Boundary value problems for the one-dimensional Willmore equation – Almost explicit solutions

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Abstract

We give closed expressions for classical symmetric solutions of boundary value problems for the one-dimensional Willmore equation. Navier as well as Dirichlet boundary conditions are considered. In the first case, one has existence of precisely two solutions for boundary data below a suitable threshold, precisely one solution on the threshold and no solution beyond the threshold. This effect reflects that we have a bending point in the corresponding bifurcation diagram and is *not* due to that we restrict ourselves to graphs. Under Dirichlet boundary conditions we always have existence of precisely one symmetric solution.

Parts of the material can already be found in Euler's work. It's the goal of the present report to make Euler's observations more accessible and to develop them under the point of view of boundary value problems.

1 Introduction

Recently, the Willmore functional and the associate L^2–gradient flow, the so–called Willmore flow, have attracted a lot of attention. Given a smooth immersed surface $f : M \to \mathbb{R}^3$, the Willmore functional is defined by

$$W(f) := \int_{f(M)} H^2 dA,$$

where $H = (\kappa_1 + \kappa_2)/2$ denotes the mean curvature of $f(M)$. Apart from being of geometric interest, the functional W is a model for the elastic energy of thin shells or biological membranes. Furthermore, it is used in image processing for problems of surface restoration and image inpainting. In these applications one is usually concerned with minima, or more generally with critical points of the Willmore functional. It is well–known that the corresponding surface Γ has to satisfy the Willmore equation

$$\Delta H + 2H(H^2 - K) = 0 \qquad \text{on } \Gamma, \tag{1}$$

where Δ denotes the Laplace–Beltrami operator on Γ and K is Gauss curvature. A solution of (1) is called a Willmore surface. Existence of closed Willmore surfaces of prescribed genus has been proved by Simon [Sn] and Bauer & Kuwert [BK]. Also, local and global existence results for the

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Willmore flow of closed surfaces are available, see e.g. [KS1, KS2, KS3, St]. The Willmore flow for one dimensional closed curves was studied by [P, DKS].

If one is interested in surfaces with boundaries, then appropriate boundary conditions have to be added to (1). Since this equation is of fourth order one requires two sets of conditions and a discussion of possible choices can be found in [Nit] along with corresponding existence results. These results however, are based on perturbation arguments and hence require severe smallness conditions on the data, which are by no means explicit. Thus the question arises whether it is possible to specify more general conditions on the boundary data that will guarantee the existence of a solution to (1). Such a task seems to be quite difficult since the problem is highly nonlinear and in addition lacks a maximum principle.
In order to gain some insight it is natural to look at the one–dimensional case first, where in some cases, almost explicit solutions can be found for suitable boundary value problems. It is the goal of the present report to collect this material, which in parts goes already back to Euler's work [E], and to highlight some geometric properties of solutions and some global properties of the corresponding energy functional and the bifurcation diagrams.

In the one dimensional case, critical points of the total squared curvature functional $\int_\gamma \kappa^2 ds$ are called elastic curves and the analogue of (1) reads

$$\kappa_{ss} + \frac{1}{2}\kappa^3 = 0 \qquad \text{on } \gamma. \tag{2}$$

Here, s denotes arclength of γ. In view of the scaling properties of the total squared curvature functional one often considers $\int_\gamma (\kappa^2 + \lambda) ds$, at least in the case of closed curves leading to an additional term $\lambda\kappa$ in (2). It is possible to describe the solutions of (2) in terms of elliptic integrals. Many results and closed parametric expressions for solutions can already be found in the "Additamentum, De Curvis Elasticis" in Vol. 24 of the first series of Euler's "Opera omnia" [E, pp. 231–297], cf. also [Nit, pp. 381–383]. More recently, Langer and Singer [LS] gave explicit descriptions for closed curves in manifolds of constant sectional curvature. Formulae for non–closed curves in the plane are derived in [LI]. Depending on the conditions prescribed at the fixed endpoints, a nonlinear system of up to three equations needs to be solved in order to determine the parameters that appear in the elliptic integrals.
In the above papers the solutions are given in terms of arclength parametrizations on a–priori unknown parameter domains. In the present work we take a different point of view: the curves under consideration are given as graphs over a fixed domain which we take for simplicity as the unit interval $[0,1]$. In this case the Willmore functional becomes

$$W(u) = \int_{\text{graph}(u)} \kappa(x)^2 \, ds(x) = \int_0^1 \kappa(x)^2 \sqrt{1+u'(x)^2} \, dx, \tag{3}$$

where

$$\kappa(x) = \frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right) = \frac{u''(x)}{(1+u'(x)^2)^{3/2}} \tag{4}$$

is the curvature of the graph of u at the point $(x, u(x))$. The Willmore equation now takes the form

$$\frac{1}{\sqrt{1+u'(x)^2}} \frac{d}{dx}\left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right) + \frac{1}{2}\kappa^3(x) = 0, \quad x \in (0,1), \tag{5}$$

see Lemma 2 below.
We shall focus on two different sets of boundary conditions. Firstly, for a given parameter $\alpha \in \mathbb{R}$ we shall consider Navier boundary conditions

$$u(0) = u(1) = 0, \quad \kappa(0) = \kappa(1) = -\alpha. \tag{6}$$

Note that critical points of (3) in $H^2(0,1)\cap H^1_0(0,1)$ will satisfy $\kappa(0)=\kappa(1)=0$ as natural boundary conditions. In order to obtain the corresponding property for nonzero α one has to replace W by the functional

$$\tilde{W}_\alpha(u)=\int_{\text{graph}(u)}\left(\kappa(x)^2+2\alpha\kappa(x)\right)\,ds(x)=\int_0^1\left(\kappa(x)^2+2\alpha\kappa(x)\right)\sqrt{1+u'(x)^2}\,dx, \tag{7}$$

see Corollary 1 below. As a second set of boundary conditions we shall examine Dirichlet boundary conditions

$$u(0)=u(1)=0,\quad u'(0)=-u'(1)=\beta, \tag{8}$$

where again $\beta\in\mathbb{R}$ is a given real parameter.
Note that due to the presence of the length element $\sqrt{1+u'(x)^2}$ the nice structure of (2) is lost. Nevertheless it is still possible to obtain formulas for the solutions of the above boundary value problems. The key observation in deriving these expressions is that if u is a solution of (5), then the auxiliary function

$$v(x):=\kappa(x)\left(1+u'(x)^2\right)^{1/4}$$

satisfies a *second* order differential equation of the form

$$-\left(a(x)v'(x)\right)'+b(x)v'(x)=0. \tag{9}$$

Here, the coefficients $a(x),b(x)$ depend on the solution u. A similar observation was already made by Euler, see [E, p. 234, line 13]. In the case of a symmetric solution we shall be able to conclude that v is a constant, more precisely we have the following result:

Lemma 1. *Let $u\in C^4([0,1])$ be a function being symmetric around $x=1/2$ and define*

$$c_0:=\int_{\mathbb{R}}\frac{1}{\left(1+\tau^2\right)^{5/4}}=\mathcal{B}\left(\frac{1}{2},\frac{3}{4}\right)=\sqrt{\pi}\frac{\Gamma(3/4)}{\Gamma(5/4)}.$$

Then u solves the Willmore equation (5) iff there exists a constant $c\in(-c_0,c_0)$ such that

$$\forall x\in[0,1]:\qquad \kappa(x)\left(1+u'(x)^2\right)^{1/4}=-c. \tag{10}$$

Having found the above representation we can then examine the abovementioned boundary value problems. Solvability of the Navier boundary value problem (5, 6) strongly depends on whether the boundary datum $|\alpha|$ is below a threshold α_0 or not.

Theorem 1. *There exists $\alpha_0=1.343799725\ldots$ such that for $0<|\alpha|<\alpha_0$, the Navier boundary value problem (5, 6) for the Willmore equation has precisely two smooth solutions u being symmetric around $x=\frac{1}{2}$. If $|\alpha|=\alpha_0$ one has precisely one such solution, for $\alpha=0$ one only has the trivial solution and no such solutions exist for $|\alpha|>\alpha_0$.*

The small solutions are ordered with respect to α while the large ones become smaller for increasing α. See Figures 1, 2 and 3. For the bifurcation diagram, see Figures 4 and 5.

As for the Dirichlet problem the situation is – somehow surprisingly – simpler:

Theorem 2. *For every $\beta\in\mathbb{R}$, the Dirichlet boundary value problem (5, 8) for the Willmore equation has precisely one smooth solution u being symmetric around $x=\frac{1}{2}$. This solution is the unique minimum of the Willmore functional in the class $M_\beta:=\{v\in H^2(0,1)\cap H^1_0(0,1)\,|\,v'(0)=-v'(1)=\beta\}$.*

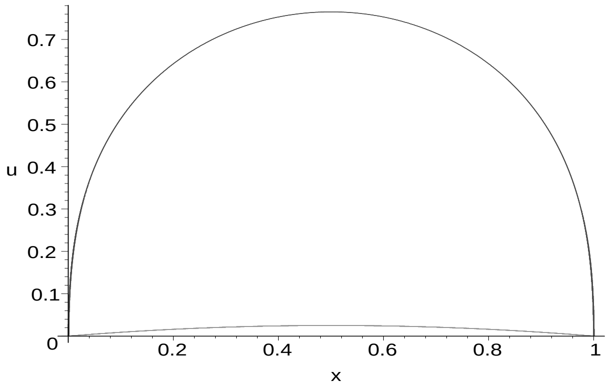

Figure 1: Solutions of the Navier boundary value problem for $\alpha = 0.2$

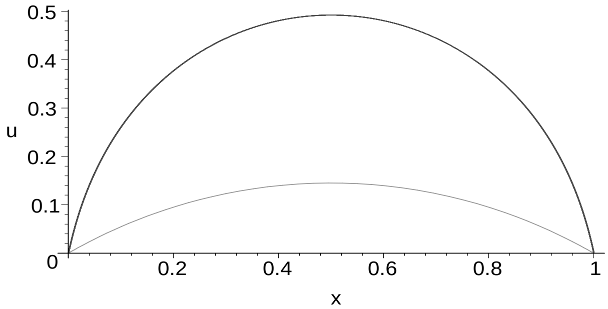

Figure 2: Solutions of the Navier boundary value problem for $\alpha = 1$

The solutions are ordered with respect to β, cf. Lemma 5. This means that we have a comparison principle for the Dirichlet problem (5, 8). For the bifurcation diagram, see Figure 6.

This report is organized as follows. In §2 we shall consider the Euler–Lagrange equations for W and $\tilde{W}_\alpha$. §3 is devoted to the derivation of (9) and to provide a proof of Lemma 1. This result is subsequently used in §4 to obtain a representation for the solution u itself from which we can easily deduce several qualitative properties. Finally, §5 contains the proofs of Theorems 1 and 2.

2 Euler-Lagrange equation

For the reader's convenience, we calculate the first variation of the Willmore functional.

Lemma 2. *Let $u \in C^4([0,1])$ and κ denote the corresponding curvature. Then, for all $\varphi \in C_0^\infty(0,1)$, we have*

$$\frac{d}{dt} W(u + t\varphi)|_{t=0} = \int_0^1 \left(\frac{2}{\sqrt{1+u'(x)^2}} \frac{d}{dx} \left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}} \right) + \kappa^3(x) \right) \varphi(x)\, dx. \tag{11}$$

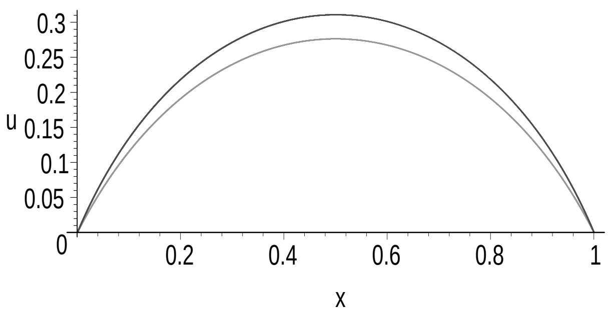

Figure 3: Solutions of the Navier boundary value problem for $\alpha = 1.34$

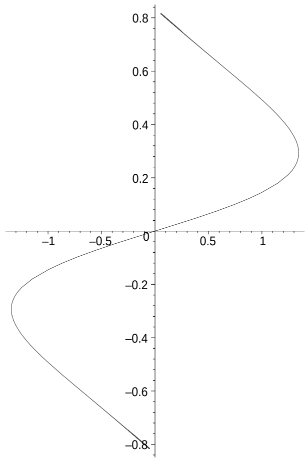

Figure 4: Bifurcation diagram for (5, 6): The extremal value of the solution $u(1/2)$ plotted over α

Proof. We assume only that $\varphi(0) = \varphi(1) = 0$.

$$\frac{d}{dt}W(u+t\varphi)|_{t=0} = \frac{d}{dt}\left(\int_0^1 \frac{(u''(x)+t\varphi''(x))^2}{(1+(u'(x)+t\varphi'(x))^2)^{5/2}}\,dx\right)_{|t=0}$$

$$= 2\int_0^1 \kappa(x)\frac{\varphi''(x)}{1+u'(x)^2}\,dx - 5\int_0^1 \kappa(x)^2\frac{u'(x)}{\sqrt{1+u'(x)^2}}\varphi'(x)\,dx$$

$$= 2\left[\frac{\kappa(x)\varphi'(x)}{1+u'(x)^2}\right]_0^1 - 2\int_0^1 \kappa'(x)\frac{\varphi'(x)}{1+u'(x)^2}\,dx + 4\int_0^1 \kappa(x)\varphi'(x)\frac{u'(x)u''(x)}{(1+u'(x)^2)^2}$$

$$+5\int_0^1 \kappa(x)^3\varphi(x)\,dx + 10\int_0^1 \kappa(x)\kappa'(x)\frac{u'(x)}{\sqrt{1+u'(x)^2}}\varphi(x)\,dx$$

$$= 2\left[\frac{\kappa(x)\varphi'(x)}{1+u'(x)^2}\right]_0^1 + 2\int_0^1 \left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right)'\frac{\varphi(x)}{\sqrt{1+u'(x)^2}}$$

$$-2\int_0^1 \frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\varphi(x)\frac{u'(x)u''(x)}{(1+u'(x)^2)^{3/2}} + 4\int_0^1 \kappa(x)^2\varphi'(x)\frac{u'(x)}{\sqrt{1+u'(x)^2}}\,dx$$

$$+5\int_0^1 \kappa(x)^3\varphi(x)\,dx + 10\int_0^1 \kappa(x)\kappa'(x)\frac{u'(x)}{\sqrt{1+u'(x)^2}}\varphi(x)\,dx$$

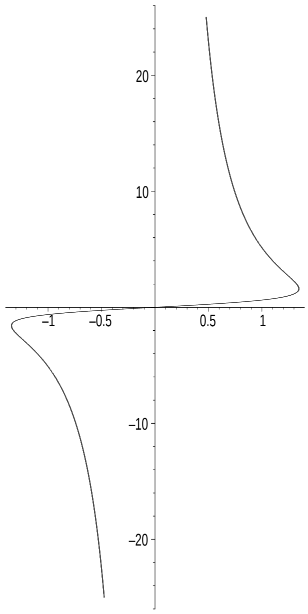

Figure 5: Bifurcation diagram for (5, 6): The extremal value of the derivative $u'(0)$ plotted over α

$$\begin{aligned}
&= 2\left[\frac{\kappa(x)\varphi'(x)}{1+u'(x)^2}\right]_0^1 + 2\int_0^1 \frac{1}{\sqrt{1+u'(x)^2}}\left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right)'\varphi(x)\,dx \\
&\quad +8\int_0^1 \kappa(x)\kappa'(x)\frac{u'(x)}{\sqrt{1+u'(x)^2}}\varphi(x)\,dx - 8\int_0^1 \kappa(x)\kappa'(x)\frac{u'(x)}{\sqrt{1+u'(x)^2}}\varphi(x)\,dx \\
&\quad -4\int_0^1 \kappa(x)^3\varphi(x)\,dx + 5\int_0^1 \kappa(x)^3\varphi(x)\,dx \\
&= 2\left[\frac{\kappa(x)\varphi'(x)}{1+u'(x)^2}\right]_0^1 + 2\int_0^1 \frac{1}{\sqrt{1+u'(x)^2}}\left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right)'\varphi(x)\,dx + \int_0^1 \kappa(x)^3\varphi(x)\,dx.
\end{aligned}$$

□

Corollary 1 *Let $u \in C^4([0,1]) \cap H_0^1(0,1)$ and κ denote the corresponding curvature. We assume that u is a critical point of the modified Willmore functional $\tilde{W}_\alpha$ for some fixed $\alpha \in \mathbb{R}$. So for all $\varphi \in C^\infty[0,1]$ with $\varphi(0) = \varphi(1) = 0$ one has*

$$\frac{d}{dt}\tilde{W}_\alpha(u + t\varphi)_{|t=0} = 0.$$

Then u is a solution of the Willmore equation (5), subject to the Navier boundary conditions:

$$u(0) = u(1) = 0, \qquad \kappa(0) = \kappa(1) = -\alpha.$$

Proof. It remains to calculate

$$\frac{d}{dt}\left(\int_0^1 \frac{u''(x) + t\varphi''(x)}{1 + (u'(x) + t\varphi'(x))^2}\,dx\right)_{|t=0} = \int_0^1 \left(\frac{\varphi''(x)}{1+u'(x)^2} - 2\frac{u''(x)u'(x)\varphi'(x)}{(1+u'(x)^2)^2}\right)dx = \left[\frac{\varphi'(x)}{1+u'(x)^2}\right]_0^1.$$

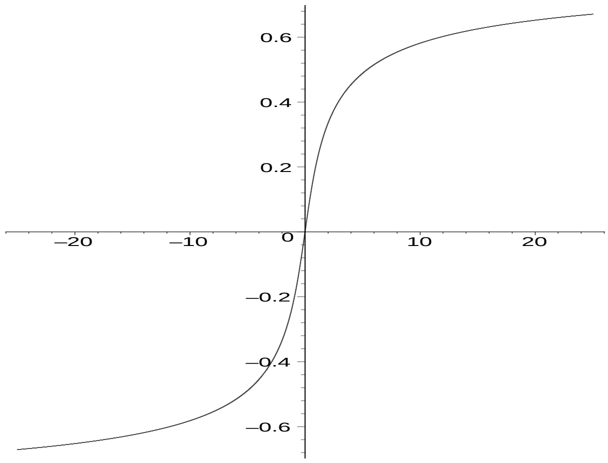

Figure 6: Bifurcation diagram for (5, 8): The extremal value of the solution $u(1/2)$ plotted over β

Combining this result with Lemma 2, we obtain

$$\begin{aligned} 0 &= \frac{d}{dt}\tilde{W}_\alpha(u+t\varphi)_{|t=0} \\ &= 2\left[\frac{(\kappa(x)+\alpha)\varphi'(x)}{1+u'(x)^2}\right]_0^1 + 2\int_0^1 \frac{1}{\sqrt{1+u'(x)^2}}\left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right)'\varphi(x)\,dx + \int_0^1 \kappa(x)^3\varphi(x)\,dx. \end{aligned}$$

Taking first arbitrary $\varphi \in C_0^\infty(0,1)$, we see that u solves the Euler–Lagrange equation (5) and that for all $\varphi \in C^\infty[0,1]$ with $\varphi(0)=\varphi(1)=0$:

$$\left[\frac{(\kappa(x)+\alpha)\varphi'(x)}{1+u'(x)^2}\right]_0^1 = 0.$$

This implies

$$\kappa(0)=\kappa(1)=-\alpha.$$

□

3 The differential equation for the auxiliary function v

We assume that $u \in C^4([0,1])$ solves the Willmore equation (5) and recall the definition of the fundamental auxiliary function

$$v(x) := \kappa(x)\left(1+u'(x)^2\right)^{1/4}. \tag{12}$$

The crucial point is that v satisfies a second order differential equation without term of order zero, see also [E, pp. 233–234].

Lemma 3. *For $x \in [0,1]$ we have:*

$$-\frac{d}{dx}\left((1+u'(x)^2)^{-3/4}v'(x)\right) + \frac{\kappa(x)u'(x)}{(1+u'(x)^2)^{1/4}}v'(x) = 0.$$

Proof. By straightforward calculations we obtain:

$$\begin{aligned} v'(x) &= \kappa'(x)\left(1+u'(x)^2\right)^{1/4} + \frac{1}{2}\kappa(x)^2 u'(x)\left(1+u'(x)^2\right)^{3/4};\\ \left(1+u'(x)^2\right)^{-3/4} v'(x) &= \frac{\kappa'(x)}{\sqrt{1+u'(x)^2}} + \frac{1}{2}\kappa(x)^2 u'(x). \end{aligned}$$

By making use of the Willmore equation (5) we conclude:

$$\begin{aligned} \frac{d}{dx}\left(\left(1+u'(x)^2\right)^{-3/4} v'(x)\right) &= \frac{d}{dx}\left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right) + \frac{1}{2}\kappa(x)^2 u''(x) + \kappa(x)\kappa'(x)u'(x)\\ &= \frac{1}{2}\kappa(x)^3 u'(x)^2\sqrt{1+u'(x)^2} + \kappa(x)\kappa'(x)u'(x) = \frac{\kappa(x)u'(x)}{(1+u'(x)^2)^{1/4}} v'(x). \end{aligned}$$

□

Corollary 2. *By the preceding lemma we know that v satisfies a maximum principle. In particular we may conclude for any solution u of the Willmore equation (5): If $\kappa(0), \kappa(1) < 0$, then $\kappa < 0$ in $[0,1]$. If $\kappa(0) = \kappa(1) = 0$, then $\kappa = 0$ in $[0,1]$ and the solution u is a straight line segment. If we additionally assume $u(0) = u(1) = 0$ then $u(x) \equiv 0$ in $[0,1]$. That means that we have uniqueness for the homogeneous Navier boundary value problem (5, 6) without assuming a priori any smallness on the solution.*

Proof of Lemma 1. Let again $v(x) := \kappa(x)\left(1+u'(x)^2\right)^{1/4}$.

In order to prove necessity of condition (10), we observe first that $v(0) = v(1)$ by our symmetry assumption on u. Since v solves a second order (linearized) differential equation without term of order zero, we conclude that there exists $c \in \mathbb{R}$ such that $\forall x \in [0,1]:\ v(x) = -c$. The additional statement on the admissible range $c \in (-c_0, c_0)$ follows from Lemma 4 below.

For proving sufficiency, we start with (10)

$$\kappa(x) = -\frac{c}{(1+u'(x)^2)^{1/4}}$$

and obtain by differentiating

$$\begin{aligned} \kappa'(x) &= \frac{c}{2}\kappa(x)u'(x)\left(1+u'(x)^2\right)^{1/4} = -\frac{1}{2}\kappa(x)^2 u'(x)\sqrt{1+u'(x)^2};\\ \frac{1}{\sqrt{1+u'(x)^2}}\left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right)' &= -u'(x)\kappa(x)\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}} - \frac{1}{2}\kappa(x)^2\frac{u''(x)}{\sqrt{1+u'(x)^2}}\\ &= \frac{1}{2}u'(x)^2\kappa(x)^3 - \frac{1}{2}\kappa(x)^3\left(1+u'(x)^2\right) = -\frac{1}{2}\kappa(x)^3 \end{aligned}$$

so that the Willmore equation (5) is satisfied. □

4 Explicit form of symmetric solutions of the Willmore equation

In what follows, the function

$$G:\mathbb{R} \to \left(-\frac{c_0}{2}, \frac{c_0}{2}\right), \qquad G(s) := \int_0^s \frac{1}{(1+\tau^2)^{5/4}}\, d\tau \tag{13}$$

plays a crucial role. It's straightforward to see that G is strictly increasing, bijective with $G'(s) > 0$. So, also the inverse function

$$G^{-1} : \left(-\frac{c_0}{2}, \frac{c_0}{2}\right) \to \mathbb{R} \tag{14}$$

is strictly increasing, bijective and smooth with $G^{-1}(0) = 0$.

Lemma 4. *Let $u \in C^4([0,1])$ be a function symmetric around $x = 1/2$. Then u solves the Willmore equation (5) iff there exists $c \in (-c_0, c_0)$ such that*

$$\forall x \in [0,1] : \qquad u'(x) = G^{-1}\left(\frac{c}{2} - cx\right). \tag{15}$$

For the curvature, one has that

$$\kappa(x) = -\frac{c}{\sqrt[4]{1 + G^{-1}\left(\frac{c}{2} - cx\right)^2}}. \tag{16}$$

Moreover, if we additionally assume that $u(0) = u(1) = 0$, then one has

$$u(x) = \frac{2}{c\sqrt[4]{1 + G^{-1}\left(\frac{c}{2} - cx\right)^2}} - \frac{2}{c\sqrt[4]{1 + G^{-1}\left(\frac{c}{2}\right)^2}} \quad (c \neq 0). \tag{17}$$

Proof. First we show the necessity of the representation formula (15). Let $u \in C^4([0,1])$ be a symmetric solution of (5). By Lemma 1 we know that there exists a constant $c \in \mathbb{R}$ such that for all $x \in [0,1]$:

$$-c = \frac{u''(x)}{(1 + u'(x)^2)^{5/4}}. \tag{18}$$

Integration yields

$$-c\left(x - \frac{1}{2}\right) = \int_{1/2}^{x} \frac{u''(\xi)}{(1 + u'(\xi)^2)^{5/4}}\, d\xi = \int_0^{u'(x)} \frac{d\tau}{(1 + \tau^2)^{5/4}} = G(u'(x)).$$

Now it becomes apparent that necessarily $(-|c|/2, |c|/2) \subset G(\mathbb{R})$ so that $|c| < \int_{\mathbb{R}} \frac{d\tau}{(1+\tau^2)^{5/4}} = c_0$.

The sufficiency of (15) is obtained by direct calculation and with help of Lemma 1.

Formula (16) follows by differentiating (15), while for (17) we perform several changes of variables:

$$\begin{aligned} u(x) &= \int_0^x G^{-1}\left(\frac{c}{2} - cs\right) ds = \frac{1}{c}\int_{c/2-cx}^{c/2} G^{-1}(\sigma)\, d\sigma \\ &= \frac{1}{c}\int_{G^{-1}(c/2-cx)}^{G^{-1}(c/2)} \frac{t}{(1 + t^2)^{5/4}}\, dt = \frac{2}{c\sqrt[4]{1 + G^{-1}\left(\frac{c}{2} - cx\right)^2}} - \frac{2}{c\sqrt[4]{1 + G^{-1}\left(\frac{c}{2}\right)^2}}. \end{aligned}$$

□

The explicit formulae of the preceding lemma allow for precise statements on the qualitative behaviour of solutions:

Corollary 3. *Let u be a solution in $[0,1]$ of the Willmore equation (5), being symmetric around $x = 1/2$. For any nonconstant solution we have that κ is of fixed sign,*

$$\forall x \in (0,1): \qquad |\kappa(x)| > |\kappa(0)|,$$

$x \mapsto u'(x)$ is either strictly decreasing or strictly increasing. The following bound for u is independent of c and is valid for all symmetric solutions of the boundary value problems (5,6), (5,8), independent of the data α or β:

$$\forall x \in [0,1]: \qquad |u(x)| < \frac{2}{c_0} = \frac{\sqrt{2\pi}}{2\Gamma(3/4)^2} = 0.8346268420\ldots.$$

Proof. Only the last statement requires a proof. It suffices to show that for $c \in (0, c_0)$

$$0 \le u(\frac{1}{2}) = \frac{2}{c} - \frac{2}{c\sqrt[4]{1 + G^{-1}\left(\frac{c}{2}\right)^2}} < \frac{2}{c_0}.$$

By changing the variable $c = 2G(d)$, this is equivalent to showing for all $d \in (0,\infty)$ that

$$H(d) := \frac{2}{c_0}G(d) + (1+d^2)^{-1/4} - 1 > 0.$$

We have $H(0) = 0$, $\lim_{d\to\infty} H(d) = 0$ and

$$H'(d) = \left(\frac{2}{c_0} - \frac{d}{2}\right)\frac{1}{(1+d^2)^{5/4}}.$$

This shows that H is increasing first and then decreasing so that $\forall d \in (0,\infty):\ H(d) > 0$. □

Moreover, from Lemma 4, we obtain that the symmetric solutions are ordered with respect to $c \in (-c_0, c_0)$ and hence with respect to $\beta = u'(0) = G^{-1}(c/2) \in \mathbb{R}$:

Lemma 5. *For the solutions $u = u_c$ in Lemma 4, we have that for $c \in (-c_0, c_0)$*

$$\forall x \in (0,1): \qquad \frac{\partial}{\partial c}u_c(x) > 0.$$

Proof. We obtain from the proof of Lemma 4

$$\frac{\partial}{\partial c}u(x) \;=\; \int_0^x \frac{\partial}{\partial c}G^{-1}\left(\frac{c}{2} - cs\right)ds = \int_0^x \left(\frac{1}{2} - s\right)\left(1 + G^{-1}\left(\frac{c}{2} - cs\right)^2\right)^{5/4} ds > 0,$$

since the integrand is odd with respect to $s = 1/2$, positive first and negative then. □

5 Boundary value problems

The Navier boundary value problem

We obtain all solutions $u = u_c$ to (5) being symmetric around $1/2$ and satisfying $u_c(0) = u_c(1) = 0$ by formulas (17), (15), (16). This family is parametrised over $c \in (-c_0, c_0)$. It remains to consider the dependence of

$$\alpha = -\kappa_c(0) = \frac{c}{\sqrt[4]{1 + G^{-1}\left(\frac{c}{2}\right)^2}}$$

on c. For this purpose, it is enough to study the function

$$h : (-c_0, c_0) \to \mathbb{R}, \qquad h(c) = \frac{c}{\sqrt[4]{1 + G^{-1}\left(\frac{c}{2}\right)^2}}. \tag{19}$$

The range of h is precisely the set of α, for which the Navier boundary value problem (5, 6) has a solution. The number of solutions c of the equation $\alpha = h(c)$ is the number of symmetric solutions of the boundary value problem.

Lemma 6. *We have $h > 0$ in $(0, c_0)$, $h < 0$ in $(-c_0, 0)$, $\lim_{c \nearrow c_0} h(c) = \lim_{c \searrow -c_0} h(c) = 0$. The function h is odd and has precisely one local maximum in $c_{\max} = 1.840428142\ldots$ and one local minimum in $c_{\min} = -c_{\max}$. The corresponding value is $\alpha_0 = h(c_{\max}) = 1.343799725\ldots$.*

Proof. First of all we observe that

$$\lim_{c \nearrow c_0} h(c) = -\lim_{c \searrow -c_0} h(c) = \frac{c_0}{\sqrt[4]{1 + \lim_{c \nearrow c_0} G^{-1}\left(\frac{c}{2}\right)^2}} = 0$$

by definition of $G(\infty) = c_0/2$. Secondly we calculate

$$h'(c) = \frac{1}{\sqrt[4]{1 + G^{-1}\left(\frac{c}{2}\right)^2}} - \frac{c}{4} G^{-1}\left(\frac{c}{2}\right), \tag{20}$$

$$h''(c) = -\frac{1}{2} G^{-1}\left(\frac{c}{2}\right) - \frac{c}{8}\left(1 + G^{-1}\left(\frac{c}{2}\right)^2\right)^{5/4} < 0 \text{ in } (0, c_0),$$

so that h is strictly convex in $(-c_0, 0)$ and strictly concave in $(0, c_0)$. This shows that there exists precisely one local maximum $c_{\max}$ in $(0, c_0)$ and precisely one local minimum $c_{\min} = -c_{\max}$ in $(-c_0, 0)$. These are determined as zeroes of (20):

$$h'(c_{\max}) = h'(c_{\min}) = 0,$$

hence

$$c_{\max} = -c_{\min} = 1.840428142\ldots, \qquad \alpha_0 = h(c_{\max}) = 1.343799725\ldots$$

Thanks to convexity and concavity we see that every number $\alpha \in (-\alpha_0, \alpha_0) \setminus \{0\}$ has precisely two preimages under h. □

Remark 1. From the previous calculations we see that for the extremal parameter α_0 the boundary slope of the corresponding solution is $u'(0) = -u'(1) = G^{-1}(c_{\max}/2) = 1.586926484\ldots$.

Energy of small and large solution

We consider α such that $|\alpha| \le \alpha_0$ and determine the corresponding values of c describing the solutions u_c to (5, 6) according to

$$\alpha = \frac{c}{\sqrt[4]{1 + G^{-1}\left(\frac{c}{2}\right)^2}}.$$

Then the modified Willmore energy can be easily calculated as:

$$\tilde{W}_\alpha(u_c) = c^2 - 4\alpha \arctan G^{-1}\left(\frac{c}{2}\right). \tag{21}$$

In Figure 8, we display the Willmore energy of the small and the large solution as a function of the boundary datum α.

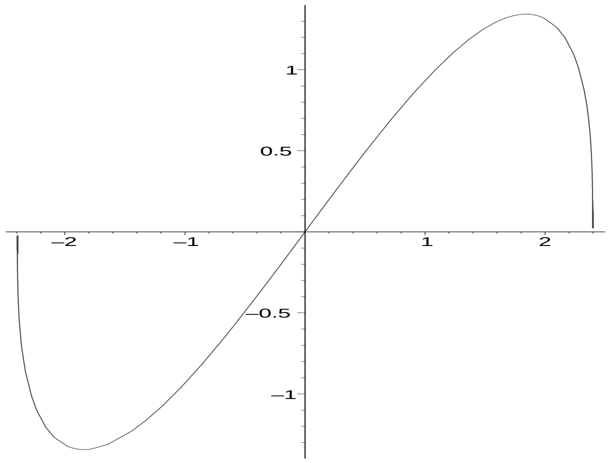

Figure 7: Admissible values of the boundary datum α plotted over the parameter c

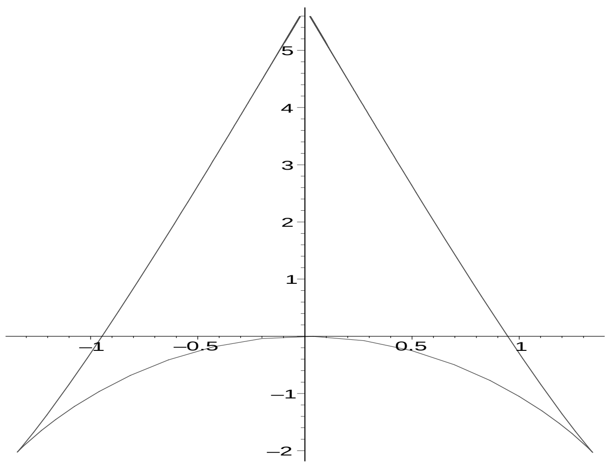

Figure 8: Energy of small and large solution of the Navier boundary value problem plotted over α

Looking at the Willmore energy (21) from a different point of view shows also a somehow unexpected feature: Let us consider some $\alpha < \alpha_0$ relatively close to α_0 and keep this α fixed. Then it turns out that even in the family $\{u_c, c \in [0, u_c)\}$, the energy of the small solution with boundary datum α is not a global minimum. The infimum is approached by $\tilde{W}_\alpha(u_c)$ when $c \nearrow c_0$ and hence, it is not attained. See Figure 9 for $\alpha = 1.2$.

The Dirichlet boundary value problem

The first part of Theorem 2 follows from (15):

$$u'(0) = -u'(1) = G^{-1}\left(\frac{c}{2}\right),$$

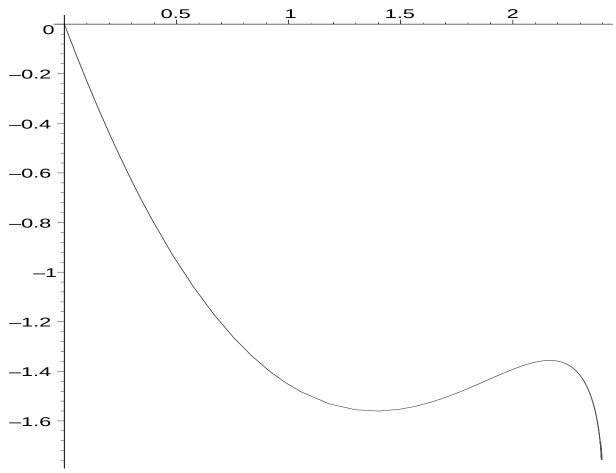

Figure 9: Energy of the functions u_c for $\alpha = 1.2$ plotted over $c \in [0, c_0)$

the strict monotonicity and continuity of G^{-1} and from $G^{-1}(-c_0/2, c_0/2) = \mathbb{R}$. Next, let $v \in M_\beta$ be arbitrary. Recalling (18) we have

$$\begin{aligned}
&W(v) - W(u)\\
&= \int_0^1 \kappa_v(x)^2 \sqrt{1+v'(x)^2}dx - \int_0^1 \kappa_u(x)^2 \sqrt{1+u'(x)^2}dx\\
&= \int_0^1 |\kappa_v(x)(1+v'(x)^2)^{\frac14} - \kappa_u(x)(1+u'(x)^2)^{\frac14}|^2\,dx - 2c\int_0^1 \frac{v''(x)}{(1+v'(x)^2)^{\frac54}}dx - 2c^2\\
&= \int_0^1 |\kappa_v(x)(1+v'(x)^2)^{\frac14} - \kappa_u(x)(1+u'(x)^2)^{\frac14}|^2\,dx - 2c\big(G(v'(1)) - G(v'(0))\big) - 2c^2\\
&= \int_0^1 |\kappa_v(x)(1+v'(x)^2)^{\frac14} - \kappa_u(x)(1+u'(x)^2)^{\frac14}|^2\,dx
\end{aligned}$$

since $G(v'(1)) - G(v'(0)) = -2G(\beta) = -c$. From this we infer $W(v) > W(u)$ unless $v \equiv u$.

Open Problem 1. Can one show that suitable symmetry of Navier or Dirichlet boundary data implies symmetry of the solution?

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