_\Omega G_x(y)\left(\Delta^2\varphi(y) + a(y)\varphi(y)\right)\,dy. \tag{3.1}$$

If $R > 0$ is such that $\Omega \subset B_R(0)$ and $\lambda > 0$ such that $\forall \varphi \in W^{2,2}_0(\Omega):\ \int_\Omega \left((\Delta\varphi)^2 + a\varphi^2\right)\,dy \geq \lambda \int_\Omega \varphi^2\,dy$ then, the following estimate for the Green's function holds true:

$$|G_x(y)| \leq C(\lambda, R, n, \|a\|_{C^{0,\alpha}}, \Omega) \cdot \begin{cases} \left(|x-y|^{4-n} + \max\{d(x,\partial\Omega), d(y,\partial\Omega)\}^{4-n}\right), & \text{if } n > 4, \\ 1 + |\log|x-y|| + |\log\left(\max\{d(x,\partial\Omega), d(y,\partial\Omega)\}\right)|, & \text{if } n = 4, \\ 1, & \text{if } n = 3. \end{cases} \tag{3.2}$$

If $n = 3, 4$, we further prove the following gradient estimates:

$$|\nabla_{(x,y)} G_x(y)| \leq C(\lambda, R, n, \|a\|_{C^{0,\alpha}}, \Omega) \cdot \begin{cases} \left(|x-y|^{-1} + \max\{d(x,\partial\Omega), d(y,\partial\Omega)\}^{-1}\right), & \text{if } n = 4, \\ 1, & \text{if } n = 3. \end{cases} \tag{3.3}$$

The dependence of the constants C on Ω is explicit via the $C^{4,\alpha}$-properties of $\partial\Omega$.

Proof. We first prove extensively the generic case $n > 4$. At the end we comment on the changes and additional arguments which have to be made, if $n = 4$ or $n = 3$.

Case $n > 4$. We introduce the fundamental solution Γ_0 of the biharmonic operator

$$\Gamma_0(x,y) := \frac{1}{2(n-4)(n-2)ne_n}|x-y|^{4-n} \tag{3.4}$$

so that $\Gamma_0 \in C^\infty\left(\overline{\Omega} \times \overline{\Omega}\right) \setminus \{(x,y) : x = y\}$. We define recursively for $j \geq 0$

$$\Gamma_{j+1}(x,y) := -\int_\Omega \Gamma_j(x,z)a(z)\Gamma_0(z,y)\,dz$$

and have that $\Gamma_j \in C^{4,\alpha}\left(\overline{\Omega} \times \overline{\Omega} \setminus \{(x,y) : x = y\}\right)$ is well defined and, according to a Lemma of Giraud [9], that for $j \geq 1$

$$|\Gamma_j(x,y)| \leq \begin{cases} C_j|x-y|^{4(j+1)-n}, & \text{if } (j+1) < \frac{n}{4}, \\ C_j\left(1 + |\log|x-y||\right), & \text{if } (j+1) = \frac{n}{4}, \\ C_j, & \text{if } (j+1) > \frac{n}{4}. \end{cases} \tag{3.5}$$

Here, $C_j = C_j\left(n, R, \|a\|_\infty\right)$, where $R > 0$ is chosen such that $\Omega \subset B_R(0)$. We fix some $\ell > \frac{n}{4}$, $x \in \Omega$ and for $u_x \in C^{4,\alpha}\left(\overline{\Omega}\right)$ to be suitably determined below, we put

$$G_x(y) := \Gamma_0(x,y) + \sum_{j=1}^{\ell} \Gamma_j(x,y) + u_x(y). \tag{3.6}$$

One should observe that $\sum_{j=0}^\infty \Gamma_j$ is the Neumann-series for the fundamental solution for the perturbed differential operator. We have that $G_x \in C^{4,\alpha}\left(\overline{\Omega} \setminus \{x\}\right)$. In order to calculate the boundary value problem solved by G_x we consider testing functions $\varphi \in C^4\left(\overline{\Omega}\right)$ with $\varphi|_{\partial\Omega} = \frac{\partial\varphi}{\partial\nu}|_{\partial\Omega} = 0$.

$$\begin{aligned}
\int_\Omega G_x(y)\left(\Delta^2\varphi + a\varphi\right)\,dy &= \int_\Omega \Gamma_0(x,y)\Delta^2\varphi(y)\,dy + \sum_{j=1}^{\ell}\int_\Omega \Gamma_j(x,y)\Delta^2\varphi(y)\,dy \\
&\quad + \int_\Omega a(y)\Gamma_0(x,y)\varphi(y)\,dy + \sum_{j=1}^{\ell}\int_\Omega a(y)\Gamma_j(x,y)\varphi(y)\,dy + \int_\Omega u_x(y)\left(\Delta^2\varphi + a\varphi\right)\,dy \\
&= \int_{\partial\Omega}\left(G_x(y)\frac{\partial}{\partial\nu}\Delta\varphi - \frac{\partial}{\partial\nu_y}G_x(y)\Delta\varphi\right)dS(y) + \int_\Omega\left(\Delta^2 u_x(y) + a u_x(y)\right)\varphi\,dy \\
&\quad + \varphi(x) - \sum_{j=1}^{\ell}\int_\Omega a(y)\Gamma_{j-1}(x,y)\varphi(y)\,dy + \int_\Omega a(y)\Gamma_0(x,y)\varphi(y)\,dy + \sum_{j=1}^{\ell}\int_\Omega a(y)\Gamma_j(x,y)\varphi(y)\,dy \\
&= \varphi(x) + \int_{\partial\Omega}\left(G_x(y)\frac{\partial}{\partial\nu}\Delta\varphi - \frac{\partial}{\partial\nu_y}G_x(y)\Delta\varphi\right)dS(y) \\
&\quad + \int_\Omega\left(\Delta^2 u_x(y) + a(y)u_x(y) + a(y)\Gamma_\ell(x,y)\right)\varphi(y)\,dy.
\end{aligned}$$

In order that G_x becomes indeed a Green's function for the Dirichlet problem for $\Delta^2 + a$, i.e. that indeed formula (3.1) is satisfied, we need u_x to be a solution of the following Dirichlet problem

$$\left\{\begin{array}{ll}
\Delta^2 u_x(y) + a(y)u_x(y) = -a(y)\Gamma_\ell(x,y) & \text{in } \Omega \\
u_x(y) = -\Gamma_0(x,y) - \sum_{j=1}^{\ell}\Gamma_j(x,y) & \text{for } y \in \partial\Omega, \\
\frac{\partial}{\partial\nu}u_x(y) = -\frac{\partial}{\partial\nu_y}\Gamma_0(x,y) - \sum_{j=1}^{\ell}\frac{\partial}{\partial\nu_y}\Gamma_j(x,y) & \text{for } y \in \partial\Omega.
\end{array}\right. \tag{3.7}$$

Since $\ell > \frac{n}{4}$, the right hand side $-a\cdot\Gamma_\ell(x,\,.\,)$ is Hölder continuous with Hölder norm bounded by a constant $C(n,R,\|a\|_{C^{0,\alpha}})$. The $C^{1,\alpha}$-norm of the datum for $u_x|_{\partial\Omega}$ and the $C^{0,\alpha}$-norm of the datum for $\frac{\partial}{\partial\nu}u_x|_{\partial\Omega}$ are bounded by $C(n,R,\partial\Omega)d(x,\partial\Omega)^{3-n-\alpha}$. The dependence of the constant C on $\partial\Omega$ is in principle constructive and explicit via its curvature properties and their derivatives. According to $C^{1,\alpha}$–estimates for boundary value problems in variational form like (3.7) – see [1, Thm. 9.3] – we see that

$$\|u_x\|_{C^{1,\alpha}(\overline{\Omega})} \le C(n,R,\lambda,\|a\|_{C^{0,\alpha}},\lambda,\partial\Omega)d(x,\partial\Omega)^{3-n-\alpha}. \tag{3.8}$$

One should observe that the differential operators are uniformly coercive, so that no u_x-terms need to appear on the right-hand-side.

As long as $d(y,\partial\Omega) \le d(x,\partial\Omega)$, (3.8) shows that

$$|u_x(y)| \le C(C_0,n,R,\lambda,\|a\|_{C^{0,\alpha}},\lambda,\partial\Omega)d(x,\partial\Omega)^{4-n}$$

and hence

$$|G_x(y)| \le C(C_0,n,R,\lambda,\|a\|_{C^{0,\alpha}},\lambda,\partial\Omega)\left(|x-y|^{4-n} + d(x,\partial\Omega)^{4-n}\right). \tag{3.9}$$

If $d(y,\partial\Omega) > d(x,\partial\Omega)$ we conclude from (3.9) by exploiting the symmetry of the Green's function:

$$|G_x(y)| = |G_y(x)| \le C(C_0,n,R,\lambda,\|a\|_{C^{0,\alpha}},\lambda,\partial\Omega)\left(|x-y|^{4-n} + d(y,\partial\Omega)^{4-n}\right). \tag{3.10}$$

Combining (3.9) and (3.10) yields (3.2) for $n > 4$.

Case $n = 4$. Here the fundamental solution we start with is

$$\Gamma_0(x,y) := -\frac{1}{16e_4}\log|x-y|. \tag{3.11}$$

We proceed with the iterated kernels Γ_j. In view of the mild singularity of Γ_0, however, it is sufficient to choose $\ell = 1$. As above we find that

$$\|u_x\|_{C^{1,\alpha}(\overline{\Omega})} \le C(R,\lambda,\|a\|_{C^{0,\alpha}},\lambda,\partial\Omega)d(x,\partial\Omega)^{-1-\alpha}. \tag{3.12}$$

As long as $d(y,\partial\Omega) \le d(x,\partial\Omega)$, (3.12) shows that

$$|\nabla_y G_x(y)| \le C(C_0,R,\lambda,\|a\|_{C^{0,\alpha}},\lambda,\partial\Omega)\left(|x-y|^{-1} + d(x,\partial\Omega)^{-1}\right). \tag{3.13}$$

In order to exploit the symmetry of $G_x(y)$ we need a similar estimate also for $|\nabla_x G_x(y)|$. To this end one has to differentiate (3.7) with respect to the parameter (!) x and obtains as before that for $d(y,\partial\Omega) \le d(x,\partial\Omega)$

$$|\nabla_x G_x(y)| \le C(C_0,R,\lambda,\|a\|_{C^{0,\alpha}},\lambda,\partial\Omega)\left(|x-y|^{-1} + d(x,\partial\Omega)^{-1}\right). \tag{3.14}$$

By symmetry $G_x(y) = G_y(x)$, and (3.14) shows that for $d(x,\partial\Omega) \le d(y,\partial\Omega)$, one has

$$|\nabla_y G_x(y)| \le C(C_0, R, \lambda, \|a\|_{C^{0,\alpha}}, \lambda, \partial\Omega)\left(|x-y|^{-1} + d(y,\partial\Omega)^{-1}\right) \tag{3.15}$$

while (3.13) yields

$$|\nabla_x G_x(y)| \le C(C_0, R, \lambda, \|a\|_{C^{0,\alpha}}, \lambda, \partial\Omega)\left(|x-y|^{-1} + d(y,\partial\Omega)^{-1}\right). \tag{3.16}$$

Combining (3.13)-(3.16) proves (3.3) and hence (3.2) in the case $n = 4$.

Case $n = 3$. Here, we simply work with the bounded Lipschitz continuous fundamental solution

$$\Gamma_0(x,y) := -\frac{1}{8\pi}|x-y| \tag{3.17}$$

so that no iterative procedure is needed and we may directly work with $\ell = 0$. One comes up with

$$\|u_x\|_{C^{1,\alpha}(\overline{\Omega})} \le C(R, \lambda, \|a\|_{C^{0,\alpha}}, \lambda, \partial\Omega) d(x,\partial\Omega)^{-\alpha}. \tag{3.18}$$

Proceeding as for $n = 4$ yields (3.3) and hence (3.2) also in the case $n = 3$. □

Let us now show that assuming certain uniform estimates on the Green's functions H_k for the biharmonic operators on a family (Ω_k) of domains according to Definition 1 implies similar uniform estimates for the Green's functions of the perturbed biharmonic operators $\Delta^2 + a_k$ on Ω_k:

Proposition 3.2. *Let $n \ge 4$ and $(\Omega_k)_{k\in\mathbb{N}}$ be a $C^{4,\alpha}$-smooth perturbation of the bounded $C^{4,\alpha}$-smooth domain Ω according to Definition 1 and $R > 0$ such that $\Omega_k \subset B_R(0)$. Let $H_k \in C^4\left(\overline{\Omega_k} \times \overline{\Omega_k} \setminus \{(x,y) : x = y\}\right)$ denote the Green's functions for Δ^2 in Ω_k and assume that there exists a uniform constant C_1 such that*

$$\forall k \quad \forall x,y \in \Omega_k (x \ne y): \qquad |H_k(x,y)| \le C_1 \cdot \begin{cases} |x-y|^{4-n}, & \text{if } n > 4,\\ (1 + |\log|x-y||), & \text{if } n = 4.\end{cases} \tag{3.19}$$

Let $a_k \in C^{0,\alpha}\left(\overline{\Omega_k}\right)$ and $\Lambda > 0$ such that $\forall k : \|a_k\|_{C^{0,\alpha}(\overline{\Omega_k})} \le \Lambda$ and let $\lambda > 0$ be such that

$$\forall k \quad \forall \varphi \in C_c^\infty(\Omega_k): \qquad \int_{\Omega_k} \left((\Delta\varphi)^2 + a_k\varphi^2\right) dy \ge \lambda \int_{\Omega_k} \varphi^2\, dy.$$

We denote by G_k the Green's functions for $\Delta^2 + a_k$ in Ω_k. Then, there exists a constant $C_2 = C_2(R, C_1, \lambda, \Lambda, \Omega)$ such that one has the following estimate:

$$\forall x,y \in \Omega_k, x \ne y: \qquad |G_k(x,y)| \le C_2 \cdot \begin{cases} |x-y|^{4-n}, & \text{if } n > 4,\\ (1 + |\log|x-y||), & \text{if } n = 4.\end{cases} \tag{3.20}$$

Moreover, assuming

$$\forall x,y \in \Omega_k, x \ne y: \qquad |\nabla_{(x,y)} H_k(x,y)| \le C_1\ |x-y|^{-1}, \qquad \text{if } n = 4, \tag{3.21}$$

in dimension $n = 4$ implies that

$$\forall x,y \in \Omega_k, x \ne y: \qquad |\nabla_{(x,y)} G_k(x,y)| \le C_2\ |x-y|^{-1}, \qquad \text{if } n = 4. \tag{3.22}$$

The dependence on $(\Omega_k)_k$ as regular perturbations of Ω is explicit via the geometric properties of $\partial\Omega$. As long as these properties are uniformly satisfied, the same constant may be chosen.

The case $n = 3$ need not be covered here, since in this case, Proposition 3.1 already provides strong enough information for our purposes.

Proof. We proceed quite similarly as in the proof of Proposition 3.1, but now using the biharmonic Green's functions H_k instead of Γ_0. That means that in Ω_k, we define inductively

$$\Gamma_{k,1}(x,y) := -\int_{\Omega_k} H_k(x,z) a_k(z) H_k(z,y)\, dz;$$

$$\Gamma_{k,j+1}(x,y) := -\int_{\Omega_k} \Gamma_{k,j}(x,z) a_k(z) H_k(z,y)\, dz.$$

Moreover, as above, we make the ansatz with $u_{k,x} \in C^{4,\alpha}(\overline{\Omega_k})$

$$G_k(x,y) := H_k(x,y) + \sum_{j=1}^{\ell} \Gamma_{k,j}(x,y) + u_{k,x}(y). \tag{3.23}$$

We choose $\ell > \frac{n}{4} + 1$ so that

$$|\Gamma_{k,\ell}| \le C(R, n, \Lambda), \qquad |\nabla\Gamma_{k,\ell}| \le C(R, n, \Lambda), \tag{3.24}$$

while for the other Γ_j, we have in particular that

$$|\Gamma_{k,j}(x,y)| \le C(R,n,\Lambda)\cdot \begin{cases} |x-y|^{4-n}, & \text{if } n>4,\\ (1+|\log|x-y||)\,, & \text{if } n=4, \end{cases} \tag{3.25}$$

and assuming (3.21) that

$$\forall x,y\in\Omega_k, x\ne y: \qquad |\nabla_{(x,y)}\Gamma_{k,j}(x,y)| \le C(R,n,\Lambda)\cdot |x-y|^{-1}\,, \quad \text{if } n=4. \tag{3.26}$$

With precisely the same calculations as before we see that G_k is indeed the Green's function for the Dirichlet problem for $\Delta^2 + a_k$ in Ω_k, iff the $u_{k,x}$ solve the following boundary value problems:

$$\begin{cases} \Delta^2 u_{k,x}(y) + a_k(y)u_{k,x}(y) = -a_k(y)\Gamma_{k,\ell}(x,y) & \text{in } \Omega_k\\ u_{k,x}(y) = \frac{\partial}{\partial\nu}u_{k,x} = 0 & \text{for } y\in\partial\Omega_k. \end{cases} \tag{3.27}$$

The right hand side is uniformly bounded, the operators are uniformly coercive. Hence, L^p-theory (see [1]) combined with Sobolev embedding theorems and differentiating (3.27) with respect to the parameter x yields

$$|u_{k,x}(y)| \le C(R,C_1,\lambda,\Lambda,(\Omega_k)_{k\in\mathbb{N}}), \qquad |\nabla_{(x,y)}u_{k,x}(y)| \le C(R,C_1,\lambda,\Lambda,(\Omega_k)_{k\in\mathbb{N}}). \tag{3.28}$$

The dependence on $(\Omega_k)_k$ is uniform in the sense described before the present proof. Inserting (3.24), (3.25), (3.28) and (3.26) into (3.23) proves the claim. ☐

Finally, we need a more precise statement concerning the smoothness of the Green's functions simultaneously with respect to *both* variables.

Proposition 3.3. *Under the assumptions of Proposition 3.2 we have in addition that*

$$G_k \in C^{4,\alpha}\left(\overline{\Omega_k}\times\overline{\Omega_k}\setminus\{(x,y):\ x\ne y\}\right).$$

Proof. We let $i\in\{0,\dots,3\}$ and $p\in(n,n+1)$ so that in particular $4-i-\frac{n}{p}>0$. We let $\varphi\in C_c^\infty(\Omega_k)$ and consider $\psi\in C^{4,\alpha}(\overline{\Omega_k})$ such that $\Delta^2\psi + a_k\psi = \varphi$ in Ω_k and $\psi = \partial_\nu\psi = 0$ on $\partial\Omega_k$. It follows from regularity theory (see [1]) and Sobolev's embedding theorem that

$$\|\psi\|_{C^{i,\beta}(\overline{\Omega_k})} \le C\|\psi\|_{W^{4,p}(\Omega_k)} \le C\|\varphi\|_{L^p(\Omega_k)}$$

with $\beta\le 4-i-\frac{n}{p}$, $\beta\in(0,1)$. Here $W^{4,p}$ denotes the Sobolev space of order 4 in differentiability and of order p in integrability. Since $\psi(x) = \int_{\Omega_k} G_k(x,y)\varphi(y)\,dy$, we get that $\nabla_x^i G_k$ makes sense and that

$$\left|\int_{\Omega_k}(\nabla_x^i G_k(x,y) - \nabla_x^i G_k(x',y))\varphi(y)\,dy\right| \le C_2\|\varphi\|_{L^p(\Omega_k)}|x-x'|^\beta.$$

By duality, we then get that $y\mapsto \nabla_x^i G_k(x,y)\in L^q(\Omega_k)$ for all $q\in(\frac{n+1}{n},\frac{n}{n-1})$ and that

$$\|\nabla_x^i G_k(x,\cdot) - \nabla_x^i G_k(x',\cdot)\|_q \le C(q)|x-x'|^\beta \text{ for all } x,x'\in\Omega_k.$$

It follows from the equation satisfied by $G_k(x,\cdot)$ that we have that $\Delta^2\nabla_x^i G_k(x,\cdot) + a\nabla_x^i G_k(x,\cdot) = 0$ in $\mathcal{D}'(\Omega_k\setminus\{x\})$ and $\nabla_x^i G_k(x,\cdot) = 0$, $\partial_\nu\nabla_x^i G_k(x,\cdot) = 0$ on $\partial\Omega_k$. It then follows from regularity theory that $\nabla_x^i G_k(x,\cdot)\in C^{4,\alpha}(\overline{\Omega_k}\setminus\{x\})$. Moreover, for all $\delta>0$, there exists $C(\delta)>0$ such that

$$\|\nabla_x^i G_k(x,\cdot) - \nabla_x^i G_k(x',\cdot)\|_{C^{4,\alpha}(\overline{\Omega_k}\setminus(B_\delta(x)\cup B_\delta(x')))} \le C(\delta)|x-x'|^\beta \text{ for all } x,x'\in\Omega_k.$$

This is valid for $i\le 3$; using the symmetry of the Green's function, the above result extends to $i\in\{0,\dots,4\}$. It then follows that $G_k\in C^{4,\alpha}(\overline{\Omega_k}\times\overline{\Omega_k}\setminus\{(x,x):\ x\in\overline{\Omega_k}\})$. This proves the proposition. ☐

4. Uniform bounds for the Green's functions

As before, we consider a family of bounded regular domains $(\Omega_k)_{k\in\mathbb{N}}$ being a smooth perturbation of a fixed bounded regular domain Ω according to Definition 1. We focus on proving

$$|H_k(x,y)| \le C_1 \begin{cases} |x-y|^{4-n}, & \text{if } n>4,\\ (1+|\log|x-y||)\,, & \text{if } n=4,\\ 1, & \text{if } n=3; \end{cases}$$

$$|\nabla_{(x,y)}H_k(x,y)| \le C_1 \begin{cases} |x-y|^{-1}, & \text{if } n=4,\\ 1, & \text{if } n=3; \end{cases}$$

with the constant $C_1 = C_1(\Omega)$ being uniform for the whole family $(\Omega_k)_{k\in\mathbb{N}}$. For individual domains, such an estimate was proved by Krasovskiĭ [17] (cf. also [16]) even for very general boundary value problems for even order elliptic operators. Here, it remains to prove that these estimates are uniform in the domain, while we keep the operator Δ^2 fixed. Since we consider a rather special situation, we are able to provide an independent and relatively short proof for the required estimates, being uniform on the family $(\Omega_k)_{k\in\mathbb{N}}$.

Theorem 3. *Let Ω be a bounded $C^{4,\alpha}$-smooth domain of $\mathbb{R}^n$, $n\geq 3$ and $(\Omega_k)_{k\in\mathbb{N}}$ a $C^{4,\alpha}$-smooth perturbation of Ω. We denote by H_k the Green's functions for Δ^2 in Ω_k under Dirichlet boundary conditions.*

Then, there exists a constant $C_1>0$ such that for all k and all $x,y\in\Omega_k$ with $x\neq y$ one has that

$$|H_k(x,y)|\leq C_1\cdot\begin{cases}|x-y|^{4-n}, & \text{if } n>4,\\ (1+|\log|x-y||)\,, & \text{if } n=4,\\ 1, & \text{if } n=3.\end{cases} \tag{4.1}$$

Moreover, for $n=3,4$ we prove that

$$\forall x,y\in\Omega_k, x\neq y: \qquad |\nabla_{(x,y)}H_k(x,y)|\leq C_1\cdot\begin{cases}|x-y|^{-1}\,, & \text{if } n=4,\\ 1, & \text{if } n=3.\end{cases} \tag{4.2}$$

Proof. If $n=3$, the statement of Proposition 3.1 is already strong enough and nothing remains to be proved. We postpone the case $n=4$ and start with proving the theorem in the generic case $n>4$. We argue by contradiction and assume that there exist two sequences $(x_k)_{k\in\mathbb{N}},(y_k)_{k\in\mathbb{N}}$ with $x_k,y_k\in\Omega_{\ell_k}$ such that $x_k\neq y_k$ for all $k\in\mathbb{N}$ and such that

$$\lim_{k\to+\infty}|x_k-y_k|^{n-4}|H_{\ell_k}(x_k,y_k)|=+\infty. \tag{4.3}$$

It is enough to consider $\ell_k=k$; other situations may be reduced to this by relabelling or are even more special. After possibly passing to a subsequence, it follows from (3.2) that there exists $x_\infty\in\partial\Omega$ such that

$$\lim_{k\to+\infty}x_k=x_\infty \text{ and } \lim_{k\to+\infty}\frac{d(x_k,\partial\Omega_k)}{|x_k-y_k|}=0. \tag{4.4}$$

We remark that the constant in (3.2) can be chosen uniformly for the family $(\Omega_k)_{k\in\mathbb{N}}$.

Lemma 4.1. *Assume that $n\geq 4$. For any $q\in\left(\frac{n}{n-3},\frac{n}{n-4}\right)$, there exists $C(q)>0$ such that for all k and all $x\in\Omega_k$ we have*

$$\|H_k(x,\,.\,)\|_{L^q(\Omega_k)}\leq Cd(x,\partial\Omega_k)^{4-n+\frac{n}{q}}. \tag{4.5}$$

The constant C can be chosen uniformly for the family $(\Omega_k)_{k\in\mathbb{N}}$.

Proof. We proceed with the help of a duality argument. Let $\psi\in C_c^\infty(\Omega_k)$ and let $\varphi\in C^{4,\alpha}(\overline{\Omega_k})$ be a solution of

$$\begin{cases}\Delta^2\varphi=\psi & \text{in } \Omega_k,\\ \varphi=\partial_\nu\varphi=0 & \text{on } \partial\Omega_k.\end{cases}$$

Let $q\in\left(\frac{n}{n-3},\frac{n}{n-4}\right)$ and denote $q'=\frac{q}{q-1}$ the dual exponent, so that in particular $\frac{n}{4}<q'<\frac{n}{3}$. It follows from elliptic estimates [1, Thm. 15.2] that there exists $C_3>0$ independent of φ,ψ and k such that

$$\|\varphi\|_{W^{4,q'}(\Omega_k)}\leq C_3\|\psi\|_{L^{q'}(\Omega_k)}.$$

The embedding $W^{4,q'}(\Omega_k)\hookrightarrow C^{0,\beta}(\overline{\Omega_k})$ with $\beta=4-\frac{n}{q'}=4-n+\frac{n}{q}$ being continuous uniformly in k shows that there exists $C_4>0$ independent of φ and k such that $\|\varphi\|_{C^{0,\beta}(\overline{\Omega_k})}\leq C_4\|\varphi\|_{W^{4,q'}(\Omega_k)}$. Let $x\in\Omega_k$ and $x'\in\partial\Omega_k$. We then get that

$$|\varphi(x)|=|\varphi(x)-\varphi(x')|\leq\|\varphi\|_{C^{0,\beta}(\overline{\Omega_k})}|x-x'|^\beta\leq C_3C_4\|\psi\|_{L^{q'}(\Omega_k)}|x-x'|^\beta.$$

Moreover, it follows from Green's representation formula that $\varphi(x)=\int_{\Omega_k}H_k(x,y)\psi(y)\,dy$ for all $x\in\Omega_k$. Therefore, taking the infimum with respect to $x'\in\partial\Omega_k$, we have that

$$\left|\int_{\Omega_k}H_k(x,y)\psi(y)\,dy\right|\leq C_3C_4\|\psi\|_{L^{q'}(\Omega_k)}d(x,\partial\Omega_k)^\beta$$

for all $\psi\in C_c^\infty(\Omega_k)$. Inequality (4.5) then follows. ☐

Lemma 4.2. *Assuming $n>4$ and (4.3), one has that* $\lim_{k\to+\infty}|x_k-y_k|=0$.

Proof. Assume by contradiction that $|x_k-y_k|$ does not converge to 0. After extracting a subsequence we may then assume that there exists $\delta>0$ such that all $x_k\in B_\delta(x_\infty)$ and all $y_k\in\Omega_k\setminus\overline{B_{3\delta}(x_\infty)}$. We consider q as in Lemma 4.1. In particular we know that $\|H_k(x,\,.\,)\|_{L^q(\Omega_k)}\leq C$ uniformly in k. By applying local elliptic estimates (cf. [1, Theorem 15.3]) combined with Sobolev embeddings in $\Omega_k\setminus\overline{B_{2\delta}(x_\infty)}$ we find that

$$\|H_k(x_k,\,.\,)\|_{L^\infty\left(\Omega_k\setminus\overline{B_{3\delta}(x_\infty)}\right)}\leq C(q,\delta)$$

uniformly in k. In particular, we would have

$$|H_k(x_k,y_k)|\leq C(q,\delta)\quad\text{ and }\quad|x_k-y_k|^{n-4}|H_k(x_k,y_k)|\leq C(q,\delta)$$

independent of k. This contradicts our hypothesis (4.3). □

Concluding the proof of Theorem 3, case $n > 4$. In what follows we may work in one fixed coordinate domain U_i; for this reason we drop the index i. Let $\Phi_k : U \to \mathbb{R}^n$ be coordinate charts of Ω_k at x_∞ as in Definition 1. We recall that

$$\Phi_k(U \cap \{x_1 < 0\}) = \Phi_k(U) \cap \Omega_k \text{ and } \Phi_k(U \cap \{x_1 = 0\}) = \Phi_k(U) \cap \partial\Omega_k.$$

Without loss of generality we may assume that $\Phi_k(0) = x_\infty$ and $B_\delta(x_\infty) \subset \Phi_k(U)$.

We let $x_k = \Phi_k(x'_k)$ and $y_k = \Phi_k(y'_k)$. Therefore, (4.4) rewrites as

$$\lim_{k\to+\infty} x'_k = 0 \text{ and } \lim_{k\to+\infty} \frac{x'_{k,1}}{|x'_k - y'_k|} = 0. \tag{4.6}$$

We define for R large enough

$$\tilde{H}_k(z) = |x'_k - y'_k|^{n-4} H_k(\Phi_k(x'_k), \Phi_k(x'_k + |x'_k - y'_k|(z - \rho_k \vec{e}_1)))$$

in $B_R(0) \cap \{x_1 < 0\}$, where $\rho_k := \frac{x'_{k,1}}{|x'_k - y'_k|}$. The biharmonic equation $\Delta^2 H_k(x, .) = 0$ complemented with Dirichlet boundary conditions is rewritten as

$$\Delta^2_{g_k} \tilde{H}_k = 0 \text{ in } (B_R(0) \cap \{z_1 < 0\}) \setminus \{\rho_k \vec{e}_1\}, \qquad \tilde{H}_k = \partial_1 \tilde{H}_k = 0 \text{ on } \{z_1 = 0\}.$$

Here, $g_k(z) = \Phi_k^*(\mathcal{E})(x'_k + |x'_k - y'_k|(z - \rho_k \vec{e}_1))$, $\mathcal{E} = (\delta_{ij})$ the Euclidean metric, and Δ_{g_k} denotes the Laplace-Beltrami operator with respect to this rescaled and translated pull back of the Euclidean metric under Φ_k. Then, for $\tau > 0$ being chosen suitably small, it follows from elliptic estimates (see [1, Theorem 15.3]) and Sobolev embeddings that there exists $C(R, \tau, q) > 0$ such that

$$|\tilde{H}_k(z)| \leq C(R, q, \tau) \|\tilde{H}_k\|_{L^q(B_R(0) \setminus B_\tau(0))} \tag{4.7}$$

for all $z \in B_{R/2}(0) \setminus B_{2\tau}(0)$, $z_1 \leq 0$. In order to estimate the L^q-norm on the right-hand side we use (4.5) and obtain that

$$\begin{aligned} \int_{B_R(0)\cap\{\zeta_1<0\}} |\tilde{H}_k(\zeta)|^q \, d\zeta &\leq C|x'_k - y'_k|^{q(n-4)-n} \int_{\Omega_k} |H_k(x_k, y)|^q \, dy \\ &\leq C|x'_k - y'_k|^{q(n-4)-n} d(x_k, \partial\Omega_k)^{(4-n)q+n} \\ &\leq C \left(\frac{d(x_k, \partial\Omega_k)}{|x'_k - y'_k|} \right)^{n-q(n-4)}. \end{aligned}$$

Therefore, with (4.4), we get that $\lim_{k\to+\infty} \|\tilde{H}_k\|_{L^q(B_R(0)\setminus B_\tau(0))} = 0$, and (4.7) yields

$$\lim_{k\to+\infty} \tilde{H}_k = 0 \text{ in } C^0((B_{R/2}(0) \setminus B_{2\tau}(0)) \cap \{z_1 \leq 0\}).$$

In particular, since $\lim_{k\to+\infty} \rho_k = 0$, we have that

$$\lim_{k\to+\infty} \tilde{H}_k \left(\frac{y'_k - x'_k}{|y'_k - x'_k|} + \rho_k \vec{e}_1 \right) = 0.$$

This limit rewrites as

$$\lim_{k\to+\infty} |x_k - y_k|^{n-4} |H_k(x_k, y_k)| = 0,$$

contradicting (4.3). The proof of Theorem 3, $n > 4$, is complete. □

Proof of Theorem 3, case $n = 4$. Here it is enough to prove (4.2) for ∇_y, exploiting the symmetry of the Green's function. We argue by contradiction and as in the proof for $n > 4$, we may assume that there exist two sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}$ with $x_k, y_k \in \Omega_k$ such that $x_k \neq y_k$ and

$$\lim_{k\to+\infty} |x_k - y_k| \, |\nabla_y H_k(x_k, y_k)| = +\infty. \tag{4.8}$$

After possibly passing to a subsequence, it follows from (3.3) that there exists $x_\infty \in \partial\Omega$ such that

$$\lim_{k\to+\infty} x_k = x_\infty \text{ and } \lim_{k\to+\infty} \frac{d(x_k, \partial\Omega_k)}{|x_k - y_k|} = 0. \tag{4.9}$$

Lemma 4.1 may be applied with some $q > 4$. The analogue of Lemma 4.2 is proved in exactly the same way as above. Like above we now put for R large enough

$$\tilde{H}_k(z) = H_k(\Phi_k(x'_k), \Phi_k(x'_k + |x'_k - y'_k|(z - \rho_k \vec{e}_1)))$$

in $B_R(0)\cap\{z_1<0\}$, where $x_k=\Phi_k(x_k')$, $y_k=\Phi_k(y_k')$, $\rho_k:=\frac{x'_{k,1}}{|x_k'-y_k'|}$. As above we find for $\tau>0$ small enough that there exists $C(R,\tau,q)>0$ such that

$$|\nabla\tilde H_k(z)|\le C(R,q,\tau)\|\tilde H_k\|_{L^q(B_R(0)\setminus B_\tau(0))}$$

for all $z\in B_{R/2}(0)\setminus B_{2\tau}(0)$, $z_1\le 0$. Using (4.5) we obtain that

$$\begin{aligned}\int_{B_R(0)\cap\{\zeta_1<0\}}|\tilde H_k(\zeta)|^q\,d\zeta &\le C|x_k'-y_k'|^{-4}\int_{\Omega_k}|H_k(x_k,y)|^q\,dy\\ &\le C\left(\frac{d(x_k,\partial\Omega_k)}{|x_k'-y_k'|}\right)^4.\end{aligned}$$

In the same way as in the generic case $n>4$, this yields first that

$$\lim_{k\to+\infty}\nabla\tilde H_k=0 \text{ in } C^0((B_{R/2}(0)\setminus B_{2\tau}(0))\cap\{z_1\le 0\})$$

and back in the original coordinates

$$\lim_{k\to\infty}|x_k-y_k|\,|\nabla_y H_k(x_k,y_k)|=0.$$

So, we achieve a contradiction also if $n=4$. This proves (4.2). Integrating (4.2), we get (4.1). The proof of Theorem 3 is complete. □

5. Convergence of the Green's functions

As before, we consider a family of bounded regular domains (Ω_k) being a $C^{4,\alpha}$-smooth perturbation of a fixed bounded $C^{4,\alpha}$-smooth domain Ω according to Definition 1. We consider the operators Δ^2+a_k in Ω_k and assume that

$$\exists U_0\supset\overline{\Omega_k}: a_k\in C^{0,\alpha}(U_0);$$
$$\exists a_\infty\in C^{0,\alpha}(U_0):\lim_{k\to\infty}a_k=a_\infty \text{ in } C^{0,\alpha}_{loc}(U_0).$$

As before, we denote by G_k the Green's functions corresponding to Δ^2+a_k in Ω_k and by G the Green's functions corresponding to Δ^2+a_∞ in Ω and show the following convergence result. As for the diffeomorphisms $\Phi_{k,i}$, Φ_i we refer to Definition 1.

Proposition 5.1. *Let $x_k\in\Omega_k$ and assume that $\lim_{k\to\infty}x_k=x_\infty\in\Omega$. Then, we have:*

$$\begin{aligned}G_k(x_k,\,.\,)&\to G(x_\infty,\,.\,) \text{ in } C^4_{loc}(\Omega\setminus\{x_\infty\}),\\ G_k(x_k,\,.\,)&\to G(x_\infty,\,.\,) \text{ in } L^1(\mathbb{R}^n),\\ G_k(x_k,\,.\,)\circ\Phi_{k,i}&\to G(x_\infty,\,.\,)\circ\Phi_i \text{ in } C^4_{loc}(U_i\cap\{z_1\le 0\}\setminus\{\Phi_i^{-1}(x_\infty)\}).\end{aligned}$$

If $n=3$ we have in addition that

$$G_k(\,.\,,\,.\,)\to G(\,.\,,\,.\,) \text{ in } C^0_{loc}(\Omega\times\Omega).$$

Proof. According to Theorem 3 and Proposition 3.2 we know that

(5.1)
$$|H_k(x,y)|\le C\cdot\begin{cases}|x-y|^{4-n}, & \text{if } n>4,\\ (1+|\log|x-y||), & \text{if } n=4,\\ 1, & \text{if } n=3;\end{cases}\qquad |G_k(x,y)|\le C\cdot\begin{cases}|x-y|^{4-n}, & \text{if } n>4,\\ (1+|\log|x-y||), & \text{if } n=4,\\ 1, & \text{if } n=3;\end{cases}$$

uniformly in k. This shows that in particular

$$\|G_k(x,\,.\,)\|_{L^1(\Omega_k)} \text{ uniformly in } k. \tag{5.2}$$

Moreover, since $x_k\to x_\infty$, we may assume that all x_k are in a small neighbourhood around x_∞. Refering to the construction in the proof of Proposition 3.1 we see that the u_{k,x_k} arising there are uniformly bounded in $C^{4,\alpha}\left(\overline{\Omega_k}\right)$. After selecting a suitable subsequence we see that for each $\Omega_0\subset\subset\Omega$ one has $G_k(x_k,\,.\,)\to\varphi$ in $C^4_{loc}\left(\overline{\Omega_0}\setminus\{x_\infty\}\right)$ and $G_k(x_k,\,.\,)\circ\Phi_{k,i}\to\varphi\circ\Phi_i$ in $C^4_{loc}(U_i\cap\{z_1\le 0\}\setminus\{\Phi_i^{-1}(x_\infty)\})$ with a suitable $\varphi\in C^{4,\alpha}(\overline{\Omega}\setminus\{x_\infty\})$. Thanks to this compactness and the fact that in any case the limit is the uniquely determined Green's function, we have convergence on the whole sequence towards $G(x_\infty,\,.\,)$.

Finally, since we have pointwise convergence, (5.1) allows for applying Vitali's convergence theorem to show that

$$G_k(x_k,\,.\,)\to G(x_\infty,\,.\,) \text{ in } L^1(\mathbb{R}^n).$$

The statement concerning $C^0_{loc}(\Omega\times\Omega)$-convergence in $n=3$ follows from $|\nabla G_k(\,.\,,\,.\,)|\le C$, cf. (3.3). □

In order to prove Lemma 6.4 below we also need a convergence result simultaneous in both variables.

Proposition 5.2. *We have that*

$$G_k(\,.\,,\,.\,)\circ(\Phi_{k,i}\times\Phi_{k,j})\to G(\,.\,,\,.\,)\circ(\Phi_i\times\Phi_j)\ \ in\ C^4_{loc}\left((U_i\cap\{x_1\le 0\})\times(U_j\cap\{x_1\le 0\})\setminus D_{ij}\right),$$

where

$$D_{ij}=\{(x,y)\in U_i\times U_j:\ \Phi_i(x)=\Phi_j(y)\}.$$

Proof. We combine the ideas of the proofs of Propositions 5.1 and 3.3. One should observe that Theorem 3 and Proposition 3.2 guarantee uniform L^1-bounds for H_k and G_k as in the proof of Proposition 5.1. □

6. The limit of the zeros of the Green's functions

We keep the notations of the previous sections. In order to prove Theorem 2, we assume that for every k, there exist

$$x_k,y_k\in\Omega_k,\ x_k\ne y_k:\qquad G_k(x_k,y_k)=0. \tag{6.1}$$

After passing to subsequences there exist $x_\infty=\lim_{k\to\infty}x_k, y_\infty=\lim_{k\to\infty}y_k$. Using Definition 1, one sees that $x_\infty,y_\infty\in\overline{\Omega}$.

As for the location of these limit points, we distinguish several cases.

6.1. Both points in the interior.

Here, we consider the case that $x_\infty,y_\infty\in\Omega$. Once it is shown that $x_\infty\ne y_\infty$ we conclude directly from Proposition 5.1 that

$$G(x_\infty,y_\infty)=0. \tag{6.2}$$

So, we are left with proving:

Lemma 6.1. $x_\infty\ne y_\infty$.

Proof. Assume for contradiction that $x_\infty=y_\infty$. We consider first the case $n>4$ and here, the rescaled Green's function:

$$\tilde G_k(z):=|x_k-y_k|^{n-4}G_k(x_k,x_k+|x_k-y_k|z). \tag{6.3}$$

Let $\varepsilon>0$ be such that $\overline{B_{2\varepsilon}(x_\infty)}\subset\Omega\cap\Omega_k$ for all k. Then, for k large enough, $|x_k-x_\infty|<\varepsilon$ and $\tilde G_k(z)$ is certainly defined for $|z|<\frac{\varepsilon}{|x_k-y_k|}$, where one has by Theorem 3 and Proposition 3.2 that

$$|\tilde G_k(z)|\le C|z|^{4-n} \tag{6.4}$$

uniformly in k. Since

$$\Delta^2\tilde G_k+|x_k-y_k|^4a_k(x_k+|x_k-y_k|z)\tilde G_k=0\text{ on }\overline{B_{\varepsilon/|x_k-y_k|}(0)}\setminus\{0\},$$

by elliptic Schauder theory we may assume that after possibly passing to a subsequence that

$$\tilde G_k\to\tilde G\text{ in }C^4_{loc}(\mathbb{R}^n\setminus\{0\}),\text{ where }|\tilde G(z)|\le C|z|^{4-n}. \tag{6.5}$$

Moreover,

$$\Delta^2\tilde G=0\text{ in }\mathbb{R}^n\setminus\{0\}.$$

In order to compute the differential equation satisfied by $\tilde G$ near $z=0$, let $\varphi\in C^\infty_c(\mathbb{R}^n)$ with supp $\varphi\subset B_R(0)$ and define for k large enough

$$\Omega_k\ni x\mapsto\varphi_k(x):=\varphi\left(\frac{x-x_k}{|x_k-y_k|}\right),\quad \varphi_k\in C^\infty_c(\Omega_k).$$

$$\begin{aligned}
\varphi(0)&=\varphi_k(x_k)=\int_{\Omega_k}G_k(x_k,y)(\Delta^2\varphi_k+a_k\varphi_k)\,dy\\
&=\int_{B_{|x_k-y_k|R}(x_k)}G_k(x_k,y)|x_k-y_k|^{-4}\left((\Delta^2\varphi)\left(\frac{y-x_k}{|x_k-y_k|}\right)+|x_k-y_k|^4a_k(y)\varphi\left(\frac{y-x_k}{|x_k-y_k|}\right)\right)dy\\
&=\int_{B_R(0)}|x_k-y_k|^{n-4}G_k(x_k,x_k+|x_k-y_k|z)\left(\Delta^2\varphi(z)+|x_k-y_k|^4a_k(x_k+|x_k-y_k|z)\varphi(z)\right)dz\\
&=\int_{\mathbb{R}^n}\tilde G_k(z)\left(\Delta^2\varphi(z)+|x_k-y_k|^4a_k(x_k+|x_k-y_k|z)\varphi(z)\right)dz.
\end{aligned}$$

We put $\gamma_n=\frac{1}{2(n-4)(n-2)ne_n}$ and obtain, letting $k\to\infty$:

$$\int_{\mathbb{R}^n}\tilde G(z)\Delta^2\varphi(z)\,dz=\varphi(0)=\int_{\mathbb{R}^n}\gamma_n|z|^{4-n}\Delta^2\varphi(z)\,dz.$$

This shows that we have in the sense of distributions that

$$\Delta^2\left(\tilde{G}(z)-\gamma_n|z|^{4-n}\right)=0 \text{ in } \mathbb{R}^n.$$

Hence,

$$\tilde{G}(z)=\gamma_n|z|^{4-n}+\psi(z),\quad \psi\in C^\infty(\mathbb{R}^n),\quad \Delta^2\psi=0.$$

Thanks to (6.4) we know further that

$$|\psi(z)|\le C(1+|z|)^{4-n}.$$

Also for entire bounded biharmonic (even more generally polyharmonic) functions, Liouville's theorem holds true, i.e. these are constant, see [19, p. 19]. Hence $\psi(z)\equiv 0$ showing that

$$\tilde{G}(z)=\gamma_n|z|^{4-n},\qquad z\in\mathbb{R}^n\setminus\{0\}.$$

On the other hand we have according to the choice (6.1) of x_k, y_k and the definition (6.3) of $\tilde{G}_k$ that

$$\tilde{G}_k\left(\frac{y_k-x_k}{|x_k-y_k|}\right)=|x_k-y_k|^{n-4}G_k\left(x_k,y_k\right)=0.$$

Hence there exists at least one point $\zeta\in\mathbb{R}^n$ with

$$|\zeta|=1 \text{ and } 0=\tilde{G}(\zeta)=\gamma_n|\zeta|^{4-n},$$

which is absurd. This proves the statement for the case $n>4$. One should observe that when looking just at the biharmonic operator, a proof for the previous lemma would directly follow from the local positivity results in general domains, which are proved in [12]. This observation will be useful in what follows.

Let us now consider the case $n=4$. Since $x_\infty\in\Omega$, according to [12], there exists (a small) $\delta_1>0$ such that for all k and all $x,y\in\Omega_k$ we have that

$$x,y\in B_{\delta_1}(x_\infty)\quad\Rightarrow\quad H_k(x,y)\ge-\frac{1}{c_3}\log|x-y|. \tag{6.6}$$

We estimate the difference between G_k and H_k. For arbitrary but fixed $x\in\Omega$, we have that with respect to the y-variable, $\left(H_k-G_k\right)(x,\,.\,)$ solves the following Dirichlet problem:

$$\begin{cases}\Delta_y^2\left(H_k-G_k\right)(x,y)+a_k(y)\left(H_k-G_k\right)(x,y)=a_k(y)H_k(x,y), & y\in\Omega_k\\ \left(H_k-G_k\right)(x,y)=\frac{\partial}{\partial\nu_y}\left(H_k-G_k\right)(x,y)=0, & y\in\partial\Omega_k.\end{cases}$$

According to Theorem 3, we have that $\|a_k(\,.\,)H_k(x,\,.\,)\|_{L^2(\Omega_k)}\le c_4$ uniformly in k and x. Since Δ^2+a_k is assumed to be uniformly coercive, elliptic estimates [1] show that

$$\|\left(H_k-G_k\right)(x,\,.\,)\|_{L^\infty(\Omega_k)}\le C\|\left(H_k-G_k\right)(x,\,.\,)\|_{W^{4,2}(\Omega_k)}\le c_5,$$

uniformly in x and k. Together with (6.6), this gives that there exist a $\delta_2>0$ and a constant $c_6>0$ such that

$$x,y\in B_{\delta_2}(x_\infty)\quad\Rightarrow\quad G_k(x,y)\ge-\frac{1}{c_6}\log|x-y|. \tag{6.7}$$

This proves the claim also for $n=4$, since by (6.7), it is impossible that $G_k(x_k,y_k)=0$, where $x_k,y_k\to x_\infty\in\Omega$.

Finally, we consider $n=3$. Since here, according to Proposition 5.1, also $G_k\to G$ in $C^0_{loc}(\Omega\times\Omega)$ we have by assumption that $G(x_\infty,x_\infty)=0$. On the other hand, testing the boundary value problem for $G(x_\infty,\,.\,)$ with $G(x_\infty,\,.\,)$ itself yields by virtue of the uniform coercivity that

$$G(x_\infty,x_\infty)\ge\lambda\int_\Omega G(x_\infty,y)^2\,dy>0.$$

We obtain a contradiction also in the case $n=3$. So, the proof of Lemma 6.1 is complete. ☐

6.2. **One point in the interior, one point on the boundary.** After possibly interchanging x_∞ and y_∞ we may consider the case that $x_\infty\in\Omega$, $y_\infty\in\partial\Omega$.

Lemma 6.2.

$$\Delta_y G(x_\infty,y_\infty)=0.$$

Proof. We may fix a neighbourhood $B_\delta(p_i)$ such that $y_\infty\in\partial\Omega\cap B_\delta(p_i)$ so that for k large enough $y_k\in\Omega_k\cap B_\delta(p_i)$. We denote $y_k':=\Phi_{k,i}^{-1}(y_k)$, $y_\infty':=\Phi_i^{-1}(y_\infty)$ and observe that $(y_k')_1<0$, $(y_\infty')_1=0$, $y_k'\to y_\infty'$ in U_i. Writing

$$\tilde{G}_{k,i}:=G_k(x_k,\,.\,)\circ\Phi_{k,i},\quad \tilde{G}_i:=G_k(x_\infty,\,.\,)\circ\Phi_i$$

we see by means of Taylor's expansion that with suitable $\theta_k \in (0,1)$:

$$\begin{aligned} 0 &= G_k(x_k, y_k) = \tilde{G}_{k,i}(y_k') \\ &= \tilde{G}_{k,i}(0, \bar{y_k'}) + \partial_1 \tilde{G}_{k,i}(0, \bar{y_k'}) y_{k,1}' + \frac{1}{2} \partial_{11} \tilde{G}_{k,i}(\theta_k y_{k,1}', \bar{y_k'})(y_{k,1}')^2 \\ &= \frac{1}{2} \partial_{11} \tilde{G}_{k,i}(\theta_k y_{k,1}', \bar{y_k'})(y_{k,1}')^2 \end{aligned}$$

due to the boundary conditions on G_k. According to Proposition 5.1 this yields $\partial_{11}\tilde{G}_i(y_\infty') = 0$. Since $G_k(x_k, \,.\,)|_{\partial\Omega} = \frac{\partial}{\partial\nu} G_k(x_k, \,.\,)|_{\partial\Omega} = 0$, we obtain back in the original coordinates that $\Delta_y G(x_\infty, y_\infty) = 0$ as stated. □

6.3. **Both points on the boundary.** So, here we have to consider the case that both $x_\infty \in \partial\Omega$ and $y_\infty \in \partial\Omega$. The most delicate part will be to prove that both points have to be distinct:

Lemma 6.3. $x_\infty \neq y_\infty$.

The proof is rather technical and will be postponed to Subsection 6.4. Assuming now Lemma 6.3 being proved it is not too difficult that in this case an additional zero of the Green's function can be observed on the boundary:

Lemma 6.4. $\Delta_x \Delta_y G(x_\infty, y_\infty) = 0$.

Proof. According to Proposition 3.3 we have that $G \in C^{4,\alpha}$ in a neighbourhood of (x_∞, y_∞). This proof is similar to that of Lemma 6.2. We fix neighbourhoods such that $x_\infty \in B_\delta(p_i), y_\infty \in B_\delta(p_j)$; without loss of generality we may assume that $B_\delta(p_i) \cap B_\delta(p_j) = \emptyset$. Moreover we may assume that $\forall k: \quad x_k \in B_\delta(p_i), y_k \in B_\delta(p_j)$. To work in local charts we define

$$x_k' := \Phi_{k,i}^{-1}(x_k), \quad x_\infty' := \Phi_i^{-1}(x_\infty), \quad y_k' := \Phi_{k,j}^{-1}(y_k), \quad y_\infty' := \Phi_j^{-1}(y_\infty).$$

Hence we have

$$x_k' \in U_i \cap \{x_1 < 0\}, \quad x_k' \to x_\infty' \in U_i \cap \{x_1 = 0\}, \qquad y_k' \in U_j \cap \{y_1 < 0\}, \quad y_k' \to y_\infty' \in U_j \cap \{y_1 = 0\}.$$

Defining

$$\tilde{G}_k : U_i \cap \{x_1 \le 0\} \times U_j \cap \{y_1 \le 0\} \to \mathbb{R}, \qquad \tilde{G}_k(x', y') := G_k\left(\Phi_{k,i}(x'), \Phi_{k,j}(y')\right);$$
$$\tilde{G} : U_i \cap \{x_1 \le 0\} \times U_j \cap \{y_1 \le 0\} \to \mathbb{R}, \qquad \tilde{G}(x', y') := G_k\left(\Phi_i(x'), \Phi_j(y')\right);$$

we see that by assumption

$$0 = G_k(x_k, y_k) = \tilde{G}_k(x_k', y_k').$$

Taylor's expansion with respect to y' and exploiting the boundary conditions for $\tilde{G}_k$ with respect to y' shows that for each k there exists a suitable $\theta_k \in (0,1)$ such that

$$\partial_{y_1}^2 \tilde{G}_k(x_{k,1}', \bar{x}_k', \theta_k y_{k,1}', \bar{y}_k') = 0.$$

Now, we use Taylor's expansion for this expression with respect to x' and obtain with suitable $\tau_k \in (0,1)$:

$$\begin{aligned} 0 &= \partial_{y_1}^2 \tilde{G}_k(x_{k,1}', \bar{x}_k', \theta_k y_{k,1}', \bar{y}_k') \\ &= \partial_{y_1}^2 \tilde{G}_k(0, \bar{x}_k', \theta_k y_{k,1}', \bar{y}_k') + \partial_{x_1} \partial_{y_1}^2 \tilde{G}_k(0, \bar{x}_k', \theta_k y_{k,1}', \bar{y}_k') x_{k,1}' \\ &\quad + \frac{1}{2} \partial_{x_1}^2 \partial_{y_1}^2 \tilde{G}_k(\tau_k x_{k,1}', \bar{x}_k', \theta_k y_{k,1}', \bar{y}_k')(x_{k,1}')^2 \\ &= \frac{1}{2} \partial_{x_1}^2 \partial_{y_1}^2 \tilde{G}_k(\tau_k x_{k,1}', \bar{x}_k', \theta_k y_{k,1}', \bar{y}_k')(x_{k,1}')^2 \end{aligned}$$

so that

$$\partial_{x_1}^2 \partial_{y_1}^2 \tilde{G}_k(\tau_k x_{k,1}', \bar{x}_k', \theta_k y_{k,1}', \bar{y}_k') = 0.$$

Since by Proposition 5.2 we have C^4 convergence of $\tilde{G}_k$ to $\tilde{G}$ it follows that

$$\partial_{x_1}^2 \partial_{y_1}^2 \tilde{G}(x_\infty', y_\infty') = 0.$$

Taking into account the boundary conditions of G and of $\tilde{G}$, back in the original variables we see that

$$\Delta_x \Delta_y G(x_\infty, y_\infty) = 0$$

thereby proving the claim. □

6.4. **Proof of Lemma 6.3.** We assume for contradiction that $\lim_{k\to\infty} x_k = x_\infty = y_\infty = \lim_{k\to\infty} y_k$. We choose a neighbourhood $B_\delta(p_i) \ni x_\infty$ and may assume that $\forall k : x_k, y_k \in B_\delta(p_i) \cap \Omega_k$. As before we introduce

$$x'_k := \Phi_{k,i}^{-1}(x_k), \quad y'_k := \Phi_{k,i}^{-1}(y_k), \quad x'_\infty := \Phi_i^{-1}(x_\infty)$$

so that we have

$$x'_k, y'_k \in U_i \cap \{x_1 < 0\}, \quad x'_k \to x'_\infty, \; y'_k \to x'_\infty \in U_i \cap \{x_1 = 0\}.$$

We distinguish two further cases according to whether the distance between x_k and y_k converges faster to 0 than the distance of these points to the boundary or vice verca.

First case: $|x_k - y_k| = o\left(\max(d(x_k, \partial\Omega_k), d(y_k, \partial\Omega_k))\right)$. After possibly interchanging x_k and y_k and passing to a subsequence we may assume that

$$|x_k - y_k| = o(d(x_k, \partial\Omega_k)).$$

This case is much simpler than the second case below and quite similar to the case where both points converge in the interior treated in Subsection 6.1. Like there we treat the case $n > 4$ first. In this case, we consider the rescaled Green's functions:

$$\tilde{G}_k(z) := |x_k - y_k|^{n-4} G_k(x_k, x_k + |x_k - y_k| z).$$

These are is certainly defined for $|z| < \frac{d(x_k, \partial\Omega_k)}{|x_k - y_k|}$, which converges to ∞ as $k \to \infty$. For this reason, we may now directly copy the reasoning of Subsection 6.1 and obtain that

$$\tilde{G}_k \to \tilde{G} \text{ in } C^4_{loc}(\mathbb{R}^n \setminus \{0\}) \text{ with } \tilde{G}(z) = \gamma_n |z|^{4-n}.$$

One should observe that also here the property of the Green's functions to be uniformly bounded by $C|x-y|^{4-n}$ is used. According to the choice (6.1) of x_k, y_k and the definition of $\tilde{G}_k$ we have that

$$\tilde{G}_k\left(\frac{y_k - x_k}{|x_k - y_k|}\right) = |x_k - y_k|^{n-4} G_k(x_k, y_k) = 0.$$

Hence there exists at least one point $\zeta \in \mathbb{R}^n$ with

$$|\zeta| = 1 \text{ and } 0 = \tilde{G}(\zeta) = \gamma_n |\zeta|^{4-n},$$

which is absurd.

We now treat the case $n = 4$ and proceed similarly as in the proof of Lemma 6.1. Rescaling the result of [12] shows the existence of $\delta > 0$, $c_3 > 0$ such that for $x, y \in \Omega_k$ with $|x - y| \le \delta d(x, \partial\Omega_k)$, one has (uniformly in k) that

$$H_k(x, y) \ge -\frac{1}{c_3} \log \frac{|x - y|}{d(x, \partial\Omega_k)}. \tag{6.8}$$

As it was shown in the proof of Lemma 6.1, $G_k - H_k$ is bounded uniformly in k. Hence, there exists a constant c_4 such that for $x, y \in \Omega_k$ we have

$$|x - y| \le \delta \operatorname{dist}(x, \partial\Omega_k) \quad \Rightarrow \quad G_k(x, y) \ge -\frac{1}{c_3} \log \frac{|x - y|}{d(x, \partial\Omega_k)} - c_4.$$

Since $|x_k - y_k| = o(d(x_k, \partial\Omega_k))$ we obtain

$$0 = G_k(x_k, y_k) \to \infty \qquad (k \to \infty).$$

This is again absurd and proves the claim for $n = 4$.

Finally we discuss the case $n = 3$. Rescaling the result of Nehari [18] shows the existence of $\delta > 0$, $\varepsilon > 0$ such that for $x, y \in \Omega_k$ with $|x - y| \le \delta d(x, \partial\Omega_k)$, one has (uniformly in k) that

$$H_k(x, y) \ge \varepsilon d(x, \partial\Omega_k). \tag{6.9}$$

Making use of elliptic theory as in the proof of Lemma 6.1 and exploiting the fact that $n = 3$ yields that $\|\left(G_k(\,.\,, y_k) - H_k(\,.\,, y_k)\right)\|_{C^2(\overline{\Omega_k})} \le c_5$ uniformly in k. Since $|x_k - y_k| \le \delta d(x_k, \partial\Omega_k)$ for k large enough we conclude that

$$0 = G_k(x_k, y_k) \ge \varepsilon d(x_k, \partial\Omega_k) - c_6 d(x_k, \partial\Omega_k)^2,$$

which becomes again absurd for $k \to \infty$.

Second case: $|x_k - y_k| \ne o(\max(d(x_k, \partial\Omega_k), d(y_k, \partial\Omega_k))$. After selecting a subsequence we may assume that there is $\tau > 0$ such that

$$|x_k - y_k| \ge \tau d(x_k, \partial\Omega_k) \text{ and } |x_k - y_k| \ge \tau d(y_k, \partial\Omega_k).$$

We define

$$\rho_k := \frac{(x'_k)_1}{|x'_k - y'_k|} < 0 \text{ and } O(1)$$

and after selecting a further subsequence we may assume that

$$\lim_{k\to\infty} \rho_k =: \rho \leq 0.$$

Again, we will introduce a rescaled family of Green's functions. For any $R>0$ and $z,\zeta \in B_R \cap \mathbb{R}^n_-$,

$$\tilde{G}_k(z,\zeta) := |x_k' - y_k'|^{n-4} G_k\left(\Phi_{k,i}(x_k' + |x_k'-y_k'|(z-\rho_k\vec{e}_1)), \Phi_{k,i}(x_k'+|x_k'-y_k'|(\zeta-\rho_k\vec{e}_1))\right). \tag{6.10}$$

Moreover, $\tilde{G}_k(z,\cdot) = \partial_{\zeta_1}\tilde{G}_k(z,\cdot) = 0$ on $B_R(0)\cap\partial\mathbb{R}^n_-$. According to (4.1) and Proposition 3.2, we see that uniformly in k, z and ζ

$$\left|\tilde{G}_k(z,\zeta)\right| \leq C|z-\zeta|^{4-n}, \qquad \text{provided that } n>4. \tag{6.11}$$

If $n=3,4$ we conclude first that

$$\left|\nabla\tilde{G}_k(z,\zeta)\right| \leq C\cdot\begin{cases} |z-\zeta|^{-1}, & \text{if } n=4,\\ 1, & \text{if } n=3.\end{cases}$$

Upon integration we obtain that

$$\left|\tilde{G}_k(z,\zeta)\right| \leq C\cdot\begin{cases} (1+|\log|z-\zeta||+\log(1+|z|)+\log(1+|\zeta|)), & \text{if } n=4,\\ (1+|z|+|\zeta|), & \text{if } n=3.\end{cases} \tag{6.12}$$

The points x_k and y_k were chosen such that $G_k(x_k,y_k)=0$, which reads in new coordinates

$$\tilde{G}_k\left(\rho_k\vec{e}_1, \frac{y_k'-x_k'}{|x_k'-y_k'|}+\rho_k\vec{e}_1\right) = 0. \tag{6.13}$$

In order to formulate the differential equation satisfied by $\tilde{G}_k$, we denote by $\mathcal{E}=(\delta_{ij})$ the Euclidean metric and

$$g_{k,i}(z) := \Phi^*_{k,i}(\mathcal{E})(x_k'+|x_k'-y_k'|(z-\rho_k\vec{e}_1))$$

its translated and rescaled pullback with respect to the coordinate charts $\Phi_{k,i}$. Moreover, we introduce its limit, the constant metric

$$g_{\infty,i} := \Phi^*_i(\mathcal{E})(x_\infty).$$

First, we keep $z\in\mathbb{R}^n_-$ fixed and consider $\tilde{G}_k(z,\,.\,) =: \tilde{G}_{k,z}(\,.\,)$ as function in the second variable. For $\zeta \in B_R(0)\cap\mathbb{R}^n_-\setminus\{z\}$ we have that for k large enough, the following boundary value problem is satisfied:

$$\begin{cases} \Delta^2_{g_{k,i},\zeta}\tilde{G}_k(z,\zeta) + |x_k'-y_k'|^4 (a_k\circ\Phi_{k,i})\,(x_k'+|x_k'-y_k'|(\zeta-\rho_k\vec{e}_1))\,\tilde{G}_k(z,\zeta) = 0 & \text{for } \zeta_1<0,\ \zeta\neq z,\\ \tilde{G}_k(z,\zeta) = \partial_{\zeta_1}\tilde{G}_k(z,\zeta) = 0 & \text{for } \zeta_1 = 0.\end{cases} \tag{6.14}$$

For $k\to\infty$, using [1], we find $\tilde{G}_z = \tilde{G}(z,\,.\,) \in C^4\left(\overline{\mathbb{R}^n_-}\setminus\{z\}\right)$ such that

$$\tilde{G}_k(z,\,.\,) \to \tilde{G}_z \text{ in } C^4_{loc}\left(\overline{\mathbb{R}^n_-}\setminus\{z\}\right), \quad \Delta^2_{g_\infty,\zeta}\tilde{G}(z,\zeta) = 0\ (z\neq\zeta); \tag{6.15}$$

$$\left|\tilde{G}(z,\zeta)\right| \leq C\cdot\begin{cases} |z-\zeta|^{4-n}, & \text{if } n>4,\\ (1+|\log|z-\zeta||+\log(1+|z|)+\log(1+|\zeta|)), & \text{if } n=4,\\ (1+|z|+|\zeta|), & \text{if } n=3;\end{cases} \tag{6.16}$$

$$\left|\nabla\tilde{G}(z,\zeta)\right| \leq C\cdot\begin{cases} |z-\zeta|^{-1}, & \text{if } n=4,\\ 1, & \text{if } n=3.\end{cases} \tag{6.17}$$

In order to calculate the differential equation satisfied by $\tilde{G}$ near $\zeta=z$, we introduce

$$\varphi\in C^\infty_c\left(\overline{\mathbb{R}^n_-}\right), \qquad \varphi=\partial_1\varphi=0 \text{ on } \partial\mathbb{R}^n_-$$

and let $\varphi_k\in C^{4,\alpha}\left(\overline{\Omega_k}\right)$ such that

$$\varphi(z) = \varphi_k\circ\Phi_{k,i}\left(x_k'+|x_k'-y_k'|(z-\rho_k\vec{e}_1)\right) \text{ for } z\in\tilde{\Omega}_k$$

$$\varphi_k = \partial_\nu\varphi_k = 0 \text{ on } \partial\Omega_k;$$

where we denote

$$\tilde{\Omega}_k := \rho_k\vec{e}_1 - \frac{x_k'}{|x_k'-y_k'|} + \frac{1}{|x_k'-y_k'|}\left(U_i\cap\mathbb{R}^n_-\right).$$

By means of the representation formula and the corresponding Green's function we see that for $z \in \mathbb{R}^n_-$ and k large enough

$$\begin{aligned}
\varphi(z) &= \varphi_k\left(\Phi_{k,i}\left(x'_k + |x'_k - y'_k|(z - \rho_k \vec{e}_1)\right)\right)\\
&= \int_{\Omega_k} G_k\left(\Phi_{k,i}\left(x'_k + |x'_k - y'_k|(z - \rho_k \vec{e}_1)\right), y\right)\left(\Delta^2 \varphi_k + a_k \varphi_k\right)\, dy\\
&= \int_{\Phi_{k,i}(U_i \cap \{\eta_1 < 0\})} G_k\left(\Phi_{k,i}\left(x'_k + |x'_k - y'_k|(z - \rho_k \vec{e}_1)\right), y\right)\left(\Delta^2 \varphi_k + a_k \varphi_k\right)\, dy\\
&= \int_{U_i \cap \{\eta_1 < 0\}} G_k\left(\Phi_{k,i}\left(x'_k + |x'_k - y'_k|(z - \rho_k \vec{e}_1)\right), \Phi_{k,i}(\eta)\right)\\
&\qquad \cdot \left(\Delta^2_{\Phi^*_{k,i}(\mathcal{E})}(\varphi_k \circ \Phi_{k,i}) + (a_k \circ \Phi_{k,i})(\varphi_k \circ \Phi_{k,i})\right)(\eta)\, |\mathrm{Jac}\, \Phi_{k,i}(\eta)|\, d\eta\\
&= \int_{\tilde{\Omega}_k} |x'_k - y'_k|^{4-n} \tilde{G}_k(z, \zeta) |x'_k - y'_k|^{-4}\left(\Delta^2_{g_{k,i}} + |x'_k - y'_k|^4 a_k\left(\Phi_{k,i}\left(x'_k + |x'_k - y'_k|(\zeta - \rho_k \vec{e}_1)\right)\right)\right)\varphi(\zeta)\\
&\qquad \cdot |x'_k - y'_k|^n\, |\mathrm{Jac}\, \Phi_{k,i}\left(x'_k + |x'_k - y'_k|(\zeta - \rho_k \vec{e}_1)\right)|\, d\zeta\\
&= \int_{\tilde{\Omega}_k} \tilde{G}_k(z, \zeta)\left(\Delta^2_{g_{k,i}} + |x'_k - y'_k|^4 a_k\left(\Phi_{k,i}\left(x'_k + |x'_k - y'_k|(\zeta - \rho_k \vec{e}_1)\right)\right)\right)\varphi(\zeta)\\
&\qquad \cdot |\mathrm{Jac}\, \Phi_{k,i}\left(x'_k + |x'_k - y'_k|(\zeta - \rho_k \vec{e}_1)\right)|\, d\zeta.
\end{aligned}$$

Observing (6.11), (6.12) and passing to the limit we obtain for $z \in \mathbb{R}^n_-$:

$$\varphi(z) = \int_{\mathbb{R}^n_-} \tilde{G}(z, \zeta) \Delta^2_{g_\infty, i} \varphi(\zeta)\, |\mathrm{Jac}\, \Phi_i(x'_\infty)|\, d\zeta.$$

We introduce the linear bijection $L = d\Phi_i(x'_\infty)$, the half space $P := L\left(\mathbb{R}^n_-\right)$ and obtain for $z \in \mathbb{R}^n_-$:

$$\varphi(z) = \int_{\mathbb{R}^n_-} \tilde{G}(z, \zeta) \Delta^2_{L^*\mathcal{E}} \varphi(\zeta)\, |\det(L)|\, d\zeta = \int_P \tilde{G}(z, L^{-1}(\eta)) \Delta^2\left(\varphi \circ L^{-1}\right)\, d\eta. \tag{6.18}$$

Finally we consider a rotation $\sigma \in O(n)$ such that $\sigma(P) = \mathbb{R}^n_-$ so that $\sigma \circ L(\mathbb{R}^n_-) = \mathbb{R}^n_-$. For arbitrary

$$\psi \in C^\infty_c\left(\overline{\mathbb{R}^n_-}\right), \qquad \text{with } \psi = \partial_1 \psi = 0 \text{ on } \partial \mathbb{R}^n_-$$

and $\tilde{x} \in \mathbb{R}^n_-$ we may take $\varphi = \psi \circ \sigma \circ L$ and $z = (\sigma \circ L)^{-1}(\tilde{x})$. We obtain from (6.18) since the Laplacian is invariant under orthogonal transformations that for $\tilde{x} \in \mathbb{R}^n_-$:

$$\begin{aligned}
\psi(\tilde{x}) &= (\psi \circ \sigma \circ L)\left((\sigma \circ L)^{-1}(\tilde{x})\right) = \int_{P = \sigma^{-1}(\mathbb{R}^n_-)} \tilde{G}\left((\sigma \circ L)^{-1}(\tilde{x}), L^{-1}(\eta)\right) \Delta^2\left(\psi \circ \sigma\right)(\eta)\, d\eta\\
&= \int_{\mathbb{R}^n_-} \tilde{G}\left((\sigma \circ L)^{-1}(\tilde{x}), (\sigma \circ L)^{-1}(\eta)\right) \Delta^2 \psi(\eta)\, d\eta.
\end{aligned}$$

This shows that in the sense of distributions

$$\Delta^2_{\tilde{y}} \bar{G}(\tilde{x}, \,.\,) = \delta_{\tilde{x}}, \tag{6.19}$$

where we have defined

$$\bar{G}(\tilde{x}, \tilde{y}) := \tilde{G}\left((\sigma \circ L)^{-1}(\tilde{x}), (\sigma \circ L)^{-1}(\tilde{y})\right). \tag{6.20}$$

Moreover, for fixed $\tilde{x} \in \mathbb{R}^n_-$ one concludes with the help of (6.16) and (6.17) that

$$|\bar{G}(\tilde{x}, \tilde{y})| \leq C \cdot \begin{cases} |\tilde{x} - \tilde{y}|^{4-n}, & \text{if } n > 4,\\ \left(1 + |\log|\tilde{x} - \tilde{y}|\,| + \log(1 + |\tilde{x}|) + \log(1 + |\tilde{y}|)\right), & \text{if } n = 4,\\ \left(1 + |\tilde{x}| + |\tilde{y}|\right), & \text{if } n = 3; \end{cases} \tag{6.21}$$

$$|\nabla \bar{G}(\tilde{x}, \tilde{y})| \leq C \cdot \begin{cases} |\tilde{x} - \tilde{y}|^{-1}, & \text{if } n = 4,\\ 1, & \text{if } n = 3. \end{cases} \tag{6.22}$$

We denote by H the biharmonic Green's function in $\mathbb{R}^n_-$, which thanks to Boggio [3] is known explicitly and known to be positive – see Lemma 6.6 below. We prove:

Lemma 6.5. $\forall x, y \in \mathbb{R}^n_-, x \neq y: \qquad \bar{G}(x, y) = H(x, y).$

Proof. In what follows we keep $x \in \mathbb{R}^n_-$ fixed. Both $\bar{G}(x,\,.\,)$ and $H(x,\,.\,)$ satisfy the biharmonic equation with the δ-distribution δ_x as right hand side and zero Dirichlet boundary conditions on $\{y_1 = 0\}$. We let $\psi_x := \bar{G}(x,\,.\,) - H(x,\,.\,)$. Hence,

$$\psi = \psi_x \in C^\infty\left(\overline{\mathbb{R}^n_-}\right)$$

solves

$$\left\{\begin{array}{ll} \Delta^2\psi = 0 & \text{in } \mathbb{R}^n_-, \\ \psi = \partial_1\psi = 0 & \text{on } \{y_1 = 0\}. \end{array}\right. \tag{6.23}$$

Moreover, according to (6.21-6.22) and (6.31) below we have that

$$\forall y \in \mathbb{R}^n_-: \qquad |\psi(y)| \le C \cdot \left\{\begin{array}{ll} |y|^{4-n}, & \text{if } n > 4, \\ (1 + |\log|y||), & \text{if } n = 4, \\ (1 + |y|), & \text{if } n = 3; \end{array}\right. \tag{6.24}$$

$$|\nabla\psi(y)| \le C \cdot \left\{\begin{array}{ll} |y|^{-1}, & \text{if } n = 4, \\ 1, & \text{if } n = 3; \end{array}\right. \tag{6.25}$$

where $C = C(x)$. According to [7, 15]

$$\psi^*(y) := \left\{\begin{array}{ll} \psi(y) & \text{if } y_1 \le 0, \\ -\psi(-y_1, \bar{y}) - 2y_1 \frac{\partial}{\partial y_1}\psi(-y_1, \bar{y}) - y_1^2 \Delta\psi(-y_1, \bar{y}), & \text{if } y_1 > 0, \end{array}\right.$$

$\psi^* \in C^4(\mathbb{R}^n)$ is an entire biharmonic function. We consider now first the case $n > 4$. Below we will prove that (6.23) and (6.24) imply that also

$$\forall j = 1, 2: \quad \forall y \in \mathbb{R}^n_-: \qquad |\nabla^j\psi(y)| \le C|y|^{4-n-j}, \quad \text{where } C = C(x). \tag{6.26}$$

This immediately gives that $|\psi^*(y)| \le C|y|^{4-n}$ and in particular that ψ^* is a bounded entire biharmonic function. Again, Liouville's theorem for biharmonic functions [19, p. 19] yields that $\psi^*(y) \equiv 0$ so that the claim of the lemma follows, provided $n > 4$.

If $n = 3, 4$ we shall prove below that for $j = 0, 1, 2$

$$\forall y \in \mathbb{R}^n_-: \qquad |D^{2+j}\psi(y)| \le C|y|^{2-n-j}, \quad \text{where } C = C(x). \tag{6.27}$$

As above ψ^* is an entire biharmonic function and so are $D\psi^*$ and $D^2\psi^*$. Since $|D^2\psi^*(y)| \le C(1 + |y|)^{2-n}$, it follows that $D^2\psi^*(x) \equiv 0$. In view of the boundary conditions in (6.23) we come up with $\psi^*(y) \equiv 0$ also in the case $n = 3, 4$.

It remains to prove (6.26) and (6.27). We consider first $n > 4$. Assume for contradiction that there exists a sequence $(y_\ell) \subset \mathbb{R}^n_-$ such that $|\nabla^j\psi(y_\ell)| \cdot |y_\ell|^{n+j-4} \to \infty$ for $\ell \to \infty$. Then

$$\tilde{\psi}_\ell(y) := |y_\ell|^{n-4}\psi\left(y_\ell - y_{\ell,1}\vec{e}_1 + |y_\ell|y\right)$$

would solve

$$\left\{\begin{array}{ll} \Delta^2\tilde{\psi}_\ell = 0 & \text{in } \mathbb{R}^n_-, \\ \tilde{\psi}_\ell = \partial_1\tilde{\psi}_\ell = 0 & \text{on } \{y_1 = 0\}. \end{array}\right. \tag{6.28}$$

From the assumption we conclude that

$$\left|\nabla^j\tilde{\psi}_\ell\left(\frac{y_{\ell,1}}{|y_\ell|}\vec{e}_1\right)\right| = |y_\ell|^{n+j-4}|\nabla^j\psi(y_\ell)| \to \infty. \tag{6.29}$$

On the other hand,

$$|\tilde{\psi}_\ell(y)| \le C|y_\ell|^{n-4}\left|y_\ell - y_{\ell,1}\vec{e}_1 + |y_\ell|y\right|^{4-n} \le C\left|\frac{y_\ell}{|y_\ell|} + y - \frac{y_{\ell,1}}{|y_\ell|}\vec{e}_1\right|^{4-n}, \tag{6.30}$$

so that $\tilde{\psi}_\ell$ remains bounded in a neighbourhood of $\frac{y_{\ell,1}}{|y_\ell|}\vec{e}_1$ in $\overline{\mathbb{R}^n_-}$. Local Schauder estimates [1, Theorem 7.3] yield

$$\left|\nabla^j\tilde{\psi}_\ell\left(\frac{y_{\ell,1}}{|y_\ell|}\vec{e}_1\right)\right| \le C,$$

thereby contradicting (6.29). This proves (6.26).

As for (6.27), i.e. in particular $n = 3, 4$, the proof is quite similar since we can already make use of the gradient estimates (6.25). Instead of (6.30) one has to make use of

$$|\nabla\tilde{\psi}_\ell(y)| \le C|y_\ell|^{n-3}\left|y_\ell - y_{\ell,1}\vec{e}_1 + |y_\ell|y\right|^{3-n} \le C\left|\frac{y_\ell}{|y_\ell|} + y - \frac{y_{\ell,1}}{|y_\ell|}\vec{e}_1\right|^{3-n},$$

so that $\nabla\tilde{\psi}_\ell$ remains bounded uniformly outside $\frac{y_\ell}{|y_\ell|}-\frac{y_{\ell,1}}{|y_\ell|}\vec{e}_1$. Therefore, since $\tilde{\psi}_\ell$ vanishes on $\partial\mathbb{R}^n_-$, we get that $\tilde{\psi}_\ell$ is bounded in a neighbourhood of $\frac{y_{\ell,1}}{|y_\ell|}\vec{e}_1$ in $\overline{\mathbb{R}^n_-}$. The proof of the present lemma is complete. □

In order to show that the present case $x_\infty = y_\infty \in \partial\Omega$ can indeed not occur we collect some basic facts on the biharmonic Green's function in the half space; modulo a simple conformal transformation, cf. [3, p. 126]:

Lemma 6.6. *The biharmonic Green's function in $\mathbb{R}^n_-$ is given by*

$$\forall x,y\in\mathbb{R}^n_-: H(x,y)=\frac{1}{4ne_n}|x-y|^{4-n}\int_1^{|x^*-y|/|x-y|}(v^2-1)v^{1-n}\,dv;\quad x^*=(-x_1,\bar{x}). \tag{6.31}$$

From this it follows by direct calculation:

$$\forall x,y\in\mathbb{R}^n_-,\ x\neq y: \qquad H(x,y)>0; \tag{6.32}$$

$$\forall x\in\mathbb{R}^n_-, y\in\partial\mathbb{R}^n_-: \qquad \Delta_y H(x,y)>0; \tag{6.33}$$

$$\forall x,y\in\partial\mathbb{R}^n_-,\ x\neq y: \qquad \Delta_x\Delta_y H(x,y)>0. \tag{6.34}$$

We proceed now showing that $x_\infty = y_\infty \in \partial\Omega$ is indeed impossible and recall that by assumption we chose x_k, y_k such that $G_k(x_k,y_k)=0$. In terms of the transformed Green's functions this reads

$$\tilde{G}_k\left(\rho_k\vec{e}_1, \frac{y_k'-x_k'}{|x_k'-y_k'|}+\rho_k\vec{e}_1\right)=0, \tag{6.35}$$

cf. (6.13). After possibly extracting a further subsequence we find a point

$$\theta=\lim_{k\to\infty}\frac{y_k'-x_k'}{|x_k'-y_k'|}$$

and may conclude that

$$\tilde{G}\left(\rho\vec{e}_1,\theta+\rho\vec{e}_1\right)=0. \tag{6.36}$$

According to the possible location of the limit points we have to distinguish four cases:

Case (a): $\rho<0$ and $(\theta+\rho\vec{e}_1)_1<0$. We put $\tilde{x}=(\sigma\circ L)(\rho\vec{e}_1)\in\mathbb{R}^n_-$, $\tilde{y}=(\sigma\circ L)(\theta+\rho\vec{e}_1)\in\mathbb{R}^n_-$. According to (6.20) and Lemma 6.5 we could conclude that

$$H(\tilde{x},\tilde{y})=\bar{G}(\tilde{x},\tilde{y})=0,$$

which is impossible in view of (6.32).

Case (b): $\rho=0$ and $(\theta+\rho\vec{e}_1)_1<0$. As in the proof of Lemma 6.2 we conclude from (6.35) that $\partial^2_{x_1}\tilde{G}(0,\theta)=0$. Together with the Dirichlet boundary conditions satisfied by $\tilde{G}$ this yields $\tilde{G}(0,\theta)=0, D_x\tilde{G}(0,\theta)=0, D_x^2\tilde{G}(0,\theta)=0$. If we put $\tilde{y}=(\sigma\circ L)(\theta)\in\mathbb{R}^n_-$ this implies due to (6.20) that also $D_x^2\bar{G}(0,\tilde{y})=0$. In particular, we have that $\Delta_x H(0,\tilde{y})=\Delta_x\bar{G}(0,\tilde{y})=0$, which is impossible in view of (6.33).

Case (c): $\rho<0$ and $(\theta+\rho\vec{e}_1)_1=0$. Due to symmetry of the Green's function, this case is completely analogous to the previous one and hence impossible in view of (6.33).

Case (d): $\rho=0$ and $(\theta+\rho\vec{e}_1)_1=0$. As in the proof of Lemma 6.4 we conclude from (6.35) that $\partial^2_{x_1}\partial^2_{y_1}\tilde{G}(0,\theta)=0$. Here $\theta_1=0, |\theta|=1$. Thanks to the boundary conditions satisfied by $\tilde{G}$ this gives $\forall|\alpha|\le 2, |\beta|\le 2:$ $D_x^\alpha D_y^\beta\tilde{G}(0,\theta)=0$. Using again (6.20), we see that also $\forall|\alpha|\le 2, |\beta|\le 2:\quad D_x^\alpha D_y^\beta\bar{G}(0,\tilde{y})=0$, where $\tilde{y}=(\sigma\circ L)(\theta)\neq 0$. In particular, we come up with $\Delta_x\Delta_y H(0,\tilde{y})=\Delta_x\Delta_y\bar{G}(0,\tilde{y})=0$. This is impossible in view of (6.34).

Conclusion. In each case we finally deduced a contradiction so that $x_\infty=y_\infty\in\partial\Omega$ is indeed impossible. The proof of Lemma 6.3 is complete. □

6.5. **Proof of Theorems 1 and 2.** .

Theorem 2 follows from the conclusions made in Subsections 6.1, 6.2 and 6.3. In order to prove Theorem 1, we assume that no such $\varepsilon_0>0$ exists. In view of the remark after Theorem 1, we would have a neighbourhood U of $\overline{B}$, $C^{4,\alpha}$-smooth diffeomorphisms $\psi_k:\overline{U}\to\psi_k(\overline{U})$ and smooth domains $\Omega_k=\psi_k(B)$ with sign changing biharmonic Green's functions H_k. Hence, one of the alternatives described in Theorem 2 would occur for the biharmonic Green's function H in the ball B. Since H enjoys precisely the analogous properties of Lemma 6.6 (cf. [3, p. 126]), this is absurd; Theorem 1 follows. □

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