Hardy inequalities with optimal constants and remainder terms

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Abstract

We show that in the classical Hardy inequalities with optimal constants in $W_0^{1,p}(\Omega)$, $W_0^{2,2}(\Omega)$, $W^{2,2}\cap W_0^{1,2}(\Omega)$, $W^{2,p}\cap W_0^{1,p}(\Omega)$ and also in further higher order Sobolev spaces remainder terms may be added. Here Ω is any bounded domain. For the Hardy inequality in $W_0^{1,p}$ $(1<p<\infty)$ a further L^p-norm appears. The corresponding estimation constant behaves differently in the cases $p\geq 2$ and $1<p<2$. In higher order Sobolev spaces besides the L^2-norm (L^p-norm resp.) further singularly weighted L^2-norms (L^p-norms resp.) arise.

1 Introduction

Hardy's inequality in dimensions $n>2$

$$\forall u \in W_0^{1,2}(\Omega): \qquad \int_\Omega |\nabla u|^2\,dx \geq \frac{(n-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2}\,dx \tag{1}$$

is one of the really classical Sobolev embedding inequalities, see [H, HLP]. Here and in what follows, $\Omega\subset\mathbb{R}^n$ is a bounded domain. Although we do not explicitly assume that $0\in\Omega$, we always have this particularly interesting case in mind. Closely related and proven analogously is the L^p-version

$$\forall u \in W_0^{1,p}(\Omega): \qquad \int_\Omega |\nabla u|^p\,dx \geq \left(\frac{n-p}{p}\right)^p\int_\Omega \frac{|u|^p}{|x|^p}\,dx; \tag{2}$$

here we assume $n>p\geq 1$. With Rellich's inequality in dimensions $n>4$, see [R]:

$$\forall u \in W_0^{2,2}(\Omega): \qquad \int_\Omega (\Delta u)^2\,dx \geq \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4}\,dx \tag{3}$$

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and its generalizations to estimates for $\int_\Omega |x|^{-\sigma} (\Delta u)^2 \, dx$, see Lemma 2 below, it is not difficult to obtain Hardy inequalities in higher order Sobolev spaces $W_0^{k,2}(\Omega)$, where $n > 2k$. See e.g. [DH, Mi]. One has, if $k = 2m$ even

$$\int_\Omega (\Delta^m u)^2 \, dx \geq \left(\prod_{\ell=1}^{2m} \frac{(n+4m-4\ell)^2}{4} \right) \int_\Omega \frac{u^2}{|x|^{4m}} \, dx \tag{4}$$

and if $k = 2m+1$ is odd:

$$\int_\Omega |\nabla \Delta^m u|^2 \, dx \geq \left(\prod_{\ell=1}^{2m+1} \frac{(n+4m+2-4\ell)^2}{4} \right) \int_\Omega \frac{u^2}{|x|^{4m+2}} \, dx. \tag{5}$$

Also for Rellich's inequality there is an extension to the L^p-case (see [DH, p. 520], [Mi]), if $n > 2p$:

$$\forall u \in W_0^{2,p}(\Omega): \qquad \int_\Omega |\Delta u|^p \, dx \geq \left(\frac{(n-2p)(p-1)n}{p^2} \right)^p \int_\Omega \frac{|u|^p}{|x|^{2p}} \, dx. \tag{6}$$

All the constants given here cannot be enlarged. For a more extensive survey and bibliography and also for historical remarks we refer again to [DH].

Refined versions of Hardy inequalities like (1) seem to have appeared first in [Ma, Sect. 2.1.6, Corr. 3]. They were applied by Brezis and Vazquez in [BV] among others to the Gelfand problem

$$-\Delta u = \lambda \exp(u), \quad u \geq 0, \quad u \in W_0^{1,2}(\Omega), \tag{7}$$

where λ is a positive parameter and for the dimension, $n > 2$ is assumed. There exists a $\lambda^* > 0$, such that for $0 \leq \lambda \leq \lambda^*$, this problem is solvable, while for $\lambda > \lambda^*$ it is not. Further, for $\lambda = 2(n-2)$ and $\Omega = B$ the unit ball, one has the singular solution

$$u_{\text{sing}} = -2 \log |x| \in W_0^{1,2}(B).$$

The linearization of (7) around this singular solution leads to the "Hardy-type" operator

$$L_{\text{lin}} \varphi = -\Delta \varphi - \frac{2(n-2)}{|x|^2} \varphi.$$

This operator is studied in [BV] in order to find out among others whether or not the singular solution is also extremal in the sense that it corresponds to λ^*. For this purpose Brezis and Vazquez employ the following Hardy inequality with optimal constant and a remainder term:

$$\forall u \in W_0^{1,2}(\Omega): \qquad \int_\Omega |\nabla u|^2 \, dx \geq \frac{(n-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} \, dx + \Lambda_2 \left(\frac{e_n}{|\Omega|} \right)^{2/n} \int_\Omega u^2 \, dx. \tag{8}$$

Here $n \geq 2$, and Λ_2 denotes the first eigenvalue of the Laplace operator (the subscript 2 refers to the p-Laplacian with $p = 2$, see Definition 1 below) in the two dimensional unit disk. Further e_n and $|\Omega|$ denote the n-dimensional Lebesgue measure of the unit ball $B \subset \mathbb{R}^n$ and the domain Ω respectively. The inequality (8) is shown by means of a "reduction of dimension"-technique, which was implicitly used also already in [Ma, Sect. 2.1.6]. For further applications and variants of (8) we refer to [VZ].

Remainder terms appear also in other Sobolev inequalities, and in the context of nonlinear eigenvalue problems, see e.g. [BL, GG].

The goal of the present paper is to find remainder terms for all the Hardy inequalities (2), (3), (4), (5) and (6) mentioned above. In a future paper we also want to address applications of these refined Hardy inequalities.

In Section 2 the $W_0^{1,p}$-case is studied, corresponding to the p-Laplace operator. Carefully exploiting the convexity properties of the function $\xi \mapsto \xi^p$, we show that an extra L^p-norm may be added. While for $p \geq 2$ the constant for this remainder term is simply an eigenvalue of a certain quasilinear boundary value problem, this constant has to be smaller for $p < 2$, and it vanishes, when $p \searrow 1$.

In Sections 3 and 4 we show that remainder terms

$$\int_\Omega u^2\,dx, \ldots, \int_\Omega \frac{u^2}{|x|^{2k-2}}\,dx$$

may be added in (4) and (5), in Section 3 under Navier boundary conditions $0 = u = \Delta u = \ldots$ on $\partial\Omega$ and in Section 4 under Dirichlet conditions $0 = u = \nabla u = \Delta u = \ldots$ on $\partial\Omega$. In Section 3 also a refinement of the L^p-version (6) of Rellich's inequality is discussed for the case $p \geq 2$. For $1 < p < 2$ we expect a similar phenomenon as in $W_0^{1,p}$, concerning the behaviour of the estimation constants in the remainder terms.

As the class of functions is larger in Section 3, the constants in front of the remainder terms are smaller than in Section 4. The constants which appear in Section 4 seem to be more natural, when the order k of the Sobolev space becomes large. On the other hand, the reduction to the radially symmetric case is more involved, because the symmetrization argument of Section 3 fails and because iterative procedures will not give largest possible constants. For this reason it may be also of interest in its own, how the result in $W_0^{k,2}(B)$ for radial functions is extended to nonsymmetric functions.

2 In $W_0^{1,p}$, $1 < p < n$

We are always restricted to $1 < p < \infty$, if not stated differently.

Definition 1. *Let* $X := \{v \in C^1([0,1]):\ v'(0) = v(1) = 0\} \setminus \{0\}$ *and*

$$\Lambda_p = \inf_X \frac{\int_0^1 r^{p-1}|v'(r)|^p\,dr}{\int_0^1 r^{p-1}|v(r)|^p\,dr}.$$

Remark. For all $p > 1$ we have $\Lambda_p > 0$ and if $p \in \mathbb{N}$, then Λ_p is the first eigenvalue of $-\Delta_p$ in $\mathbb{R}^p$.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \in (1,n)$, and let $e_n = |B_1(0)|$ and $|\Omega|$ denote the n-dimensional Lebesgue measure of the unit ball and of the domain Ω resp.*

(a) If $p \geq 2$, then for every $u \in W_0^{1,p}(\Omega)$ we have

$$\int_\Omega |\nabla u|^p\,dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p}\,dx + \Lambda_p \left(\frac{e_n}{|\Omega|}\right)^{p/n} \int_\Omega |u|^p\,dx. \tag{9}$$

(b) If $1 < p < 2$, then there exists a constant $C = C(n,p) > 0$ such that for every $u \in W_0^{1,p}(\Omega)$ there holds

$$\int_\Omega |\nabla u|^p\,dx \geq \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p}\,dx + C \cdot \left(\frac{e_n}{|\Omega|}\right)^{p/n} \int_\Omega |u|^p\,dx. \tag{10}$$

Remarks.

(i) Following the generalizations in [BV] and [VZ] we expect that also the p-th power of any $W^{1,q}$-norm with $q < p$ or of the L^r-norm of u with $r < \frac{np}{n-p}$ may serve as a remainder term. In that case the constants will be not so simple and less natural also for $p \geq 2$.

(ii) If $p = 1$ and $\Omega = B_1(0)$ the Hardy constant

$$n - 1 = \inf_{u \in W_0^{1,1}(B_1(0)) \setminus \{0\}} \frac{\int |\nabla u| \, dx}{\int \frac{|u|}{|x|} \, dx}$$

is attained on any positive smooth and radially decreasing function u with $u|_{|x|=1} = 0$. This means that for $p = 1$ no remainder term may be added and shows that necessarily $C(n,p) \searrow 0$ as $p \searrow 1$.

(iii) For related inequalities with weights being the inverse of the distance from the boundary of $\partial\Omega$ we refer to [BM, MS].

To prove Theorem 1 we need the following elementary inequalities, which give a quantitative estimate from below for the convexity behaviour of power-like functions. Similar inequalities in a more general context but with smaller constants were obtained in [L].

Lemma 1. *Let $p \geq 1$ and ξ, η be real numbers such that $\xi \geq 0$ and $\xi - \eta \geq 0$. Then*

$$(\xi - \eta)^p + p\,\xi^{p-1}\eta - \xi^p \geq \begin{cases} |\eta|^p, & \text{if } p \geq 2, \\ \frac{1}{2}p(p-1)\frac{\eta^2}{(\xi + |\eta|)^{2-p}}, & \text{if } 1 \leq p \leq 2. \end{cases}$$

Proof. By means of integration by parts or Taylor's formula we find

$$(\xi - \eta)^p + p\,\xi^{p-1}\eta - \xi^p = p(p-1)\eta^2 \int_0^1 (1-t)\,(\xi - t\,\eta)^{p-2}\,dt. \tag{11}$$

We assume first that $p \geq 2$. If $\eta \leq 0$ one has $\xi - t\,\eta \geq t\,|\eta|$ and the remainder term in (11) becomes

$$p(p-1)\eta^2 \int_0^1 (1-t)\,(\xi - t\,\eta)^{p-2}\,dt \geq p(p-1)|\eta|^p \int_0^1 (1-t)\,t^{p-2}\,dt = |\eta|^p.$$

For $\eta \geq 0$ we estimate $\xi - t\,\eta \geq (1-t)\,|\eta|$ and obtain

$$p(p-1)\eta^2 \int_0^1 (1-t)\,(\xi - t\,\eta)^{p-2}\,dt \geq p(p-1)|\eta|^p \int_0^1 (1-t)^{p-1} = (p-1)\,|\eta|^p \geq |\eta|^p.$$

Let now $1 \leq p < 2$, that means that $p - 2 < 0$. Here we estimate $(\xi - t\,\eta)^{p-2} \geq (\xi + |\eta|)^{p-2}$, and there follows:

$$p(p-1)\eta^2 \int_0^1 (1-t)\,(\xi - t\,\eta)^{p-2}\,dt \geq p(p-1)\frac{\eta^2}{(\xi + |\eta|)^{2-p}} \int_0^1 (1-t)\,dt = \frac{1}{2}p(p-1)\frac{\eta^2}{(\xi + |\eta|)^{2-p}}.$$

□

Proof of Theorem 1. After (Schwarz) symmetrization and rescaling we may assume that Ω is the unit ball, which we simply denote with $B = B_1(0)$, and that $u \in W_0^{1,p}(\Omega)$ is nonnegative, radially symmetric and nonincreasing. We recall that symmetrization leaves L^q-norms of functions itself unchanged, increases L^p-norms with the singular weight $|x|^{-p}$ (see e.g. [AL, Theorem 2.2]) and decreases L^p-norms of the gradient (see [AL, Theorem 2.7]). By density we may further assume that u is as smooth as needed.

So, in what follows we may write $u(r)$, $u'(r) = \frac{d}{dr}u(r)$; it holds $|\nabla u(x)| = |u'(r)|$ with $r = |x|$ and we have to find a lower bound for

$$I := \int_0^1 r^{n-1}|u'(r)|^p\, dr - \left(\frac{n-p}{p}\right)^p \int_0^1 r^{n-p-1} u(r)^p\, dr.$$

Similarly as in [BV] a suitable transformation "reduces" the dimension and replaces not only the partial integration, which is usually performed to show Hardy inequalities, but also uncovers a remainder term. We set

$$v(r) := r^{(n/p)-1}\, u(r), \qquad u(r) = r^{1-(n/p)}\, v(r), \tag{12}$$

so that

$$u'(r) = -\frac{n-p}{p} r^{-n/p}\, v(r) + r^{1-(n/p)}\, v'(r).$$

Since u is radially non-increasing we have for $r \in (0,1]$:

$$\frac{n-p}{p} \cdot \frac{v(r)}{r} \geq v'(r), \tag{13}$$

and the expression I to be estimated becomes

$$I = \int_0^1 r^{p-1} \left(\frac{n-p}{p} \cdot \frac{v(r)}{r} - v'(r)\right)^p dr - \left(\frac{n-p}{p}\right)^p \int_0^1 \frac{v(r)^p}{r}\, dr.$$

By (13) we may apply Lemma 1 with

$$\xi = \frac{n-p}{p} \cdot \frac{v(r)}{r} \qquad \text{and} \qquad \eta = v'(r).$$

For $p \geq 2$ we conclude by virtue of $v(0) = v(1) = 0$:

$$\begin{aligned} I &\geq -p\left(\frac{n-p}{p}\right)^{p-1} \int_0^1 v(r)^{p-1} \cdot v'(r)\, dr + \int_0^1 r^{p-1}|v'(r)|^p\, dr \\ &\geq \Lambda_p \int_0^1 r^{p-1} v(r)^p\, dr = \Lambda_p \int_0^1 r^{n-1} u(r)^p\, dr \end{aligned}$$

and (9) follows.

The case $1 < p < 2$ requires greater effort. Again by Lemma 1 and observing $\int_0^1 v^{p-1} \cdot v'\, dr = 0$, it follows that

$$I \geq \frac{1}{2} p(p-1) \int_0^1 \frac{r^{p-1}|v'(r)|^2}{\left(\frac{n-p}{p} \cdot \frac{v(r)}{r} + |v'(r)|\right)^{2-p}}\, dr. \tag{14}$$

In order to estimate this term from below we have to introduce a further regularizing factor r^{ε}, where $\varepsilon > 0$ will be chosen below in dependence on n and p. As a consequence we will miss Λ_p as a constant for the remainder term. Application of Hölder's inequality yields

$$
\begin{aligned}
&\left(\int_0^1 r^{\varepsilon+p-1}|v'(r)|^p\,dr\right)^{2/p}\\
&= \left(\int_0^1 \frac{r^{p(p-1)/2}|v'(r)|^p}{\left(\frac{n-p}{p}\cdot\frac{v(r)}{r}+|v'(r)|\right)^{(2-p)p/2}}\left(r^{\varepsilon+(p-1)(2-p)/2}\left(\frac{n-p}{p}\cdot\frac{v(r)}{r}+|v'(r)|\right)^{(2-p)p/2}\right)dr\right)^{\frac{2}{p}}\\
&\le \left(\int_0^1 \frac{r^{p-1}|v'(r)|^2}{\left(\frac{n-p}{p}\cdot\frac{v(r)}{r}+|v'(r)|\right)^{2-p}}\,dr\right)\left(\int_0^1 r^{2\varepsilon/(2-p)}r^{p-1}\left(\frac{n-p}{p}\cdot\frac{v(r)}{r}+|v'(r)|\right)^p dr\right)^{\frac{2-p}{p}} \qquad (15)
\end{aligned}
$$

For the second integral we find by means of a Hardy inequality in "dimension $(p+\varepsilon)$", which is proved for radial functions as in any "integer dimension":

$$
\begin{aligned}
&\int_0^1 r^{2\varepsilon/(2-p)}r^{p-1}\left(\frac{n-p}{p}\cdot\frac{v(r)}{r}+|v'(r)|\right)^p dr\\
&\le 2^{p-1}\left(\frac{n-p}{p}\right)^p\int_0^1 r^{\varepsilon+p-1}\left(\frac{v(r)}{r}\right)^p dr+2^{p-1}\int_0^1 r^{\varepsilon+p-1}|v'(r)|^p\,dr\\
&\le 2^{p-1}\left(\left(\frac{n-p}{\varepsilon}\right)^p+1\right)\int_0^1 r^{\varepsilon+p-1}|v'(r)|^p\,dr
\end{aligned}
$$

where we have also used the inequality $(a+b)^p\le 2^{p-1}(a^p+b^p)$ $(\forall a,b\ge 0)$ and the fact that $r^{2\varepsilon/(2-p)}<r^{\varepsilon}$ for all $r\in(0,1)$. Combining this estimate with (15), inserting the result into (14) and choosing for the sake of simplicity

$$\varepsilon := 2-p, \qquad (16)$$

we arrive at:

$$I \ge \frac{1}{2}p(p-1)2^{(p-1)(p-2)/p}\left(\left(\frac{n-p}{2-p}\right)^p+1\right)^{(p-2)/p}\int_0^1 r|v'(r)|^p\,dr. \qquad (17)$$

On the other hand, by means of the embedding

$$C_0(p) := \inf_X \frac{\int_0^1 r|v'(r)|^p\,dr}{\left(\int_0^1 r|v(r)|^{p(3-p)}\,dr\right)^{1/(3-p)}} \qquad (18)$$

we have the following inequalities:

$$
\begin{aligned}
\int_0^1 r^{n-1}u(r)^p\,dr &= \int_0^1 r^{p-1}v(r)^p\,dr=\int_0^1 r^{-(p-2)^2/(3-p)}\left(r^{1/(3-p)}v(r)^p\right)dr\\
&\le \left(\int_0^1 r^{p-2}\,dr\right)^{(2-p)/(3-p)}\left(\int_0^1 rv^{p(3-p)}\,dr\right)^{1/(3-p)}\\
&\le \left(\frac{1}{p-1}\right)^{(2-p)/(3-p)}C_0(p)^{-1}\int_0^1 r|v'(r)|^p\,dr.
\end{aligned}
$$

We insert this estimate into (17) and obtain finally

$$\int_B |\nabla u|^p\,dx \ge \left(\frac{n-p}{p}\right)^p\int_B\frac{|u|^p}{|x|^p}\,dx + C(n.p)\int_B |u|^p\,dx$$

with

$$C(n.p) = \frac{1}{2} p\,(p-1)^{1+(2-p)/(3-p)}\,2^{(p-1)(p-2)/p}\left(\left(\frac{n-p}{2-p}\right)^p+1\right)^{(p-2)/p} C_0(p). \qquad (19)$$

□

Remark. Applying the a-priori estmates for the p-Laplace operator of [To] and [E] to the Euler-Lagrange equations corresponding to (18) one could show that $C_0(p) \to \Lambda_2$ as $p \nearrow 2$. Together with the formula (19) for $C(n,p)$ this would yield the convergence of $C(n,p)$ in Theorem 1 to Λ_2 as $p \nearrow 2$, provided $n \geq 3$.

The following result shows that Theorem 1 (b) is sharp in so far, as Λ_p cannot be chosen as a constant for the remainder term, if $p \in (1,2)$.

Proposition 1. *Let $\Omega = B \subset \mathbb{R}^n$ $(n \geq 2)$ be the unit ball. For any $p \in (1,2)$ there is a nonnegative radially nonincreasing function $u_p \in W_0^{1,p}(B)$ such that*

$$\int_B |\nabla u_p|^p\,dx < \left(\frac{n-p}{p}\right)^p \int_B \frac{|u_p|^p}{|x|^p}\,dx + \Lambda_p \int_B |u_p|^p\,dx.$$

Proof. We fix $p \in (1,2)$, $n \geq 2$. Let $v \in C^1([0,1])$ be a nonnegative radially nonincreasing "first eigenfunction" for Λ_p defined in Definition 1, i.e.

$$\int_0^1 r^{p-1}|v'(r)|^p\,dr = \Lambda_p \int_0^1 r^{p-1} v(r)^p\,dr, \qquad v'(0)=v(1)=0.$$

According to (12) we consider

$$u(r) := r^{1-(n/p)} v(r);$$

this function, however, will not be in $W_0^{1,p}(B)$. So, for $\varepsilon \in \left(0, \frac{n-p}{p}\right)$ we introduce functions, which are regularized around 0:

$$v_\varepsilon(r) := r^\varepsilon v(r); \qquad u_\varepsilon(r) := r^{1-(n/p)} v_\varepsilon(r) = r^{\varepsilon+1-(n/p)}\,v(r);$$

and we want to estimate the following expression from *above*:

$$\begin{aligned} &\int_0^1 r^{n-1}|u_\varepsilon'(r)|^p\,dr - \left(\frac{n-p}{p}\right)^p \int_0^1 r^{n-p-1} u_\varepsilon(r)^p\,dr \\ &= \int_0^1 r^{p-1}\left(\frac{n-p}{p}\cdot\frac{v_\varepsilon(r)}{r} - v_\varepsilon'(r)\right)^p dr - \left(\frac{n-p}{p}\right)^p \int_0^1 r^{-1} v_\varepsilon(r)^p\,dr =: I_\varepsilon. \end{aligned}$$

Here we observe that for $\varepsilon \leq (n-p)/p$

$$v_\varepsilon'(r) = r^\varepsilon\,v'(r) + \varepsilon r^\varepsilon \frac{v(r)}{r} \leq \varepsilon\frac{v_\varepsilon(r)}{r} \leq \frac{n-p}{p}\cdot\frac{v_\varepsilon(r)}{r}.$$

That means first that u_ε is nonincreasing and further that for all r, $\xi := \frac{n-p}{p}\cdot\frac{v_\varepsilon(r)}{r}$ and $\eta := v_\varepsilon'(r)$ are in the region which was considered in Lemma 1. A close look at its proof shows that for $\xi \geq 0$, $\xi \geq \eta$ and $p < 2$:

$$(\xi-\eta)^p \leq -p\xi^{p-1}\eta + \xi^p + |\eta|^p,$$

and equality holds only if $\xi = 0$ or $\eta = 0$. As on any compact interval $\subset (0,1]$, v_ε and v_ε' converge uniformly to v and v' resp., we find an interval $[r_0, r_1] \subset (0,1]$ such that we have $\xi = \frac{n-p}{p}\cdot\frac{v_\varepsilon(r)}{r}$ and

$\eta = v'_\varepsilon(r)$ in a suitable compact subset of $(0,\infty)\times(-\infty,0)$ for ε small enough, say $\varepsilon<\bar\varepsilon$. Hence there is a number $\delta>0$ such that we have on $[r_0,r_1]$ uniformly in $\varepsilon<\bar\varepsilon$:

$$\left(\frac{n-p}{p}\cdot\frac{v_\varepsilon(r)}{r}-v'_\varepsilon(r)\right)^p+p\left(\frac{n-p}{p}\right)^{p-1}\left(\frac{v_\varepsilon(r)}{r}\right)^{p-1}v'_\varepsilon(r)-\left(\frac{n-p}{p}\right)^p\left(\frac{v_\varepsilon(r)}{r}\right)^p-|v'_\varepsilon(r)|^p\le-\delta.$$

On the whole interval $[0,1]$ the l.h.s. of this expression is at least nonpositive. We conclude that we have a positive constant $c_0=c_0(p,r_0,r_1)$ such that for all $\varepsilon<\bar\varepsilon$:

$$\begin{aligned}
I_\varepsilon &\le -\delta\int_{r_0}^{r_1} r^{p-1}\,dr-p\left(\frac{n-p}{p}\right)^{p-1}\int_0^1 v'_\varepsilon(r)v_\varepsilon(r)^{p-1}\,dr+\int_0^1 r^{p-1}|v'_\varepsilon(r)|^p\,dr\\
&\le -c_0\delta-\left(\frac{n-p}{p}\right)^{p-1}(v_\varepsilon(1)^p-v_\varepsilon(0)^p)+\int_0^1 r^{p-1}|v'(r)|^p\,dr\\
&\quad +p\int_0^1 r^{p-1}\left(|v'(r)|^{p-1}+|v'_\varepsilon(r)|^{p-1}\right)|v'_\varepsilon(r)-v'(r)|\,dr\\
&\le -c_0\delta+\Lambda_p\int_0^1 r^{p-1}|v_\varepsilon(r)|^p\,dr+C\int_0^1 r^{p-1}\left(1-r^{\varepsilon p}\right)\,dr\\
&\quad +C\int_0^1 r^{p-1}\left(1+\varepsilon^{p-1}r^{(\varepsilon-1)(p-1)}\right)\left((1-r^\varepsilon)+\varepsilon r^{\varepsilon-1}\right)\,dr\\
&\le -c_0\delta+\Lambda_p\int_0^1 r^{n-1}|u_\varepsilon(r)|^p\,dr+\mathcal{O}\left(\varepsilon^{p-1}\right)\\
&< \Lambda_p\int_0^1 r^{n-1}|u_\varepsilon(r)|^p\,dr
\end{aligned}$$

for ε small enough; $u_p(x):=u_\varepsilon(|x|)$ satisfies the stated reverse inequality. □

3 In higher order Sobolev spaces under Navier boundary conditions

Also here a similar "reduction of dimension"-technique as in [BV] and as in the preceding section will be employed.

Let $p\ge 2$; in the sequel we make use of some "lower dimensional" Laplacians: for all smooth radially symmetric function $v=v(r)$ we denote

$$\Delta_\vartheta v:=v''+\frac{\vartheta-1}{r}v'\qquad\text{where}\quad \vartheta=\vartheta(p)=4+\frac{n(p-2)}{p}.$$

Then we define the "generalized eigenvalues"

Definition 2. *Let $X=\{v\in C^1([0,1]):\ v'(0)=v(1)=0,\ v\not\equiv 0\}$ and let*

$$\lambda=\inf_X\frac{\int_0^1 r|v'(r)|^2\,dr}{\int_0^1 rv(r)^2\,dr}\qquad\textit{and}\qquad \Lambda_\vartheta=\inf_X\frac{\int_0^1 r^{2p-1}|\Delta_\vartheta v(r)|^p\,dr}{\int_0^1 r^{2p-1}|v(r)|^p\,dr}.$$

This notation should not be confused with that of Section 2: we also point out that λ *does not* depend on p.

In the $W^{2,p}$ setting, the best Hardy constant is C_p^p (see [DH, Theorem 12], [Mi, Theorem 3.1]), where

$$C_p=\frac{(n-2p)(p-1)n}{p^2}.$$

We can now state

Theorem 2. *Let $p \geq 2$, let $\Omega \subset \mathbb{R}^n$ $(n \geq 2p)$ be a bounded domain and denote as above by $|\Omega|$ its n-dimensional Lebesgue measure and by $e_n = |B|$, where $B = B_1(0)$ is the unit ball. Then for any $u \in W^{2,p} \cap W_0^{1,p}(\Omega)$ one has:*

$$\begin{aligned}\int_\Omega |\Delta u|^p \, dx \quad &\geq \quad \left(\frac{(n-2p)(p-1)n}{p^2}\right)^p \int_\Omega \frac{|u|^p}{|x|^{2p}} \, dx \qquad (20)\\ &+ \frac{4(p-1)^p(n-2p)^{p-1}n^{p-1}}{p^{2p-1}} \lambda \left(\frac{e_n}{|\Omega|}\right)^{2/n} \int_\Omega \frac{|u|^p}{|x|^{2p-2}} \, dx + \Lambda_\vartheta \left(\frac{e_n}{|\Omega|}\right)^{2p/n} \int_\Omega |u|^p \, dx.\end{aligned}$$

Proof. STEP 1. By scaling it suffices to consider $|\Omega| = e_n$. We show first that we may restrict ourselves to $\Omega = B$ and radial superharmonic functions $u(r)$. For this purpose we assume that (20) has already been shown in this setting, and let now Ω be arbitrary with $|\Omega| = e_n$ and $u \in W^{2,p} \cap W_0^{1,p}(\Omega)$. We define $f = -\Delta u$, let $w \in W^{2,p} \cap W_0^{1,p}(B)$ be a (radial) strong solution of

$$\begin{cases} -\Delta w = f^* & \text{in } B, \\ w = 0 & \text{on } \partial B, \end{cases}$$

where $f^* \geq 0$ denotes the Schwarz symmetrization of f. By [Ta, Theorem 1] we know that $w \geq u^* \geq 0$, so that

$$\begin{aligned}\int_\Omega |\Delta u|^p \, dx \quad &= \quad \int_\Omega |f|^p \, dx = \int_B (f^*)^p \, dx = \int_B |\Delta w|^p \, dx \\ \int_B \frac{|w|^p}{|x|^\alpha} \, dx \quad &\geq \quad \int_B \frac{|u^*|^p}{|x|^\alpha} \, dx \geq \int_\Omega \frac{|u|^p}{|x|^\alpha} \, dx, \qquad \alpha \in \{0, 2p-2, 2p\};\end{aligned}$$

for the last inequality we again refer to [AL, Theorem 2.2]. Assuming (20) in the radial superharmonic setting, the same follows for any domain Ω and any $u \in W^{2,p} \cap W_0^{1,p}(\Omega)$.

STEP 2. We assume now that $\Omega = B$ and that u is superharmonic radially symmetric: $u = u(r)$, $r = |x|$. Similarly as in (12) we introduce the transformation

$$v(r) := r^{(n/p)-2} u(r), \qquad u(r) = r^{2-(n/p)} v(r),$$

so that

$$-\Delta u(r) = r^{-n/p} \left(-r^2 \Delta_\vartheta v(r) + C_p v(r)\right) \geq 0,$$

and a "lower dimensional Laplacian" of v arises. This allows us to apply Lemma 1 (with $\xi = C_p v$ and $\eta = r^2 \Delta_\vartheta v$) and to obtain

$$\begin{aligned} I_p \quad &:= \quad \int_0^1 r^{n-1} (-\Delta u)^p \, dr - C_p^p \int_0^1 r^{n-2p-1} |u|^p \, dr \\ &= \quad \int_0^1 r^{-1} \left[\left(C_p v - r^2 \Delta_\vartheta v\right)^p - (C_p v)^p\right] dr \\ &\geq \quad \int_0^1 r^{-1} \left(-p C_p^{p-1} v^{p-1} r^2 \Delta_\vartheta v + r^{2p} |\Delta_\vartheta v|^p\right) dr \\ &= \quad -p C_p^{p-1} \int_0^1 r v^{p-1} v'' \, dr + \int_0^1 r^{2p-1} |\Delta_\vartheta v|^p \, dr, \end{aligned}$$

where we have used the boundary datum $u(1) = v(1) = 0$. With an integration by parts and using the identity $v^{p-2}|v'|^2 = \frac{4}{p^2}|(v^{p/2})'|^2$ we get

$$-\int_0^1 rv^{p-1}v''\,dr = (p-1)\int_0^1 rv^{p-2}|v'|^2\,dr = \frac{4(p-1)}{p^2}\int_0^1 r|(v^{p/2})'|^2\,dr.$$

Therefore, recalling the definition of λ and Λ_ϑ, we infer

$$\begin{aligned} I_p &\geq \frac{4(p-1)}{p}C_p^{p-1}\lambda\int_0^1 r|v|^p\,dr + \Lambda_\vartheta\int_0^1 r^{2p-1}|v|^p\,dr \\ &= \frac{4(p-1)}{p}C_p^{p-1}\lambda\int_0^1 r^{n-1}\frac{|u|^p}{r^{2p-2}}\,dr + \Lambda_\vartheta\int_0^1 r^{n-1}|u|^p\,dr, \end{aligned}$$

thereby completing the proof of Theorem 2. □

Remark. In the case $1 < p < 2$, by arguing as in the previous section, one obtains a statement similar to Theorem 1 (b) with a “smaller” remainder term.

In order to avoid tedious calculations, from now on we deal only with the Hilbert case $p = 2$, the case $p > 2$ being similar. To make more precise the statements we introduce some further notations.

Definition 3. *Let* $X = \{v \in C^1([0,1]) :\ v'(0) = v(1) = 0,\ v \not\equiv 0\}$ *and for* $n \in \mathbb{N}$ *let*

$$\Lambda(n) = \inf_X \frac{\int_0^1 r^{n-1}|v'(r)|^2\,dr}{\int_0^1 r^{n-1}v(r)^2\,dr}.$$

Again, this notation should not be confused with the previous ones.

Proposition 2. *Let* $B \subset \mathbb{R}^n$ *be the unit ball. The first eigenvalue* λ_0 *of the following Navier boundary value problem*

$$\begin{cases} \Delta^2 u = \lambda u \ \text{in } B, \\ u = \Delta u = 0 \ \text{on } \partial B, \end{cases}$$

satisfies

$$\Lambda(n)^2 = \lambda_0 = \inf_{u \in W^{2,2}\cap W_0^{1,2}(B)\setminus\{0\}} \frac{\int_B (\Delta u)^2\,dx}{\int_B u^2\,dx}.$$

Proof. The first equality is due to the fact that the biharmonic operator $u \mapsto \Delta^2 u$ under Navier boundary conditions $u = \Delta u = 0$ on ∂B is actually the square of the Laplacian under Dirichlet conditions. For the second equality, see e.g. [V, Lemma B3]. □

With these notations we obtain directly from Theorem 2:

Corollary 1. *Let* $\Omega \subset \mathbb{R}^n$*,* $n \geq 4$*, be a bounded domain and denote as above by* $|\Omega|$ *its* n*-dimensional Lebesgue measure and by* $e_n = |B|$*, where* $B = B_1(0)$ *is the unit ball. Then for any* $u \in W^{2,2} \cap W_0^{1,2}(\Omega)$ *one has:*

$$\begin{aligned} \int_\Omega (\Delta u)^2\,dx \ &\geq\ \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4}\,dx \\ &\quad + \frac{n(n-4)}{2}\Lambda(2)\left(\frac{e_n}{|\Omega|}\right)^{2/n}\int_\Omega \frac{u^2}{|x|^2}\,dx + \Lambda(4)^2\left(\frac{e_n}{|\Omega|}\right)^{4/n}\int_\Omega u^2\,dx. \end{aligned}$$

Combining this result with the original extended Hardy inequality of Brezis-Vazquez (8) already mentioned in the introduction, we have for third order Sobolev spaces:

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 6$, be a bounded domain. Then for any $u \in W^{3,2} \cap W_0^{1,2}(\Omega)$ with $\Delta u = 0$ on $\partial\Omega$, i.e. $\Delta u \in W_0^{1,2}(\Omega)$, one has:*

$$\begin{aligned}\int_\Omega |\nabla \Delta u|^2 \, dx \quad \geq \quad & \frac{(n+2)^2(n-2)^2(n-6)^2}{64} \int_\Omega \frac{u^2}{|x|^6} \, dx \\ & + \frac{1}{16} \left(3(n-2)^4 + 8(n-2)^2 + 16\right) \Lambda(2) \left(\frac{e_n}{|\Omega|}\right)^{2/n} \int_\Omega \frac{u^2}{|x|^4} \, dx \\ & + \left(\frac{(n-2)^2}{4} \Lambda(4)^2 + \frac{n(n-4)}{2} \Lambda(2)^2\right) \left(\frac{e_n}{|\Omega|}\right)^{4/n} \int_\Omega \frac{u^2}{|x|^2} \, dx \\ & + \Lambda(2)\, \Lambda(4)^2 \left(\frac{e_n}{|\Omega|}\right)^{6/n} \int_\Omega u^2 \, dx.\end{aligned}$$

Proof. First we may reduce to $\Omega = B$ and the radial setting by combining the argument from the proof of Theorem 2 and [AL, Theorem 2.7].

The application of the Brezis-Vazquez inequality (8) reduces the order of terms by one:

$$\int_0^1 r^{n-1} |\nabla \Delta u|^2 \, dr \geq \frac{(n-2)^2}{4} \int_0^1 r^{n-3} (\Delta u)^2 \, dr + \Lambda(2) \int_0^1 r^{n-1} (\Delta u)^2 \, dr.$$

The second term will be estimated directly with help of Corollary 1, while for the singular term

$$\int_0^1 r^{n-3} (\Delta u)^2 \, dr$$

we have to extend that proof by means of the transformation

$$v(r) := r^{(n/2)-3} u(r), \qquad u(r) = r^{3-(n/2)} v(r),$$

so that

$$\begin{aligned} & \int_0^1 r^{n-3} (\Delta u)^2 \, dr - \frac{(n+2)^2(n-6)^2}{16} \int_0^1 r^{n-7} u^2 \, dr \\ = \quad & \int_0^1 r^3 \left(\Delta_4 v + \frac{2}{r} v'\right)^2 dr - \frac{(n+2)(n-6)}{2} \int_0^1 r \left(v'' + \frac{5}{r} v'\right) v \, dr \\ = \quad & \int_0^1 r^3 (\Delta_4 v)^2 \, dr + 4 \int_0^1 r^2 (\Delta_4 v)\, v' \, dr \\ & + 4 \int_0^1 r \left(v'\right)^2 dr + \frac{(n+2)(n-6)}{2} \int_0^1 r \left(-\Delta_2 v\right) v \, dr \\ = \quad & \int_0^1 r^3 (\Delta_4 v)^2 \, dr + 2 \left[r^2 v'(r)^2\right]_0^1 + \frac{1}{2} \left((n-2)^2 + 8\right) \int_0^1 r \left(-\Delta_2 v\right) v \, dr \\ \geq \quad & \Lambda(4)^2 \int_0^1 r^3 v^2 \, dr + \frac{1}{2} \left((n-2)^2 + 8\right) \Lambda(2) \int_0^1 r v^2 \, dr \\ = \quad & \Lambda(4)^2 \int_0^1 r^{n-3} u^2 \, dr + \frac{1}{2} \left((n-2)^2 + 8\right) \Lambda(2) \int_0^1 r^{n-5} u^2 \, dr. \end{aligned}$$

Note that only $u(1) = v(1) = 0$ was needed. Collecting terms yields the stated inequality. $\square$

In order to iterate further we quote from [DH, Theorem 12], [Mi, Theorem 3.3]:

Lemma 2. *Let $\Omega \subset \mathbb{R}^n$ be a sufficiently smooth bounded domain and $\sigma < n-4$. Then for every $u \in C^2(\bar{\Omega})$ with $u|_{\partial\Omega} = 0$ we have*

$$\int_\Omega \frac{(\Delta u)^2}{|x|^\sigma}\, dx \geq \frac{(n-4-\sigma)^2(n+\sigma)^2}{16} \int_\Omega \frac{u^2}{|x|^{\sigma+4}}\, dx.$$

We emphasize that it is sufficient to have $u|_{\partial\Omega} = 0$.

With help of this result it is not difficult to see that there are remainder terms in Hardy inequalities of arbitrary order.

Corollary 2. *Let $\Omega \subset \mathbb{R}^n$ be a sufficiently smooth domain and let $k \in \mathbb{N}$, $2k \leq n$. Then there exist constants $c_1, \ldots, c_k$, depending only on k, n and $|\Omega|$, such that for every $u \in W^{2,k}(\Omega)$ with $\Delta^j u|_{\partial\Omega} = 0$ for $j \in \mathbb{N}_0$ and $2j < k$ there holds:*

(a) if k even, $k = 2m$, $m \in \mathbb{N}$:

$$\int_\Omega (\Delta^m u)^2\, dx \geq \frac{1}{4^{2m}} \left(\prod_{j=1}^{2m} (n + 4(m-j))^2 \right) \int_\Omega \frac{u^2}{|x|^{4m}}\, dx + \sum_{\ell=1}^{2m} c_\ell \int_\Omega \frac{u^2}{|x|^{4m-2\ell}}\, dx;$$

(b) if k odd, $k = 2m+1$, $m \in \mathbb{N}_0$:

$$\begin{aligned}\int_\Omega |\nabla \Delta^m u|^2\, dx \quad &\geq \quad \frac{1}{4^{2m+1}} \left(\prod_{j=1}^{2m+1} (n + 4m + 2 - 4j)^2 \right) \int_\Omega \frac{u^2}{|x|^{4m+2}}\, dx \\ &\quad + \sum_{\ell=1}^{2m+1} c_\ell \int_\Omega \frac{u^2}{|x|^{4m+2-2\ell}}\, dx.\end{aligned}$$

The existence of some constants $c_1, \ldots, c_k$ as above is easily shown, of particular interest would be their respective largest possible value. But already the proof of Theorem 3 indicates that iterative methods under Navier boundary conditions $0 = u = \Delta u = \ldots$ on $\partial\Omega$ yield constants which cannot be easily given in closed form and which presumably will not be optimal.

One may hope that things will improve under the more restrictive Dirichlet conditions, i.e. $u \in W_0^{k,2}(\Omega)$. This will be discussed in the following section.

4 In higher order Sobolev spaces under Dirichlet boundary conditions

Our first goal will be to adapt Corollary 1, i.e. Theorem 2 in the case $p = 2$, to Dirichlet conditions, i.e. to the space $W_0^{2,2}(\Omega)$ instead of $W^{2,2} \cap W_0^{1,2}(\Omega)$. At least when Ω is a ball or "close" to a ball the estimation constants will be considerably larger. On the other hand the symmetrization argument in the proof of Theorem 2 fails: the function w used in that proof is not in the class $W_0^{2,2}(B)$ to be considered here. We overcome this difficulty by a relatively involved argument, which bases on positivity properties of Green functions and seems to work only in the Hilbert space case $p = 2$ and if Ω is replaced with the *circumscribed* ball.

As the technical difficulties to obtain best possible constants increase rapidly with the order k of the space $W_0^{k,2}(\Omega)$, we give a complete discussion only for the improved Hardy inequality in

$W_0^{2,2}(\Omega)$. For the general case we only give a conjecture to show the expected behaviour of the remainder terms, when the order of spaces become arbitrarily large.

In this section also eigenvalues of biharmonic and more general polyharmonic operators will play an important role.

Definition 4. *Let $B \subset \mathbb{R}^n$ denote again the unit ball and let*

$$\Lambda\left((-\Delta)^k, n\right) = \begin{cases} \inf\limits_{W_0^{k,2}(B)\setminus\{0\}} \dfrac{\int_B (\Delta^m u)^2\, dx}{\int_B u^2\, dx}, & k = 2m,\ m \in \mathbb{N}; \\ \inf\limits_{W_0^{k,2}(B)\setminus\{0\}} \dfrac{\int_B |\nabla \Delta^m u|^2\, dx}{\int_B u^2\, dx}, & k = 2m+1,\ m \in \mathbb{N}_0. \end{cases}$$

Remark. The notation $\Lambda(n)$ of Section 3 is a special case by $\Lambda(n) = \Lambda\left((-\Delta), n\right)$.

Taking into account Proposition 2 we immediately see $\Lambda\left((-\Delta)^2, n\right) \geq \Lambda(-\Delta, n)^2$. With help of qualitative properties of first eigenfunctions in particular under Navier boundary conditions, also strict inequality can be shown. Moreover the ratio of these eigenvalues is rather large, by elementary calculations one finds e.g. $\Lambda\left((-\Delta)^2, 1\right) = 31.285243\ldots$ while $\Lambda\left((-\Delta), 1\right)^2 = \pi^4/16 = 6.088068\ldots$. For $n = 4$, the case which is needed below, a rough estimate according to [PS, p. 57] gives $\Lambda\left((-\Delta)^2, 4\right) \geq j_1^2 j_2^2 = 387.23\ldots$ while $\Lambda\left((-\Delta), 4\right)^2 = j_1^4 = 215.56\ldots$.

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 4$, be a bounded domain, $\Omega \subset B_R(0)$. Then for every $u \in W_0^{2,2}(\Omega)$ one has:*

$$\begin{aligned} \int_\Omega (\Delta u)^2\, dx \quad &\geq \quad \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4}\, dx \\ &\quad + \frac{n(n-4)}{2} \Lambda(-\Delta, 2)\, R^{-2} \int_\Omega \frac{u^2}{|x|^2}\, dx + \Lambda\left((-\Delta)^2, 4\right) R^{-4} \int_\Omega u^2\, dx. \end{aligned} \tag{21}$$

Proof. By trivial extension we have $W_0^{2,2}(\Omega) \hookrightarrow W_0^{2,2}(B_R(0))$, after suitable scaling we may assume that $\Omega = B$.

If u is additionally radially symmetric, we proceed as in Step 2 of the proof of Theorem 2 with $p = 2$. In its last conclusion we may instead exploit that u satisfies homogeneous Dirichlet boundary conditions and replace $\Lambda_\vartheta|_{\vartheta=4} = \Lambda(4)^2 = \Lambda(-\Delta, 4)^2$ with $\Lambda\left((-\Delta)^2, 4\right)$, thereby proving (21) for radial u.

It remains to extend (21) to arbitrary functions $u \in W_0^{2,2}(B)$. The idea is that the formal Euler-Lagrange equation for a minimum problem associated with (21) and with the eigenvalue parameter in front of the $\int_B u^2\, dx$-term is linear and has only radial coefficients. Hence radialization of any solution would give again a solution. That the latter solution is not the trivial one has to be avoided e.g. with starting with a positive solution. Hence a minimizer could be chosen radial. However, the mentioned eigenvalue problem is "critical" in so far, as the infimum may not be attained.

For that reason we consider for $\ell \in \mathbb{N}$ a sequence of relaxed minimum problems:

$$\mu_\ell := \inf_{W_0^{2,2}(B)\setminus\{0\}} \frac{F_\ell(v)}{\int_B v^2\, dx},$$

where

$$\begin{aligned} F_\ell(v) &= \int_B (\Delta v)^2\, dx - \frac{n^2(n-4)^2}{16}\left(1-\frac{1}{\ell}\right)\int_B \frac{v^2}{|x|^4}\, dx \\ &\quad - \frac{n(n-4)}{2}\,\Lambda\left(-\Delta,2\right)\left(1-\frac{1}{\ell}\right)\int_B \frac{v^2}{|x|^2}\, dx. \end{aligned}$$

As we may already use Corollary 1 we have $F_\ell(v) \geq \frac{1}{\ell}\int_B (\Delta v)^2\, dx$ for every $v \in W_0^{2,2}(B)$. That means that for every minimizing sequence $v_k \in W_0^{2,2}(B)$ with $\int_B v_k^2\, dx = 1$ and $\lim_{k\to\infty} F_\ell(v_k) = \mu_\ell$, we have boundedness of $\left(\|u_k\|_{W_0^{2,2}(B)}\right)_{k\in\mathbb{N}}$. After selecting a subsequence we may assume that $u_k \rightharpoonup u \in W_0^{2,2}(B)$ and $u_k \to u$ strongly in $L^2(B)$. The bilinear form

$$\begin{aligned} F_\ell(v,w) &:= \int_B \Delta v\cdot\Delta w\, dx - \frac{n^2(n-4)^2}{16}\left(1-\frac{1}{\ell}\right)\int_B \frac{v\cdot w}{|x|^4}\, dx \\ &\quad - \frac{n(n-4)}{2}\,\Lambda\left(-\Delta,2\right)\left(1-\frac{1}{\ell}\right)\int_B \frac{v\cdot w}{|x|^2}\, dx \end{aligned}$$

associated with $F_\ell(v)$ defines a scalar product on $W_0^{2,2}(B)$, which by

$$\frac{1}{\ell}\int_B (\Delta v)^2\, dx \leq F_\ell(v,v) \leq \int_B (\Delta v)^2\, dx = \|v\|_{W_0^{2,2}(B)}$$

gives an equivalent norm. If we consider just for the following argument $W_0^{2,2}(B)$ with $F_\ell(\,.\,,\,.\,)$ as scalar product, then lower semicontinuity of the corresponding norm in the weak topology gives

$$\mu_\ell = \liminf_{k\to\infty} F_\ell(v_k) \geq F_\ell(v).$$

We have further

$$1 = \lim_{k\to\infty}\int_B v_k^2\, dx = \int_B v^2\, dx.$$

Hence $F_\ell(v) = \mu_\ell$, and $v \in W_0^{2,2}(B)$ is an optimal (nontrivial) function for μ_ℓ. Consequently v is a weak solution of the Euler-Lagrange equation

$$\begin{cases} \Delta^2 v = \left(1-\frac{1}{\ell}\right)\frac{n^2(n-4)^2}{16}\frac{v}{|x|^4} + \left(1-\frac{1}{\ell}\right)\frac{n(n-4)}{2}\Lambda(-\Delta,2)\frac{v}{|x|^2} + \mu_\ell v & \text{in } B, \\ v = \nabla v = 0 & \text{on } \partial B. \end{cases} \tag{22}$$

Next we show that v is of fixed sign. Assume by contradiction that there are subsets $B^+, B^- \subset B$ both with positive measure such that $v > 0$ on B^+ and $v < 0$ on B^-. We may now apply a decomposition method explained in detail in [GG, Section 3], cf. also [Mo]. Let

$$\mathcal{K} = \left\{u \in W_0^{2,2}(B):\ u \geq 0\right\}$$

be the closed convex cone of nonnegative functions and

$$\mathcal{K}^* = \left\{u^* \in W_0^{2,2}(B):\ (u^*, u) \leq 0 \text{ for all } u \in \mathcal{K}\right\}$$

the dual cone. Here $(u^*, u) = \int_B (\Delta u^*)\,(\Delta u)\, dx$ is the standard scalar product in $W_0^{2,2}(B)$. $\mathcal{K}^*$ is the cone of all weak subsolutions of the biharmonic equation in B under Dirichlet conditions. By a comparison result of Boggio [Bo, p. 126] the elements of $\mathcal{K}^*$ are nonpositive. We decompose

$$v = v_1 + v_2, \qquad v_1 \in \mathcal{K}, \qquad v_2 \in \mathcal{K}^*, \qquad v_1 \perp v_2.$$

As $0 \not\equiv v_1 \geq 0$, $0 \not\equiv v_2 \leq 0$, replacing $v = v_1 + v_2$ with $v_1 - v_2 \geq 0$ yields

$$F_\ell(v_1 + v_2) > F_\ell(v_1 - v_2), \qquad \int_B (v_1 + v_2)^2 \, dx < \int_B (v_1 - v_2)^2 \, dx,$$

contradicting the minimality of $F_\ell(v) / \int_B v^2 \, dx$.

Hence we may assume $0 \not\equiv v \geq 0$. Considering polar coordinates $x = r\xi$, $r \in [0,1]$, $|\xi| = 1$ and integrating the Euler-Lagrange equation (22) over $\{|\xi| = 1\}$ shows that

$$w(r) := \frac{1}{n\, e_n} \int_{|\xi|=1} v(r\,\xi)\, d\omega(\xi) \in W_0^{2,2}(B)$$

is a *radial* weak solution of (22). By virtue of $0 \not\equiv v \geq 0$, we also have $0 \not\equiv w \geq 0$. As we may already use the Hardy inequality (21) for the radial function w we may conclude:

$$\begin{aligned} \mu_\ell &= \frac{F_\ell(w)}{\int_B w^2 \, dx} > \frac{\int_B (\Delta w)^2 \, dx - \frac{n^2(n-4)^2}{16} \int_B \frac{w^2}{|x|^4} \, dx - \frac{n(n-4)}{2} \Lambda(-\Delta, 2) \int_B \frac{w^2}{|x|^2} \, dx}{\int_B w^2 \, dx} \\ &\geq \lambda\left((-\Delta)^2, 4\right). \end{aligned}$$

To sum up, we have shown that for any $\ell \in \mathbb{N}$ one has for every $u \in W_0^{2,2}(B)$:

$$\begin{aligned} \int_B (\Delta u)^2 \, dx \;&\geq\; \left(1 - \frac{1}{\ell}\right) \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} \, dx \\ &\quad + \left(1 - \frac{1}{\ell}\right) \frac{n(n-4)}{2} \Lambda(-\Delta, 2) \int_B \frac{u^2}{|x|^2} \, dx + \Lambda\left((-\Delta)^2, 4\right) \int_B u^2 \, dx. \end{aligned}$$

Letting $\ell \to \infty$, the stated Hardy inequality (21) follows. □

We conclude with some remarks concerning general higher order Sobolev spaces. We recall that $W_0^{k,2}(\Omega)$ is equipped with the norm

$$\|u\|^2_{W_0^{k,2}(\Omega)} = \begin{cases} \displaystyle\int_\Omega (\Delta^m u)^2 \, dx, & \text{if } k = 2m, \; m \in \mathbb{N}; \\ \displaystyle\int_\Omega |\nabla \Delta^m u|^2 \, dx, & \text{if } k = 2m+1, \; m \in \mathbb{N}_0. \end{cases}$$

We expect the following result:

Conjecture 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\Omega \subset B_R(0)$. Let $k \in \mathbb{N}$, $n \geq 2k$. Then for all $u \in W_0^{k,2}(\Omega)$ there holds:*

$$\begin{aligned} \|u\|^2_{W_0^{k,2}(\Omega)} \;&\geq\; \sum_{j=0}^{k} \frac{1}{4^j} \binom{k}{j} \left\{ \prod_{\ell=1}^{j} \left((n + 2k - 4\ell)(n - 2k - 4 + 4\ell)\right) \right\} \\ &\quad \cdot \Lambda\left((-\Delta)^{k-j}, 2k - 2j\right) R^{2j-2k} \int_\Omega \frac{u^2}{|x|^{2j}} \, dx \end{aligned} \tag{23}$$

with the convention that $\Lambda((-\Delta)^0, 0) = 1$.

In the radial setting the terms

$$\Lambda\left((-\Delta)^{k-j}, 2k - 2j\right) \int_B \frac{u^2}{|x|^{2j}} \, dx$$

originate from

$$ne_n \int_0^1 r^{2k-2j-1} \left(\Delta_{2k-2j}^{(k-j)/2} v\right)^2 \, dr$$

if $k-j$ is even, and from

$$ne_n \int_0^1 r^{2k-2j-1} \left(\left(\Delta_{2k-2j}^{(k-j-1)/2} v\right)'\right)^2 \, dr$$

if $k-j$ is odd. Here Δ_{2k-2j} is the radial Laplacian in dimension $2k-2j$. The radial functions u and v are related by means of the transformation

$$v(r) = r^{(n/2)-k} u(r), \qquad u(r) = r^{k-(n/2)} v(r).$$

In Conjecture 1 as well as in the proof of Theorem 4 we simply used for these terms eigenvalue estimates. One may also wish to use Hardy-like inequalities in order to have largest possible constants in front of the most singular remainder terms. The following Lemma may be applied, because we assume every function to satisfy homogeneous Dirichlet boundary conditions.

Lemma 3.

(a) Let $k \geq 2$, $j \geq 1$, $v \in C^2([0,1])$ with $v'(1) = 0$. Then:

$$\int_0^1 r^k \left(\Delta_j v\right)^2 \, dr \geq \frac{(k+1-2j)^2}{4} \int_0^1 r^{k-2} \left(v'\right)^2 \, dr.$$

(b) Let $k \geq 2$, $v \in C^1([0,1])$ with $v(1) = 0$. Then:

$$\int_0^1 r^k \left(v'\right)^2 \, dr \geq \frac{(k-1)^2}{4} \int_0^1 r^{k-2} v^2 \, dr.$$

To illustrate how the application of this elementary and well-known Lemma shifts less singular remainder terms to more singular ones, we modify the proof of Theorem 4 and obtain:

Corollary 3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 4$, be a bounded domain, $\Omega \subset B_R(0)$. Then for every $u \in W_0^{2,2}(\Omega)$ one has:*

$$\begin{aligned}\int_\Omega (\Delta u)^2 \, dx \quad \geq \quad & \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} \, dx \\ & + \frac{1}{2} \left(n(n-4)+8\right) \Lambda\left(-\Delta, 2\right) R^{-2} \int_\Omega \frac{u^2}{|x|^2} \, dx.\end{aligned}$$

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