Decay and eventual local positivity for biharmonic parabolic equations *

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Abstract

We study existence and positivity properties for solutions of Cauchy problems for both linear and semilinear parabolic equations with the biharmonic operator as elliptic principal part. The self-similar kernel of the parabolic operator $\partial_t + \Delta^2$ is a sign changing function and the solution of the evolution problem with a positive initial datum may display almost instantaneous change of sign. We determine conditions on the initial datum for which the corresponding solution exhibits some kind of positivity behaviour. We prove eventual local positivity properties both in the linear and semilinear case. At the same time, we show that negativity of the solution may occur also for arbitrarily large given time, provided the initial datum is suitably constructed.

1 Introduction and results

Contrary to the second order heat equation, no general positivity preserving property (ppp in the sequel) holds for the following Cauchy problem for fourth order parabolic equations

$$\begin{cases} u_t + \Delta^2 u = 0 & \text{in } \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times [0,\infty) \\ u\,(x,0) = u_0(x) & \text{in } \mathbb{R}^n\ , \end{cases} \tag{1}$$

where $n \geq 1$ and $u_0 \in C^0 \cap L^\infty(\mathbb{R}^n)$. By ppp, we mean here that positivity of the initial datum u_0 implies positivity (in space and time) for the solution $u = u(x,t)$ of (1). In fact, no global ppp for (1) may be expected at all, see [2, 4, 11]. This common feeling is based on observing the oscillatory behaviour of the biharmonic heat kernel. However, a more careful analysis shows that, under suitable assumptions, some restricted and somehow hidden versions of the ppp can be observed, see [10]. A better understanding of the behaviour of the kernels will certainly allow to reach stronger versions of the ppp. It is precisely the first goal of the present paper to analyze in some detail the biharmonic heat kernels: we determine explicitly their power series expansion, we obtain a third order differential equation satisfied by the kernels and we show that the kernels have infinitely many damped oscillations. With these tools we are then able to reach the second goal of the present paper, a rather complete description of versions of the ppp for (1).
All these observations are then applied to the corresponding nonlinear problem

$$\begin{cases} u_t + \Delta^2 u = |u|^{p-1}u & \text{in } \mathbb{R}^{n+1}_+ \\ u\,(x,0) = u_0(x) & \text{in } \mathbb{R}^n \end{cases} \tag{2}$$

where $p > 1 + 4/n$; the exponent $1 + 4/n$ is the analogue of the "Fujita"-exponent (see [6, 15] and the references therein), arising in second order semilinear Cauchy problems. Our third purpose is to extend

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the global existence result and the decay estimates obtained in [9] to the case where the initial datum u_0 has *supercritical decay* at infinity in order to show that in these cases, when u_0 is positive and has a suitable asymptotic profile, global solutions of (2) are eventually locally positive.

For the somehow related equation with *positive* nonlinearity $u_t + \Delta^2 u = |u|^p$, it is known [5] that solutions blow up in finite time in the sub-Fujita case $1 < p \leq 1 + \frac{4}{n}$, provided the initial datum has nonnegative integral over $\mathbb{R}^n$. For the very same equation in the super-Fujita case $p > 1 + \frac{4}{n}$, interesting and useful techniques were introduced by Galaktionov-Pohožaev [7] and Caristi-Mitidieri [3]. Although the nonlinearity in (2) may change sign, in some parts of our proofs we take advantage of these techniques.
The corresponding second order semilinear parabolic problem has been widely studied; in this case and in contrast with our situation, powerful tools like strong maximum principles and construction of auxiliary functions satisfying suitable differential inequalities are available. For an overview of these results, we refer to the introduction in [9], to [15] and to references therein.
Let us now explain in some more detail, what is known about (1) and (2) and what we are going to prove in the present work.

1.1 Positivity in the linear problem

For the biharmonic heat equation,

$$u_t + \Delta^2 u = 0 \qquad (t > 0), \tag{3}$$

either in $\mathbb{R}^n$ or in a bounded smooth domain (then complemented with suitable boundary conditions) no positivity preserving of the solution with respect to the initial datum holds true. In general, one even has to expect instantaneous change of sign, see e.g. [2, 4, 11], which is a property of the differential equation and can be observed independently of a possible choice of boundary data and independently of whether it is considered in bounded or unbounded domains.
On the other hand, there exist bounded domains with boundary conditions such that the corresponding elliptic first eigenvalue is simple and the first eigenfunction is of fixed sign and displays a nondegenerate behaviour at the boundary, see [12] and references therein. In these domains and for each positive initial datum, in dependence of this datum the solution of the initial boundary value problem is eventually positive. This positivity comes up almost immediately, since the higher modes in the expansion with respect to the eigenfunctions decay much faster than the fundamental mode.
In a previous note [10], a related result - i.e. eventual local positivity - was proved for the Cauchy problem (1), assuming that the initial datum u_0 is nonnegative and compactly supported. We consider here initial data u_0 which are not compactly supported and which display a given decay behaviour as $|x| \to \infty$. Not only we prove eventual local positivity for the solution to (1), but we also give quantitative lower bounds.
We fix some arbitrary $\beta \geq 0$ and consider the functional set

$$\mathcal{C}_\beta := \{g \in C^0(\mathbb{R}^n; \mathbb{R}_+) : \, g(0) > 0 \, , \, g(x) = o(|x|^\beta) \text{ as } |x| \to \infty\} \, .$$

Our main positivity result for the linear Cauchy problem (1) is the following

Theorem 1. *Let $\beta \geq 0$ and let $g \in \mathcal{C}_\beta$. Let*

$$u_0(x) = \frac{1}{g(x) + |x|^\beta}.$$

Let $u = u(x,t)$ be the corresponding solution of (1) *and $K \subset \mathbb{R}^n$ be a compact set.*
(i) If $\beta < n$, then there exists $\widetilde{C}_{n,\beta} > 0$ such that

$$\lim_{t\to+\infty} t^{\beta/4} u(x,t) = \widetilde{C}_{n,\beta}\,,$$

uniformly with respect to $x \in K$.
(ii) If $\beta \geq n$ and $g(x) \equiv 1$, then there exists $\widetilde{D}_{n,\beta} > 0$ such that

$$\begin{aligned} &\lim_{t\to+\infty} t^{n/4} (\log t)^{-1}\, u(x,t) = \widetilde{D}_{n,n} && \text{if } \beta = n \\ &\lim_{t\to+\infty} t^{n/4}\, u(x,t) = \widetilde{D}_{n,\beta} && \text{if } \beta > n\,, \end{aligned} \tag{4}$$

uniformly with respect to $x \in K$.

The constants $\widetilde{C}_{n,\beta}$ and $\widetilde{D}_{n,\beta}$ in Theorem 1 *do not* depend on K. What does depend on K is the "speed of convergence", namely how fast $t^{\beta/4}u(x,t) - \widetilde{C}_{n,\beta}$ converges to 0 (and similarly for $\widetilde{D}_{n,\beta}$). Let us also mention that when $\beta \geq n$, then for any $g \in \mathcal{C}_\beta$ (not necessarily constant) one still has that $\lim_{t\to+\infty} t^{\beta/4}u(x,t) = +\infty$ uniformly with respect to $x \in K$.

The quantitative positivity result of Theorem 1 provides strong enough information to be applied also to the semilinear problem (2). At a first glance, this appears somehow unexpected, since the techniques connected with the proof of Theorem 1 seem to be purely linear.

Theorem 1 does not clarify whether the eventual positivity for solutions of (1) is global or only local. It is shown in [10] that if the initial datum u_0 is compactly supported, then the negativity for the solution of (1) always exists and shifts to infinity. This suggests that a similar phenomenon might occur for β sufficiently large.
On the other hand, if $u_0 \equiv 1$ then the solution of (1) is $u\,(x,t) \equiv 1$. This trivial example shows that if $\beta = 0$, presumably one has *global* eventual positivity for (1). We show that – at least in the case $n = 1$ – this is also true if β is positive but sufficiently small, see Proposition 4 in the appendix.

In any case, the following result shows that in general, we cannot expect neither global positivity nor uniform bounds for eventual positivity.

Theorem 2. *Let $\beta \in (0,n)$. For any $T > 1$ there exists $g \in \mathcal{C}_\beta$ such that if*

$$u_0(x) = \frac{1}{g(x) + |x|^\beta}$$

then, the corresponding solution $u = u\,(x,t)$ of (1) *satisfies $u\,(x_T, T) < 0$ for some $x_T \in \mathbb{R}^n$.*

1.2 Global existence, decay and positivity in the nonlinear problem

We denote by b the biharmonic heat kernel, i.e. the fundamental solution of (3). Then, we introduce the definition of solution of (2):

Definition 1. *We say that u is a* **solution** *of* (2) *over $[0,T)$ if $u \in C^{4,1}(\mathbb{R}^n \times (0,T))$, u is bounded in $\mathbb{R}^n \times [0,t]$ for any $t \in (0,T)$, u solves the equation in* (2) *in the classical sense in $\mathbb{R}^n \times (0,T)$ and $\|u\,(t) - b(t) * u_0\|_{L^\infty(\mathbb{R}^n)} \to 0$ as $t \to 0$. We denote by T^* the supremum of all T's such that u is a solution of* (2) *in $[0,T)$. Finally, we say that u is a* **global** *solution of* (2) *if $T^* = +\infty$.*

For the nonlinear Cauchy problem (2) with initial data u_0 as specified in Theorem 1, we shall construct global solutions, where on the long term, the contribution of the initial datum dominates the nonlinear term, provided $\beta \in (4/(p-1), n)$. This observation allows us to prove an eventual local positivity property even for solutions of the nonlinear problem (2):

Theorem 3. *Let $p > 1 + 4/n$, $4/(p-1) < \beta < n$ and $g \in \mathcal{C}_\beta$. Then, for $a > 0$ small enough the initial datum*

$$u_0\,(x) = \frac{a}{g(x) + |x|^\beta}$$

admits a global solution, which is eventually locally positive. More precisely, for any compact set $K \subset \mathbb{R}^n$ there exists $T_K > 0$ such that $u\,(x,t) > 0$ for any $x \in K$ and $t \geq T_K$.

In order to show this eventual local positivity result, we prove first an existence and decay result for solutions of (2) covering the presumably full ranges for the exponent p and for the asymptotic decay of the initial datum, where global existence may be expected. The limiting case $\beta = 4/(p-1)$ was treated in [9]. The precise estimates in the range $\beta \in \left(\frac{4}{p-1}, n\right)$, however, are crucial for proving positivity in Theorem 3.

Theorem 4. *Let $p > 1 + 4/n$. There exists $\alpha > 0$ such that if u_0 satisfies*

$$|u_0\,(x)| \leq \frac{\alpha}{1 + |x|^\beta}$$

for some $\beta \geq 4/(p-1)$, then there exists a global solution of (2). Moreover, there exists a constant $A = A\,(n,p,\beta) > 0$ such that for any $(x,t) \in \mathbb{R}^{n+1}_+$ we have

$$|u(x,t)| \leq \begin{cases} \frac{A}{1+|x|^\beta + t^{\beta/4}} & \text{if } \beta \in \left[\frac{4}{p-1}, n\right) \\[2ex] \frac{A}{1+\frac{|x|^n}{\log(1+|x|)}+\frac{t^{n/4}}{\log(1+t)}} & \text{if } \beta = n \\[2ex] \frac{A}{1+|x|^n + t^{n/4}} & \text{if } \beta > n. \end{cases} \tag{5}$$

For any $\varepsilon > 0$ the constant $A > 0$ may be chosen independently of $\beta \in [4/(p-1), n-\varepsilon] \cup [n+\varepsilon, +\infty)$.

Remark 1. The shape of u_0 in the statements of Theorems 1 and 3 can be slightly modified. By using the results in [10] one can see that the proof still works if we add to any such u_0 some nonnegative continuous compactly supported function.

2 Basic properties of the heat kernels

The kernel of the linear operator $v \mapsto v_t + \Delta^2 v$ in $\mathbb{R}^n$ is given by

$$\begin{aligned} b(x,t) &= \alpha_n \frac{f_n(\eta)}{t^{n/4}}\,, \qquad \eta = \frac{|x|}{t^{1/4}}\,, \\ f_n(\eta) &= \eta^{1-n} \int_0^\infty e^{-s^4} (\eta s)^{n/2} J_{(n-2)/2}(\eta s)\, ds\,, \end{aligned} \tag{6}$$

where J_ν denotes the ν-th Bessel function and $\alpha_n > 0$ is a normalization constant. More precisely, if ω_n denotes the surface measure of the n-dimensional unit ball (so that $\omega_1 = 2$), then α_n is given by

$$\alpha_n^{-1} = \omega_n \int_0^\infty r^{n-1} f_n(r)\, dr.$$

Thanks to Galaktionov-Pohožaev [7], we know that these f-functions have exponential decay at infinity. More precisely, for any integer $n \geq 1$ there exist $K = K_n > 0$, $\mu = \mu_n > 0$ such that

$$|f_n(\eta)| \leq K \exp\left(-\mu\eta^{4/3}\right) \qquad \text{for all } \eta \geq 0. \tag{7}$$

Let us now recall the definition of the Gamma function and the power series expansion of the Bessel function J_ν:

$$\Gamma(y) = \int_0^\infty e^{-s}\, s^{y-1}\, ds \quad (y > 0)\,, \qquad J_\nu(y) = \sum_{k=0}^\infty \frac{(-1)^k (y/2)^{2k+\nu}}{k!\,\Gamma(k+\nu+1)} \quad (\nu > -1). \tag{8}$$

For these definitions and for further properties of Γ and J_ν we refer to [1]. Here, we just recall the last formula on p.13 in [1]:

$$\Gamma(x+\ell) = (x+\ell-1)(x+\ell-2)\cdots x\Gamma(x) \qquad \text{for all } x > 0\,,\ \ell \in \mathbb{N}\ . \tag{9}$$

Moreover, we will need the following property, obtained through the change of variable $z = s^4$:

$$\int_0^\infty e^{-s^4}\, s^\alpha\, ds = \frac{1}{4}\int_0^\infty e^{-z}\, z^{-1+(\alpha+1)/4}\, dz = \frac{1}{4}\Gamma\left(\frac{\alpha+1}{4}\right) \qquad (\alpha > -1). \tag{10}$$

The f-functions obey the following recurrence formula:

$$f_n'(\eta) = -\eta\, f_{n+2}(\eta)\,, \qquad \text{for all } n \geq 1. \tag{11}$$

This follows by direct computation:

$$\begin{aligned} \frac{d}{d\eta} f_n(\eta) \quad &= \quad \frac{d}{d\eta}\left[\int_0^\infty e^{-s^4} s^{n-1} (\eta s)^{(2-n)/2} J_{(n-2)/2}(\eta s)\, ds\right] \\ \{\text{by [1, (4.6.2)]}\} \quad &= \quad -\int_0^\infty e^{-s^4} s^n (\eta s)^{(2-n)/2} J_{n/2}(\eta s)\, ds = -\eta f_{n+2}(\eta). \end{aligned}$$

As a consequence of (11) we have the following recursion formula, which relates integrals of f-functions in annuli of the corresponding space:

$$\int_{r_1}^{r_2} r^{n-1} f_n(r)\, dr = \left[\frac{r^n}{n} f_n(r)\right]_{r_1}^{r_2} + \frac{1}{n}\int_{r_1}^{r_2} r^{n+1} f_{n+2}(r)\, dr.$$

Next, we show that a representation of f_n through power series is available:

Theorem 5. *For any integer $m \geq 1$, we have*

$$f_{2m}(\eta) \quad = \quad \sum_{k=0}^\infty (-1)^k \ \frac{\Gamma\left(\frac{k+m}{2}\right)}{2^{2k+m+1}\, k!\ (k+m-1)!}\ \eta^{2k}\ . \tag{12}$$

For any nonnegative integer m, we have

$$f_{2m+1}(\eta) \quad = \quad \frac{2^m}{\sqrt{8\pi}} \sum_{k=0}^\infty (-1)^k\ \frac{(k+m)!\ \Gamma\left(\frac{2k+2m+1}{4}\right)}{k!\ (2k+2m)!}\ \eta^{2k}\ . \tag{13}$$

In particular, $f_n(0) > 0$ for all n and

$$f_1(\eta) = \frac{1}{\sqrt{8\pi}}\sum_{k=0}^\infty (-1)^k\ \frac{\Gamma\left(\frac{2k+1}{4}\right)}{(2k)!}\ \eta^{2k}\ , \qquad f_2(\eta) = \frac{1}{4}\sum_{k=0}^\infty (-1)^k\ \frac{\Gamma\left(\frac{k+1}{2}\right)}{[2^k\, k!]^2}\ \eta^{2k}\ .$$

Proof. Throughout this proof we use the convention that for any real numbers a_k and any integer i: $\prod_{k=i}^{i-1} a_k = 1$. By exploiting the power series expansion (8), for any integer n we find

$$f_n(\eta) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\;\Gamma\left(k+\frac{n}{2}\right)\;2^{2k-1+n/2}} \left(\int_0^\infty e^{-s^4} s^{2k+n-1}\,ds\right) \eta^{2k}\;.$$

By (9) we obtain

$$\Gamma\left(k+\frac{n}{2}\right) \;=\; 2^{-k}\;\Gamma\left(\frac{n}{2}\right)\prod_{h=0}^{k-1}(n+2h) \qquad \text{for all } k\in\mathbb{N}$$

which, combined with (10), yields

$$f_n(\eta) \;=\; \sum_{k=0}^{\infty}(-1)^k\,\frac{\Gamma\left(\frac{2k+n}{4}\right)}{k!\;\Gamma\left(\frac{n}{2}\right)\;2^{k+1+n/2}\;\prod_{h=0}^{k-1}(n+2h)}\;\eta^{2k}\;. \tag{14}$$

Assume now that $n = 2m+1$ for some integer $m \ge 0$. Then, recalling that $\Gamma(1/2) = \sqrt{\pi}$ and using again (9), we obtain

$$\Gamma\left(\frac{n}{2}\right) \;=\; \Gamma\left(m+\frac{1}{2}\right) \;=\; 2^{-m}\;\sqrt{\pi}\prod_{h=0}^{m-1}(2h+1).$$

Moreover, as may be proved by induction over $(m+k)$, one has

$$2^k\prod_{h=0}^{m+k-1}(2h+1) \;=\frac{(2k+2m)!}{2^m\,(k+m)!} \qquad \text{for any integer } m\ge 0\;.$$

Using these two identities, the denominator of the coefficient of η^{2k} in (14) becomes

$$k!\;\Gamma\left(\frac{n}{2}\right)\;2^{k+1+n/2}\prod_{h=0}^{k-1}(n+2h) \;=\; \sqrt{8\pi}\;k!\;2^k\prod_{h=0}^{m+k-1}(2h+1) \;=\; \sqrt{8\pi}\;\frac{k!\;(2k+2m)!}{2^m\;(k+m)!}\;.$$

This proves (13).
On the other hand, if $n = 2m$ for some integer $m \ge 1$, then

$$\Gamma\left(\frac{n}{2}\right) = \Gamma(m) = (m-1)! \qquad \text{and} \qquad \prod_{h=0}^{k-1}(n+2h) = 2^k\prod_{h=0}^{k-1}(m+h)$$

so that

$$\Gamma\left(\frac{n}{2}\right)\prod_{h=0}^{k-1}(n+2h) \;=\; 2^k\;(k+m-1)!$$

and (12) follows by replacing into (14). □

Basing upon Bessel's equation, we deduce a third order ordinary differential equation for the function f_n, which will be essential for proving their positivity/oscillatory properties.

Theorem 6. *For any integer $n \ge 1$, the function f_n solves the equation*

$$f_n'''(\eta) + \frac{n-1}{\eta}f_n''(\eta) - \frac{n-1}{\eta^2}f_n'(\eta) - \frac{\eta}{4}f_n(\eta) = 0. \tag{15}$$

Proof. In the proof of this lemma we use the following notation: keeping $n \in \mathbb{N}$ as space dimension fixed, for any function $g \in C^2((0,\infty))$ we write

$$\Delta g(\rho) = g''(\rho) + \frac{n-1}{\rho} g'(\rho) \qquad \text{for all } \rho > 0. \tag{16}$$

One may check by direct computation that the Bessel function $J_{(n-2)/2}$ satisfies the identity

$$\Delta\left(\rho^{1-n/2} J_{(n-2)/2}\right) = -\rho^{1-n/2} J_{(n-2)/2}. \tag{17}$$

If we put

$$F(\rho) = \rho^{1-n/2} J_{(n-2)/2}(\rho) \qquad \text{for all } \rho > 0$$

then, by (6) and (17) we find

$$\begin{aligned} \Delta f_n(\eta) &= \Delta\left(\int_0^\infty s^{n-1} e^{-s^4} F(\eta s)\ ds\right) = \int_0^\infty s^{n-1} e^{-s^4} s^2 (\Delta F)(\eta s)\ ds \\ &= -\int_0^\infty s^{n-1} e^{-s^4} s^2 F(\eta s)\ ds. \end{aligned}$$

Using the divergence formula and (17) we obtain

$$\begin{aligned} \left(\Delta f_n(\eta)\right)' &= -\int_0^\infty s^{n-1} e^{-s^4} s^3 F'(\eta s)\ ds = \frac{1}{4\omega_n} \int_{\mathbb{R}^n} \nabla\left(e^{-|y|^4}\right) \nabla F(\eta |y|)\ dy \\ &= -\frac{1}{4\omega_n} \int_{\mathbb{R}^n} e^{-|y|^4} \eta \Delta F(\eta |y|)\ dy = \frac{1}{4\omega_n} \int_{\mathbb{R}^n} e^{-|y|^4} \eta F(\eta |y|)\ dy \\ &= \frac{\eta}{4} \int_0^\infty s^{n-1} e^{-s^4} F(\eta s)\ ds = \frac{\eta}{4} f_n(\eta). \end{aligned}$$

By (16) we immediately obtain (15). □

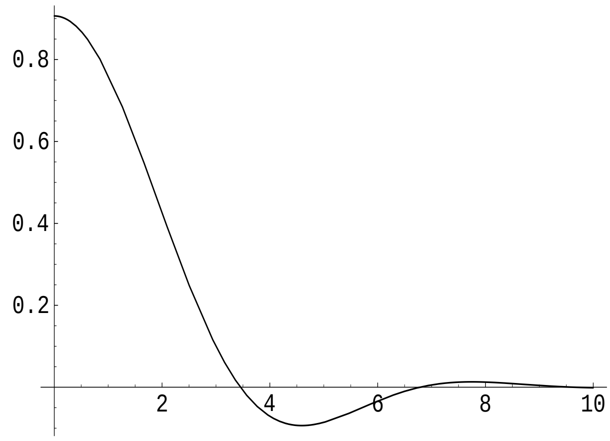

Figure 1: The function f_1

Finally, we show that the f_n and so, the biharmonic heat kernel, has infinitely many oscillations:

Theorem 7. *As $\eta \to \infty$, the function $\eta \mapsto f_n(\eta)$ changes sign infinitely many times.*

Proof. Assume first that $n = 1$ so that (15) reads

$$f_1'''(\eta) = \frac{\eta}{4} f_1(\eta) . \tag{18}$$

For contradiction assume that $f_1(\eta) \geq 0$ for η sufficiently large, say $\eta \geq \overline{\eta}$. Then, (18) states that also $f_1'''(\eta) \geq 0$ for $\eta \geq \overline{\eta}$. This implies that $f_1''(\eta)$ admits a limit as $\eta \to \infty$. In view of (7) this limit is necessarily zero. And since f_1'' is non-decreasing, this means that $f_1''(\eta) \leq 0$ for $\eta \geq \overline{\eta}$. Therefore, f_1 is concave, nonnegative and vanishes at infinity in view of (7): this is impossible. Similarly, we reach a contradiction if we assume that $f_1(\eta) \leq 0$ for η sufficiently large.
Assume now that $n = 2$. By putting $g(t) = f_2(e^t)$, (15) becomes

$$\frac{d}{dt}\left(e^{-2t} g''(t)\right) = \frac{e^{2t}}{4} g(t) . \tag{19}$$

For contradiction, assume that $g(t) \geq 0$ for t sufficiently large, say $t \geq T$. Then, with the same argument used above we exclude that $g''(t) \leq 0$ for all $t \geq T$. Take $\tau > T$ such that $g''(\tau) > 0$ and integrate (19) over $[\tau, t]$ for any $t \geq \tau$:

$$e^{-2t} g''(t) \geq e^{-2\tau} g''(\tau) \implies g''(t) \geq e^{-2\tau} g''(\tau) e^{2t}$$

showing that $g''(t) \to +\infty$ as $t \to \infty$. But then also $g(t) \to +\infty$ as $t \to \infty$, contradicting (7). Similarly one reaches a contradiction by assuming that $g(t) \leq 0$ for t sufficiently large.
The statement is so proved for $n = 1$ and $n = 2$. This means that f_1 and f_2 have an increasing sequence of local maxima and of local minima. Hence, for larger values of n it suffices to recall (11) and to argue by induction. □

Remark 2. In order to gain a more precise impression of the asymptotic behaviour of the f-function for $\eta \to \infty$, we define

$$g(y) := f_n\left(y^{3/4}\right), \qquad f_n(\eta) = g\left(\eta^{4/3}\right), \qquad \eta = y^{3/4} \geq 0.$$

So, we have

$$\begin{aligned}
f_n'(\eta) &= \frac{4}{3}\eta^{1/3} g'\left(\eta^{4/3}\right) \\
f_n''(\eta) &= \frac{16}{9}\eta^{2/3} g''\left(\eta^{4/3}\right) + \frac{4}{9}\eta^{-2/3} g'\left(\eta^{4/3}\right) \\
f_n'''(\eta) &= \frac{64}{27}\eta g'''\left(\eta^{4/3}\right) + \frac{16}{9}\eta^{-1/3} g''\left(\eta^{4/3}\right) - \frac{8}{27}\eta^{-5/3} g'\left(\eta^{4/3}\right) .
\end{aligned}$$

In terms of g, the differential equation (15) for f_n reads:

$$g(y) = \frac{256}{27} g'''(y) + \frac{64}{9} n \frac{g''(y)}{y} - \frac{32}{27}(3n-2)\frac{g'(y)}{y^2}. \tag{20}$$

According to [13, Chapter X], the asymptotic behaviour of solutions to the "almost autonomous" equation (20) is to a "certain extent" determined by the corresponding autonomous equation

$$h(y) = \frac{256}{27} h'''(y). \tag{21}$$

Bounded solutions to (21) are given by

$$h(y) = \gamma \exp\left(-\frac{3}{16}\sqrt[3]{2} y\right) \cos\left(\frac{3}{16}\sqrt{3}\sqrt[3]{2} y + \delta\right)$$

where $\gamma > 0$ and $\delta \in \mathbb{R}$. So, we *expect* the following asymptotic behaviour of $f_n(\eta)$:

$$(\gamma + o(1)) \exp\left(-\frac{3}{16}\sqrt[3]{2}\eta^{4/3}\right) \cos\left(\frac{3}{16}\sqrt{3}\sqrt[3]{2}\eta^{4/3} + \delta + o(1)\right)$$

for suitable constants γ and δ. Numerical experiments strongly support validity of such an asymptotic expansion while on the other hand, [13, Chapter X.13–X.17] does not directly yield such a statement.

3 Linear decay and proof of Theorem 1

The solution u of the linear Cauchy problem (1) is given by

$$u(x,t) = \alpha_n t^{-n/4} \int_{\mathbb{R}^n} u_0(x-y) f_n\left(\frac{|y|}{t^{1/4}}\right) dy \;, \qquad (x,t) \in \mathbb{R}^{n+1}_+. \tag{22}$$

With the change of variables $y = t^{1/4}z$, we obtain

$$u(x,t) = \alpha_n \int_{\mathbb{R}^n} u_0(x - t^{1/4}z) f_n(|z|)\, dz = \alpha_n t^{-\beta/4} \int_{\mathbb{R}^n} \frac{f_n(|z|)}{t^{-\beta/4} g(x - t^{1/4}z) + |z - t^{-1/4}x|^\beta}\, dz \;. \tag{23}$$

We distinguish two cases.

3.1 The case $\beta < n$

We first prove

Lemma 1. *Let $K \subset \mathbb{R}^n$ be a compact subset and let $0 \le \beta < n$. Then, there exists $C_{n,\beta} \in \mathbb{R}$ such that*

$$\lim_{t \to +\infty} t^{\beta/4} u(x,t) = \alpha_n \omega_n C_{n,\beta} \,,$$

the limit being uniform with respect to $x \in K$. In particular, $C_{n,0} = (\omega_n \alpha_n)^{-1}$.

Proof. By (7) and since $\beta < n$ and $g \in \mathcal{C}_\beta$, we may apply Lebesgue's Theorem and exploit local uniform convergence in $\mathbb{R}^n \setminus \{0\}$ to obtain

$$\lim_{t \to +\infty} \left(\int_{\mathbb{R}^n} \frac{f_n(|z|)}{t^{-\beta/4} g(x - t^{1/4}z) + |z - t^{-1/4}x|^\beta}\, dz \right) = \int_{\mathbb{R}^n} \frac{f_n(|z|)}{|z|^\beta}\, dz = \omega_n \int_0^\infty \eta^{n-1-\beta} f_n(\eta)\, d\eta.$$

Hence, if we put

$$C_{n,\beta} := \int_0^\infty \eta^{n-1-\beta} f_n(\eta)\, d\eta \;, \tag{24}$$

the statement follows by replacing into (23). ☐

By Lemma 1, the proof of Theorem 1 (in the case $\beta < n$) will be complete once we show:

Proposition 1. *For all integer $n \ge 1$ and all $\beta \in [0,n)$ we have $C_{n,\beta} > 0$.*

Since the proof of Proposition 1 is quite involved we postpone it to the next section.

3.2 The case $\beta \geq n$

Here we cannot apply Lebesgue's Theorem as in Lemma 1 because we must isolate the singularity. More precisely, if $\gamma > 0$ denotes the least zero of f_n, we split the integral in (23) as follows:

$$\frac{t^{\beta/4}u(x,t)}{\alpha_n} = \int_{\mathbb{R}^n} \frac{f_n(|z|)}{t^{-\beta/4} + |z - t^{-1/4}x|^\beta}\, dz = \int_{|z-t^{-1/4}x|>\gamma} + \int_{|z-t^{-1/4}x|<\gamma} =: I_1(t) + I_2(t)$$

and we estimate the two integrals. In view of (7), to I_1 we may apply Lebesgue's Theorem and obtain

$$\lim_{t\to+\infty} I_1(t) = \int_{|z|>\gamma} \frac{f_n(|z|)}{|z|^\beta}\, dz = \delta_{n,\beta} \in \mathbb{R}\ . \tag{25}$$

The estimate of I_2 appears slightly more delicate. With the change of variables $y = t^{-1/4}x - z$, we obtain

$$I_2(t) = \int_{|y|<\gamma} \frac{f_n(|t^{-1/4}x - y|)}{t^{-\beta/4} + |y|^\beta}\, dy = \int_{|y|<\gamma} \frac{f_n(|y|) + o(1)}{t^{-\beta/4} + |y|^\beta}\, dy = \omega_n \int_0^\gamma r^{n-1} \frac{f_n(r)}{t^{-\beta/4} + r^\beta}\, dr + o(I_2(t))$$

so that, with the change of variables $r = t^{-1/4}s$, we obtain

$$(1 + o(1))I_2(t) = \omega_n\, t^{(\beta-n)/4} \int_0^{\gamma t^{1/4}} s^{n-1} \frac{f_n(t^{-1/4}s)}{1 + s^\beta}\, ds\ .$$

If $\beta = n$ the last equality implies $I_2(t) \sim D_{n,n} \log t$ as $t \to \infty$, whereas if $\beta > n$ it implies $I_2(t) \sim D_{n,\beta} t^{(\beta-n)/4}$ as $t \to \infty$, for suitable positive constants $D_{n,\beta}$. By combining these estimates with (25), we readily obtain (4).

4 Proof of Proposition 1

The proof of Proposition 1 is somehow inductive. It needs a statement for large β in order to be started and two recursion formulae in order to be continued.

Lemma 2. *For all integer $n \geq 4$ and all $\beta \in (\frac{n+1}{2}, n)$ we have $C_{n,\beta} > 0$.*

Proof. According to (6), we may rewrite $C_{n,\beta}$ in (24) as

$$C_{n,\beta} = \int_0^\infty \eta^{-\beta} \int_0^\infty e^{-s^4} (\eta s)^{n/2} J_{(n-2)/2}(\eta s)\, ds\, d\eta\ . \tag{26}$$

Absolute integrability with respect to η near ∞ is ensured by the condition $\beta > (n+1)/2$. Observe that $\sqrt{\rho}J_{(n-2)/2}(\rho)$ is bounded at ∞ and that $\rho^{1-n/2}J_{(n-2)/2}(\rho)$ is bounded near $\rho = 0$. For every $\eta > 0$, with the change of variable $s = z/\eta$ we obtain

$$\int_0^\infty e^{-s^4} (\eta s)^{n/2} J_{(n-2)/2}(\eta s)\, ds = \frac{1}{\eta} \int_0^\infty e^{-z^4/\eta^4} z^{n/2} J_{(n-2)/2}(z)\, dz$$

which, inserted into (26), yields

$$C_{n,\beta} = \int_0^\infty \eta^{-\beta-1} \int_0^\infty e^{-z^4/\eta^4} z^{n/2} J_{(n-2)/2}(z)\, dz\, d\eta\ .$$

By Fubini's Theorem we then get

$$C_{n,\beta} = \int_0^\infty z^{n/2} J_{(n-2)/2}(z) \int_0^\infty \frac{e^{-z^4/\eta^4}}{\eta^{\beta+1}}\, d\eta\, dz\ . \tag{27}$$

For every $z > 0$, with the change of variable $\eta = z/s$ we obtain

$$\int_0^\infty \frac{e^{-z^4/\eta^4}}{\eta^{\beta+1}}\, d\eta = \frac{1}{z^\beta} \int_0^\infty e^{-s^4} s^{\beta-1}\, ds\ .$$

Inserting this into (27) shows that

$$C_{n,\beta} = \gamma_{n,\beta} \int_0^\infty z^{\frac{n-1}{2}-\beta} \sqrt{z}\, J_{(n-2)/2}(z)\, dz \tag{28}$$

with

$$\gamma_{n,\beta} = \int_0^\infty e^{-s^4} s^{\beta-1}\, ds = \frac{1}{4}\Gamma\left(\frac{\beta}{4}\right) > 0$$

where the second equality follows from (10). Then, we argue as in [10, Lemma 1]: since the map $z \mapsto z^{\frac{n-1}{2}-\beta}$ is decreasing, we may apply the Lorch-Szegö Theorem [1, Corollary 4.15.2] to (28) in order to show that $C_{n,\beta} > 0$. Here we use $n \geq 4$ so that for the index of the Bessel function, we have that $\frac{n-2}{2} > \frac{1}{2}$. □

Lemma 3. *For any $n \geq 1$ and $\beta \in [0, n)$ we have*

$$C_{n+2,\beta} = (n-\beta) C_{n,\beta}. \tag{29}$$

For any $n \geq 3$ and $\beta \in [0, n-2)$ we have

$$C_{n,\beta} = 4(\beta+2)(n-2-\beta) C_{n+2,\beta+4}. \tag{30}$$

Proof. By (11) and integration by parts (recall (7)) we have

$$\begin{aligned} C_{n+2,\beta} &= \int_0^\infty \eta^{n+1-\beta} f_{n+2}(\eta)\ d\eta = -\int_0^\infty \eta^{n-\beta} f_n'(\eta)\ d\eta \\ &= (n-\beta) \int_0^\infty \eta^{n-\beta-1} f_n(\eta)\ d\eta = (n-\beta) C_{n,\beta}. \end{aligned}$$

This proves (29).
By exploiting the differential equation for f_n, see Theorem 6, we have

$$\begin{aligned} C_{n,\beta} &= \int_0^\infty \eta^{n-1-\beta} f_n(\eta)\ d\eta \\ &= 4\int_0^\infty \eta^{n-2-\beta} f_n'''(\eta)\ d\eta \\ &\quad +4(n-1)\int_0^\infty \eta^{n-3-\beta} f_n''(\eta)\ d\eta - 4(n-1)\int_0^\infty \eta^{n-4-\beta} f_n'(\eta)\ d\eta. \end{aligned} \tag{31}$$

Note that the last integral in (31) is well defined since $\beta < n-2$ and by Theorem 5 we have $f_n'(\eta) \sim c\eta$ as $\eta \to 0$.

By (11) we have

$$\int_0^\infty \eta^{n-4-\beta} f_n'(\eta)\ d\eta = -\int_0^\infty \eta^{n-3-\beta} f_{n+2}(\eta)\ d\eta = -C_{n+2,\beta+4}. \tag{32}$$

Hence, after an integration by parts and using again (7), (11) and the fact that $f_n'(\eta) \sim c\eta$ as $\eta \to 0$, we get

$$\begin{aligned} \int_0^\infty \eta^{n-3-\beta} f_n''(\eta)\ d\eta &= \left[\eta^{n-3-\beta} f_n'(\eta)\right]_0^\infty - (n-3-\beta)\int_0^\infty \eta^{n-4-\beta} f_n'(\eta)\ d\eta \\ &= (n-3-\beta)\, C_{n+2,\beta+4}. \end{aligned} \tag{33}$$

With the aid of (33) and integration by parts we obtain

$$\begin{aligned} \int_0^\infty \eta^{n-2-\beta} f_n'''(\eta)\ d\eta &= \left[\eta^{n-2-\beta} f_n''(\eta)\right]_0^\infty - (n-2-\beta)\int_0^\infty \eta^{n-3-\beta} f_n''(\eta)\ d\eta \\ &= -(n-2-\beta)(n-3-\beta)\, C_{n+2,\beta+4}. \end{aligned} \tag{34}$$

Combining (31)-(34) the proof of (30) follows. □

We are now ready to prove Proposition 1. We prove it separately in the cases where n is odd or even.

Proof of Proposition 1 for odd integers. By Lemma 2 we have

$$C_{9,\beta} > 0 \qquad \text{for all } \beta \in (5,9). \tag{35}$$

In turn, by (35) and (30) we have $C_{7,\beta} > 0$ for any $\beta \in (1,5)$ while by Lemma 2 it follows that $C_{7,\beta} > 0$ for any $\beta \in (4,7)$. This yields

$$C_{7,\beta} > 0 \qquad \text{for all } \beta \in (1,7). \tag{36}$$

Then, according to (36) and (30), we obtain $C_{5,\beta} > 0$ for any $\beta \in [0,3)$ and according to (36) and (29) we obtain $C_{5,\beta} > 0$ for any $\beta \in (1,5)$. This yields

$$C_{5,\beta} > 0 \qquad \text{for all } \beta \in [0,5). \tag{37}$$

By (37) and (29) we also have for $n = 1, 3$

$$C_{n,\beta} > 0 \qquad \text{for all } \beta \in [0,n).$$

It remains to consider odd dimensions $n \geq 5$. It is already proved for $n = 5$ in view of (37). Suppose by induction that the conjecture is true for some odd integer $n \geq 5$. By (29) we have $C_{n+2,\beta} > 0$ for any $\beta \in [0,n)$ and by Lemma 2 we deduce that $C_{n+2,\beta} > 0$ for any $\beta \in \left(\frac{n+3}{2}, n+2\right)$. Since $n > \frac{n+3}{2}$ for any $n \geq 4$, this proves that

$$C_{n+2,\beta} > 0 \qquad \text{for all } \beta \in [0, n+2).$$

The proof of Proposition 1 for odd integers is now complete. □

Proof of Proposition 1 for even integers. By Lemma 2 we have

$$C_{8,\beta} > 0 \qquad \text{for all } \beta \in (9/2, 8). \tag{38}$$

By (38) and (30) we have $C_{6,\beta} > 0$ for any $\beta \in (1/2, 4)$ while by Lemma 2 it follows that $C_{6,\beta} > 0$ for any $\beta \in (7/2, 6)$. This yields

$$C_{6,\beta} > 0 \qquad \text{for all } \beta \in (1/2, 6). \tag{39}$$

According to (39) and (30) we obtain $C_{4,\beta} > 0$ for any $\beta \in [0,2)$ while according to (39) and (29) we obtain $C_{4,\beta} > 0$ for any $\beta \in (1/2, 4)$. This yields

$$C_{4,\beta} > 0 \qquad \text{for all } \beta \in [0,4)\,. \tag{40}$$

By (40) and (29) we also have

$$C_{2,\beta} > 0 \qquad \text{for all } \beta \in [0,2)\,.$$

It remains to consider even dimensions $n \geq 4$. This proof by induction is performed exactly as in the case of odd n. The proof of Proposition 1 is so complete also for even integers. □

5 Proof of Theorem 2

Let $\beta \in (0,n)$ and $T > 1$ be fixed. According to Theorem 7 we know that f_n is changes sign infinitely many times. Let $\gamma_1 < \gamma_2$ be the first two zeroes of f_n so that we have

$$f_n(\eta) > 0 \quad \text{for all } \eta \in (0,\gamma_1) \quad \text{and} \quad f_n(\eta) < 0 \quad \text{for all } \eta \in (\gamma_1,\gamma_2)\,.$$

Let $d > 0$ and $0 < \delta < (\gamma_2 - \gamma_1)/2$ be such that

$$f_n(\eta) < -d \qquad \text{for all } \eta \in (\gamma_1 + \delta, \gamma_2 - \delta)\,. \tag{41}$$

Put $b = (\gamma_2 - \gamma_1)/2$ and $a = b - \delta$ and define for a fixed $M > 1$

$$g(x) = \begin{cases} 1 & \text{if } |x| \leq a \\ 1 + \frac{M-1}{b-a}(|x| - a) & \text{if } a < |x| < b \\ M & \text{if } |x| \geq b. \end{cases}$$

If

$$u_0(x) = \frac{1}{g(x) + |x|^\beta}$$

then by (22) the corresponding solution $u = u(x,t)$ of (1) reads

$$\frac{u(x,t)}{\alpha_n} = t^{-n/4} \int_{\mathbb{R}^n} u_0(x-y) f_n\left(\frac{|y|}{t^{1/4}}\right) dy = \int_{\mathbb{R}^n} \frac{f_n(|z|)}{g\left(|x - t^{1/4}z|\right) + |x - t^{1/4}z|^\beta}\, dz.$$

By (7) and (41) we have

$$\begin{aligned} \frac{u(x,t)}{\alpha_n} &< \int_{\{\gamma_1+\delta<|z|<\gamma_2-\delta\}} \frac{-d}{g\left(|x - t^{1/4}z|\right) + |x - t^{1/4}z|^\beta}\, dz \\ &\quad + K \int_{\{|z|<\gamma_1\}\cup\{|z|>\gamma_2\}} \frac{e^{-\mu|z|^{4/3}}}{g\left(|x - t^{1/4}z|\right) + |x - t^{1/4}z|^\beta}\, dz. \end{aligned} \tag{42}$$

Take $x_0 \in \mathbb{R}^n$ such that $|x_0| = (\gamma_1 + \gamma_2)/2$ and put $R = T^{-1/4}a$ and $x_T = T^{1/4}x_0$. Then by (42) we have

$$\begin{aligned} \frac{u(x_T,T)}{\alpha_n} &< \int_{B_R(x_0)} \frac{-d}{g\left(T^{1/4}|x_0 - z|\right) + T^{\beta/4}|x_0 - z|^\beta}\, dz \\ &\quad + K \int_{\{|z|<\gamma_1\}} \frac{1}{g\left(|x - t^{1/4}z|\right)}\, dz + K \int_{\{|z|>\gamma_2\}} \frac{e^{-\mu|z|^{4/3}}}{g\left(|x - t^{1/4}z|\right)}\, dz. \end{aligned} \tag{43}$$

Note that if $z \in B_R(x_0)$, then

$$T^{1/4}|x_0 - z| < RT^{1/4} = a \qquad \text{and} \qquad g\left(T^{1/4}|x_0 - z|\right) = 1. \tag{44}$$

On the other hand if $|z| < \gamma_1$, then

$$|x_0 - z| > |x_0| - |z| > \frac{\gamma_1 + \gamma_2}{2} - \gamma_1 = b.$$

Since $T > 1$ we also have $T^{1/4}|x_0 - z| > b$ and

$$g\left(T^{1/4}|x_0 - z|\right) = M. \tag{45}$$

Similarly for $|z| > \gamma_2$ we have

$$|x_0 - z| > |z| - |x_0| > \gamma_2 - \frac{\gamma_1 + \gamma_2}{2} = b$$

and

$$g\left(T^{1/4}|x_0 - z|\right) = M. \tag{46}$$

Inserting (44)-(46) into (43) we obtain

$$\frac{u(x_T, T)}{\alpha_n} < -\frac{\omega_n R^n d}{n(1 + a^\beta)} + \frac{K\omega_n \gamma_1^n}{nM} + \frac{K}{M}\int_{\{|z|>\gamma_2\}} e^{-\mu|z|^{4/3}}\, dz.$$

It is clear that if we choose M sufficiently large then $u(x_T, T) < 0$.

6 Proof of Theorems 3 and 4

Let u_0 be a function such that

$$|u_0(x)| \le \frac{\alpha}{1 + |x|^\beta} \tag{47}$$

for some $\alpha > 0$. Let u be a solution of the integral equation

$$u(t) = b(t) * u_0 + \int_0^t b(t-s) * |u|^{p-1} u(s)\ ds. \tag{48}$$

By a contraction mapping argument (see [14] Section 3.3), we know that (48) admits a unique solution $u(t)$ defined in the maximal interval of existence $[0, T^*)$ with $0 < T^* \le +\infty$ and such that $u \in C^0 \cap L^\infty(\mathbb{R}^n \times [0,T])$ for any $T \in (0, T^*)$. Moreover, it is well known (see Proposition A4 in [6]) that any bounded solution of the integral equation (48) is a solution of (2).
In order to overcome the difficulties which arise from the oscillatory behaviour of the kernel b (see also Theorem 7 above), Galaktionov and Pohožaev [7] introduced the following majorizing kernel $\widetilde{b}(x,t)$:

$$\widetilde{b}(x,t) = \theta_n t^{-n/4} \exp\left(-\mu\eta^{4/3}\right)$$

with μ as in (7) and

$$\theta_n^{-1} = \omega_n \int_0^\infty r^{n-1} \exp\left(-\mu r^{4/3}\right)\ dr.$$

From this definition we have

$$|b(x,t)| \le D_n \widetilde{b}(x,t) \tag{49}$$

for a suitable constant $D_n > 0$. Let v_0 be defined by

$$v_0\left(x\right) = D_n\left|u_0\left(x\right)\right|. \tag{50}$$

Besides (48), we also consider the integral equation

$$v(t) = \widetilde{b}\left(t\right) * v_0 + D_n \int_0^t \widetilde{b}\left(t-s\right) * v^p\left(s\right)\ ds. \tag{51}$$

If v is a solution of (51), by (49) and (50) we infer that

$$|u(x,t)| \leq v\left(x,t\right) \tag{52}$$

as long as $v(t)$ exists. In particular, the maximal interval of existence for (48) contains the corresponding interval for (51).
We define the operator

$$\widetilde{B}v(x,t) = \widetilde{b}\left(t\right) * v_0 + D_n \int_0^t \widetilde{b}\left(t-s\right) * |v\left(s\right)|^{p-1} v\left(s\right)\ ds \tag{53}$$

over the space $C^0 \cap L^\infty\left(\mathbb{R}^n \times [0,T]\right) =: C_b^0\left(\mathbb{R}^n \times [0,T]\right)$ for any $T > 0$. Since the initial datum $v_0 = v_0(x)$ is nonnegative, any fixed point $v = v(x,t)$ of the operator $\widetilde{B}$ is a nonnegative function so that v also satisfies (51).
As for the "linear" contribution of the initial datum to $\widetilde{B}$ in (53), we have

$$\begin{aligned} 0 &\leq \widetilde{b}\left(t\right) * v_0 = \theta_n \int_{\mathbb{R}^n} t^{-n/4} e^{-\mu\left(\frac{|x-y|^4}{t}\right)^{1/3}} D_n\left|u_0(y)\right|\ dy \\ &\leq D_n\theta_n\alpha \int_{\mathbb{R}^n} t^{-n/4} e^{-\mu\left(\frac{|y|^4}{t}\right)^{1/3}} \frac{1}{1+|x-y|^\beta}\ dy =: D_n\theta_n\alpha G_1\left(x,t\right). \end{aligned} \tag{54}$$

Here, any $\beta \geq 0$ makes sense. Next, we estimate the second term in the right hand side of (53). Let $M > 0$ and let $v \in C^0\left(\mathbb{R}_+^{n+1}\right)$ be such that

$$0 \leq v\left(x,t\right) \leq \frac{M}{1+|x|^\beta + t^{\beta/4}} \qquad \text{for all } (x,t) \in \mathbb{R}_+^{n+1}.$$

Then, we have

$$\begin{aligned} 0 &\leq \int_0^t \widetilde{b}\left(t-s\right) * v^p\left(s\right)\ ds = \theta_n \int_0^t \int_{\mathbb{R}^n} s^{-n/4} e^{-\mu\left(\frac{|y|^4}{s}\right)^{1/3}} v^p(x-y,t-s)\ dyds \\ &\leq \theta_n M^p \int_0^t \int_{\mathbb{R}^n} s^{-n/4} e^{-\mu\left(\frac{|y|^4}{s}\right)^{1/3}} \frac{1}{\left[1+(t-s)^{\beta/4} + |x-y|^\beta\right]^p}\ dyds =: \theta_n M^p G_2\left(x,t\right). \end{aligned} \tag{55}$$

In the following two propositions we formulate precise decay estimates for $G_1\left(x,t\right)$ and $G_2\left(x,t\right)$.

Proposition 2. *Let $\beta \geq 0$. Let G_1 be the function introduced in (54). There exists a constant $C_1 = C_1\left(n,\beta\right) > 0$ such that for all $(x,t) \in \mathbb{R}_+^n$ we have*

$$G_1\left(x,t\right) \leq \begin{cases} \frac{C_1}{1+|x|^\beta + t^{\beta/4}} & \text{if } \beta \in [0,n) \\ \frac{C_1}{1+\frac{|x|^n}{\log(1+|x|)} + \frac{t^{n/4}}{\log(1+t)}} & \text{if } \beta = n \\ \frac{C_1}{1+|x|^n + t^{n/4}} & \text{if } \beta > n. \end{cases} \tag{56}$$

Moreover, for any $\varepsilon > 0$ the constant C_1 may be chosen independently of $\beta \in [0, n-\varepsilon] \cup [n+\varepsilon, +\infty)$.

Proposition 3. *Let $p > 1 + 4/n$ and $\beta \geq \frac{4}{p-1}$. Let G_2 be the function introduced in (55). There exists a constant $C_2 = C_2(n, p, \beta) > 0$ such that for all $(x,t) \in \mathbb{R}^n_+$ we have*

$$G_2(x,t) \leq \begin{cases} \frac{C_2}{1+|x|^{\beta p-4}+t^{(\beta p/4)-1}} & \text{if } \beta \in \left[\frac{4}{p-1}, \frac{n+4}{p}\right) \\ \frac{C_2}{1+\frac{|x|^n}{\log(1+|x|)}+\frac{t^{n/4}}{\log(1+t)}} & \text{if } \beta = \frac{n+4}{p} \\ \frac{C_2}{1+|x|^n+t^{n/4}} & \text{if } \beta > \frac{n+4}{p}. \end{cases} \tag{57}$$

Moreover, for any $\varepsilon > 0$ the constant C_2 may be chosen independently of $\beta \in \left[\frac{4}{p-1}, \frac{n+4}{p} - \varepsilon\right] \cup \left[\frac{n+4}{p} + \varepsilon, +\infty\right)$.

The proofs of Propositions 2 and 3 are postponed to Sections 8 and 9, respectively.

6.1 Proof of Theorem 4 in the case $\beta \in \left[\frac{4}{p-1}, n\right)$

We fix $\beta_0 \in [4/(p-1), n)$ arbitrary and remark that the following arguments are uniform in $\beta \in [4/(p-1), \beta_0]$. We proceed as in [9]. Let $M > 0$ to be fixed below. For any $T > 0$ we introduce the set

$$S_T = \left\{ v \in C^0(\mathbb{R}^n \times [0,T]) : 0 \leq v(x,t) \leq \frac{M}{1+|x|^\beta + t^{\beta/4}} \right\}.$$

It is clear that the set S_T is a nonempty, closed and convex subset of the Banach space $C^0_b(\mathbb{R}^n \times [0,T])$ of continuous and bounded functions in $\mathbb{R}^n \times [0,T]$.
We show that, for any $T > 0$, S_T is an invariant set for $\widetilde{B}$, i.e.

$$\widetilde{B} S_T \subset S_T$$

for a suitable choice of $M > 0$ and $\alpha > 0$ in (47).
Note that for any $\beta \in \left(\frac{4}{p-1}, n\right)$ the exponents of t which appear in (57) are strictly larger than $\beta/4$ which appears in (56), while they are equal for $\beta = 4/(p-1)$. Hence, we have

$$G_2(x,t) \leq \frac{\widetilde{C}_2}{1+|x|^\beta + t^{\beta/4}} \tag{58}$$

with a suitable constant $\widetilde{C}_2 = \widetilde{C}_2(n,p,\beta) > 0$. To see that $\widetilde{C}_2$ may be chosen independently of $\beta \in \left[\frac{4}{p-1}, \beta_0\right]$, we observe that in (55), β may be replaced by $\tilde{\beta} := \frac{\beta+4}{p} \leq \frac{\beta_0+4}{p} < \frac{n+4}{p}$. Applying Proposition 3 to this $\tilde{\beta}$ yields (58) with $\widetilde{C}_2$ to be chosen uniformly for $\tilde{\beta} \in \left[\frac{4}{p-1}, \frac{\beta_0+4}{p}\right]$, i.e. independently of $\beta \in \left[\frac{4}{p-1}, \beta_0\right]$.
Therefore, inserting (56) and (58) into (54) and (55), we obtain

$$\widetilde{B} v(x,t) \leq \frac{D_n \theta_n \alpha C_1 + \theta_n D_n M^p \widetilde{C}_2}{1+|x|^\beta + t^{\beta/4}} = \frac{M}{1+|x|^\beta + t^{\beta/4}}$$

for

$$M=\left(\widetilde{C}_2\theta_n D_n p\right)^{-1/(p-1)}$$

and

$$\alpha=\frac{p-1}{D_n^{p/(p-1)}\theta_n^{p/(p-1)}C_1\widetilde{C}_2^{1/(p-1)}p^{p/(p-1)}}.$$

The rest of the proof is based on an application of the Schauder fixed point Theorem as in the proof of Theorem 1 in [9]. With this procedure we find a function $v\in C^0\left(\mathbb{R}_+^{n+1}\right)$ such that $\widetilde{B}v=v$ and

$$0\le v\left(x,t\right)\le\frac{M}{1+|x|^\beta+t^{\beta/4}}\qquad\text{for all }\ (x,t)\in\mathbb{R}_+^{n+1}.$$

By (52) we conclude that (48) admits a solution $u\in C^0(\mathbb{R}_+^{n+1})$ such that

$$0\le|u(x,t)|\le v\left(x,t\right)\qquad\text{for all }\ (x,t)\in\mathbb{R}_+^{n+1}.$$

Finally by Propositions 2 and (58) we infer that the constants C_1 and $\widetilde{C}_2$ may be chosen independently of $\beta\in\left[\frac{4}{p-1},\beta_0\right]$ so that also α and M may be chosen independently of $\beta\in\left[\frac{4}{p-1},\beta_0\right]$.
So, Theorem 4 is proved for $\beta\in[4/(p-1),n)$.

6.2 Proof of Theorem 4 in the case $\beta\ge n$

If $\beta>n$, for suitable M we work within the set

$$S_T=\left\{v\in C^0\left(\mathbb{R}^n\times[0,T]\right):0\le v\left(x,t\right)\le\frac{M}{1+|x|^n+t^{n/4}}\right\},$$

while for $\beta=n$ we put

$$S_T=\left\{v\in C^0\left(\mathbb{R}^n\times[0,T]\right):0\le v\left(x,t\right)\le\frac{M}{1+\frac{|x|^n}{\log(1+|x|)}+\frac{t^{n/4}}{\log(1+t)}}\right\}.$$

This yields the proof of Theorem 4 in the case $\beta\ge n$ exactly as in the case $\frac{4}{p-1}\le\beta<n$. □

6.3 Proof of Theorem 3

Let $u\left(x,t\right)$ be the global solution of (2) with the initial condition u_0 as in the statement of Theorem 3. Provided a is sufficiently small, this global solution exists in view of Theorem 4. Moreover, by (5), (55) and Proposition 3 we infer that for any $\varepsilon>0$ there exists $t_1>0$ (independent of x) such that

$$\begin{aligned}\left|\int_0^t b\left(t-s\right)*|u|^{p-1}u\left(s\right)\,ds\right| &\le D_n\int_0^t\widetilde{b}\left(t-s\right)*|u(s)|^p\,ds\\ &\le D_n\theta_nA^pG_2\left(x,t\right)\le\varepsilon t^{-\beta/4}\qquad\text{for all }t>t_1\ \text{ and }x\in\mathbb{R}^n.\end{aligned}\tag{59}$$

This fact occurs since for any $4/(p-1)<\beta<n$ we have $\beta/4<\min\left\{(\beta p/4)-1,n/4\right\}.$
On the other hand, by Theorem 1 we deduce that for any compact set $K\subset\mathbb{R}^n$ and for any $\varepsilon>0$ there exists $t_2>0$ (independent of $x\in K$) such that

$$b\left(t\right)*u_0\ge\left(a\widetilde{C}_{n,\beta}-\varepsilon\right)t^{-\beta/4}\qquad\text{for all }t>t_2\ \text{ and }x\in K.\tag{60}$$

Combining (59)-(60) we infer that

$$u(x,t) = b(t) * u_0 + \int_0^t b(t-s) * |u|^{p-1} u(s)\ ds \geq \left(a\widetilde{C}_{n,\beta} - 2\varepsilon\right) t^{-\beta/4} \qquad \text{for all } t > t_3 \text{ and } x \in K$$

with $t_3 = \max\{t_1, t_2\}$. Since $\widetilde{C}_{n,\beta} > 0$ we may choose $\varepsilon > 0$ small enough such that $a\widetilde{C}_{n,\beta} - 2\varepsilon > 0$. This proves the eventual local positivity of u. □

7 Some technical lemmas

We recall here the trivial inequalities

$$\min\left\{\frac{1}{a}, \frac{1}{b}\right\} \leq \frac{2}{a+b} \qquad \text{for all } a, b > 0. \tag{61}$$

For any $m \in \mathbb{N}$ and $q > 0$ there exist $\gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 \left(\sum_{i=1}^{m} \alpha_i\right)^q \leq \sum_{i=1}^{m} \alpha_i^q \leq \gamma_2 \left(\sum_{i=1}^{m} \alpha_i\right)^q \qquad \text{for all } \alpha_i \geq 0,\ i = 1, ..., m. \tag{62}$$

We provide some technical estimates which will be fundamental for our results.

Lemma 4. *Let $p > 1 + 4/n$ and $\beta \in \left[\frac{4}{p-1}, \frac{n+4}{p}\right)$. There exists a constant $C = C(n,p,\beta)$ such that for $n \geq 2$ we have*

$$\Gamma_{1,n} := \int_0^\infty \frac{1}{(1+w^2)^{n/2}} \int_0^{3/T} \rho^{n-5} e^{-\mu\rho^{4/3}} \int_0^{T\rho} \frac{\sigma^3}{\left[\frac{T^3}{\rho}(T\rho - \sigma) + (\sigma-1)^4 + \frac{\sigma^2}{(1+w^2)^2}\right]^{\frac{\beta p}{4}}} d\sigma\, d\rho\, dw \leq C. \tag{63}$$

For $n = 1$ and any $T > 0$ we have

$$\Gamma_{1,1} := \int_0^{3/T} z^{-4} e^{-\mu z^{4/3}} \int_0^{Tz} \frac{\sigma^3}{\left[\frac{T^3}{z}(Tz - \sigma) + (\sigma-1)^4\right]^{\frac{\beta p}{4}}}\ d\sigma\, dz \leq C. \tag{64}$$

Moreover, for any $\beta_1 \in \left(\frac{4}{p-1}, \frac{n+4}{p}\right)$ the constant C may be chosen independently of $\beta \in \left[\frac{4}{p-1}, \beta_1\right]$.

Proof. The estimates (63) and (64) may be obtained with the same procedure introduced in the proof of Lemmas 1 to 4 in [9]. There, the integral was split as follows:

$$\int_0^\infty \int_0^{1/(2T)} \int_0^{T\rho} + \int_0^\infty \int_{1/(2T)}^{3/T} \int_0^{T\rho/2} + \int_0^\infty \int_{1/(2T)}^{1/T} \int_{T\rho/2}^{T\rho} + \int_0^\infty \int_{1/T}^{3/T} \int_{T\rho/2}^{T\rho}.$$

Most delicate are the integrals

$$\int_0^\infty \int_{1/(2T)}^{1/T} \int_{T\rho/2}^{T\rho} \quad \text{and} \quad \int_0^\infty \int_{1/T}^{3/T} \int_{T\rho/2}^{T\rho},$$

where the σ-integral is divided further by a ρ dependent polynomial of order 8. This enables us to find suitable estimates for terms which become strongly singular for certain values of ρ. The condition $\beta < \frac{n+4}{p}$ is used in order to ensure finiteness of certain integrals which arise in the course of our estimates. □

Lemma 5. *Let $p > 1 + 4/n$ and $\beta \in \left[\frac{4}{p-1}, n\right)$. There exists a constant $C = C(n, p, \beta)$ such that:*
– for $n \geq 2$ and for any $T > 0$ we have

$$\Gamma_{2,n} := \int_0^\infty \frac{1}{(1+w^2)^{n/2}} \int_{3/T}^\infty \rho^{n-5} e^{-\mu\rho^{4/3}} \int_0^{T\rho} \frac{\sigma^3}{\left[\frac{T^3}{\rho}(T\rho - \sigma) + (\sigma-1)^4 + \frac{\sigma^2}{(1+w^2)^2}\right]^{\frac{\beta p}{4}}} \, d\sigma \, d\rho \, dw \leq C; \tag{65}$$

– for $n = 1$ and for any $T > 0$ we have

$$\Gamma_{2,1} := \int_{3/T}^\infty z^{-4} e^{-\mu z^{4/3}} \int_0^{Tz} \frac{\sigma^3}{\left[\frac{T^3}{z}(Tz - \sigma) + (\sigma-1)^4\right]^{\frac{\beta p}{4}}} \, d\sigma \, dz \leq C. \tag{66}$$

Moreover, for any $\beta_1 \in \left(\frac{4}{p-1}, n\right)$ the constant C may be chosen independently of $\beta \in \left[\frac{4}{p-1}, \beta_1\right]$.

Proof. Proceeding as in the proof of Lemma 5 in [9] we arrive at

$$\Gamma_{2,n} \leq C + \frac{C}{T^{n-4+\beta p}} \int_3^\infty \tau^{n-5+\frac{\beta p}{4}} e^{-\mu\left(\frac{\tau}{T}\right)^{4/3}} d\tau =: C + Cg(T) \tag{67}$$

for any $n \geq 1$. We need to prove that the function $g = g(T)$ defined above is bounded for $T \in (0, \infty)$. Since g is continuous in $(0, \infty)$ we have only to prove that g remains bounded as $T \to 0$ and $T \to \infty$. If $T < 1$ we have

$$\begin{aligned} g(T) &= \frac{1}{T^{n-4+\beta p}} \int_3^\infty \tau^{n-5+\frac{\beta p}{4}} e^{-\frac{\mu}{2}\left(\frac{\tau}{T}\right)^{4/3}} e^{-\frac{\mu}{2}\left(\frac{\tau}{T}\right)^{4/3}} d\tau \\ &\leq \frac{e^{-\frac{\mu}{2}\left(\frac{3}{T}\right)^{4/3}}}{T^{n-4+\beta p}} \int_3^\infty \tau^{n-5+\frac{\beta p}{4}} e^{-\frac{\mu}{2}\tau^{4/3}} d\tau = C \frac{e^{-\frac{\mu}{2}\left(\frac{3}{T}\right)^{4/3}}}{T^{n-4+\beta p}}. \end{aligned}$$

The last term tends to 0 as $T \to 0$.
If $T > 1$ we split the integral in (67) as follows

$$g(T) = \frac{1}{T^{n-4+\beta p}} \left[\int_3^{3T^\alpha} \tau^{n-5+\frac{\beta p}{4}} e^{-\mu\left(\frac{\tau}{T}\right)^{4/3}} d\tau + \int_{3T^\alpha}^\infty \tau^{n-5+\frac{\beta p}{4}} e^{-\mu\left(\frac{\tau}{T}\right)^{4/3}} d\tau\right]$$

for some $\alpha > 1$ which will be fixed below. Since

$$\tau \leq 3T^\alpha \Longrightarrow \frac{1}{T} \leq \frac{C}{\tau^{1/\alpha}}$$

and

$$\tau \geq 3T^\alpha \Longrightarrow \frac{\tau}{T} \geq 3^{1/\alpha} \tau^{\frac{\alpha-1}{\alpha}},$$

we have

$$g(T) \leq C \int_3^{3T^\alpha} \frac{\tau^{n-5+\frac{\beta p}{4}}}{\tau^{\frac{n-4+\beta p}{\alpha}}} d\tau + \frac{C}{T^{n-4+\beta p}} \int_3^\infty \tau^{n-5+\frac{\beta p}{4}} e^{-\mu C \tau^{\frac{4(\alpha-1)}{3\alpha}}} d\tau.$$

If we choose $\alpha > 1$ sufficiently close to 1 then $\tau^{n-5+\frac{\beta p}{4}-\frac{n-4+\beta p}{\alpha}} \in L^1(3, \infty)$ so that $g(T)$ remains bounded when $T \to \infty$. This completes the proof of the lemma. ☐

8 Proof of Proposition 2

We proceed along the lines of the proof of [9, Proposition 2]. We make the change of variables $y = t^{1/4}z$ to obtain

$$G_1(x,t) = \int_{\mathbb{R}^n} \frac{e^{-\mu|z|^{4/3}}}{1 + \left|x - t^{1/4}z\right|^{\beta}}\, dz.$$

We start with the following

Lemma 6. *There exists a constant $K_1 = K_1(n, \beta) > 0$ such that for all $|x| \le 1$ and $t \ge 0$ we have*

$$G_1(x,t) \le \begin{cases} \frac{K_1}{1+|x|^{\beta}+t^{\beta/4}} & \text{if } \beta \in [0,n) \\ \frac{K_1}{1+|x|^{n}+\frac{t^{n/4}}{\log(1+t)}} & \text{if } \beta = n \\ \frac{K_1}{1+|x|^{n}+t^{n/4}} & \text{if } \beta > n. \end{cases} \tag{68}$$

Moreover, for any $\varepsilon > 0$ the constant K_1 may be chosen independently of $\beta \in [0, n-\varepsilon] \cup [n+\varepsilon, +\infty)$.

Proof. For $|x| \le 1$ and $t \ge 0$ we immediately obtain

$$G_1(x,t) \le \int_{\mathbb{R}^n} e^{-\mu|z|^{4/3}} dz \le \frac{A_1}{1+|x|^{\beta}}. \tag{69}$$

with a suitable constant $A_1 = A_1(n, \beta) > 0$.
If $\beta \in [0, n)$ then following closely the proof of Lemma 6 in [9] we obtain

$$G_1(x,t) \le A_2 t^{-\beta/4} \qquad \text{for all } (x,t) \in \mathbb{R}^n_+, \tag{70}$$

where $A_2 = A_2(n, \beta) > 0$.
If $\beta = n$ then for any $(x,t) \in \mathbb{R}^n_+$ we have

$$\begin{aligned} G_1(x,t) &\le t^{-n/4}\left[\int_{|z-t^{-1/4}x|\le 1} \frac{1}{t^{-n/4} + \left|z - t^{-1/4}x\right|^{n}}\, dz + \int_{|z-t^{-1/4}x|>1} e^{-\mu|z|^{4/3}} dz\right] \\ &\le t^{-n/4}\left[\int_{|w|\le 1} \frac{1}{t^{-n/4} + |w|^{n}}\, dw + \int_{\mathbb{R}^n} e^{-\mu|z|^{4/3}} dz\right] \le A_3 t^{-n/4}\left[1 + \log(1+t)\right], \end{aligned} \tag{71}$$

where $A_3 = A_3(n) > 0$.
If $\beta > n$ then for $x \in \mathbb{R}^n$ and $t > 1$, we have

$$\begin{aligned} G_1(x,t) &\le t^{-\beta/4}\left[\int_{|z-t^{-1/4}x|\le 1} \frac{1}{t^{-\beta/4} + \left|z - t^{-1/4}x\right|^{\beta}}\, dz + \int_{|z-t^{-1/4}x|>1} e^{-\mu|z|^{4/3}} dz\right] \\ &\le t^{-\beta/4}\left[\int_{|w|\le 1} \frac{1}{t^{-\beta/4} + |w|^{\beta}}\, dw + \int_{\mathbb{R}^n} e^{-\mu|z|^{4/3}} dz\right]. \end{aligned}$$

After the change of variables $w = t^{-1/4}\eta$ we obtain

$$G_1(x,t) \le t^{-\beta/4}\left[t^{\beta/4-n/4}\int_{\mathbb{R}^n} \frac{1}{1+|\eta|^{\beta}}\, d\eta + \int_{\mathbb{R}^n} e^{-\mu|z|^{4/3}} dz\right] \le A_4 t^{-n/4}$$

with $A_4 = A_4(n,\beta) > 0$.
On the other hand, for $|x| \le 1$ and $t \in [0,1]$ the first inequality in (69) holds true so that for any $x \in \mathbb{R}^n$ and $t > 0$ we have

$$G_1(x,t) \le A_5 t^{-n/4}. \tag{72}$$

for $A_5 = A_5(n,\beta) > 0$.
Combining (69) with (70) if $\beta \in [0,n)$, with (71) if $\beta = n$ and with (72) if $\beta > n$, by (61) we immediately obtain (68). □

Assuming now $|x| > 1$ we prove:

Lemma 7. *There exists a constant $K_2 = K_2(n,\beta) > 0$ such that for all $|x| > 1$ and $t \ge 0$ we have*

$$G_1(x,t) \le \begin{cases} \frac{K_2}{1+|x|^\beta + t^{\beta/4}} & \text{if } \beta \in [0,n) \\ \frac{K_2}{1+\frac{|x|^n}{\log(1+|x|)}+\frac{t^{n/4}}{\log(1+t)}} & \text{if } \beta = n \\ \frac{K_2}{1+|x|^n+t^{n/4}} & \text{if } \beta > n. \end{cases} \tag{73}$$

Moreover, for any $\varepsilon > 0$ the constant K_2 may be chosen independently of $\beta \in [0, n-\varepsilon] \cup [n+\varepsilon, +\infty)$.

Proof. For $\beta \in [0,n)$ the proof of (73) is exactly the same as in the proof of Lemma 7 in [9].
In the rest of this proof we assume $\beta \ge n$. We proceed as in the proof of [3, Lemma 2.2]. Given $R > 1/2$, let $|x| > R$ such that

$$\begin{aligned} G_1(x,t) &= \int_{\mathbb{R}^n} t^{-n/4} \frac{e^{-\mu\left(\frac{|x-y|^4}{t}\right)^{1/3}}}{1+|y|^\beta}\, dy \\ &= \int_{|y|>R} t^{-n/4} \frac{e^{-\mu\left(\frac{|x-y|^4}{t}\right)^{1/3}}}{1+|y|^\beta}\, dy + \int_{|y|\le R} t^{-n/4} \frac{e^{-\mu\left(\frac{|x-y|^4}{t}\right)^{1/3}}}{1+|y|^\beta}\, dy \\ &\le \frac{C(n)}{1+R^\beta} + \int_{|y|\le R} \frac{1}{|x-y|^n} \left(\frac{|x-y|^4}{t}\right)^{n/4} \frac{e^{-\mu\left(\frac{|x-y|^4}{t}\right)^{1/3}}}{1+|y|^\beta}\, dy. \end{aligned} \tag{74}$$

Since we have

$$|x-y| \ge |x| - |y| \ge |x| - R > 0 \qquad \text{for } |y| \le R,\ |x| > R$$

and

$$z^{n/4} e^{-\mu z^{1/3}} \le C(n) e^{-(\mu/2) z^{1/3}} \qquad \text{for all } z \ge 0,$$

by (74) we obtain

$$\begin{aligned} G_1(x,t) &\le \frac{C(n)}{1+R^n} + \frac{C(n)}{(|x|-R)^n} \int_{|y|\le R} \frac{e^{-(\mu/2)\left(\frac{|x-y|^4}{t}\right)^{1/3}}}{1+|y|^\beta}\, dy \\ &\le \frac{C(n)}{1+R^n} + \frac{C(n)}{(|x|-R)^n} \int_{|y|\le R} \frac{1}{1+|y|^\beta}\, dy. \end{aligned}$$

If $\beta = n$, we infer

$$G_1(x,t) \le \frac{C(n)}{1+R^n} + \frac{C(n)}{(|x|-R)^n} \log(1+R).$$

If $\beta > n$, we find that

$$G_1(x,t) \le \frac{C(n)}{1+R^n} + \frac{C(n,\beta)}{(|x|-R)^n}.$$

Choosing $R = |x|/2$, then for any $|x| > 1$ and $t \ge 0$ we obtain with a suitable constant $C(n,\beta)$

$$G_1(x,t) \le \begin{cases} \frac{C(n,\beta)}{1+|x|^n} \log(1+|x|) & \text{if } \beta = n \\ \\ \frac{C(n,\beta)}{1+|x|^n} & \text{if } \beta > n. \end{cases}$$

Combining this estimate with (71), (72) and (61), we obtain (73) also for $\beta \ge n$. □

The proof of Proposition 2 follows combining the estimates of Lemmas 6 and 7.

9 Proof of Proposition 3

With the change of variables $y = s^{1/4}z$ we obtain

$$G_2(x,t) = \int_0^t \int_{\mathbb{R}^n} \frac{e^{-\mu|z|^{4/3}}}{\left[1+(t-s)^{\beta/4} + \left|x - s^{1/4}z\right|^\beta\right]^p} \, dzds. \tag{75}$$

We start with the following time decay estimate.

Lemma 8. *Let $p > 1+4/n$. Then, there exists a constant $B_1 = B_1(n,p,\beta) > 0$ such that for all $x \in \mathbb{R}^n$ and $t > 0$ we have*

$$G_2(x,t) \le \begin{cases} B_1 t^{-\frac{\beta p}{4}+1} & \text{if } \beta \in \left[\frac{4}{p-1}, \frac{n+4}{p}\right) \\ \\ B_1 t^{-n/4}\left[1+\log(1+t)\right] & \text{if } \beta = \frac{n+4}{p} \\ \\ B_1 t^{-n/4} & \text{if } \beta > \frac{n+4}{p}. \end{cases} \tag{76}$$

Moreover, for any $\varepsilon > 0$ the constant B_1 may be chosen independently of $\beta \in \left[\frac{4}{p-1}, \frac{n+4}{p} - \varepsilon\right] \cup \left[\frac{n+4}{p} + \varepsilon, +\infty\right)$.

Proof. We split the integral in the right hand side of (75) in the following way

$$G_2(x,t) \le \int_0^t \int_{\left|z-s^{-1/4}x\right| \le 1/2} + \int_0^t \int_{\left|z-s^{-1/4}x\right| > 1/2} = I_1 + I_2.$$

In the rest of this proof, we denote $C = C(n,p,\beta)$. If we put $w = z - s^{-1/4}x$ then, making use of (62), we find for the integral defined by I_1:

$$\begin{aligned} I_1 &= \int_0^t \int_{|z-s^{-1/4}x|\le 1/2} \frac{e^{-\mu|z|^{4/3}}}{\left[1+(t-s)^{\beta/4}+s^{\beta/4}\left|z-s^{-1/4}x\right|^{\beta}\right]^p}\,dzds \\ &\le \int_0^t \int_{|w|\le 1/2} \frac{1}{\left[1+(t-s)^{\beta/4}+s^{\beta/4}|w|^{\beta}\right]^p}\,dw\,ds \\ &\le C\int_0^t \int_{|w|\le 1/2} \frac{1}{\left(1+t-s+s|w|^4\right)^{\frac{\beta p}{4}}}\,dwds = C\int_{|w|\le 1/2}\int_0^t \frac{1}{\left(1+t-s+s|w|^4\right)^{\frac{\beta p}{4}}}\,dsdw \\ &\le C\int_{|w|\le 1/2} \frac{1}{\left(1-|w|^4\right)\left(1+t|w|^4\right)^{\frac{p\beta}{4}-1}}\,dw. \end{aligned} \tag{77}$$

We distinguish the three cases $\frac{4}{p-1} \le \beta < \frac{n+4}{p}$, $\beta = \frac{n+4}{p}$, $\beta > \frac{n+4}{p}$.
If $\frac{4}{p-1} \le \beta < \frac{n+4}{p}$, then $|w|^{-p\beta+4}/\left(1-|w|^4\right) \in L^1\left(B_{1/2}(0)\right)$ so that by (77) we obtain

$$I_1 \le \frac{C}{t^{\frac{\beta p}{4}-1}}\int_{|w|\le 1/2}\frac{|w|^{-\beta p+4}}{\left(1-|w|^4\right)}\,dw \le \frac{C}{t^{\frac{\beta p}{4}-1}}. \tag{78}$$

If $\beta = \frac{n+4}{p}$, by (77) we have

$$I_1 \le C\int_{|w|\le 1/2}\frac{1}{\left(1+t|w|^4\right)^{n/4}}\,dw \le \frac{C}{t^{n/4}}\int_{|w|\le 1/2}\frac{1}{\left(\frac{1}{t}+|w|^4\right)^{n/4}}\,dw \le \frac{C\left[1+\log(1+t)\right]}{t^{n/4}}. \tag{79}$$

Finally, if $\beta > \frac{n+4}{p}$ by (77) and the change of variables $\eta = t^{1/4}w$ we infer that

$$I_1 \le \frac{C}{t^{n/4}}\int_{|\eta|\le t^{1/4}/2}\frac{1}{\left(1+|\eta|^4\right)^{\frac{\beta p}{4}-1}}\,d\eta \le \frac{C}{t^{n/4}}\int_{\mathbb{R}^n}\frac{1}{\left(1+|\eta|^4\right)^{\frac{\beta p}{4}-1}}\,d\eta. \tag{80}$$

Next, we consider the integral I_2. By (62) we have

$$\begin{aligned} I_2 &\le \int_0^t \int_{|z-s^{-1/4}x|>1/2} \frac{e^{-\mu|z|^{4/3}}}{\left[(t-s)^{\beta/4}+s^{\beta/4}\left|z-s^{-1/4}x\right|^{\beta}\right]^p}\,dzds \\ &\le \int_0^t \int_{|z-s^{-1/4}x|>1/2} \frac{e^{-\mu|z|^{4/3}}}{\left[(t-s)^{\beta/4}+\left(\frac{s}{16}\right)^{\beta/4}\right]^p}\,dzds \\ &\le C\int_{\mathbb{R}^n} e^{-\mu|z|^{4/3}}dz \cdot \int_0^t \frac{1}{\left(t-s+\frac{s}{16}\right)^{\frac{\beta p}{4}}}\,ds \le \frac{C}{t^{\frac{\beta p}{4}-1}}. \end{aligned}$$

Combining this estimate with (78), (79), (80) we obtain (76). □

We now study the decay of $G_2(x,t)$ with respect to x.

Lemma 9. *Let $p > 1 + 4/n$ and $\beta \geq 4/(p-1)$. Then, there exists a constant $B_2 = B_2(n, p, \beta) > 0$ such that*

$$G_2(x,t) \leq \frac{B_2}{1+|x|^{\beta p-4}} \qquad \textit{for all } |x| \leq 1, \quad \textit{for all } t \geq 0.$$

Moreover, the constant B_2 may be chosen independently of β.

Proof. See the proof of Lemma 9 in [9]. □

Lemma 10. *Let $p > 1 + 4/n$ and $\beta \in \left[\frac{4}{p-1}, \frac{n+4}{p}\right)$. Then, there exists a constant $B_3 = B_3(n, p, \beta) > 0$ such that*

$$G_2(x,t) \leq \frac{B_3}{1+|x|^{\beta p-4}} \qquad \textit{for all } |x| > 1, \quad \textit{for all } t \geq 0. \tag{81}$$

Moreover, for any $\beta_1 \in \left(\frac{4}{p-1}, \frac{n+4}{p}\right)$ the constant B_3 may be chosen independently of $\beta \in \left[\frac{4}{p-1}, \beta_1\right]$.

Proof. Define $F_2(x,t) = \left(1+|x|^{\beta p-4}\right) G_2(x,t)$. We will show that F_2 is bounded in $\mathbb{R}^{n+1}_+$. We distinguish the cases $n \geq 2$ and $n = 1$.

The case $n \geq 2$. The argument is almost the same as in the proof of Lemma 10 in [9]. We obtain for any $|x| > 1$ and $t \geq 0$

$$F_2(x,t) \leq C \int_0^\infty \frac{1}{(1+w^2)^{n/2}} \int_0^\infty \rho^{n-5} e^{-\mu\rho^{4/3}} \int_0^{T\rho} \frac{\sigma^3}{\left[\frac{T^3}{\rho}(T\rho-\sigma) + (\sigma-1)^4 + \frac{\sigma^2}{(1+w^2)^2}\right]^{\frac{\beta p}{4}}} \, d\sigma \, d\rho \, dw. \tag{82}$$

We split the integral in the right hand side of (82) in the following way

$$\int_0^\infty \int_0^{3/T} \int_0^{T\rho} + \int_0^\infty \int_{3/T}^\infty \int_0^{T\rho}.$$

By Lemmas 4 and 5 we infer that $F_2(x,t)$ is bounded for $|x| > 1$ and $t \geq 0$. This implies (81).

The case $n = 1$. We argue again as in Lemma 10 in [9] to obtain for any $|x| > 1$ and $t \geq 0$

$$F_2(x,t) \leq C \int_0^\infty z^{-4} e^{-\mu z^{4/3}} \int_0^{Tz} \frac{\sigma^3}{\left[\frac{T^3}{z}(Tz-\sigma) + (\sigma-1)^4\right]^{\frac{\beta p}{4}}} \, d\sigma \, dz. \tag{83}$$

As above, we split the integral in the right hand side of (83) into

$$\int_0^{3/T} \int_0^{Tz} + \int_{3/T}^\infty \int_0^{Tz}.$$

Again, by Lemmas 4 and 5 we infer that $F_2(x,t)$ is bounded for $|x| > 1$ and $t \geq 0$. This implies (81) also for $n = 1$. □

Lemma 11. *Let $p > 1 + 4/n$ and $\beta \geq \frac{n+4}{p}$. Then, there exists a constant $B_4 = B_4(n, p, \beta) > 0$ such that for all $|x| > 1$ and $t \geq 0$, we have:*

$$G_2(x,t) \leq \begin{cases} \dfrac{B_4}{1+|x|^n} & \textit{if } \beta > \frac{n+4}{p} \\ B_4 \dfrac{\log(1+|x|)}{1+|x|^n} & \textit{if } \beta = \frac{n+4}{p}. \end{cases}$$

Moreover, for any $\beta_1 > \frac{n+4}{p}$ *the constant* B_4 *may be chosen independently of* $\beta \in \left[\beta_1, \infty\right)$*.*

Proof. Define $F_2(x,t) = (1+|x|^n)\, G_2(x,t)$. We split the corresponding integral expression as follows:

$$\begin{aligned} F_2(x,t) &= \int_0^t \int_{|x-s^{1/4}z|\geq \frac{|x|}{2}} \frac{(1+|x|^n)\exp\left(-\mu|z|^{4/3}\right)}{\left(1+(t-s)^{\beta/4}+|x-s^{1/4}z|^{\beta}\right)^p}\, dz\, ds \\ &\quad + \int_0^t \int_{|x-s^{1/4}z|< \frac{|x|}{2}} \frac{(1+|x|^n)\exp\left(-\mu|z|^{4/3}\right)}{\left(1+(t-s)^{\beta/4}+|x-s^{1/4}z|^{\beta}\right)^p}\, dz\, ds =: F_{2,1}(x,t) + F_{2,2}(x,t)\,. \end{aligned}$$

In what follows, we repeatedly use (62). As for the the first term, we have

$$\begin{aligned} F_{2,1}(x,t) &\leq C\int_0^t \int_{|x-s^{1/4}z|\geq \frac{|x|}{2}} \frac{(1+|x|^n)\exp\left(-\mu|z|^{4/3}\right)}{(1+t-s+|x|^4)^{\beta p/4}}\, dz\, ds \\ &\leq C(1+|x|^n)\left[1+t-s+|x|^4)^{1-\beta p/4}\right]_0^t \\ &\leq C(1+|x|)^{n+4-\beta p} \leq C, \end{aligned}$$

where we used the fact that $n+4-\beta p \leq 0$.
As for the second integral, we observe that

$$|x-s^{1/4}z| < \frac{|x|}{2} \Rightarrow |z| > \frac{|x|}{2s^{1/4}}.$$

Taking into account that for any $\alpha \geq 0$ there exists $C_\alpha > 0$ such that $\eta^\alpha \exp(-\eta) \leq C_\alpha$ for all $\eta \geq 0$, we conclude that

$$\begin{aligned} F_{2,2}(x,t) &\leq C\int_0^t \left(\frac{|x|^4}{s}\right)^{n/4} \int_{|w|<|x|/2} \exp\left(-\frac{\mu}{2^{4/3}}\left(\frac{|x|^4}{s}\right)^{1/3}\right) \frac{1}{(1+t-s+|w|^4)^{\beta p/4}}\, dw\, ds \\ &\leq C\int_0^{|x|/2}\int_0^t \frac{\rho^{n-1}}{(1+t-s+\rho^4)^{\beta p/4}}\, ds\, d\rho \\ &= C\int_0^{|x|/2} \rho^{n-1}\left[\left(1+t-s+\rho^4\right)^{1-\beta p/4}\right]_0^t d\rho \\ &\leq C\int_0^{|x|/2} \rho^{n-1}(1+\rho^n)^{(4-\beta p)/n}\, d\rho \\ &= \begin{cases} C\left[\log(1+\rho^n)\right]_0^{|x|/2} \leq C\log(1+|x|) & \text{if } \beta = \dfrac{n+4}{p} \\ C\left[-(1+\rho^n)^{(n+4-\beta p)/n}\right]_0^{|x|/2} \leq C < \infty & \text{if } \beta > \dfrac{n+4}{p}, \end{cases} \end{aligned}$$

thereby proving the claim. ∎

The proof of Proposition 3 follows combining the estimates of Lemmas 8-11.

A Appendix: polynomial approximations of the heat kernels

By Theorem 5, we may rewrite the f-function as

$$f_n(\eta) = c_n \sum_{k=0}^{\infty} (-1)^k a_k \eta^{2k},$$

where

$$a_k = \begin{cases} \frac{(k+m)!\Gamma\left(\frac{2k+2m+1}{4}\right)}{k!(2k+2m)!} & \text{if } n = 2m+1 \text{ is odd,} \\ \frac{\Gamma\left(\frac{k+m}{2}\right)}{2^{2k+m+1}k!(k+m-1)!} & \text{if } n = 2m \text{ is even;} \end{cases}$$

and

$$c_n = \begin{cases} \frac{2^m}{\sqrt{8\pi}} & \text{if } n = 2m+1 \text{ is odd,} \\ 1 & \text{if } n = 2m \text{ is even.} \end{cases}$$

Moreover, for our purposes, the quotient of two consecutive coefficients

$$b_k := \frac{a_{k+1}}{a_k}$$

will be important. We first prove

Lemma 12. *For any integer $n \geq 1$, the sequence $(b_k)_{k\in\mathbb{N}}$ is strictly decreasing.*

Proof. Consider first the case where the space dimension is odd, namely $n = 2m+1$ for some integer $m \geq 0$. Then, we have

$$\begin{aligned} \frac{b_{k+1}}{b_k} &= \frac{a_{k+2}a_k}{a_{k+1}^2} = \frac{(k+1)(2k+2m+1)\Gamma\left(\frac{2k+2m+5}{4}\right)\Gamma\left(\frac{2k+2m+1}{4}\right)}{(k+2)(2k+2m+3)\Gamma\left(\frac{2k+2m+3}{4}\right)^2} \\ &= \frac{(k+1)\Gamma\left(\frac{2k+2m+5}{4}\right)^2}{(k+2)\Gamma\left(\frac{2k+2m+3}{4}\right)\Gamma\left(\frac{2k+2m+7}{4}\right)} < \frac{k+1}{k+2} < 1 \end{aligned}$$

due to the log–convexity of the Gamma–function.
Similarly, when the space dimension is even, namely $n = 2m$ for some integer $m \geq 1$, we have

$$\frac{b_{k+1}}{b_k} = \frac{(k+1)\Gamma\left(\frac{k+m+2}{2}\right)^2}{(k+2)\Gamma\left(\frac{k+m+1}{2}\right)\Gamma\left(\frac{k+m+3}{2}\right)} < \frac{k+1}{k+2} < 1$$

where we used again the log–convexity of the Gamma–function. □

For $K \in \mathbb{N}_0$, we define the following approximations for the f-functions:

$$F_K(\eta) = c_n \sum_{k=0}^{K} (-1)^k a_k \eta^{2k} .$$

We do not indicate the dependence of F_K on n by an additional index. We are now ready to prove polynomial approximations of the f-functions:

Lemma 13. *Let $n \geq 1$ and assume that $K \in \mathbb{N}$ is even. Then, for $0 \leq \eta \leq \frac{1}{\sqrt{b_K}}$ one has*

$$F_{K-1}(\eta) \leq f_n(\eta) \leq F_K(\eta).$$

Moreover, we have the following error estimates:

$$\text{for all } |\eta| \leq \frac{1}{\sqrt{b_K}}: \quad \max\{|f_n(\eta) - F_K(\eta)|, |f_n(\eta) - F_{K-1}(\eta)|\} \leq |F_{K-1}(\eta) - F_K(\eta)| = c_n a_K \eta^{2K}.$$

Proof. One certainly has $f_n(\eta) \le F_K(\eta)$, if for all even $k \ge K$ one has that

$$a_{k+2}\eta^{2k+4} - a_{k+1}\eta^{2k+2} \le 0 \quad \Leftrightarrow \quad \eta^2 \le \frac{1}{b_{k+1}}$$

which, thanks to Lemma 12, is equivalent to

$$\eta^2 \le \frac{1}{b_{K+1}}.$$

Similarly, one has $F_{K-1}(\eta) \le f_n(\eta)$, provided for all odd $k \ge K-1$

$$a_{k+1}\eta^{2k+2} - a_{k+2}\eta^{2k+4} \ge 0 \quad \Leftrightarrow \quad \eta^2 \le \frac{1}{b_{k+1}} \quad \Leftrightarrow \quad \eta^2 \le \frac{1}{b_K}.$$

This completes the proof. □

We now show how this polynomial approximation may be used to determine some quantitative properties of the f-function in space dimension $n = 1$, where it takes a particularly simple form:

$$f_1(\eta) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-s^4} \cos(\eta s)\, ds.$$

We first refine (7) by estimating its modulus in a quantitative way:

Lemma 14. *For any $\eta \ge 0$, the following holds*

$$\frac{1}{c_1}|f_1(\eta)| = 4\left|\int_0^\infty e^{-s^4} \cos(\eta s)\, ds\right| \le 4.4 \exp\left(-0.15\eta^{4/3}\right).$$

Proof. We have

$$\frac{1}{c_1} f_1(\eta) = 2\int_{\mathbb{R}} e^{-s^4} \cos(\eta s)\, ds = \int_{\mathbb{R}} e^{-s^4} e^{i\eta s}\, ds + \int_{\mathbb{R}} e^{-s^4} e^{-i\eta s}\, ds = 2\int_{\mathbb{R}} e^{-s^4} e^{i\eta s}\, ds,$$

where we used the change of variables $s \mapsto -s$. By Cauchy's Theorem we have for any real $\varepsilon > 0$:

$$\int_{\mathbb{R}} e^{-s^4} e^{i\eta s}\, ds = \int_{\mathbb{R}} e^{-(s+i\varepsilon)^4} e^{i\eta(s+i\varepsilon)}\, ds\ ,$$

$$\left|\int_{\mathbb{R}} e^{-s^4} e^{i\eta s}\, ds\right| \le e^{-\eta\varepsilon} \int_{\mathbb{R}} e^{-s^4} e^{6\varepsilon^2 s^2} e^{-\varepsilon^4} \le e^{-\eta\varepsilon} e^{17\varepsilon^4} \int_{\mathbb{R}} e^{-s^4/2}\, ds.$$

Choosing $\varepsilon = (\eta/34)^{1/3}$ so that $17\varepsilon^3 = \eta/2$ yields

$$\left|\int_{\mathbb{R}} e^{-s^4} e^{i\eta s}\, ds\right| \le \left(\int_{\mathbb{R}} e^{-s^4/2}\, ds\right) \exp\left(-\eta^{4/3}/\left(2\sqrt[3]{34}\right)\right) \le 2.2 \exp\left(-0.15\eta^{4/3}\right).$$

The last inequality follows from the fact that

$$\int_{\mathbb{R}} e^{-s^4/2}\, ds \;=\; 2\int_0^\infty e^{-s^4/2}\, ds \;=\; \frac{\sqrt[4]{2}}{2}\,\Gamma\left(\frac{1}{4}\right) \;=\; 2.1558...$$

which, in turn, follows by arguing as for (10). □

We prove a crucial positivity property for f_1:

Lemma 15. *For any positive nonincreasing function* $h : (0,\infty) \to [0,\infty)$ *satisfying*

$$\int_0 h(\eta) d\eta < +\infty$$

we have

$$\int_0^\infty h(\eta) f_1(\eta)\, d\eta > 0.$$

Proof. Note first that by Lemma 14 and under the assumptions on h, we have $hf_1 \in L^1(0,\infty)$. Then, we apply Lemma 13 with $K = 6$ so that, for $|\eta| \le 10$:

$$F_5(\eta) \le f_1(\eta) \le F_6(\eta).$$

F_5 has its first zero beyond 3.43, F_6 before 3.46, so that f_1 has its first zero in the interval $[3.43, 3.46]$. We conclude by means of explicit calculations and Lemma 14

$$\begin{aligned}
\frac{1}{c_1}\int_0^\infty h(\eta) f_1(\eta)\, d\eta &= \frac{1}{c_1}\int_0^{3.43} h(\eta) f_1(\eta)\, d\eta + \frac{1}{c_1}\int_{3.43}^\infty h(\eta) f_1(\eta)\, d\eta \\
&\ge h(3.43)\left(\frac{1}{c_1}\int_0^{3.43} F_5(\eta)\, d\eta - \frac{1}{c_1}\int_{3.43}^\infty |f_1(\eta)|\, d\eta\right) \\
&\ge h(3.43)\left(6.93 - 4.4\int_{3.43}^\infty \exp\left(-0.15\eta^{4/3}\right)\, d\eta\right) \\
&\ge h(3.43)\left(6.93 - \frac{4.4}{\sqrt[3]{3.43}}\int_{3.43}^\infty \eta^{1/3}\exp\left(-0.15\eta^{4/3}\right)\, d\eta\right) \\
&\ge h(3.43)\,(6.93 - 6.72) > 0.
\end{aligned}$$

□

For suitable profiles h, the monotonicity assumption in the previous lemma may be relaxed.

Lemma 16. *There exists* $\beta_0 > 0$ *such that if* $0 < \beta < \beta_0$, *then*

$$\int_{\mathbb{R}} f_1(|\eta|)|\xi - \eta|^{-\beta}\, d\eta > 0 \qquad \text{for all } \xi \in \mathbb{R}. \tag{84}$$

Proof. Let γ_1 be the smallest positive zero of f_1 and take $\beta \le 1/2$. Let $g(\eta) = 4.4c_1 \cdot \exp\left(-0.15\eta^{4/3}\right)$ so that by Lemma 14, we have $|f_1(|\eta|)| \le g(\eta)$. We choose $\xi_0 > 0$ large enough so that

$$\xi_0 \ge 2\gamma_1 \quad \text{and for all } \xi \ge \xi_0: \ 4\xi g(\xi/2) \le 0.1c_1.$$

First, we consider $\xi \ge \xi_0$ and split the integral over $\mathbb{R}$ in (84) into

$$\int_{-\infty}^{-\gamma_1} + \int_{-\gamma_1}^{\gamma_1} + \int_{\gamma_1}^{\xi/2} + \int_{\xi/2}^{2\xi} + \int_{2\xi}^\infty .$$

We obtain by making use of the approximations for $\int f_1$ and $\int g$ from Lemma 15:

$$\begin{aligned}
\int_{\mathbb{R}} f_1(|\eta|)|\xi - \eta|^{-\beta}\, d\eta &\ge -|\xi|^{-\beta}\int_{-\infty}^{-\gamma_1} g(\eta)\, d\eta + 2(3/2)^{-\beta}|\xi|^{-\beta}\int_0^{\gamma_1} f_1(\eta)\, d\eta \\
&\quad -(|\xi|/2)^{-\beta}\int_{\gamma_1}^\infty g(\eta)\, d\eta - 2g(\xi/2)\int_0^\xi \eta^{-\beta}\, d\eta \\
&\ge \left(2(3/2)^{-\beta}\int_0^{\gamma_1} f_1(\eta)\, d\eta - (1+2^\beta)\int_{\gamma_1}^\infty g(\eta)\, d\eta - 4|\xi| g(\xi/2)\right)|\xi|^{-\beta} \\
&\ge c_1\left(2(3/2)^{-\beta} 6.93 - (1+2^\beta)6.72 - 0.1\right)|\xi|^{-\beta} > 0
\end{aligned}$$

uniformly in $\xi \geq \xi_0$ for β close enough to 0. The same estimate holds for $\xi \leq -\xi_0$. Finally, for $\xi \in [-\xi_0, \xi_0]$, we have uniform convergence of

$$\int_{\mathbb{R}} f_1(|\eta|)|\xi - \eta|^{-\beta}\, d\eta \to \int_{\mathbb{R}} f_1(|\eta|) d\eta = \alpha_1^{-1} > 0 \quad \text{as} \quad \beta \to 0.$$

Therefore, the existence of $\beta_0 > 0$ as in the statement follows. □

As an immediate consequence we have even global positivity for a special class of initial data:

Proposition 4. *We assume that $n = 1$ and $u_0(x) = |x|^{-\beta}$. For $\beta > 0$ sufficiently small, the corresponding solution of (1) given by (23) is positive in $\mathbb{R}^2_+$.*

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