Supercritical biharmonic equations with power-type nonlinearity

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Abstract

The biharmonic supercritical equation $\Delta^2 u = |u|^{p-1}u$, where $n > 4$ and $p > (n+4)/(n-4)$, is studied in the whole space $\mathbb{R}^n$ as well as in a modified form with $\lambda(1+u)^p$ as right-hand-side with an additional eigenvalue parameter $\lambda > 0$ in the unit ball, in the latter case together with Dirichlet boundary conditions. As for entire regular radial solutions we prove oscillatory behaviour around the explicitly known radial *singular* solution, provided $p \in ((n+4)/(n-4), p_c)$, where $p_c \in ((n+4)/(n-4), \infty]$ is a further critical exponent, which was introduced in a recent work by Gazzola and the second author. The third author proved already that these oscillations do not occur in the complementing case, where $p \geq p_c$.

Concerning the Dirichlet problem we prove existence of at least one singular solution with corresponding eigenvalue parameter. Moreover, for the extremal solution in the bifurcation diagram for this nonlinear biharmonic eigenvalue problem, we prove smoothness as long as $p \in ((n+4)/(n-4), p_c)$.

1 Introduction and main results

In the present paper we consider qualitative properties of entire radial solutions (defined and regular in the whole space) of the supercritical biharmonic equation

$$\Delta^2 u = |u|^{p-1}u \qquad \text{in } \mathbb{R}^n, \tag{1}$$

where $n \geq 5$ and $p > \frac{n+4}{n-4}$. An important role is played by the explicitly known entire solution

$$u_s(r) = K_0^{1/(p-1)} r^{-4/(p-1)}, \tag{2}$$

where

$$K_0 = \frac{4}{p-1}\left(\frac{4}{p-1}+2\right)\left(n-2-\frac{4}{p-1}\right)\left(n-4-\frac{4}{p-1}\right). \tag{3}$$

It was shown in [5, 8] that positive regular entire solutions to (1) exist and that asymptotically they behave like the singular solution u_s:

$$\lim_{r\to\infty} \frac{u(r)}{u_s(r)} = 1.$$

Moreover, for $n > 12$ a further critical exponent $p_c \in \left(\frac{n+4}{n-4}, \infty\right)$ was introduced being in that interval the unique solution of the following polynomial equation:

$$p_c \cdot \frac{4}{p_c-1}\cdot\left(\frac{4}{p_c-1}+2\right)\cdot\left(n-2-\frac{4}{p_c-1}\right)\cdot\left(n-4-\frac{4}{p_c-1}\right) = \frac{n^2(n-4)^2}{16}. \tag{4}$$

The third author [9] proved in particular that in the "supercritical case", i.e

$$p \geq p_c$$

the convergence of u to u_s is monotone, i.e. $\forall r: \ u(r) < u_s(r)$. Here, we study the reverse case:

Theorem 1. *Let $p_c \in ((n+4)/(n-4), \infty)$ be the number, which is defined by (4) for $n \geq 13$. We assume that*

$$\frac{n+4}{n-4} < p < p_c \text{ if } n \geq 13, \qquad \frac{n+4}{n-4} < p < \infty \text{ if } 5 \leq n \leq 12.$$

Let $r \mapsto u(r)$ be a radial entire solution to (1). Then, as $r \to \infty$, $u(r)$ oscillates infinitely many times around the singular solution $u_s(r)$.

We study also existence of singular solutions as well as qualitative properties of positive solutions of the corresponding Dirichlet problem

$$\begin{cases} \Delta^2 u = \lambda(1+u)^p & \text{in } B, \\ u > 0 & \text{in } B, \\ u = |\nabla u| = 0 & \text{on } \partial B, \end{cases} \tag{5}$$

where $B \subset \mathbb{R}^n$ is the unit ball, $\lambda > 0$ is an eigenvalue parameter and again $n \geq 5$ and $p > \frac{n+4}{n-4}$. In [7] (see also [2]) it was proved that there exists an extremal parameter λ^* such that for $\lambda \in [0, \lambda^*)$ one has a minimal solution which is regular, while not even a weak solution does exist for $\lambda > \lambda^*$. On the extremal parameter $\lambda = \lambda^*$, an extremal solution $u^* \in H_0^2(B) \cap L^p(B)$ exists as monotone limit of the minimal solutions. It is expected that also in the Dirichlet problem, a singular (i.e. unbounded) solution u_σ corresponding to a suitable singular parameter λ_σ exists and will play an important role as far as the shape of the bifurcation diagram for (5) is concerned. However, in [7] we had to leave open even the existence of a singular solution which will be proved in the present paper:

Theorem 2. *Let $n > 4$ and $p > (n+4)/(n-4)$. Then, there exists a parameter $\lambda_\sigma > 0$ such that for $\lambda = \lambda_\sigma$, problem* (5) *admits a radial singular solution.*

Moreover, in [7] we left open whether the extremal solution u^* introduced above is singular (unbounded) or regular (bounded). The corresponding question has been settled for the exponential nonlinearity by Davila, Dupaigne, Guerra and Montenegro [6] thereby developing the previous work [1]. Here, taking advantage of an idea in [6], we prove regularity of the extremal solution of the problem with power-type nonlinearity in the "subcritical" range.

Theorem 3. *Let $p_c \in ((n+4)/(n-4), \infty)$ be the number, which is defined by (4) for $n \geq 13$. We assume that*

$$\frac{n+4}{n-4} < p < p_c \text{ if } n \geq 13, \qquad \frac{n+4}{n-4} < p < \infty \text{ if } 5 \leq n \leq 12.$$

Let $u^* \in H_0^2(B) \cap L^p(B)$ be the extremal radial solution of (5) corresponding to the extremal parameter λ^*, which is obtained as monotone limit of the minimal regular solutions for $\lambda \nearrow \lambda^*$. Then, u^* is regular.

Related results for the corresponding second order problems were obtained e.g. in [3, 4, 10, 11].

2 Entire solutions: The corresponding autonomous system

Here we study qualitative properties of entire radial solutions $r \mapsto u(r)$ to (1) and shall prove Theorem 1. We put

$$v(s) := e^{4s/(p-1)} u(e^s) \quad (s \in \mathbb{R}), \qquad u(r) = r^{-4/(p-1)} v(\log r) \quad (r > 0). \tag{6}$$

According to [8, 9], (1) is then equivalent to

$$\left(\partial_s - \frac{4}{p-1} + n - 4\right) \left(\partial_s - \frac{4}{p-1} + n - 2\right) \left(\partial_s - \frac{4}{p-1} - 2\right) \left(\partial_s - \frac{4}{p-1}\right) v(s) = |v(s)|^{p-1} v(s), \tag{7}$$

$s \in \mathbb{R}$. In order to write this as an autonomous system, we define

$$\begin{cases} w_1(s) &= v(s) \\ w_2(s) &= \left(\partial_s - \frac{4}{p-1}\right) w_1(s) \\ w_3(s) &= \left(\partial_s - \frac{4}{p-1} - 2\right) w_2(s) \\ w_4(s) &= \left(\partial_s - \frac{4}{p-1} + n - 2\right) w_3(s). \end{cases} \tag{8}$$

Equation (7) is equivalent to the following system:

$$\begin{cases} w_1'(s) &= \frac{4}{p-1} w_1 + w_2, \\ w_2'(s) &= \left(\frac{4}{p-1} + 2\right) w_2 + w_3, \\ w_3'(s) &= \left(\frac{4}{p-1} - (n-2)\right) w_3 + w_4, \\ w_4'(s) &= |w_1(s)|^{p-1} w_1(s) + \left(\frac{4}{p-1} - (n-4)\right) w_4. \end{cases} \tag{9}$$

In order to perform the stability analysis around the singular solution $u_s(r) = K_0^{1/(p-1)} r^{-4/(p-1)}$, i.e. $v(s) = K_0^{1/(p-1)}$, we have to linearize (9) around the vector

$$w^{(0)} := K_0^{1/(p-1)} \left(1, -\frac{4}{p-1}, \frac{4}{p-1}\left(\frac{4}{p-1} + 2\right), \left(n - 2 - \frac{4}{p-1}\right) \frac{4}{p-1}\left(\frac{4}{p-1} + 2\right)\right)$$

and come up with the system $w'(s) = M \circ w(s)$ where

$$M := \begin{pmatrix} \frac{4}{p-1} & 1 & 0 & 0 \\ 0 & \frac{4}{p-1} + 2 & 1 & 0 \\ 0 & 0 & \frac{4}{p-1} - (n-2) & 1 \\ pK_0 & 0 & 0 & \frac{4}{p-1} - (n-4) \end{pmatrix}.$$

The corresponding characteristic polynomial is given by

$$P(\nu) = \left(\nu - \frac{4}{p-1} + n - 4\right) \left(\nu - \frac{4}{p-1} + n - 2\right) \left(\nu - \frac{4}{p-1} - 2\right) \left(\nu - \frac{4}{p-1}\right) - pK_0.$$

According to [8], the eigenvalues are given by

$$\nu_1 = \frac{N_1 + \sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)}, \qquad \nu_2 = \frac{N_1 - \sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)},$$

$$\nu_3 = \frac{N_1 + \sqrt{N_2 - 4\sqrt{N_3}}}{2(p-1)}, \qquad \nu_4 = \frac{N_1 - \sqrt{N_2 - 4\sqrt{N_3}}}{2(p-1)},$$

where

$$N_1 := -(n-4)(p-1)+8, \qquad N_2 := (n^2-4n+8)(p-1)^2,$$

$$\begin{aligned} N_3 &:= (9n-34)(n-2)\,(p-1)^4 + 8(3n-8)(n-6)\,(p-1)^3 \\ &\quad +(16n^2-288n+832)\,(p-1)^2 - 128(n-6)(p-1)+256. \end{aligned}$$

One has $\nu_1, \nu_2 \in \mathbb{R}$ and $\nu_2 < 0 < \nu_1$. For any $5 \le n \le 12$ we have $\nu_3, \nu_4 \notin \mathbb{R}$ and $\operatorname{Re}\nu_3 = \operatorname{Re}\nu_4 < 0$. For any $n \ge 13$ and $p < p_c$, $\nu_3, \nu_4 \notin \mathbb{R}$ and $\operatorname{Re}\nu_3 = \operatorname{Re}\nu_4 < 0$, while $\nu_3, \nu_4 \in \mathbb{R}$ and $\nu_4 \le \nu_3 < 0$ if $p \ge p_c$. In any case,

$$\nu_2 < \operatorname{Re}\nu_{3/4} < 0 < \nu_1.$$

The stable manifold of $w^{(0)}$, where the trajectory of any w corresponding to an entire regular solution is contained in, is tangential to the span of the eigenvectors corresponding to ν_2, ν_3, ν_4. In [8] the following strategy to prove Theorem 1 was outlined: in the "subcritical" setting $\frac{n+4}{n-4} < p < p_c$, any such trajectory oscillates around $w^{(0)}$ infinitely many times except those which are tangential to the eigenvector corresponding to ν_2. We show that the latter can not correspond to an entire regular solution.

Proposition 1. *Let $w(\,.\,)$ be a solution of (9) in the stable manifold of $w^{(0)}$ being tangential to the eigenvector corresponding to ν_2. Then the corresponding solution u of (1) is singular or even not defined for all $r > 0$.*

In order to prove this proposition we need the following crucial observation on the sign of the components of an eigenvector corresponding to ν_2:

Lemma 1. *One eigenvector of M corresponding to ν_2 is given by $t = (t_1, t_2, t_3, t_4)$ with*

$$\begin{aligned} t_1 &= 1 > 0, \\ t_2 &= \left(\nu_2 - \frac{4}{p-1}\right) < 0, \\ t_3 &= \left(\nu_2 - 2 - \frac{4}{p-1}\right)\left(\nu_2 - \frac{4}{p-1}\right) > 0, \\ t_4 &= \left(\nu_2 + n - 2 - \frac{4}{p-1}\right)\left(\nu_2 - 2 - \frac{4}{p-1}\right)\left(\nu_2 - \frac{4}{p-1}\right) < 0. \end{aligned}$$

Proof. Since $\nu_2 < 0$ we only have to show that

$$0 > \nu_2 + n - 2 - \frac{4}{p-1} = \frac{n}{2} - \frac{1}{2(p-1)}\sqrt{N_2 + 4\sqrt{N_3}} \tag{10}$$

the latter being equivalent to proving that

$$N_3 > (n-2)^2(p-1)^4.$$

Indeed, by using the supercriticality assumption $(n-4)(p-1) > 8$, we have

$$\begin{aligned} N_3 - (n-2)^2(p-1)^4 &= 8(n-2)(n-4)(p-1)^4 + 8(3n^2-26n+48)(p-1)^3 \\ &\quad +16(n^2-18n+52)(p-1)^2 - 128(n-6)(p-1)+256 \\ &= 8p(p+1)((n-2)(p-1)-4)((n-4)(p-1)-4) > 0. \end{aligned}$$

This proves (10) and hence the lemma. $\square$

Proof of Proposition 1. Let $w(\,.\,)$ be a solution to (9) being tangential for $s \to \infty$ to the eigenvector t from the previous lemma. We may assume that $w(\,.\,)$ exists on the whole real line $\mathbb{R}$ because otherwise, nothing is to be proved. We put $z_1(s) = w_1(s) - w_1^{(0)}$ and further

$$\begin{aligned}
z_1(s) &= w_1(s) - w_1^{(0)} = v(s) - K_0^{1/(p-1)}, \\
z_2(s) &= w_2(s) - w_2^{(0)} = \left(\partial_s - \frac{4}{p-1}\right) z_1(s), \\
z_3(s) &= w_3(s) - w_3^{(0)} = \left(\partial_s - \frac{4}{p-1} - 2\right) z_2(s), \\
z_4(s) &= w_4(s) - w_4^{(0)} = \left(\partial_s - \frac{4}{p-1} + n - 2\right) z_3(s),
\end{aligned}$$

so that

$$\left(\partial_s - \frac{4}{p-1} + n - 4\right) z_4(s) = |v(s)|^{p-1} v(s) - K_0^{p/(p-1)} = |w_1(s)|^{p-1} w_1(s) - |w_1^{(0)}|^{p-1} w_1^{(0)}.$$

Writing this more systematically yields

$$\begin{cases}
z_1'(s) = \frac{4}{p-1} z_1(s) + z_2(s), \\
z_2'(s) = \left(\frac{4}{p-1} + 2\right) z_2(s) + z_3(s), \\
z_3'(s) = \left(\frac{4}{p-1} - (n-2)\right) z_3(s) + z_4(s), \\
z_4'(s) = |w_1(s)|^{p-1} w_1(s) - |w_1^{(0)}|^{p-1} w_1^{(0)} + \left(\frac{4}{p-1} - (n-4)\right) z_4(s).
\end{cases} \tag{11}$$

According to whether $z(\,.\,)$ approaches the origin from "above" or "below" we distinguish two cases.

First case. There exists s_0 large enough such that

$$z_1(s_0) > 0, \quad z_2(s_0) < 0, \quad z_3(s_0) > 0, \quad z_4(s_0) < 0. \tag{12}$$

On any interval $[s, s_0]$ where $z_1(\,.\,) = w_1(\,.\,) - w_1^{(0)} \geq 0$, we must then have

$$\left(\partial_s + (n-4) - \frac{4}{p-1}\right) z_4(s) = |w_1(s)|^{p-1} w_1(s) - |w_1^{(0)}|^{p-1} w_1^{(0)} \geq 0.$$

This makes $e^{\left((n-4) - \frac{4}{p-1}\right)s} z_4(s)$ increasing on $[s, s_0]$, and so (12) implies that

$$e^{\left((n-4) - \frac{4}{p-1}\right)s} z_4(s) \leq e^{\left((n-4) - \frac{4}{p-1}\right)s_0} z_4(s_0) < 0$$

on $[s, s_0]$. In particular, $z_4(s) < 0$ throughout the interval, and we have

$$\left(\partial_s + (n-2) - \frac{4}{p-1}\right) z_3(s) = z_4(s) < 0.$$

This makes $e^{\left((n-2) - \frac{4}{p-1}\right)s} z_3(s)$ decreasing on $[s, s_0]$, so we similarly find that

$$e^{\left((n-2) - \frac{4}{p-1}\right)s} z_3(s) \geq e^{\left((n-2) - \frac{4}{p-1}\right)s_0} z_3(s_0) > 0$$

by (12). Since $\left(\partial_s - 2 - \frac{4}{p-1}\right) z_2(s) = z_3(s) > 0$, the exact same argument leads us to

$$e^{\left(-2-\frac{4}{p-1}\right)s} z_2(s) \leq e^{\left(-2-\frac{4}{p-1}\right)s_0} z_2(s_0) < 0$$

by (12), hence $\left(\partial_s - \frac{4}{p-1}\right) z_1(s) = z_2(s) < 0$ and we finally get

$$e^{-\frac{4}{p-1}s} z_1(s) \geq e^{-\frac{4}{p-1}s_0} z_1(s_0) > 0.$$

That is, $z_1(s) > 0$ on any interval $[s, s_0]$ where $z_1(s) \geq 0$, so it is impossible for $z_1(s)$ to become 0 at some $s < s_0$. Hence $\forall s \leq s_0 : \quad z_1(s) > 0$. For the original solution this means that for $r \leq r_0$, $u(\,.\,)$ lies above the singular solution. This means that $u(\,.\,)$ itself is singular at $r = 0$.

Second case. There exists s_0 large enough such that

$$z_1(s_0) < 0, \quad z_2(s_0) > 0, \quad z_3(s_0) < 0, \quad z_4(s_0) > 0. \tag{13}$$

On any interval $[s, s_0]$ where $z_1(\,.\,) = w_1(\,.\,) - w_1^{(0)} \leq 0$, we must then have

$$\left(\partial_s + (n-4) - \frac{4}{p-1}\right) z_4(s) = |w_1(s)|^{p-1} w_1(s) - |w_1^{(0)}|^{p-1} w_1^{(0)} \leq 0.$$

This makes $e^{\left((n-4)-\frac{4}{p-1}\right)s} z_4(s)$ decreasing on $[s, s_0]$, and so (13) implies that

$$e^{\left((n-4)-\frac{4}{p-1}\right)s} z_4(s) \geq e^{\left((n-4)-\frac{4}{p-1}\right)s_0} z_4(s_0) > 0$$

on $[s, s_0]$. In particular, $z_4(s) > 0$ throughout the interval, and we have

$$\left(\partial_s + (n-2) - \frac{4}{p-1}\right) z_3(s) = z_4(s) > 0.$$

This makes $e^{\left((n-2)-\frac{4}{p-1}\right)s} z_3(s)$ increasing on $[s, s_0]$, so we similarly find that

$$e^{\left((n-2)-\frac{4}{p-1}\right)s} z_3(s) \leq e^{\left((n-2)-\frac{4}{p-1}\right)s_0} z_3(s_0) < 0 \tag{14}$$

by (13). Following this approach, as in the first case, we eventually get

$$z_4(s) > 0, \quad z_3(s) < 0, \quad z_2(s) > 0, \quad z_1(s) < 0 \tag{15}$$

on any interval $[s, s_0]$ where $z_1(s) \leq 0$, so it is impossible for $z_1(s)$ to become 0 at some $s < s_0$. Hence $\forall s \leq s_0 : \quad z_1(s) < 0$, i.e. the corresponding $u(\,.\,)$ is always below the singular solution. In order to prove that $u(\,.\,)$ itself is singular also in this case, we show that $z_1(s) \to -\infty$ for $s \to -\infty$. Since $\forall s \leq s_0 : \quad z_1(s) < 0$, we have that (14) holds true for all $s \leq s_0$. Referring to [7, Proposition 1] would already show that also v and so u cannot be bounded. However, here it is quite easy to show this directly. For some suitable constant $\delta_1 > 0$ one has:

$$\partial_s \left(e^{-\left(2+\frac{4}{p-1}\right)s} z_2(s) \right) = e^{-\left(2+\frac{4}{p-1}\right)s} z_3(s) \leq -\delta_1 e^{-ns}$$

because of (14), and this implies that

$$\begin{aligned} e^{-\left(2+\frac{4}{p-1}\right)s} z_2(s) &\geq \frac{\delta_1}{n} e^{-ns} - \frac{\delta_1}{n} e^{-ns_0} + e^{-\left(2+\frac{4}{p-1}\right)s_0} z_2(s_0) \\ &\geq \delta_2 e^{-ns} \end{aligned}$$

for some suitable constant $\delta_2 > 0$. In particular,

$$\partial_s \left(e^{-\frac{4}{p-1}s} z_1(s) \right) = e^{-\frac{4}{p-1}s} z_2(s) \geq \delta_2 e^{-(n-2)s}$$

and this implies that

$$\begin{aligned} e^{-\frac{4}{p-1}s} z_1(s) &\leq \frac{\delta_2}{n-2} \left(e^{-(n-2)s_0} - e^{-(n-2)s} \right) + e^{-\frac{4}{p-1}s_0} z_1(s_0) \\ &\leq -\delta_3 e^{-(n-2)s} \end{aligned}$$

for some suitable constant $\delta_3 > 0$. Thus, we end up with

$$z_1(s) \leq -\delta_3 e^{-\left(n-2-\frac{4}{p-1}\right)s} \to -\infty \text{ as } s \to -\infty, \tag{16}$$

so that also in this case, the corresponding solution u of (1) becomes singular at $r = 0$. $\square$

Completing the proof of Proposition 1 also yields the proof of Theorem 1.

3 The Dirichlet problem

If we put $r = |x|$ then the equation in (5) becomes

$$u^{(4)}(r) + \frac{2(n-1)}{r} u'''(r) + \frac{(n-1)(n-3)}{r^2} u''(r) - \frac{(n-1)(n-3)}{r^3} u'(r) = \lambda(1+u)^p, \quad r \in [0,1]. \tag{17}$$

If we put

$$U(x) = 1 + u(x/\sqrt[4]{\lambda}) \qquad \text{for } x \in B_{\sqrt[4]{\lambda}}(0) \tag{18}$$

then U solves the equation

$$\Delta^2 U = U^p \qquad \text{in } B_{\sqrt[4]{\lambda}}(0). \tag{19}$$

Since the equation (19) is invariant under the rescaling

$$U_a(x) = aU(a^{\frac{p-1}{4}} x)$$

i.e. U is a solution of (19) if and only if U_a is a solution of (19), it is not restrictive to concentrate our attention on solutions U of the equation (19) which satisfy the condition $U(0) = 1$.

Next we define $U_\gamma = U_\gamma(r)$ as the unique solution of the initial value problem

$$\begin{aligned} &U_\gamma^{(4)}(r) + \frac{2(n-1)}{r} U_\gamma'''(r) + \frac{(n-1)(n-3)}{r^2} U_\gamma''(r) - \frac{(n-1)(n-3)}{r^3} U_\gamma'(r) = |U_\gamma(r)|^{p-1} U_\gamma(r), \\ &U_\gamma(0) = 1, \qquad U_\gamma'(0) = U_\gamma'''(0) = 0, \qquad U_\gamma''(0) = \gamma < 0. \end{aligned} \tag{20}$$

We report here the following fundamental result by [8]:

Lemma 2 ([8]). *Let $n > 4$ and $p > (n+4)/(n-4)$.*

(i) There exists a unique $\overline{\gamma} < 0$ such that the solution $U_{\overline{\gamma}}$ of (20) exists on the whole interval $[0, \infty)$, it is positive everywhere, it vanishes at infinity and it satisfies $U_{\overline{\gamma}}'(r) < 0$ for any $r \in (0, \infty)$.

(ii) *If* $\gamma < \overline{\gamma}$ *there exist* $0 < R_1 < R_2 < \infty$ *such that the solution* U_γ *of* (20) *satisfies* $U_\gamma(R_1) = 0$, $\lim_{r\uparrow R_2} U_\gamma(r) = -\infty$ *and* $U'_\gamma(r) < 0$ *for any* $r \in (0, R_2)$.

(iii) *If* $\gamma > \overline{\gamma}$ *there exist* $0 < R_1 < R_2 < \infty$ *such that the solution* U_γ *of* (20) *satisfies* $U'_\gamma(r) < 0$ *for* $r \in (0, R_1)$, $U'_\gamma(R_1) = 0$, $U'_\gamma(r) > 0$ *for* $r \in (R_1, R_2)$ *and* $\lim_{r\uparrow R_2} U_\gamma(r) = +\infty$.

(iv) *If* $\gamma_1 < \gamma_2 < 0$ *then the corresponding solutions* $U_{\gamma_1}, U_{\gamma_2}$ *of* (20) *satisfy* $U_{\gamma_1} < U_{\gamma_2}$ *and* $U'_{\gamma_1} < U'_{\gamma_2}$ *as long as they both exist.*

Proof. See the statement of [8, Theorem 2] and related proof and also the statement of [8, Lemma 2]. □

For any $\gamma < 0$ let U_γ be the unique local solution of (20). Thanks to Lemma 2 (iii), for $\gamma > \overline{\gamma}$ we may define R_γ as the unique value of $r > 0$ for which we have $U'_\gamma(R_\gamma) = 0$.

Lemma 3. *Let* $n > 4$, $p > (n+4)/(n-4)$ *and* $\gamma \in (\overline{\gamma}, 0)$ *with* $\overline{\gamma}$ *as in the statement of Lemma 2. Then the map* $\gamma \mapsto R_\gamma$ *is monotonically decreasing and*

$$\lim_{\gamma\downarrow\overline{\gamma}} R_\gamma = +\infty.$$

Proof. The fact that the map $\gamma \mapsto R_\gamma$ is monotonically decreasing follows immediately by Lemma 2 (iv). This shows that the function $\gamma \mapsto R_\gamma$ admits a limit as $\gamma \to \overline{\gamma}$. Suppose by contradiction that

$$\overline{R} := \lim_{\gamma\downarrow\overline{\gamma}} R_\gamma < +\infty.$$

Then, by Lemma 2 (i), (iv) we have for all $\gamma \in (\overline{\gamma}, 0)$ that

$$U_\gamma(R_\gamma) > U_{\overline{\gamma}}(R_\gamma) \geq U_{\overline{\gamma}}(\overline{R}) > 0. \tag{21}$$

Define for any $\gamma \in (\overline{\gamma}, 0)$, $r \in [0,1]$ the function

$$u_\gamma(r) = \frac{U_\gamma(R_\gamma r)}{U_\gamma(R_\gamma)} - 1. \tag{22}$$

Then, u_γ solves the Dirichlet problem

$$\begin{cases} \Delta^2 u_\gamma = R_\gamma^4 U_\gamma(R_\gamma)^{p-1}(1+u_\gamma)^p & \text{in } B, \\ u_\gamma = |\nabla u_\gamma| = 0 & \text{on } \partial B. \end{cases} \tag{23}$$

Moreover, by (21) and the fact that $U_\gamma(R_\gamma) \leq U_\gamma(r) \leq U_\gamma(0) = 1$ for any $r \in [0, R_\gamma]$, we have for all $\gamma \in (\overline{\gamma}, 0)$, $x \in B$

$$0 \leq u_\gamma(x) \leq U_{\overline{\gamma}}(\overline{R})^{-1} - 1. \tag{24}$$

This shows that the set $\{u_\gamma : \gamma \in (\overline{\gamma}, 0)\}$ is bounded in $L^\infty(B)$ and hence by a bootstrap argument, from (23) and the fact that $R_\gamma^4 U_\gamma(R_\gamma)^{p-1} \leq \lambda^*$, we deduce that there exists a sequence $\gamma_k \downarrow \overline{\gamma}$ and a function $\overline{u} \in H_0^2(B) \cap C^\infty(\overline{B})$ such that

$$u_{\gamma_k} \to \overline{u} \qquad \text{in } C^4(\overline{B}) \tag{25}$$

as $k \to \infty$. Since the sequence $U_{\gamma_k}(R_{\gamma_k})$ is monotonically decreasing and bounded from below then for any $r \in [0, \overline{R})$ we have that for sufficiently large k, $U_{\gamma_k}(r) = U_{\gamma_k}(R_{\gamma_k})\,[u_{\gamma_k}(r/R_{\gamma_k}) + 1]$ is well defined and admits a finite limit as $k \to \infty$ which will be denoted by $\overline{U}(r)$. In fact $U_{\gamma_k} \to \overline{U}$ in $C^4([0, R])$ for any $0 < R < \overline{R}$ and moreover by (25) we also have that

$$\overline{U}(x) = \left[\lim_{k\to\infty} U_{\gamma_k}(R_{\gamma_k})\right] \cdot \left[\overline{u}\left(\frac{r}{\overline{R}}\right) + 1\right].$$

Since $\overline{u} \in H_0^2(B)$ we also have

$$\lim_{r\uparrow\overline{R}} \overline{U}'(r) = 0. \tag{26}$$

On the other hand by continuous dependence on the initial conditions we also have that

$$\lim_{k\to\infty} U_{\gamma_k}(r) = U_{\overline{\gamma}}(r) \qquad \text{for all } r \in [0, \overline{R})$$

and hence $\overline{U}(r) = U_{\overline{\gamma}}(r)$ for any $r \in [0, \overline{R})$. This with (26) implies

$$\lim_{r\uparrow\overline{R}} U'_{\overline{\gamma}}(r) = 0$$

which is absurd since $U'_{\overline{\gamma}}(\overline{R}) < 0$. This completes the proof of the lemma. □

Lemma 4. *Let $n > 4$ and $p > (n + 4)/(n - 4)$ and let u be a regular solution of* (5). *Then*

$$u(x) \le \left(\frac{\lambda^*}{\lambda}\right)^{1/(p-1)} |x|^{-4/(p-1)} - 1 \qquad \textit{for all } x \in B\backslash\{0\}.$$

Proof. Let u be a regular solution of (5) for some $\lambda > 0$ and define the rescaled function

$$U(x) = \frac{1}{1 + u(0)} \left[1 + u\left(\frac{x}{\sqrt[4]{\lambda}(1 + u(0))^{\frac{p-1}{4}}}\right)\right] \tag{27}$$

so that U satisfies

$$\Delta^2 U = U^p \ \text{ in } \ B_R(0) \qquad \text{and} \qquad U(0) = 1 \tag{28}$$

where we put $R = \sqrt[4]{\lambda}(1 + u(0))^{\frac{p-1}{4}}$.

Define

$$M = \max_{r\in[0,R]} r^{4/(p-1)} U(r)$$

and let $\overline{R} \in (0, R]$ be such that $\overline{R}^{4/(p-1)} U(\overline{R}) = M$. If we define

$$w(r) = \frac{U(\overline{R}r)}{U(\overline{R})} - 1$$

then w solves the problem

$$\begin{cases} \Delta^2 w = \overline{R}^4 U(\overline{R})^{p-1}(1 + w)^p & \text{in } B \\ w = 0 & \text{on } \partial B \\ w' \le 0 & \text{on } \partial B. \end{cases}$$

This proves that $M^{p-1} = \overline{R}^4 U(\overline{R})^{p-1} \le \lambda^*$ since otherwise by the super-subsolution method (see [2, Lemma 3.3] for more details) we would obtain a solution of (5) for $\lambda = \overline{R}^4 U(\overline{R})^{p-1} > \lambda^*$. This yields for all $r \in [0, R]$ that

$$U(r) \le M r^{-4/(p-1)} \le (\lambda^*)^{1/(p-1)} r^{-4/(p-1)}. \tag{29}$$

Then reversing the identity (27), by (29) we obtain

$$u(r) = \lambda^{-1/(p-1)} R^{4/(p-1)} U(Rr) - 1 \le \left(\frac{\lambda^*}{\lambda}\right)^{1/(p-1)} r^{-4/(p-1)} - 1$$

which completes the proof of the lemma. □

Proof of Theorem 2. For $\gamma \in (\overline{\gamma}, 0)$ consider the corresponding solution U_γ of the Cauchy problem (20) and the function u_γ introduced in (22). If we put $\lambda_\gamma = R_\gamma^4 U_\gamma(R_\gamma)^{p-1}$ then by (23) we have that u_γ solves

$$\begin{cases} \Delta^2 u_\gamma = \lambda_\gamma (1+u_\gamma)^p & \text{in } B, \\ u_\gamma = |\nabla u_\gamma| = 0 & \text{on } \partial B. \end{cases} \tag{30}$$

We show that λ_γ remains bounded away from zero for $\gamma > \overline{\gamma}$ sufficiently close to $\overline{\gamma}$, which is defined in Lemma 2. By [8, Theorem 3] we infer that for a fixed $\varepsilon \in (0, K_0^{1/(p-1)})$ there exists a corresponding $r_\varepsilon > 0$ such that

$$U_{\overline{\gamma}}(r) > (K_0^{1/(p-1)} - \varepsilon) r^{-4/(p-1)} \qquad \text{for all } r > r_\varepsilon. \tag{31}$$

On the other hand, by Lemma 3, we deduce that there exists $\gamma_0 \in (\overline{\gamma}, 0)$ such that for any $\gamma \in (\overline{\gamma}, \gamma_0)$ then $R_\gamma > r_\varepsilon$. Therefore by Lemma 2 (iv) we obtain for all $\gamma \in (\overline{\gamma}, \gamma_0)$

$$U_\gamma(R_\gamma) > U_{\overline{\gamma}}(R_\gamma) > (K_0^{1/(p-1)} - \varepsilon) R_\gamma^{-4/(p-1)}$$

and this yields

$$\forall \gamma \in (\overline{\gamma}, \gamma_0): \qquad \lambda_\gamma > (K_0^{1/(p-1)} - \varepsilon)^{p-1} =: C. \tag{32}$$

Combining (32) and Lemma 4 we obtain for all $\gamma \in (\overline{\gamma}, \gamma_0)$, $x \in B \setminus \{0\}$

$$u_\gamma(x) \le \left(\frac{\lambda^*}{C}\right)^{1/(p-1)} |x|^{-4/(p-1)} - 1. \tag{33}$$

Since u_γ solves (30), by (33) we obtain

$$\int_B |\Delta u_\gamma|^2 dx = \lambda_\gamma \int_B (1+u_\gamma)^p u_\gamma dx \le \lambda^* \int_B (1+u_\gamma)^{p+1} dx \le \frac{(\lambda^*)^{\frac{2p}{p-1}}}{C^{\frac{p+1}{p-1}}} \int_B |x|^{-\frac{4(p+1)}{p-1}} dx < +\infty$$

since $p > (n+4)/(n-4)$. This proves that the set $\{u_\gamma : \gamma \in (\overline{\gamma}, \gamma_0)\}$ is bounded in $H_0^2(B)$ and hence there exists a sequence $\gamma_k \downarrow \overline{\gamma}$ and a function $u \in H_0^2(B)$ such that $u_{\gamma_k} \rightharpoonup u$ in $H_0^2(B)$. Moreover, by (33) and applying Lebesgue's theorem, u weakly solves (5) for a suitable $\widetilde{\lambda} \ge C$.

It remains to prove that the function u is unbounded. For simplicity, in the rest of the proof $u_{\gamma_k}, U_{\gamma_k}, R_{\gamma_k}, \lambda_{\gamma_k}$ will be denoted respectively by u_k, U_k, R_k, λ_k.

By compact embedding we have that $u_k \to u$ in $L^1(B)$ and hence we have

$$\lim_{r\downarrow 0} \frac{1}{|B_r(0)|} \int_{B_r(0)} u(x)dx = \lim_{r\downarrow 0} \left[\frac{1}{r^n |B|} \lim_{k\to\infty} \int_{B_r(0)} u_k(x)dx\right]$$

and passing to radial coordinates, by (22) and Lemma 2 (iv), we obtain

$$\begin{aligned} \lim_{r\downarrow 0} \frac{1}{|B_r(0)|} \int_{B_r(0)} u(x)dx &= \lim_{r\downarrow 0} \left[-1 + \frac{n}{r^n} \lim_{k\to\infty} \int_0^r \frac{U_k(R_k\rho)}{U_k(R_k)} \rho^{n-1} d\rho\right] \\ &= \lim_{r\downarrow 0} \left[-1 + \frac{n}{r^n} \lim_{k\to\infty} \frac{1}{R_k^n U_k(R_k)} \int_0^{R_k r} U_k(\rho)\rho^{n-1} d\rho\right] \\ &\ge \lim_{r\downarrow 0} \left[-1 + \frac{n}{r^n} \lim_{k\to\infty} \frac{1}{R_k^n U_k(R_k)} \int_0^{R_k r} U_{\overline{\gamma}}(\rho)\rho^{n-1} d\rho\right]. \end{aligned} \tag{34}$$

By (31) we have that there exist $C, R_0 > 0$ such that

$$\forall \rho \in (R_0, \infty): \qquad U_{\overline{\gamma}}(\rho) > C\rho^{-4/(p-1)}. \tag{35}$$

Hence we have for $k > \overline{k} = \overline{k}(r)$

$$\int_0^{R_k r} U_{\overline{\gamma}}(\rho)\rho^{n-1} d\rho \geq \int_0^{R_0} U_{\overline{\gamma}}(\rho)\rho^{n-1} d\rho + C\left(n - \frac{4}{p-1}\right)^{-1}\left(R_k^{n-\frac{4}{p-1}} r^{n-\frac{4}{p-1}} - R_0^{n-\frac{4}{p-1}}\right). \tag{36}$$

Since $p > (n+4)/(n-4) > (n+4)/n$ and since by (32), λ_k is bounded away from zero as $k \to \infty$ then

$$\lim_{k\to\infty} R_k^n U_k(R_k) = \lim_{k\to\infty} R_k^{n-\frac{4}{p-1}} \lambda_k^{\frac{1}{p-1}} = +\infty$$

and hence by (36) we obtain

$$\lim_{k\to\infty} \frac{1}{R_k^n U_k(R_k)} \int_0^{R_k r} U_{\overline{\gamma}}(\rho)\rho^{n-1} d\rho \geq \liminf_{k\to\infty} \frac{C}{\left(n - \frac{4}{p-1}\right) R_k^n U_k(R_k)} \left(R_k^{n-\frac{4}{p-1}} r^{n-\frac{4}{p-1}} - R_0^{n-\frac{4}{p-1}}\right)$$

$$= \liminf_{k\to\infty} \frac{C r^{n-\frac{4}{p-1}}}{\left(n - \frac{4}{p-1}\right)\lambda_k^{1/(p-1)}} \geq \frac{C r^{n-4/(p-1)}}{\left(n - \frac{4}{p-1}\right)(\lambda^*)^{1/(p-1)}} =: \widetilde{C} r^{n-4/(p-1)}. \tag{37}$$

Inserting (37) in (34) we obtain

$$\lim_{r\downarrow 0} \frac{1}{|B_r(0)|} \int_{B_r(0)} u(x)dx \geq \lim_{r\downarrow 0}(-1 + n\widetilde{C} r^{-4/(p-1)}) = +\infty.$$

This proves that $u \notin L^\infty(B)$. □

Proof of Theorem 3. We make use of an idea from [6]. Let u_λ denote the positive minimal regular solution of (5) for $0 \leq \lambda < \lambda^*$. According to [7, Theorem 2], these are stable so that one has in particular:

$$\forall \varphi \in C_0^\infty(B): \quad \int_B (\Delta\varphi(x))^2 \, dx - p\lambda \int_B (1 + u_\lambda(x))^{p-1} \varphi(x)^2 \, dx \geq 0.$$

By taking the monotone limit we obtain that

$$\forall \varphi \in C_0^\infty(B): \quad \int_B (\Delta\varphi(x))^2 \, dx - p\lambda^* \int_B (1 + u^*(x))^{p-1} \varphi(x)^2 \, dx \geq 0. \tag{38}$$

We assume now for contradiction that u^* is singular. Then, according to [7, Theorem 5] we have the following estimate from below:

$$u^*(x) > \left(\frac{K_0}{\lambda^*}\right)^{1/(p-1)} |x|^{-4/(p-1)} - 1. \tag{39}$$

Combining this with (38) yields

$$\forall \varphi \in C_0^\infty(B): \quad \int_B (\Delta\varphi(x))^2 \, dx \geq pK_0 \int_B |x|^{-4} \varphi(x)^2 \, dx. \tag{40}$$

However under the subcriticality assumptions made we have that $pK_0 > n^2(n-4)^2/16$. This contradicts the optimality of the constant in Hardy's inequality

$$\forall \varphi \in C_0^\infty(B): \quad \int_B (\Delta\varphi(x))^2\, dx \geq \frac{n^2(n-4)^2}{16} \int_B |x|^{-4} \varphi(x)^2\, dx,$$

so that u^* has indeed to be regular. □

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