Existence for Willmore surfaces of revolution satisfying non-symmetric Dirichlet boundary conditions

Sascha Eichmann & Hans-Christoph Grunau

August 25, 2016

Abstract

In this paper existence for Willmore surfaces of revolution is shown, which satisfy non-symmetric Dirichlet boundary conditions, if the infimum of the Willmore energy in the admissible class is strictly below 4π. Under a more restrictive but still explicit geometric smallness condition we obtain a quite interesting additional geometric information: The profile curve of this solution can be parameterised as a graph over the x-axis. By working below the energy threshold of 4π and reformulating the problem in the Poincaré half plane, compactness of a minimising sequence is garantueed, of which the limit is indeed smooth. The last step consists of two main ingredients: We analyse the Euler-Lagrange equation by an order reduction argument by Langer & Singer and modify, when necessary, our solution with the help of suitable parts of catenoids and circles.

Keywords. Dirichlet boundary conditions, Willmore surfaces of revolution, elastic curves, projectability.
MSC. 49Q10, 53C42, 34B30, 34C20, 35J62, 34L30.

1 Introduction

The Willmore energy for a two dimensional regular surface $S \subset \mathbb{R}^3$ is defined by

$$W_e(S) = \int_S H^2 \, dA. \tag{1.1}$$

Here H denotes the mean curvature of S, i.e. the mean value of the principal curvatures. Already Poisson studied this energy in [36], but Willmore revived the discussion in [42]. It can be viewed as a natural modification of the area functional, which greatly motivated the research of nonlinear elliptic differential equations of second order. Hence we think the Willmore energy will play a similar role for fourth order elliptic equations. Physical applications are e.g. in modeling the elastic energy of thin biomembranes (see [34]) or thin shells (see [20]). In these applications and even more so in mathematics itself one is interested in studying minimisers of the Willmore energy. A critical point S of the Willmore functional satisfies the following Euler-Lagrange equation

$$\Delta_S H + 2H(H^2 - G) = 0, \tag{1.2}$$

which was found by Thomsen in [41]. Here G denotes the Gauss curvature and Δ_S is the Laplace-Beltrami operator on S with respect to the first fundamental form. A surface satisfying this equation is called Willmore surface. Under an additional assumption Bryant was able in [8] to classify these surfaces to be either totally umbilic or a Moebius transformed minimal surface.
Many beautiful existence results for closed Willmore surfaces have been proven in e.g. [4, 40]. An analysis of singularities and possible branch points has been carried out in [24, 25]. In [7, 37] additional compactness results for sequences of closed Willmore surfaces by working below an energy threshold are shown.
Recently constrained Willmore surfaces have been considered. Existence for Willmore surfaces with prescribed conformal class is shown in [39] and examples are given in e.g. [32]. Especially tori are of interest in this setting. In [21] an analysis of these tori and their underlying differential equation has been carried out, which extends the results by Langer & Singer (see [26]). Even some stability issues have been considered in [23]. Other constrains are e.g. prescribing the isoperimetric ratio (see [22]) or using Lagrangian and Hamiltonian deformations (see [28]).
In contrast to considering closed surfaces, in the present paper we study Dirichlet boundary value problems, so the Willmore equation becomes a kind of frame invariant counterpart of the clamped plate equation. A discussion of suitable boundary problems was done by Nitsche in [33]. Since (1.2) is of fourth order many established techniques like the maximum principle or de Giorgi-Nash-Moser-Stampacchia-type arguments do not apply. This equation is highly nonlinear, because Δ_S depends on the unkown surface. Furthermore, the Willmore energy is invariant under Moebius transformations of $\mathbb{R}^3$ (see [41]). Schätzle proved in [38] a general existence result for branched Willmore immersions by working in $\mathbb{S}^3$, but it is not clear, whether these solutions possibly contain ∞, when they are pulled back to $\mathbb{R}^3$ by the stereographic projection.
Under free boundary conditions an existence result for constrained Willmore immersions was shown in [2]. The surface area of the immersion was prescribed, which also had to obey some smallness conditions.
One-dimensional boundary value problems were discussed in e.g. [14, 16, 29].
To obtain solutions, where rather explicit analytic and geometric properties can be shown, Dall'Acqua, Deckelnick and the second author studied the Dirichlet problem for surfaces of revolution in [11]. We will pursuit a similar approach and hope that our reasoning provides a deeper understanding for Willmore surfaces for future research, e.g. projectability issues of Willmore graphs. Pinkall and Hertrich-Jeromin already studied Willmore surfaces under suitable symmetry assumptions in [35], but were interested in closed surfaces. Here we will use an approach similar to [11] and define such a surface of revolution $S(c)$ with a sufficiently smooth profile curve $c : [0,1] \to \mathbb{H}^2 := \mathbb{R} \times (0, \infty)$ by

$$[0,1] \times [0, 2\pi] \ni (t, \varphi) \mapsto f(t, \varphi) = (c^1(t), c^2(t)\cos\varphi, c^2(t)\sin\varphi). \tag{1.3}$$

Now we can state the Dirichlet problem, which we are concerned with in this paper:

$$\begin{cases} \Delta_{S(c)} H + 2H(H^2 - G) = 0, \quad \text{in } (0,1) \\ c(0) = (-1, \alpha_-),\ c(1) = (1, \alpha_+),\ \dot{c}^2(0) = \dot{c}^2(1) = 0, \\ \dot{c}^1(0), \dot{c}^1(1) > 0, \end{cases} \tag{1.4}$$

with given $\alpha_-, \alpha_+ \in (0,\infty)$. In [11] existence for $\alpha_- = \alpha_+$ was shown by a variational approach. This result was extended in [13] to arbitrary boundary angles but the data still had to be symmetric. For natural symmetric boundary conditions existence was shown in this setting in [5] and [15] and later extended to non-symmetric boundary data in [6]. In [5] and [6] great use was made of catenoids, which were inserted into minimising sequences. The resulting curves automatically satisfy $H = 0$ at the boundary. Similar techniques cannot directly be adapted when the first derivative at the boundary has to be preserved. Even stability issues were discussed in [12] and [15]. Nevertheless we like to extend the aforementioned existence results to *non-symmetric Dirichlet* boundary data. We will proceed by a variational approach so the following objects will come in handy. The class of admissible curves is denoted by

$$\begin{aligned} M_{\alpha_-,\alpha_+} := \{c \in H^2([0,1], \mathbb{R} \times (0,\infty)) :\ & c(0) = (-1, \alpha_-),\ c(1) = (1, \alpha_+), \\ & \dot{c}^2(0) = \dot{c}^2(1) = 0, \dot{c}^1(0), \dot{c}^1(1) > 0,\ |\dot{c}| \neq 0\}. \end{aligned} \tag{1.5}$$

We call the infimum of the Willmore energy

$$W^e_{\alpha_-,\alpha_+} := \inf\{W_e(S(c)) :\ c \in M_{\alpha_-,\alpha_+}\}. \tag{1.6}$$

Our first main result is stated as follows.

Theorem 1.1. *Let $\alpha_-, \alpha_+ > 0$ and satisfy $W^e_{\alpha_-,\alpha_+} < 4\pi$. Then there exists a minimising curve $c \in M_{\alpha_-,\alpha_+} \cap C^\infty([0,1], \mathbb{R} \times (0,\infty))$ with*

$$W_e(S(c)) = W^e_{\alpha_-,\alpha_+}.$$

This minimiser does not intersect itself.

We will use the direct method of the calculus of variations. Our reasoning will consist of four major steps. In Section 2 we will reformulate the problem in the hyperbolic half plane (see (2.11) and [9]). In Section 3 we eliminate the inner invariance group by using a suitable parameterisation. This will also permit to prove compactness of a minimising sequence if and only if the hyperbolic arclength of this sequence is bounded (see Theorem 3.1). Section 4 then is dedicated to finding these bounds by working below the energy threshold of 4π (see Theorem 4.4). (In [3] a similar phenomenon was observed for minimising sequences for the Willmore energy in $\mathbb{H}^3$. In this article the loss of compactness is further analysed and a bubbling result is proven.) Finally Section 5 will provide the regularity of the minimiser. The injectivity of c is proven in Remark 6.14. This follows from an analysis of the Euler-Lagrange equation, conducted in Section 6, which is based on arguments by Langer & Singer in [26].

In the papers [11] and [13] the profile curve of a solution is a graph $u : [-1,1] \to (0,\infty)$ over the x-axis, which is an interesting geometric property, because such curves look rather special and simple. The question is widely open, whether Willmore graphs exist, satisfying general non-symmetric boundary conditions (see [17]). We hope, that our reasoning here gives some insight also for this setting. Hence we give three matlab$^{\text{TM}}$ plots of solutions with different behaviour. If α_- and α_+ are very close to each other, Figure 1 shows a sign change in the first derivative of the solution. On the other hand this does not

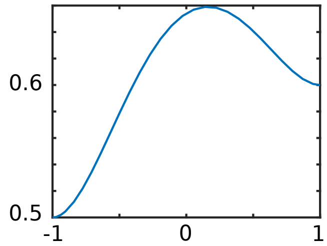

Figure 1: Solution satisfying $\alpha_- = 0.5$, $\alpha_+ = 0.6$.

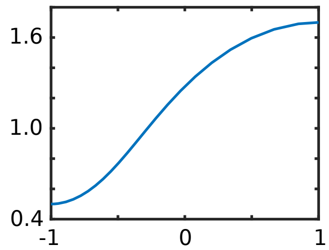

Figure 2: Solution satisfying $\alpha_- = 0.5$, $\alpha_+ = 1.7$.

seem to occur if α_- and α_+ have a moderate distance (see Figure 2). Last but not least Figure 3 showcases the failure of projectivity, which seems to start somewhere around $x = -0.5$ for α_+ and α_- being further apart.

Unfortunately projectability can in general not be expected (see Appendix A, Lemmas A.1 and A.3) for our solutions. This is quite astounding in comparison to [6], where the solution is a graph for every boundary value. This seems to be the case, since these natural boundary conditions allow the solutions to be close to catenoids. In our situation this is not the case. This is quite interesting, since the topological class does not change and only the boundary condition differs. It is remarkable, however that we can give explicit smallness conditions (see Figure 4), under which we are able to prove a similar result.

Assumption 1.2. We impose the following smallness condition on the pair $(\alpha_-, \alpha_+) \in (0,\infty) \times (0,\infty)$:

- If $\alpha_- \leq \alpha_+$, then for all $x \in \mathbb{R}$
$$p_{1,\alpha_+}(x) < \alpha_- \cosh\left(\frac{1+x}{\alpha_-}\right).$$
- If $\alpha_- > \alpha_+$, then for all $x \in \mathbb{R}$
$$p_{-1,\alpha_-}(x) < \alpha_+ \cosh\left(\frac{1-x}{\alpha_+}\right).$$

Here $p_{x_0,r}$ denotes the upper half circle with centre $(x_0, 0)$, radius r and is given by
$$p_{x_0,r}(x) = \begin{cases} \sqrt{r^2 - (x-x_0)^2}, & x \in (x_0 - r, x_0 + r) \\ 0, & \text{else.} \end{cases}$$

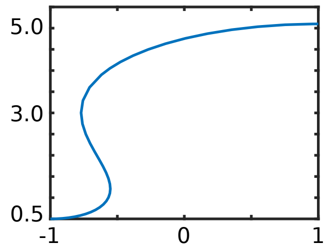

Figure 3: Solution satisfying $\alpha_- = 0.5$, $\alpha_+ = 5.1$.

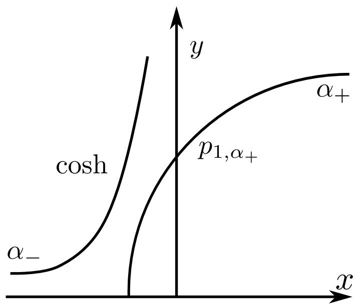

Figure 4: Smallness assumption for boundary data.

Our second main result can now be stated as follows.

Theorem 1.3. *If the pair* $(\alpha_-, \alpha_+) \in \mathbb{R}_+ \times \mathbb{R}_+$ *satisfies Assumption 1.2, the following Dirichlet problem possesses a graph* $u : [-1, 1] \to (0, \infty)$ *as solution:*

$$\begin{cases} \Delta_{S(u)} H + 2H(H^2 - G) = 0, \quad \text{in } (-1,1) \\ u(\pm 1) = \alpha_\pm, \ u'(\pm 1) = 0. \end{cases} \tag{1.7}$$

Moreover u *satisfies*

$$W_e(S(u)) = W^e_{\alpha_-,\alpha_+} < 4\pi.$$

Existence of a smooth curve solving the Dirichlet problem follows from Theorem 1.1 by combining Lemma 7.4 and (2.11). We then proceed by carefully analysing the Euler-Lagrange equation for this solution in Section 6 with an order reduction argument by Langer & Singer (see [26]). In Theorem 7.5 we use these results and combine them with an idea by Dall'Acqua, Deckelnick and the second author (see [11]) to obtain the desired result. This method was refined in [19], which inspired our reasoning here, to prove a more general existence result (see [19, Thm. 2.10]) for projectable minimisers with $\alpha_- = \alpha_+$ This result is being generalised by Theorem 1.3. The idea consists in suitably modifying a minimising sequence, or a possible minimiser respectively, with pieces of catenoids and half circles.

Since we cannot insert a catenoid into a possible minimiser and expect the derivative at the boundary to be preserved, our reasonings are more involved than in [6].

In [31] a constrained variational problem for the Willmore energy was also solved by modifying a minimising sequence.

2 Geometric background

In this section we derive the basic formulae we need for our calculations. These formulae are based on [10].
For a surface of revolution $S(c)$ given by a curve $c \in M_{\alpha_-,\alpha_+}$ (see (1.3)) the metric tensor is given by

$$(g_{ij})_{i,j=t,\varphi} = \begin{pmatrix} (\dot{c}^1)^2 + (\dot{c}^2)^2 & 0 \\ 0 & (c^2)^2 \end{pmatrix}. \tag{2.1}$$

The second fundamental form can then be calculated as

$$(h_{ij})_{i,j=t,\varphi} = \frac{1}{|\dot{c}|} \begin{pmatrix} \ddot{c}^1\dot{c}^2 - \ddot{c}^2\dot{c}^1 & 0 \\ 0 & \dot{c}^1 c^2 \end{pmatrix}. \tag{2.2}$$

with respect to the 'interior' normal

$$\nu(t,\varphi) = \frac{1}{|\dot{c}|}(\dot{c}^2(t), -\dot{c}^1(t)\cos\varphi, -\dot{c}^1(t)\sin\varphi). \tag{2.3}$$

Here we use $|\dot{c}| = \sqrt{(\dot{c}^1)^2 + (\dot{c}^2)^2}$ to denote the euclidean length of a vector. Now we can write down the mean curvature H. For this we use the sign convention that H is positive if the surface is mean convex and negative if it is mean concave with respect to the interior normal ν.

$$H = \frac{1}{2}\frac{c^2\ddot{c}^1\dot{c}^2 - c^2\ddot{c}^2\dot{c}^1 + |\dot{c}|^2\dot{c}^1}{|\dot{c}|^3 c^2}, \tag{2.4}$$

and finally the Willmore energy is

$$W_e(S(c)) = \frac{\pi}{2}\int_0^1 \frac{(c^2\ddot{c}^1\dot{c}^2 - c^2\ddot{c}^2\dot{c}^1 + |\dot{c}|^2\dot{c}^1)^2}{|\dot{c}|^5 c^2}\,dt. \tag{2.5}$$

Now we reformulate the problem by introducing the elastic energy for curves on the upper half plane $\mathbb{H}^2 = \{(x,y): \; y > 0\}$ equipped with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \tag{2.6}$$

which will also be denoted by $g(\cdot,\cdot)$. The Christoffel symbols are given by

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = 0, \; \Gamma^1_{21} = \Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}, \; \Gamma^2_{11} = \frac{1}{y}.$$

The covariant derivative of a curve c can be calculated as (see e.g. [18, Lemma 4.5])

$$\nabla_{\dot{c}}\dot{c} = \left(\ddot{c}^1 - 2\frac{\dot{c}^1\dot{c}^2}{c^2}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\ddot{c}^2 - \frac{(\dot{c}^2)^2}{c^2} + \frac{(\dot{c}^1)^2}{c^2}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.7}$$

Since the hyperbolic metric is conformal to the euclidean metric the unit normal can be choosen as

$$N = \frac{1}{\sqrt{g_c(\dot{c},\dot{c})}}\begin{pmatrix} -\dot{c}^2 \\ \dot{c}^1 \end{pmatrix}. \tag{2.8}$$

We use this sign convention because $\left(\frac{\dot{c}}{\sqrt{g_c(\dot{c},\dot{c})}}, N\right)$ becomes positively oriented. With this we obtain the geodesic curvature (please keep in mind that $|\dot{c}| = \sqrt{(\dot{c}^1)^2 + (\dot{c}^2)^2}$ denotes the usual euclidean length)

$$\kappa[c] = \frac{g_c(\nabla_{\dot{c}}\dot{c}, N)}{g_c(\dot{c},\dot{c})} = \frac{\ddot{c}^2\dot{c}^1c^2 - \ddot{c}^1\dot{c}^2c^2 + \dot{c}^1(\dot{c}^2)^2 + (\dot{c}^1)^3}{|\dot{c}|^3}. \tag{2.9}$$

and in turn can introduce the *hyperbolic elastic energy* of a curve c by

$$W_h(c) := \int_0^1 (\kappa[c](t))^2\, ds(t) = \int_0^1 \frac{(\ddot{c}^2\dot{c}^1c^2 - \ddot{c}^1\dot{c}^2c^2 + \dot{c}^1(\dot{c}^2)^2 + (\dot{c}^1)^3)^2}{|\dot{c}|^5c^2}\, dt. \tag{2.10}$$

The next observation goes back to Bryant & Griffiths in [9] and connects the Willmore energy with the hyperbolic elastic energy

$$\frac{2}{\pi}W_e(S(c)) = W_h(c) - 4\left[\frac{\dot{c}^2}{\sqrt{(\dot{c}^1)^2 + (\dot{c}^2)^2}}\right]_0^1. \tag{2.11}$$

Finally we state three equations which will be useful on several occasions: The Frenet equations for a curve c parameterised by hyperbolic arclength with geodesic curvature $\kappa[c]$ are (see (2.6))

$$\ddot{c}^1 - 2\frac{1}{c^2}\dot{c}^1\dot{c}^2 = -\kappa[c]\dot{c}^2, \tag{2.12}$$

$$\ddot{c}^2 - \frac{1}{c^2}(\dot{c}^2)^2 + \frac{1}{c^2}(\dot{c}^1)^2 = \kappa[c]\dot{c}^1. \tag{2.13}$$

If on the other hand c can be reparameterised as a smooth graph $u : [-1,1] \to (0,\infty)$, the hyperbolic elastic energy becomes

$$W_h(u) = \int_{-1}^1 \frac{u''(x)^2u(x)}{(1+u'(x)^2)^{\frac{5}{2}}} + \frac{1}{u(x)\sqrt{1+u'(x)^2}}\, dx. \tag{2.14}$$

3 Compactness assuming bounds on the hyperbolic arclength

In this section we introduce a special form of the elastic energy by reparameterising an admissible curve. Let $c \in M_{\alpha_-,\alpha_+}$ be parameterised proportionally to hyperbolic arclength and let L be the hyperbolic arclength of c. The idea to employ this special kind of parameterisation was already used by Langer & Singer in [27] but on compact manifolds in combination with the L^2-flow of $W_h(\cdot)$. Using this parameterisation we obtain

$$L^2 = g(\dot{c},\dot{c}) = \frac{(\dot{c}^1)^2 + (\dot{c}^2)^2}{(c^2)^2}, \tag{3.1}$$

and the geodesic curvature $\kappa[c]$ satisfies

$$\kappa[c]^2 = g\left(\nabla_{\frac{\dot{c}}{L}}\frac{\dot{c}}{L}, \nabla_{\frac{\dot{c}}{L}}\frac{\dot{c}}{L}\right) = \frac{1}{L^4}g(\nabla_{\dot{c}}\dot{c}, \nabla_{\dot{c}}\dot{c}). \tag{3.2}$$

Differentiating (3.1) yields

$$\dot{c}^1\ddot{c}^1 + \dot{c}^2\ddot{c}^2 = L^2 c^2 \dot{c}^2. \tag{3.3}$$

Let us now turn to the elastic energy itself:

$$\begin{aligned}
&\int_0^1 \kappa[c]^2\,ds = \frac{1}{L^3}\int_0^1 g(\nabla_{\dot{c}}\dot{c}, \nabla_{\dot{c}}\dot{c})\,dt\\
=&\frac{1}{L^3}\int_0^1 \frac{1}{(c^2)^2}\left(\left(\ddot{c}^1 - 2\frac{\dot{c}^1\dot{c}^2}{c^2}\right)^2 + \left(\ddot{c}^2 - \frac{(\dot{c}^2)^2}{c^2} + \frac{(\dot{c}^1)^2}{c^2}\right)^2\right)dt\\
=&\frac{1}{L^3}\int_0^1 \frac{1}{(c^2)^2}\Bigg((\ddot{c}^1)^2 - 4\frac{\ddot{c}^1\dot{c}^1\dot{c}^2}{c^2} + 4\left(\frac{\dot{c}^1\dot{c}^2}{c^2}\right)^2\\
&+ (\ddot{c}^2)^2 - 2\frac{\ddot{c}^2(\dot{c}^2)^2}{c^2} + 2\frac{\ddot{c}^2(\dot{c}^1)^2}{c^2} + \frac{(\dot{c}^2)^4}{(c^2)^2} - 2\left(\frac{\dot{c}^1\dot{c}^2}{c^2}\right)^2 + \frac{(\dot{c}^1)^4}{(c^2)^2}\Bigg)dt\\
=&\frac{1}{L^3}\int_0^1 \frac{1}{(c^2)^2}\Bigg((\ddot{c}^1)^2 + (\ddot{c}^2)^2 - 4\frac{\ddot{c}^1\dot{c}^1\dot{c}^2}{c^2} + 2\left(\frac{\dot{c}^1\dot{c}^2}{c^2}\right)^2\\
&- 2\frac{\ddot{c}^2(\dot{c}^2)^2}{c^2} + 2\frac{\ddot{c}^2(\dot{c}^1)^2}{c^2} + \frac{(\dot{c}^2)^4}{(c^2)^2} + \frac{(\dot{c}^1)^4}{(c^2)^2}\Bigg)dt\\
\underset{(3.3)}{\overset{(3.1)}{=}}&\frac{1}{L^3}\int_0^1 \frac{1}{(c^2)^2}\Bigg((\ddot{c}^1)^2 + (\ddot{c}^2)^2 - 4\frac{\ddot{c}^1\dot{c}^1\dot{c}^2}{c^2} + 2\left(\frac{\dot{c}^1\dot{c}^2}{c^2}\right)^2\\
&+ 2\frac{\ddot{c}^1\dot{c}^2\dot{c}^1}{c^2} - 2\frac{L^2(\dot{c}^2)^2c^2}{c^2} + 2\frac{\ddot{c}^2}{c^2}(L^2(c^2)^2 - (\dot{c}^2)^2) + \frac{(\dot{c}^2)^4}{(c^2)^2} + \frac{(\dot{c}^1)^4}{(c^2)^2}\Bigg)dt\\
\overset{(3.3)}{=}&\frac{1}{L^3}\int_0^1 \frac{1}{(c^2)^2}\Bigg((\ddot{c}^1)^2 + (\ddot{c}^2)^2 - 2\frac{\ddot{c}^1\dot{c}^1\dot{c}^2}{c^2} + 2\left(\frac{\dot{c}^1\dot{c}^2}{c^2}\right)^2\\
&- 2L^2(\dot{c}^2)^2 + 2\ddot{c}^2c^2L^2 + 2\frac{\ddot{c}^1\dot{c}^1\dot{c}^2}{c^2} - 2L^2(\dot{c}^2)^2 + \frac{(\dot{c}^2)^4}{(c^2)^2} + \frac{(\dot{c}^1)^4}{(c^2)^2}\Bigg)dt\\
=&\frac{1}{L^3}\int_0^1 \frac{1}{(c^2)^2}\Bigg((\ddot{c}^1)^2 + (\ddot{c}^2)^2 + 2\left(\frac{\dot{c}^1\dot{c}^2}{c^2}\right)^2\\
&+ 2\ddot{c}^2c^2L^2 - 4L^2(\dot{c}^2)^2 + \frac{(\dot{c}^2)^4}{(c^2)^2} + \frac{(\dot{c}^1)^4}{(c^2)^2}\Bigg)dt\\
=&\frac{1}{L^3}\Bigg(\int_0^1 \frac{1}{(c^2)^2}\Bigg((\ddot{c}^1)^2 + (\ddot{c}^2)^2 + 2\left(\frac{\dot{c}^1\dot{c}^2}{c^2}\right)^2 - 2L^2(\dot{c}^2)^2\\
&+ \frac{(\dot{c}^2)^4}{(c^2)^2} + \frac{(\dot{c}^1)^4}{(c^2)^2}\Bigg)dt + 2L^2\int_0^1 \ddot{c}^2\frac{1}{c^2} - \frac{(\dot{c}^2)^2}{(c^2)^2}\,dt\Bigg)\\
=&\frac{1}{L^3}\Bigg(\int_0^1 \frac{1}{(c^2)^2}\Bigg((\ddot{c}^1)^2 + (\ddot{c}^2)^2 - 2L^2(\dot{c}^2)^2 + \frac{(\dot{c}^2)^4}{(c^2)^2} + 2\left(\frac{\dot{c}^1\dot{c}^2}{c^2}\right)^2 + \frac{(\dot{c}^1)^4}{(c^2)^2}\Bigg)dt\\
&+ \frac{2}{L}\left[\frac{\dot{c}^2}{c^2}\right]_0^1\\
\overset{(3.1)}{=}&\frac{1}{L^3}\Bigg(\int_0^1 \frac{1}{(c^2)^2}\Big((\ddot{c}^1)^2 + (\ddot{c}^2)^2 - 2L^2(\dot{c}^2)^2 + L^4(c^2)^2\Big)dt\\
&+ \frac{2}{L}\left[\frac{\dot{c}^2}{c^2}\right]_0^1.
\end{aligned}$$

To summarize our findings we state the following equation

$$W_h(c) = \frac{1}{L^3}\left(\int_0^1 \frac{1}{(c^2)^2}\left((\ddot{c}^1)^2 + (\ddot{c}^2)^2 - 2L^2(\dot{c}^2)^2\right)\,dt + L + \frac{2}{L}\left[\frac{\dot{c}^2}{c^2}\right]_0^1\right). \quad (3.4)$$

With this equation in mind we can prove the following theorem:

Theorem 3.1. *Let $(c_n)_{n\in\mathbb{N}} \subset M_{\alpha_-,\alpha_+}$ be a minimising sequence for $W_h(c)$ with bounded hyperbolic arclength. Then there exists a curve $c \in M_{\alpha_-,\alpha_+}$ with*

$$W_h(c) = W^h_{\alpha_-,\alpha_+} := \inf\{W_h(v) : \ v \in M_{\alpha_-,\alpha_+}\}.$$

Proof. Reparameterise $c_n : [0,1] \to \mathbb{H}^2$ proportional to hyperbolic arclength. Let L_n be the hyperbolic arclength of c_n. Since L_n is bounded, c_n^2 is bounded from above and below. (3.1) then gives us upper bounds on $|\dot{c}^1|$ and $|\dot{c}^2|$. With (3.4) there exists a $C > 0$ such that

$$C > \int_0^1 (\ddot{c}_n^1)^2 + (\ddot{c}_n^2)^2\,dt.$$

Because of this we can extract a subsequence which is weakly convergent in $H^2((0,1),\mathbb{H}^2)$ and strongly convergent in $C^1([0,1],\mathbb{H}^2)$ to a curve $c : [0,1] \to \mathbb{H}^2$. The convergence in C^1 ensures the attainment of the boundary data. This, (3.1) and the bound on L_n ensure that c is parameterised proportionally to hyperbolic arclength. This in turn gives us $\dot{c} \neq 0$, since c^2 is bounded from above as well as from below. Hence c belongs to M_{α_-,α_+}. Since $\frac{1}{c_n^2}$ converges in $C^0([0,1])$, the fractions $\frac{\ddot{c}_n^1}{c_n^2}$ and $\frac{\ddot{c}_n^2}{c_n^2}$ converge weakly in $L^2((0,1),\mathbb{H}^2)$. Together with the lower semi-continuity of a norm and (3.4) this yields

$$W^h_{\alpha_-,\alpha_+} \leq W_h(c) \leq \liminf_{n\to\infty} W_h(c_n) \leq W^h_{\alpha_-,\alpha_+} = \inf\{W_h(v) : \ v \in M_{\alpha_-,\alpha_+}\}.$$

□

4 Establishing a bound on hyperbolic arclength

If we wish to apply Theorem 3.1 to obtain a solution of (1.4), we have to find an upper bound for the hyperbolic arclength of a minimising sequence. We can achieve this, if we work strictly below the threshold of 8 for the elastic energy. The main idea is contained in the following lemma.

Lemma 4.1. *Let $c \in M_{\alpha_-,\alpha_+}$ and $0 < t_1 < t_2 < 1$ with*

$$\dot{c}^1(t_1) = \dot{c}^1(t_2) = 0 \text{ and } \dot{c}^2(t_1)\cdot\dot{c}^2(t_2) < 0.$$

Then the elastic energy satisfies

$$W_h(c) \geq 8.$$

Proof. Let us distinguish two cases. The first one is $\dot{c}^2(t_1) < 0$. Equation (2.11) yields

$$\begin{aligned} W_h(c) &= W_h(c)\big|_{[0,t_1]} + W_h(c)\big|_{[t_1,t_2]} + W_h(c)\big|_{[t_2,1]} \\ &\geq W_h(c)\big|_{[t_1,t_2]} = \frac{2}{\pi} W_e(S(c))\big|_{[t_1,t_2]} + 4\left[\frac{\dot{c}^2}{\sqrt{(\dot{c}^1)^2 + (\dot{c}^2)^2}}\right]_{t_1}^{t_2} \\ &\geq 8 \end{aligned}$$

The other case is $\dot{c}^2(t_1) > 0$. The boundary data and again equation (2.11) yield

$$\begin{aligned} W_h(c) &= W_h(c)\big|_{[0,t_1]} + W_h(c)\big|_{[t_1,t_2]} + W_h(c)\big|_{[t_2,1]} \\ &\geq W_h(c)\big|_{[0,t_1]} + W_h(c)\big|_{[t_2,1]} = \frac{2}{\pi} W_e(S(c))\big|_{[0,t_1]} + \frac{2}{\pi} W_e(S(c))\big|_{[t_2,1]} \\ &+ 4\left[\frac{\dot{c}^2}{\sqrt{(\dot{c}^1)^2 + (\dot{c}^2)^2}}\right]_0^{t_1} + 4\left[\frac{\dot{c}^2}{\sqrt{(\dot{c}^1)^2 + (\dot{c}^2)^2}}\right]_{t_2}^1 \\ &\geq 8 \end{aligned}$$

□

By using the same idea we obtain a lower bound:

Lemma 4.2. *Let $(c_n)_{n\in\mathbb{N}} \subset M_{\alpha_-,\alpha_+}$ be a sequence with $\sup_n W_h(c_n) < 8$. Then there exists a constant $C > 0$ with*

$$\forall t \in [0,1],\ \forall n \in \mathbb{N}:\ C < c_n^2(t).$$

Proof. We proceed by contradiction. Let us assume that no such bound exists. Hence after possibly passing to a subsequence we find a sequence $(t_n)_{n\in\mathbb{N}} \subset (0,1)$ with

$$c_n^2(t_n) = \min_{t\in[0,1]} c_n^2(t) \to 0,\ (n \to \infty).$$

Because of the boundary data we have $\dot{c}_n^2(t_n) = 0$ for n large enough. Let us for now assume the existence of another two sequences $0 < t_n^- < t_n < t_n^+ < 1$ with

$$\left|\frac{\dot{c}_n^2}{\sqrt{(\dot{c}_n^1)^2 + (\dot{c}_n^2)^2}}(t_n^-)\right| \to 1 \text{ and } \left|\frac{\dot{c}_n^2}{\sqrt{(\dot{c}_n^1)^2 + (\dot{c}_n^2)^2}}(t_n^+)\right| \to 1.$$

We will show the existence of these sequences later. For now we can use the idea from Lemma 4.1: Let us assume for the first case that $\dot{c}_n^2(t_n^-) < 0$ with n large enough. Then with (2.11) we obtain

$$\begin{aligned} W_h(c_n)\big|_{[0,t_n]} &= W_h(c_n)\big|_{[0,t_n^-]} + W_h(c_n)\big|_{[t_n^-,t_n]} \geq W_h(c_n)\big|_{[t_n^-,t_n]} \\ &= \frac{2}{\pi} W_e(S(c_n))\big|_{[t_n^-,t_n]} + 4\left[\frac{\dot{c}_n^2}{\sqrt{(\dot{c}_n^1)^2 + (\dot{c}_n^2)^2}}\right]_{t_n^-}^{t_n} \\ &\geq 4 + o(1). \end{aligned}$$

If we assume on the other hand that $\dot{c}_n^2(t_n^-) > 0$ for large n, then

$$\begin{aligned}
W_h(c_n)\big|_{[0,t_n]} &= W_h(c_n)\big|_{[0,t_n^-]} + W_h(c_n)\big|_{[t_n^-,t_n]} \geq W_h(c_n)\big|_{[0,t_n^-]} \\
&= \frac{2}{\pi} W_e(S(c_n))\big|_{[0,t_n^-]} + 4\left[\frac{\dot{c}_n^2}{\sqrt{(\dot{c}_n^1)^2 + (\dot{c}_n^2)^2}}\right]_0^{t_n^-} \\
&\geq 4 + o(1).
\end{aligned}$$

We can work completely analogously in the interval $[t_n, 1]$ and obtain

$$W_h(c_n) = W_h(c_n)\big|_{[0,t_n]} + W_h(c_n)\big|_{[t_n,1]} \geq 8 + o(1),$$

which would be a contradiction. Now we have to show the existence of t_n^- and t_n^+: Again we proceed by contradiction and assume that such sequences t_n^-/t_n^+ do not exist. After passing to a further subsequence we can find a $\delta > 0$ such that

$$\forall t \in [0,1]: \ \left|\frac{\dot{c}_n^2}{\sqrt{(\dot{c}_n^1)^2 + (\dot{c}_n^2)^2}}(t)\right| \leq 1 - \delta. \tag{4.1}$$

Squaring the inequality gives us for all $t \in [0,1]$

$$\begin{aligned}
&\Rightarrow (\dot{c}_n^2)^2 \leq (1-\delta)^2 \cdot ((\dot{c}_n^1)^2 + (\dot{c}_n^2)^2) \\
&\Rightarrow 0 \leq (\dot{c}_n^2)^2(1 - (1-\delta)^2) \leq (1-\delta)^2(\dot{c}_n^1)^2.
\end{aligned}$$

Since the curves are regular ($\dot{c} \neq 0$) we obtain

$$\forall t \in [0,1]: \ \dot{c}_n^1 \neq 0.$$

Since this is the case, the curve c_n can be reparameterised as a graph on $[0,1]$. Hence we obtain a sequence of functions $u_n \in H^2([-1,1],(0,\infty))$ representing the curves on $[0,1]$. To obtain a contradiction we first have to show that $x_n := c_n^1(t_n) \nrightarrow -1$. Thanks to (4.1) there exists a $C > 0$ such that $\forall x \in [-1,1]$ $|u_n'(x)| \leq C$. The mean value theorem yields

$$|x_n \pm 1| \geq \frac{\alpha_-}{C} + o(1) > 0$$

for n large enough. Since the elastic energy is a geometric functional it is invariant under reparameterisation. This yields together with (2.14) and (2.6)

$$\begin{aligned}
8 \geq W_h(c_n)\big|_{[0,t_n]} &= W_h(u_n)\big|_{[-1,x_n]} \\
&= \int_{-1}^{x_n} \frac{u_n''(x)^2 u_n(x)}{(1+u_n'(x)^2)^{\frac{5}{2}}}\,dx + \int_{-1}^{x_n} \frac{1}{u_n(x)\sqrt{1+u_n'(x)^2}}\,dx \\
&\geq \int_{-1}^{x_n} \frac{1}{u_n(x)\sqrt{1+u_n'(x)^2}}\,dx. \\
&= \int_{-1}^{x_n} \frac{1}{1+u_n'(x)^2}\,ds(x) \\
&\geq \frac{1}{1+C^2}\int_{-1}^{x_n} ds(x) \to \infty,
\end{aligned}$$

since the hyperbolic arclength tends to infinity. This is the case, because $u_n(x_n) \to 0$. The proof for t_n^+ is analogous to t_n^- by working on $[t_n, 1]$. □

The proof of the next lemma is based on the previous one and provides an upper bound.

Lemma 4.3. *Let* $(c_n)_{n\in\mathbb{N}} \subset M_{\alpha_-,\alpha_+}$ *be a sequence with* $\sup_n W_h(c_n) < 8$. *Then there exists a constant* $C > 0$ *with*

$$\forall t \in [0,1],\ \forall n \in \mathbb{N}:\ C > |c_n(t)|.$$

Proof. As in Lemma 4.2 we will proceed by contradiction. After possibly passing to a subsequence we may assume the existence of a sequence $(\xi_n)_{n\in\mathbb{N}} \subset [0,1]$ with

$$\max\{|c_n(t)|,\ t \in [0,1]\} = |c_n(\xi_n)| \to \infty,\ (n \to \infty).$$

We transform this problem in such a way that we can directly apply our approach from Lemma 4.2. To do this we need the Cayley transformation $Q : \mathbb{H}^2 \to \{(x,y) \in \mathbb{R}^2 :\ |(x,y)| < 1\} =: \mathbb{D}^2$. Since it is an isometry between the Poincaré disk $\mathbb{D}^2$ and $\mathbb{H}^2$, the elastic energy remains invariant under this transformation. Now we need a rotation around $(0,0) \in \mathbb{R}^2$ with angle $\varphi > 0$. Let us denote this function by $R_\varphi : \mathbb{R}^2 \to \mathbb{R}^2$. Please keep in mind that R_φ does not change the elastic energy in $\mathbb{D}^2$, since it is an isometry of this manifold. Let us now define the transformed curves:

$$c_{n,\varphi} : [0,1] \to \mathbb{H}^2 \text{ with } c_{n,\varphi}(t) := Q^{-1}(R_\varphi(Q(c_n(t)))).$$

Figure 5 explains the transformation. Since $\varphi > 0$ we can find a sequence

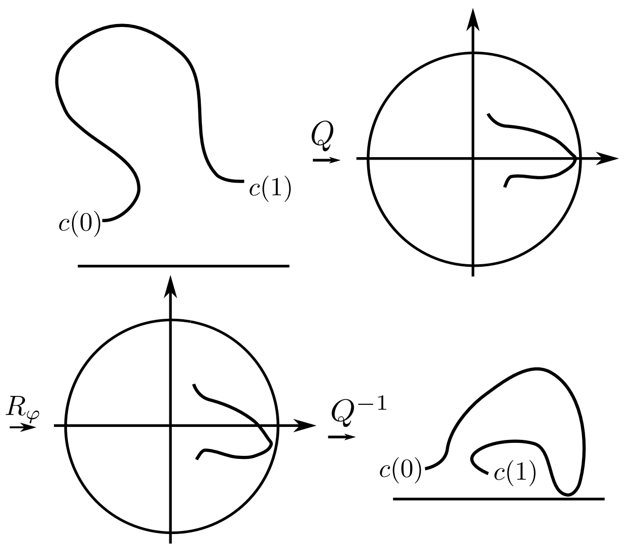

Figure 5: Transformation of curves such that an extremum tends to zero.

$(t_n^\varphi)_{n\in\mathbb{N}} \subset (0,1)$ with

$$c^2_{n,\varphi}(t_n^\varphi) = \min_{t\in[0,1]} c^2_{n,\varphi}(t) \to 0.$$

Let $\varepsilon > 0$ be fixated but arbitrary. Since R_φ, Q and Q^{-1} are smooth and c_n satisfies the boundary data, we can find a small angle $\varphi > 0$ with

$$\left|\frac{\dot{c}^2_{n,\varphi}(0)}{\dot{c}^1_{n,\varphi}(0)}\right| < \varepsilon \text{ and } \left|\frac{\dot{c}^2_{n,\varphi}(1)}{\dot{c}^1_{n,\varphi}(1)}\right| < \varepsilon.$$

This also yields

$$\left|\frac{\dot c^2_{n,\varphi}(0)}{\sqrt{(\dot c^1_{n,\varphi}(0))^2+(\dot c^2_{n,\varphi}(0))^2}}\right| \le \left|\frac{\dot c^2_{n,\varphi}(0)}{\dot c^1_{n,\varphi}(0)}\right| < \varepsilon,$$

and the same result for $t = 1$. As in in the proof of Lemma 4.2 we can find two sequences $0 < t_n^{\varphi,-} < t_n^{\varphi} < t_n^{\varphi,+} < 1$ for n large enough which satisfy

$$\left|\frac{\dot c^2_{n,\varphi}}{\sqrt{(\dot c^1_{n,\varphi})^2+(\dot c^2_{n,\varphi})^2}}(t_n^{\varphi,\pm})\right| \to 1,\ (n\to\infty).$$

Let us assume for large n that we have $\dot c^2_{n,\varphi}(t_n^{\varphi,-}) > 0$ and $\dot c^2_{n,\varphi}(t_n^{\varphi,+}) > 0$. Then the elastic energy can be estimated with (2.11) as follows

$$\begin{aligned}
W_h(c_n) &= W_h(c_{n,\varphi}) \ge W_h(c_{n,\varphi})\big|_{[0,t_n^{\varphi,-}]} + W_h(c_{n,\varphi})\big|_{[t_n^{\varphi},t_n^{\varphi,+}]}\\
&= \frac{2}{\pi}W_e(S(c_{n,\varphi}))\big|_{[0,t_n^{\varphi,-}]} + \frac{2}{\pi}W_e(S(c_{n,\varphi}))\big|_{[t_n^{\varphi},t_n^{\varphi,+}]}\\
&+4\left[\frac{\dot c^2_{n,\varphi}}{\sqrt{(\dot c^1_{n,\varphi})^2+(\dot c^2_{n,\varphi})^2}}\right]_0^{t_n^{\varphi,-}} +4\left[\frac{\dot c^2_{n,\varphi}}{\sqrt{(\dot c^1_{n,\varphi})^2+(\dot c^2_{n,\varphi})^2}}\right]_{t_n^{\varphi}}^{t_n^{\varphi,+}}\\
&\ge 8 + o(1) - \varepsilon > 8 - 2\varepsilon,
\end{aligned}$$

for n large enough. The remaining cases can be dealt with in the same way. By choosing $\varepsilon < \frac{1}{2}(8 - \sup_n W_h(c_n))$ we finally reach a contradiction. □

Now we can tackle our main estimate for the hyperbolic arclength:

Theorem 4.4. *Let $(c_n)_{n\in\mathbb{N}} \subset M_{\alpha_-,\alpha_+}$ be a sequence with $\sup_n W_h(c_n) < 8$. Let L_n be the hyperbolic arclength of c_n. Then there exists a constant $C > 0$ with*

$$\forall n\in\mathbb{N}:\ L_n \le C.$$

Proof. Again we proceed by contradiction. So let us assume that after passing to a subsequence we have that $L_n \to \infty$ for $n \to \infty$. By reparameterising the curves proportionally to hyperbolic arclength, we achieve with Lemma 4.2 and (3.1) the following uniform convergence on $[0, 1]$ for $(n \to \infty)$

$$(\dot c_n^1)^2 + (\dot c_n^2)^2 = (c_n^2)^2 L_n^2 \to \infty. \tag{4.2}$$

With Lemma 4.3 and the mean value theorem we obtain three sequences:

$$\begin{aligned}
\infty > C > \left|c_n^1\left(\frac{2}{6}\right) - c_n^1\left(\frac{1}{6}\right)\right| &= |\dot c_n^1(t_{n,1})|\frac{1}{6},\ \frac{1}{6} < t_{n,1} < \frac{2}{6},\\
\infty > C > \left|c_n^2\left(\frac{4}{6}\right) - c_n^2\left(\frac{3}{6}\right)\right| &= |\dot c_n^2(t_{n,2})|\frac{1}{6},\ \frac{3}{6} < t_{n,2} < \frac{4}{6},\\
\infty > C > \left|c_n^1\left(\frac{6}{6}\right) - c_n^1\left(\frac{5}{6}\right)\right| &= |\dot c_n^1(t_{n,3})|\frac{1}{6},\ \frac{5}{6} < t_{n,3} < \frac{6}{6}.
\end{aligned}$$

(4.2) then yields for $n \to \infty$

$$|\dot{c}_n^2(t_{n,1})| \to \infty,$$
$$|\dot{c}_n^1(t_{n,2})| \to \infty,$$
$$|\dot{c}_n^2(t_{n,3})| \to \infty.$$

This yields

$$\left|\frac{\dot{c}_n^2}{\sqrt{(\dot{c}_n^1)^2 + (\dot{c}_n^2)^2}}\right|(t_{n,1}) \to 1,$$
$$\left|\frac{\dot{c}_n^2}{\sqrt{(\dot{c}_n^1)^2 + (\dot{c}_n^2)^2}}\right|(t_{n,2}) \to 0,$$
$$\left|\frac{\dot{c}_n^2}{\sqrt{(\dot{c}_n^1)^2 + (\dot{c}_n^2)^2}}\right|(t_{n,3}) \to 1.$$

Like before we would need to distinguish some cases but we will just demonstrate the case $\dot{c}_n^2(t_{n,1}) < 0$ and $\dot{c}_n^2(t_{n,3}) < 0$.

$$\begin{aligned}
W_h(c_n) &\geq W_h(c_n)\big|_{[t_{n,1},t_{n,2}]} + W_h(c_n)\big|_{[t_{n,3},1]} \\
&= \frac{2}{\pi} W_e(S(c_n))\big|_{[t_{n,1},t_{n,2}]} + \frac{2}{\pi} W_e(S(c_n))\big|_{[t_{n,3},1]} \\
&+ 4\left[\frac{\dot{c}_n^2}{\sqrt{(\dot{c}_n^1)^2 + (\dot{c}_n^2)^2}}\right]_{t_{n,1}}^{t_{n,2}} + 4\left[\frac{\dot{c}_n^2}{\sqrt{(\dot{c}_n^1)^2 + (\dot{c}_n^2)^2}}\right]_{t_{n,3}}^{1} \\
&\geq 8 + o(1).
\end{aligned}$$

For the other cases one just needs to adjust the intervals of integration and then the desired contradiction follows. $\square$

5 Regularity of a solution

We will proceed similarly to [11, Theorem 4, step 2]. For this we need to calculate an Euler-Lagrange equation for the elastic energy $W_h(\cdot)$. Let $c \in M_{\alpha_-,\alpha_+}$ be a critical point of the hyperbolic elastic energy. We may parameterise this proportionally to its hyperbolic arclength L. With the help of (2.9) we obtain for any testing function $\varphi \in H_0^2([0,1],\mathbb{R}^2)$:

$$\begin{aligned}
\frac{d}{d\varepsilon}\kappa[c+\varepsilon\varphi]\bigg|_{\varepsilon=0} =& \frac{1}{|\dot{c}|^3}\big(\ddot{\varphi}^2\dot{c}^1c^2 + \dot{\varphi}^1\ddot{c}^2c^2 + \ddot{c}^2\dot{c}^1\varphi^2 - \ddot{\varphi}^1\dot{c}^2c^2 \\
&- \ddot{c}^1\dot{\varphi}^2c^2 - \ddot{c}^1\dot{c}^2\varphi^2 + \dot{\varphi}^1(\dot{c}^2)^2 + 2\dot{c}^1\dot{\varphi}^2\dot{c}^2 + 3\dot{\varphi}^1(\dot{c}^1)^2\big) \\
&- 3\frac{\dot{\varphi}^1\dot{c}^1 + \dot{\varphi}^2\dot{c}^2}{|\dot{c}|^5}\big(\ddot{c}^2\dot{c}^1c^2 - \ddot{c}^1\dot{c}^2c^2 + \dot{c}^1(\dot{c}^2)^2 + (\dot{c}^1)^3\big) \\
=& \frac{1}{|\dot{c}|^3}\big(\ddot{\varphi}^2\dot{c}^1c^2 + \dot{\varphi}^1\ddot{c}^2c^2 + \ddot{c}^2\dot{c}^1\varphi^2 - \ddot{\varphi}^1\dot{c}^2c^2 \\
&- \ddot{c}^1\dot{\varphi}^2c^2 - \ddot{c}^1\dot{c}^2\varphi^2 + \dot{\varphi}^1(\dot{c}^2)^2 + 2\dot{c}^1\dot{\varphi}^2\dot{c}^2 + 3\dot{\varphi}^1(\dot{c}^1)^2\big) \\
&- 3\frac{\dot{\varphi}^1\dot{c}^1 + \dot{\varphi}^2\dot{c}^2}{|\dot{c}|^2}\kappa[c].
\end{aligned}$$

The derivative of $W_h(\cdot)$ in $c(\cdot)$ then becomes

$$\begin{aligned}
0 =& \frac{d}{d\varepsilon}\int_0^1 \kappa[c+\varepsilon\varphi]^2\,ds\Big|_{\varepsilon=0}\\
=& \int_0^1 \left(2\kappa[c]\frac{d}{d\varepsilon}\kappa[c+\varepsilon\varphi]\Big|_{\varepsilon=0}\frac{|\dot c|}{c^2} + \kappa[c]^2\left(\frac{\dot c^1\dot\varphi^1+\dot c^2\dot\varphi^2}{c^2|\dot c|} - \frac{|\dot c|}{(c^2)^2}\varphi^2\right)\right)dt\\
=& \int_0^1 2\Bigg(\kappa[c]\Bigg(\frac{1}{|\dot c|^3}\Big(\dot\varphi^1\ddot c^2 c^2 - \ddot\varphi^1\dot c^2 c^2 + \dot\varphi^1(\dot c^2)^2 + 3\dot\varphi^1(\dot c^1)^2\\
&+ \ddot\varphi^2\dot c^1 c^2 + \ddot c^2\dot c^1\varphi^2 - \ddot c^1\dot\varphi^2 c^2 - \ddot c^1\dot c^2\varphi^2 + 2\dot c^1\dot\varphi^2\dot c^2\Big)\\
&- 3\kappa[c]\frac{1}{|\dot c|^2}(\dot\varphi^1\dot c^1+\dot\varphi^2\dot c^2)\Bigg) + \kappa[c]^2\left(\frac{\dot c^1\dot\varphi^1+\dot c^2\dot\varphi^2}{c^2|\dot c|} - \frac{|\dot c|}{(c^2)^2}\varphi^2\right)\Bigg)dt. \qquad (5.1)
\end{aligned}$$

Let $\eta \in C_c^\infty([0,1],\mathbb{R})$ be arbitrary. We define

$$\begin{aligned}
\mu(t) &= \int_0^t\int_0^y \eta(s)\,ds\,dy + \beta t^2 + \gamma t^3\\
\beta &= \int_0^1 \eta(s)\,ds - 3\int_0^1\int_0^y \eta(s)\,ds\,dy\\
\gamma &= 2\int_0^1\int_0^y \eta(s)\,ds\,dy - \int_0^1 \eta(s)\,ds.
\end{aligned}$$

The idea for this comparison function stems from [11, Theorem 4, step 2]. Thanks to the choice of α and β we have that $\mu(0)=\mu(1)=\dot\mu(0)=\dot\mu(1)=0$. We also observe the following estimates

$$\beta,\ \gamma,\ \|\mu\|_{C^1} \le C\|\eta\|_{L^1}.$$

If we set $\varphi(t) = (\mu(t),0)$, we get $\varphi \in H_0^2([0,1],\mathbb{R}^2)$. By inserting this into (5.1), we obtain for every $\eta \in C_c^\infty(0,1)$

$$\left|\int_0^1 \kappa[c](t)\eta(t)\frac{\ddot c^2 c^2}{|\dot c|^3}\,dt\right| \le C(c)\|\eta\|_{L^1}, \qquad (5.2)$$

since $c \in H^2([0,1],\mathbb{R}^2)$ and $\kappa[c] \in L^2([0,1])$. Because the hyperbolic arclength is fixed, $\frac{c^2}{|\dot c|}$ is bounded from below as well (see (3.1)). This and $(L^1)^* = L^\infty$ ensure that $\kappa[c]\dot c^2 \in L^\infty([0,1])$.
We have to repeat this process for $\varphi(t) = (0,\mu(t))$, to obtain a bound for $\kappa[c]$. With the same arguments as above, we get

$$\left|\int_0^1 \kappa[c](t)\eta(t)\frac{\dot c^1 c^2}{|\dot c|^3}\,dt\right| \le C(c)\|\eta\|_{L^1}, \qquad (5.3)$$

for every $\eta \in C_c^\infty([0,1])$. This means that we have $\kappa[c]\dot c^1 \in L^\infty([0,1])$. If we combine this with the bound on $\kappa[c]\dot c^2$ and (3.1), we finally get $\kappa[c] \in L^\infty([0,1])$. The Frenet equations (2.12) and (2.13) then yield $c \in W^{2,\infty}([0,1],\mathbb{R}^2)$.
Now we can show higher differentiability: For arbitrary $\eta \in C_c^\infty([0,1])$ we define

$$\nu(t) = \int_0^t \eta(s)\,ds - 3t^2\int_0^1 \eta(s)\,ds + 2t^3\int_0^1 \eta(s)\,ds. \qquad (5.4)$$

and get

$$\nu(0) = \nu(1) = \dot{\nu}(0) = \dot{\nu}(1) = 0, \ \|\nu\|_{C^0} \leq C\|\eta\|_{L^1} \text{ and } \|\dot{\nu}\|_{L^1} \leq C\|\eta\|_{L^1}.$$

If we now choose $\varphi(t) = (\nu(t), 0)$, insert it into (5.1) and combine it with the already established L^∞-bounds on κ and $\ddot{c}$, we obtain

$$\left| \int_0^1 \kappa[c](t)\dot{\eta}(t)\frac{\dot{c}^2 c^2}{|\dot{c}|^3}\, dt \right| \leq C(c)\|\eta\|_{L^1}. \tag{5.5}$$

The functional $F(\eta) := \int_0^1 \kappa[c](t)\dot{\eta}(t)\frac{\dot{c}^2 c^2}{|\dot{c}|^3}\, dt$ is defined on $W^{1,1}([0,1])$, since $\kappa[c] \in L^\infty([0,1])$. (5.5) now allows us to extend it to a functional $\tilde{F}$ on $L^1([0,1])$, since $W^{1,1}([0,1])$ is dense in $L^1([0,1])$ and F is bounded w.r.t. to $\|\cdot\|_{L^1}$. The Riesz representation theorem then yields a $g \in L^\infty([0,1])$ with

$$\forall f \in L^1([0,1]) \ \tilde{F}(f) = \int_0^1 g(t)f(t)\, dt.$$

If we go back to $C_c^\infty([0,1])$ as a domain of definition for $\tilde{F}$, we get

$$\forall \eta \in C_c^\infty([0,1]) \ \int_0^1 g(t)\eta(t)\, dt = \int_0^1 \kappa[c](t)\dot{\eta}(t)\frac{\dot{c}^2 c^2}{|\dot{c}|^3}\, dt.$$

Since $\frac{c^2}{|\dot{c}|^3} \neq 0$ the curvature satisfies $\kappa[c]\dot{c}^2 \in W^{1,\infty}([0,1])$.
Repeating this argument with $\varphi = (0, \nu(t))$, we also obtain $\kappa[c]\dot{c}^1 \in W^{1,\infty}([0,1])$. Since $\dot{c} \neq 0$ and $\dot{c}$ is continous, we finally achieve $\kappa[c] \in W^{1,\infty}([0,1])$. Then the Frenet equations (2.12) and (2.13) yield that $c \in W^{3,\infty}([0,1], \mathbb{R}^2)$.
For achieving higher regularity another form of the Euler-Lagrange equation is more appropriate. Langer & Singer deduced it in [26, p. 3]. By reparameterising c proportionally to hyperbolic arclength and denoting L as the hyperbolic arclength of c, it can be stated as follows

$$\frac{2}{L^2}\ddot{\kappa}[c](t) = -\kappa[c](t)^3 + 2\kappa[c](t). \tag{5.6}$$

Hence $\kappa[c]$ satisfies an equation with a right-hand side in $W^{1,\infty}([0,1])$, which in return gives us $\kappa[c] \in W^{3,\infty}([0,1]) = C^{2,1}([0,1])$. Hence the Frenet equations (2.12) and (2.13) yield $c \in C^4([0,1])$. By straightforward bootstrapping we finally obtain $c \in C^\infty([0,1])$.

6 Analysis of the Euler-Lagrange equation

In order to prove Theorem 1.3, we need to analyse the Euler-Lagrange equation. We will proceed by an order reduction argument introduced by Langer & Singer in [26]. We also collect results from [18], which expanded the theory provided by Langer & Singer:
First let us reparameterise our solution c by hyperbolic arclength. We call this new curve $\gamma : [0, L] \to \mathbb{H}^2$ and L is the hyperbolic length of γ. With (5.6) the geodesic curvature $K = \kappa[\gamma]$ of γ satisfies (see [26, eq. (1.2)])

$$\ddot{K} = K - \frac{1}{2}K^3. \tag{6.1}$$

Every curve in $\mathbb{H}^2$ with geodesic curvature satisfying this equation is called an elastica (see [26, p.3 bottom]). Solutions of (6.1) can be expressed with the help of Jacobian elliptic functions (see [26, table 2.7]). The following definitions of these functions are taken from [1, Section 16.1]: Let $0 < k < 1$ and

$$F_k(\varphi) = \int_0^{\varphi} \frac{1}{\sqrt{1 - k^2 \sin^2 \psi}}\, d\psi. \tag{6.2}$$

Since $F_k' > 0$ we can invert it and define

$$\mathrm{AM}(s,k) = F_k^{-1}(s). \tag{6.3}$$

The Jacobian elliptic functions can then be expressed as

$$\begin{gathered} \mathrm{sn}(s,k) = \sin(\mathrm{AM}(s,k)),\ \mathrm{cn}(s,k) = \cos(\mathrm{AM}(s,k)), \\ \mathrm{dn}(s,k) = \sqrt{1 - k^2 \,\mathrm{sn}^2(s,k)}. \end{gathered} \tag{6.4}$$

The non-trivial solutions of (6.1) are classified by the following three lemmas (see [26, p. 6] and also [18, Lemma 3.1]).

Lemma 6.1 (see [26] table 2.7c). *Let $K \in C^2(\mathbb{R})$ be a solution of (6.1) with $K(0) = K_0 \in (0,2)$ and $\dot{K}(0) = 0$, then*

$$K(s) = \begin{cases} 2r\,\mathrm{dn}(r(s+s_0),k), & \text{if } K_0 \neq \sqrt{2}, \\ \sqrt{2}, & \text{if } K_0 = \sqrt{2}, \end{cases}$$

with

$$r = \left\{ \begin{array}{ll} \frac{K_0}{2}, & \text{if } K_0 \in (\sqrt{2},2), \\ \frac{\sqrt{4-K_0^2}}{2}, & \text{if } K_0 \in (0,\sqrt{2}), \end{array} \right\} \in \left(\frac{1}{2}\sqrt{2}, 1\right),$$

$$k = \frac{\sqrt{2r^2-1}}{r} \in (0,1),\ s_0 = \begin{cases} 0, & \text{if } K_0 \in (\sqrt{2},2), \\ \frac{1}{r}F_k(\frac{\pi}{2}), & \text{if } K_0 \in (0,\sqrt{2}). \end{cases}$$

These solutions are called orbitlike or in the constant case circular.

Lemma 6.2 (see [26] table 2.7c). *Let $K \in C^2(\mathbb{R})$ be a solution of (6.1) with $K(0) = 2$ and $\dot{K}(0) = 0$, then*

$$K(s) = \frac{2}{\cosh(s)}.$$

This solution is called asymptotically geodesic.

Lemma 6.3 (see [26] table 2.7c). *Let $K \in C^2(\mathbb{R})$ be a solution of (6.1) with $K(0) = K_0 > 2$ and $\dot{K}(0) = 0$, then*

$$K(s) = K_0 \,\mathrm{cn}(rs,k)$$

with

$$r = \sqrt{-1 + \frac{1}{2}K_0^2},\ k = \frac{K_0}{2r}.$$

These solutions are called wavelike.

The following lemma provides estimates for the elastic energy of a wavelike solution. We are especially interested in these type of elastica because later we will see (see Lemma 6.9) that a non-projectable solution of (1.4) is indeed wavelike.

Lemma 6.4. *Let $K \in C^2(\mathbb{R})$ be wavelike with parameters $K_0 > 2$, k, r given as in Lemma 6.3. Additionally let $a \in \mathbb{N}$ be arbitrary. Then the following estimate holds*

$$\int_0^{a\frac{1}{r}F_k\left(\frac{\pi}{2}\right)} (K(s))^2\,ds \geq a\pi.$$

One may observe that $\frac{1}{r}F_k(\frac{\pi}{2})$ is a $\frac{1}{4}$-period of $K(\cdot)$.

Proof. By using $aF_k\left(\frac{\pi}{2}\right) = F_k\left(a\frac{\pi}{2}\right)$, we find

$$\begin{aligned}
\int_0^{\frac{1}{r}F_k\left(a\frac{\pi}{2}\right)} (K(s))^2\,ds &= K_0^2 \int_0^{\frac{1}{r}F_k\left(a\frac{\pi}{2}\right)} \mathrm{cn}^2(rs,k)\,ds \\
&= K_0^2 \int_0^{\frac{1}{r}F_k\left(a\frac{\pi}{2}\right)} \cos^2(F_k^{-1}(rs))\,ds \\
&= K_0^2 \int_0^{a\frac{\pi}{2}} \frac{1}{r}\cos^2(x)\frac{1}{\sqrt{1-k^2\sin^2(x)}}\,dx \\
&\geq \frac{K_0^2}{\sqrt{\frac{1}{2}K_0^2-1}} \int_0^{a\frac{\pi}{2}} \cos^2(x)\,dx \\
&= \frac{K_0^2}{\sqrt{\frac{1}{2}K_0^2-1}}a\frac{\pi}{4}.
\end{aligned}$$

Now we need to estimate the prefactor:

$$\begin{aligned}
&(K_0^2-4)^2 \geq 0 \\
\Rightarrow &K_0^4 \geq 8K_0^2 - 16 = 16\left(\frac{K_0^2}{2}-1\right) \\
\Rightarrow &\frac{K_0^2}{\sqrt{\frac{K_0^2}{2}-1}} \geq 4.
\end{aligned}$$

□

Now we cite the main order reduction argument of [26].

Theorem 6.5 (see [26] Prop. 2.1)**.** *Let $-\infty \leq a < b \leq \infty$ and $\gamma : (a,b) \to \mathbb{H}^2$ be an elastica parameterised by hyperbolic arclength with curvature $K : (a,b) \to \mathbb{R}$ satisfying* (6.1)*. Then the vector field*

$$J_\gamma = K^2\dot{\gamma} + 2\dot{K}\begin{pmatrix} -\dot{\gamma}^2 \\ \dot{\gamma}^1 \end{pmatrix}$$

has a unique extension to a Killing vector field J on the whole hyperbolic half plane $\mathbb{H}^2$.

The following example allows us to compare the local Killing field of the previous theorem to its corresponding extension.

Example 6.6 (see [18] Lemma 4.2)**.** *Let V be a Killing vector field on $\mathbb{H}^2$. Then there exist $a, b, c \in \mathbb{R}$, such that for all $(x, y) \in \mathbb{H}^2$*

$$V(x,y) = a \begin{pmatrix} \frac{x^2-y^2}{2} \\ xy \end{pmatrix} + b \begin{pmatrix} x \\ y \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

As mentioned before we like to analyse the behaviour of wavelike elastica. The following lemma will help us in determining the possible shape of such curves.

Lemma 6.7 (see [26] p. 8, or [18] Corollary 6.2, Lemma 6.1)**.** *Let $-\infty \leq a < b \leq \infty$ and $\gamma : (a, b) \to \mathbb{H}^2$ be an elastica parameterised by hyperbolic arclength with curvature $K : (a, b) \to \mathbb{R}$ satisfying* (6.1)*. Additionally let the elastica be wavelike (cf. Lemma 6.3). Then there exists a unique geodesic Σ, which is an integral curve of the Killing vector field J (cf. Theorem 6.5). K also changes sign precisely when γ intersects Σ. These intersections are always perpendicular.*

The following lemma will help us in finding a zero in the curvature of a solution.

Lemma 6.8. *Let $-\infty \leq a < b \leq \infty$ and $c : (a, b) \to \mathbb{H}^2$ be regular and parameterised by hyperbolic arclength with curvature $\kappa[c]$. Additionally let $a < t_1 \leq t_2 < b$ with*

$$\dot{c}^1(t_1) = \dot{c}^1(t_2) = 0, \ \dot{c}^2(t_1) \cdot \dot{c}^2(t_2) > 0, \ \ddot{c}^1(t_1) \cdot \ddot{c}^1(t_2) \leq 0.$$

Then there exists a $t_* \in [t_1, t_2]$ satisfying $\kappa[c](t_*) = 0$.

Proof. The Frenet equation (2.12) yields

$$\begin{aligned} \ddot{c}^1(t_1) &= -\kappa(t_1)\dot{c}^2(t_1) \\ \ddot{c}^1(t_2) &= -\kappa(t_2)\dot{c}^2(t_2). \end{aligned}$$

By multiplying both equations we obtain

$$0 \geq \ddot{c}^1(t_1)\ddot{c}^1(t_2) = \kappa(t_1)\kappa(t_2)\dot{c}^2(t_1)\dot{c}^2(t_2).$$

Then the intermediate value theorem gives us the desired zero. □

The next lemma shows that a non-projectable solution has to be wavelike.

Lemma 6.9. *Let $c : [0, 1] \to \mathbb{H}^2$ be a solution of* (1.4) *with $W_h(c) < 8$ and let $0 < t^* < 1$ satisfy $\dot{c}^1(t^*) = 0$. Then c is wavelike (cf. Lemma 6.3) and the geodesic from Lemma 6.7 is an upper half circle centered on the x-axis.*

Proof. At first we will show the existence of a zero in the curvature. Due to the discussion in the Lemmas 6.1, 6.2 and 6.3 the solution then has to be wavelike. For this we need to distinguish three cases:

i) $\ddot{c}^1(t^*) = 0$: Equation (2.9) yields a zero in the geodesic curvature.

ii) $\ddot{c}^1(t^*) < 0$: In this case c^1 has a local maximum in t^*. This also means there exists an $\varepsilon > 0$, such that $\forall t \in (t^*, t^*+\varepsilon) : \dot{c}^1(t) < 0$. Then $\dot{c}^1(1) > 0$ yields the existence of a $t_* \in (t^*, 1)$ in which c^1 has a local minimum. Now we can distinguish two cases (since c is regular $\dot{c}^2(t^*_*) \neq 0$):

   a) $\dot{c}^2(t^*) \cdot \dot{c}^2(t_*) < 0$: Lemma 4.1 gives us $W_h(c) > 8$, which is a contradiction.

   b) $\dot{c}^2(t^*) \cdot \dot{c}^2(t_*) > 0$: Reparameterising c by hyperbolic arclength, Lemma 6.8 yields a zero of the geodesic curvature in $[t^*, t_*]$.

iii) $\ddot{c}^1(t^*) > 0$: This case can be treated as case ii): c^1 has a local minimum in t^*. As seen above we can find a local maximum in $(0, t^*)$. With the same distinction in subcases we also obtain a zero in the curvature.

This shows that the solution has to be wavelike. Let us turn to the shape of the geodesic Σ given by Lemma 6.7. Let t_0 denote our zero of the geodesic curvature $\kappa[c]$. We proceed by contradiction and assume Σ to be a parallel line to the y-axis. Since Σ is perpendicular to $\dot{c}(t)$ iff $\kappa[c](t) = 0$, equation (2.9) yields $\ddot{c}^1(t^*) \neq 0$. The existence proof for the zero in the curvature now yields a $t_* \in (0,1) \setminus \{t^*\}$ with $\dot{c}^1(t_*) = 0$. The proof also shows that we can assume without loss of generality $t^* < t_0 < t_*$. The perpendicularity of Σ also gives us $\dot{c}^2(t_0) = 0$. Let us now put everything together and assume for a moment that $\dot{c}^2(t^*), \dot{c}^2(t_*) > 0$. Equation (2.11) yields

$$
\begin{aligned}
W_h(c) &\geq W_h(c)\big|_{[0,t^*]} + W_h(c)\big|_{[t_0,t_*]} \\
&= \frac{2}{\pi} W_e(S(c))\big|_{[0,t^*]} + \frac{2}{\pi} W_e(S(c))\big|_{[t_0,t_*]} \\
&+ 4\left[\frac{\dot{c}^2}{\sqrt{(\dot{c}^1)^2 + (\dot{c}^2)^2}}\right]_0^{t^*} + 4\left[\frac{\dot{c}^2}{\sqrt{(\dot{c}^1)^2 + (\dot{c}^2)^2}}\right]_{t_0}^{t_*} \\
&\geq 4\left[\frac{\dot{c}^2}{\sqrt{(\dot{c}^1)^2 + (\dot{c}^2)^2}}\right]_0^{t^*} + 4\left[\frac{\dot{c}^2}{\sqrt{(\dot{c}^1)^2 + (\dot{c}^2)^2}}\right]_{t_0}^{t_*} = 8.
\end{aligned}
$$

This is a contradiction to $W_h(c) < 8$. So Σ has to be a half circle centered on the x-axis.
The other cases can be treated analogously. □

The next lemma will help us to determine locally the sign of the curvature of a wavelike elastica. Langer & Singer already observed this in [26, Prop. 5.1] and they even gave a more precise statement. For the readers convenience we will point out how to obtain the proof by arguments, which are scattered throughout [18]. Figure 6 explains the geometric meaning of the next lemma.

Lemma 6.10 (see [26], Prop. 5.2 (iii))**.** *Let $\gamma : \mathbb{R} \to \mathbb{H}^2$ be a wavelike (see Lemma 6.3) elastica parameterised by hyperbolic arclength with curvature K. Let J be the Killing vector field given by Theorem 6.5 and let Φ_J be the corresponding flow of J. Let Σ denote the geodesic given by Lemma 6.7 and let $p \in \Sigma$. Let further be $s_1 < s_2$, such that $K(s_1) = K(s_2) = 0$ and $|s_1 - s_2|$ is minimal. Then there exist parameters $t_1 < t_2$ with $\Phi_J(p, t_1) = \gamma(s_1)$ and $\Phi_J(p, t_2) = \gamma(s_2)$. Moreover, the hyperbolic distance of $\gamma(s_1)$ and $\gamma(s_2)$ is the same for any such*

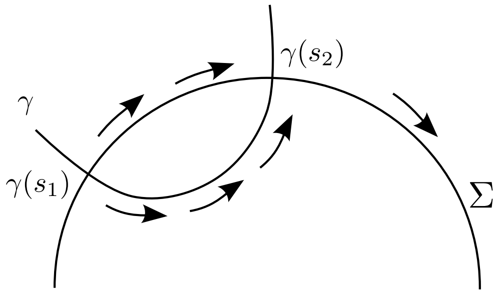

Figure 6: Constant hyperbolic increment on the geodesic Σ.

pair of points. This means, if there exists another pair $\tilde{s}_1 < \tilde{s}_2$ satisfying the same assumptions, we have

$$\operatorname{dist}_g(\gamma(s_1), \gamma(s_2)) = \operatorname{dist}_g(\gamma(\tilde{s}_1), \gamma(\tilde{s}_2)) > 0.$$

Proof. [18, Thm. 7.4] shows the existence of the parameters t_1, t_2.
Since J is a Killing vector field, every integral curve of J is parameterised proportionally to hyperbolic arclength: Let $t \in \mathbb{R}$, then

$$g_{\Phi(p,t)}\left(\frac{\partial}{\partial t}\Phi(p,t), \frac{\partial}{\partial t}\Phi(p,t)\right) = g_{\Phi(p,t)}(J(\Phi(p,t)), J(\Phi(p,t))) = g_p(J(p), J(p)).$$

Also every integral curve possesses constant geodesic curvature (see [18, Thm. A.3]). Hence they have to be euclidean circles (see [18, Thm. 4.11]). Since K is periodic, [18, Lemma 7.8] shows that the above mentioned distance of $\gamma(s_1)$ and $\gamma(s_2)$ is constant. $\square$

The next two lemmas will give us the opportunity to spot extrema of the geodesic curvature of a wavelike elastica.

Lemma 6.11. *Let $\gamma : \mathbb{R} \to \mathbb{H}^2$ be a wavelike elastica (see Lemma 6.3) parameterised by hyperbolic arclength. The corresponding Killing vector field will be denoted by J respectively J_γ (see Theorem 6.5). Let K denote the geodesic curvature of γ. Furthermore let $s_0 \in \mathbb{R}$ with $K(s_0) = 0$ and $s_1 > s_0$ minimal, such that $|K(s_1)|$ is maximal. Finally let I be an arbitrary integral curve of J. Then the number of elements in $\gamma([s_0, s_1]) \cap I$ is at most 1.*

Proof. Equation [18, (5.4)] yields

$$g(J_\gamma, J_\gamma) = 4K^2 + const.$$

Since K^2 is strictly increasing on $[s_0, s_1]$, $g(J_\gamma, J_\gamma)$ has to be strictly monotone as well. [18, Theorem 7.4] and [18, Lemma 7.7] give us a one to one correspondence between the hyperbolic length of J_γ and the set of integral curves of J, as long as K does not change sign. Thus the lemma follows. $\square$

Lemma 6.12 (see [26] p. 8 and Prop. 2.3, [18] Section 5.3)**.** *Let γ be a wavelike solution (cf. 6.3) with Σ being a half circle centered on the x-axis (cf. Lemma 6.7). Then there exist two euclidean circles $B_+, B_- \subset \mathbb{R}^2$, which are the only integral curves of J (cf. Theorem 6.5), that are tangent to γ. Moreover, γ touches B_+ iff the geodesic curvature K has a maximum and it touches B_- iff K has a minimum.*

By the next lemma a wavelike solution has to be injective.

Lemma 6.13. *As before let $\gamma : \mathbb{R} \to \mathbb{H}^2$ be a wavelike elastica (see Lemma 6.3), parameterised by hyperbolic arclength, and J the corresponding Killing vector field (cf. Theorem 6.5). We set $J^\perp := (J^2, -J^1)$ as the perpendicular vector field of J. Let now $I \subset \mathbb{H}^2$ be an integral curve of $J^\perp$ defined on its maximal interval of existence. Then I intersects γ exactly once.*

Proof. First let us fix the notation: K is the geodesic curvature of γ, Φ_J the flow of J and J_γ the local variant of J (see Theorem 6.5). [18, Theorem 4.10] and [18, Lemma 7.1] allow us without loss of generality to write J as

$$J(x,y) = a \begin{pmatrix} \frac{x^2-y^2}{2} \\ xy \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

with $a < 0$ and $c > 0$. This can be achieved by finding a point $s \in \mathbb{R}$, such that $K(s)$ is extremal. [18, Theorem 4.10] gives us a Moebius transformation $\Psi : \mathbb{H}^2 \to \mathbb{H}^2$ with $\Psi(\gamma(s)) \in \{(0,y) : y > 0\}$ and $d\Psi(\gamma(s))\dot{\gamma}(s) = (|\dot{\gamma}(s)|_g, 0)$. [18, Lemma 7.1] then yields the above mentioned form of J.
Since Φ_J is an isometry and hence preserves orthogonality, we have that for all $t \in \mathbb{R}$ the curve $\Phi_J(I,t)$ is again an integral curve of $J^\perp$. Furthermore [18, Theorem 7.4] gives us that $\Phi_J((0,\cdot),\cdot)$ is a diffeomorphism of $\mathbb{H}^2$ with $\Phi_J(\{(0,y), y > 0\}, \mathbb{R}) = \mathbb{H}^2$. If we fix a $p \in I$ then there exists a $t \in \mathbb{R}$ such that $\Phi_J(p,t) \in \{(0,y), y > 0\}$. We have achieved that $\Phi(I,t) \subset \{(0,y), y > 0\}$, because $\Phi_J(I,t)$ is an integral curve of $J^\perp$. If we assume the existence of a $q \in \{(0,y), y > 0\}$ with $q \notin \Phi_J(I,t)$, we could further extend $\Phi_J(I,t)$ as an integral curve to the whole of $\{(0,y), y > 0\}$. This would contradict I being defined on its maximal interval of existence. Hence we can assume without loss of generality that

$$I = \{(0,y),\ y > 0\}.$$

Multiplying $\dot{\gamma}$ and J_γ yields

$$g(J_\gamma, \dot{\gamma}) = (K(s))^2 \geq 0. \tag{6.5}$$

Now we proceed by contradiction and assume that $s_1 < s_2 \in \mathbb{R}$ exist and satisfy $\gamma(s_1), \gamma(s_2) \in I$. For every $y > 0$ the Killing vector field satisfies $J^1(0,y) > 0$ and $J^2(0,y) = 0$. In combination with (6.5) and Lemma 6.7 we have without loss of generality $K(s_2) = 0$. Then $K(s_1) \neq 0$. Otherwise $\gamma(s_1)$ and $\gamma(s_2)$ would both lie on the unique geodesic Σ (see Lemma 6.7). Since Σ is an upper half circle (see e.g. [18, Lemma 6.1]) this would be a contradiction. Now we can find an $\varepsilon \neq 0$ such that $\Phi_J(I,\varepsilon)$ meets γ at least twice with different signs for $g(J_\gamma, \dot{\gamma})$. This contradicts (6.5). Figure 7 explains the situation. $\square$

Remark 6.14. Lemma 6.13 together with Lemma 6.9 shows that our energy minimising solution, found in Theorem 1.1, does not intersect itself.

7 Projectability of solutions under a smallness condition

In this section we will prove that the solution, obtained in Section 4 of (1.4), is a graph $(\cdot, u(\cdot))$, if we assume the smallness condition 1.2. It will be an important

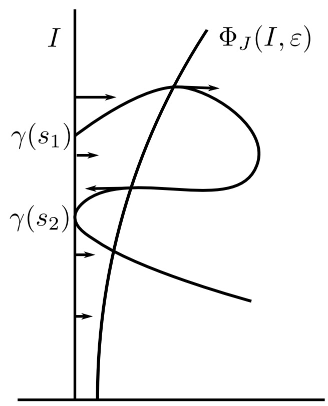

Figure 7: I meets γ exactly once.

argument to compare the energy of our solution to special comparison functions which we calculate now:

Example 7.1. *The euclidean Willmore energy of a revolved upper half circle centered on the x-axis is*

$$W_e(S(p_{x_0,r})) = 4\pi$$

for all $x_0 \in \mathbb{R}$ and $r > 0$.

Example 7.2. *The hyperbolic elastic energy of a catenoid c_{cat} is*

$$W_h(c_{cat}) = 8.$$

Proof. The geodesic curvature of c_{cat} parameterised by hyperbolic arclength can be expressed as (see Lemma 6.2)

$$K(s) = \frac{2}{\cosh(s)}.$$

Integration gives

$$W_h(c_{cat}) = \int_{-\infty}^{\infty} \left(\frac{2}{\cosh(s)}\right)^2 ds = 4\big[\tanh(x)\big]_{-\infty}^{\infty} = 8.$$

□

To get a grasp of the local elastic energy of a curve, we need the following lemma:

Lemma 7.3. *Let $c : [0,1] \to \mathbb{H}^2$ be a smooth regular curve, such that there exists a $t^* \in (0,1)$ with $\dot{c}^1(t^*) = 0$. c should also satisfy $\dot{c}^2(0) = \dot{c}^2(1) = 0$. Then its hyperbolic elastic energy satisfies*

$$W_h(c) \geq 4.$$

Proof. Without loss of generality we can assume that $\dot{c}^2(t^*) > 0$. Then we have with the help of (2.11) that

$$\begin{aligned} W_h(c) &= W_h(c)\big|_{[0,t^*]} + W_h(c)\big|_{[t^*,1]} \\ &\geq \frac{2}{\pi} W_e(S(c))\big|_{[0,t^*]} + 4\left[\frac{\dot{c}^2}{\sqrt{(\dot{c}^1)^2 + (\dot{c}^2)^2}}\right]_0^{t^*} \\ &\geq 4. \end{aligned}$$

□

To apply Theorems 3.1 and 4.4 we need a suitable comparison function:

Lemma 7.4. *If the pair (α_-, α_+) satisfies Assumption 1.2, then*

$$W^h_{\alpha_-,\alpha_+} = \inf\{W_h(v) : \; v \in M_{\alpha_-,\alpha_+}\} < 8.$$

Proof. Without loss of generality we assume $\alpha_- \leq \alpha_+$. Furthermore let

$$\cosh_-(x) = \alpha_- \cosh\left(\frac{1+x}{\alpha_-}\right) \text{ and } \cosh_+(x) = \alpha_+ \cosh\left(\frac{1-x}{\alpha_+}\right).$$

Now define $t \mapsto (x(t), r(t))$ in such a way that the upper half circle $p_{x(t),r(t)}(\cdot)$ is tangent to the graph of $\cosh_+$ at $(1-t, \cosh_+(1-t))$. Since $\cosh_+$ is smooth, $x(\cdot)$ and $r(\cdot)$ are continuous. Assumption 1.2 also ensures that $p_{x(0),r(0)} = p_{1,\alpha_+}$ and $\cosh_-$ do not meet. Since $\cosh_-$ and $\cosh_+$ do meet, we can find a first time $t_0 > 0$, such that a $\tilde{t} > t_0$ exists for which $p_{x(t_0),r(t_0)}$ is tangent to the graph of $\cosh_-$ in $(1-\tilde{t}, \cosh_-(1-\tilde{t}))$. Figure 8 gives a sketch of the situation. We can

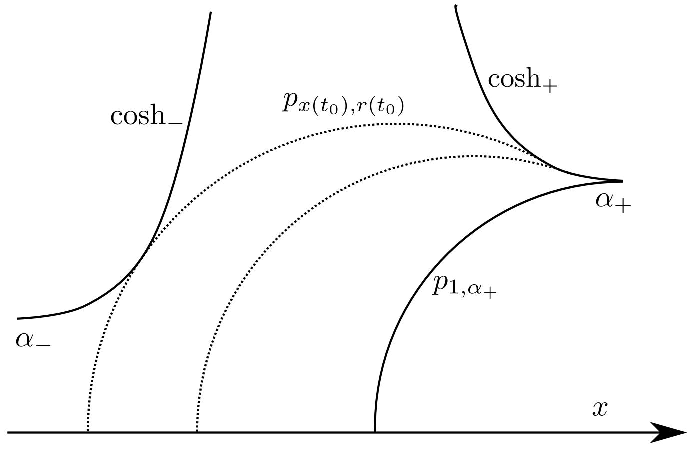

Figure 8: Comparison function for the elastic energy.

define the desired comparison function by

$$v_{\alpha_-,\alpha_+}(x) = \begin{cases} \cosh_-(x), & x \in \left[-1, 1-\tilde{t}\right] \\ p_{x(t_0),r(t_0)}(x), & x \in \left(1-\tilde{t}, 1-t_0\right] \\ \cosh_+(x), & x \in (1-t_0, 1] \end{cases}$$

Since $v_{\alpha_-,\alpha_+} \in C^{1,1}([-1,1])$, it is also an element of M_{α_-,α_+}. Since $p_{x(t_0),r(t_0)}$ is a geodesic w.r.t. to the hyperbolic metric, it does not contribute to the hyperbolic elastic energy W_h. The proof of Example 7.2 then shows

$$W^h_{\alpha_-,\alpha_+} \leq W_h(v_{\alpha_-,\alpha_+}) < 8.$$

□

We are now able to tackle the main result of this section:

Theorem 7.5. *Let the pair $(\alpha_-,\alpha_+) \in \mathbb{R}_+ \times \mathbb{R}_+$ obey Assumption 1.2. Also let $c \in C^\infty([0,1],\mathbb{H}^2) \cap M_{\alpha_-,\alpha_+}$ satisfy*

$$W_e(S(c)) = W^e_{\alpha_-,\alpha_+} = \inf\{W_e(S(v)) : v \in M_{\alpha_-,\alpha_+}\}.$$

(Existence is proven by Theorem 1.1 and Lemma 7.4). Then c can be represented as a graph $(\cdot, u(\cdot))$ with $u \in C^\infty([-1,1],(0,\infty))$.

Proof. Without loss of generality we can assume that

$$\alpha_- \leq \alpha_+.$$

Otherwise we simply reflect c at the y-axis. Let $L > 0$ be the hyperbolic arclength of c and $\gamma : [0,L] \to \mathbb{H}^2$ the reparameterisation by hyperbolic arclength with $\dot\gamma^1(0) > 0$. Then the corresponding geodesic curvature $K : [0,L] \to \mathbb{R}$ satisfies (6.1). Theorem 6.5 gives us the Killing vector field J, which is the unique extension of $J_\gamma = K^2\dot\gamma + 2\dot K(-\dot\gamma^2, \dot\gamma^1)$. We proceed by contradiction and assume that there exists a **minimal** $s^* \in (0,L)$ with $\dot\gamma^1(s^*) = 0$. Lemma 6.9 yields K to be wavelike (cf. Lemma 6.3) and the unique geodesic Σ (cf. Lemma 6.7) to be an upper half circle with center on the x-axis.
We can now state the following observation concerning the geodesic curvature of γ:

Claim 1. *K possesses at most two zeros in $[0,L]$.*

Proof. Assume the existence of at least three zeros of K in $[0,L]$. Then $[0,L]$ contains at least one period of K. Lemma 6.4 yields $W_h(\gamma) \geq 4\pi > 8$, a contradiction to Lemma 7.4. □

For the remainder of the proof we need to distinguish two major cases. For now let

$$\dot\gamma^2(s^*) > 0.$$

Now we have to prove a claim concerning Σ, which is described by Figure 9.

Claim 2. *From an euclidean viewpoint the geodesic Σ seperates $\mathbb{H}^2$ into an unbounded part Σ_- and a bounded part Σ_+. For all $s \in [0,L]$ we then have:*
$\gamma(s) \in \Sigma_-$, iff $K(s) < 0$. On the other hand $\gamma(s) \in \Sigma_+$, iff $K(s) > 0$.
We also have that $K(s^*) \geq 0$.

Proof. We need another distinction of cases:

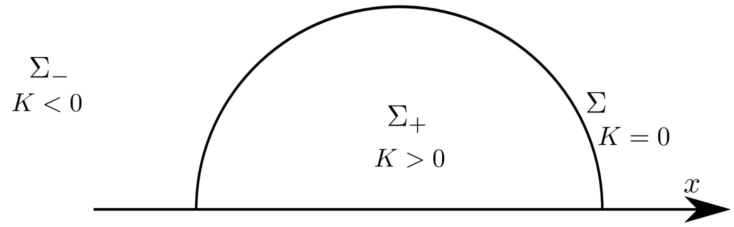

Figure 9: Decomposition of $\mathbb{H}^2$ by means of the geodesic curvature K.

1. $\ddot{\gamma}^1(s^*) \neq 0$: This means we have an extremum of γ^1 in s^*. Since s^* is minimal the boundary datum $\dot{\gamma}^1(0) > 0$ yields s^* to be a local maximum of γ^1. With $\dot{\gamma}^1(L) > 0$ we can then find an $s_* \in (s^*, L)$ minimal with $\dot{\gamma}^1(s_*) = 0$ and $\ddot{\gamma}^1(s_*) \geq 0$. The minimality ensures $\forall s \in (s^*, s_*)$ $\dot{\gamma}^1(s) < 0$. By Lemma 7.3 we also obtain $\forall s \in [s^*, s_*]$ that $\dot{\gamma}^2(s) > 0$, because $W_h(\gamma) < 8$. Let us give a look at the curvature K: Equation (2.9) yields

$$K(s^*_*) = -\frac{\ddot{\gamma}^1\dot{\gamma}^2\gamma^2}{|\dot{\gamma}|^3}(s^*_*).$$

Here s^*_* denotes that the equation is satisfied for s_* and s^*. This means that $K(s^*) > 0$ and $K(s_*) \leq 0$. On the other hand the sign conditions on $\dot{\gamma}$ ($\forall s \in (s^*, s_*)$ $\dot{\gamma}^1(s) < 0$, $\dot{\gamma}^2(s) > 0$), Σ being a geodesic circle and γ crossing Σ perpendicularly yield $\gamma(s^*) \in \Sigma_+$ and $\gamma(s_*) \in \overline{\Sigma_-}$. Figure 10 explains this situation.

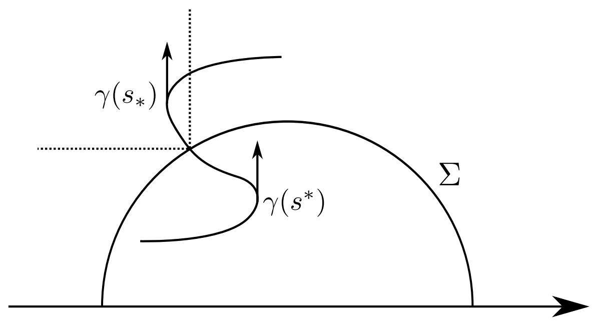

Figure 10: Proof of curvature decomposition of $\mathbb{H}^2$, part 1.

2. $\ddot{\gamma}^1(s^*) = 0$: Equation (2.9) yields $K(s^*) = 0$. Lemma 6.7 then yields $\gamma(s^*) \in \Sigma$. Since Σ is a half circle, $\gamma(s^*)$ has to be the north pole of this geodesic. Since we have choosen s^* to be minimal, we can reparameterise $\gamma\big|_{[0,s^*)}$ as a smooth graph $(x, u(x))$ with $u : [-1, \gamma^1(s^*)) \to (0, \infty)$. The geodesic curvature $\kappa[u]$, which represents K in this parameterisation, can be calculated by (cf. (2.9))

$$\frac{u''(x)u(x)}{(1 + (u'(x))^2)^{\frac{3}{2}}} + \frac{1}{\sqrt{1 + (u'(x))^2}} = \kappa[u](x). \tag{7.1}$$

We also have $\lim_{x \nearrow \gamma^1(s^*)} u'(x) = +\infty$. The mean value theorem then yields a sequence $\xi_k \nearrow \gamma^1(s^*)$ with

$$u''(\xi_k)\left(\gamma^1(s^*) - \frac{1}{k} - 1\right) = u'\left(\gamma^1(s^*) - \frac{1}{k}\right) - u'(-1) \to \infty.$$

For $k \in \mathbb{N}$ big enough we obtain $u''(\xi_k) \geq 0$. Equation (7.1) then gives us $\kappa[u](\xi_k) > 0$. This yields a sequence $s_k \nearrow s^*$ with $K(s_k) > 0$. Since $\gamma^1(s^*)$ lies on the north pole of Σ and $\dot{\gamma}^2(s^*) > 0$, we have $\gamma(s_k) \in \Sigma_+$ for $k \in \mathbb{N}$ big enough.

□

As a corollary to the proof of Claim 2 we have the following:

Claim 3. *There exists an $s_0 \in (0, L)$ with $K(s_0) = 0$ and in which the geodesic curvature changes sign from positive to negative. This point also satisfies $s^* \leq s_0$ and $\forall s \in (s^*, s_0)$ we have $\dot{\gamma}^1(s) \neq 0$.*
If $K(s^*) > 0$, there exists exactly one $s_* \in [s_0, L)$ with $\dot{\gamma}^1(s_*) = 0$ and $K(s_*) \leq 0$.
If on the other hand $K(s^*) = 0$ we do not have another point $s \in [0, L] \setminus \{s^*\}$ with $\dot{\gamma}^1(s) = 0$.

Proof. The proof of Claim 2 already showed most of the statement. All that is left is the uniqueness property of s_*. The aforementioned proof yields an $s_* \in [s^*, L)$ with $\dot{\gamma}^1(s_*) = 0$, $K(s_*) \leq 0$, $\dot{\gamma}^2(s_*) > 0$ and $\forall s \in (s^*, s_*)$ $\dot{\gamma}^1(s) \neq 0$ (If $K(s^*) = 0$ we set $s_* := s^*$). Let us proceed by contradiction and assume that a minimal $\tilde{s} \in (s_*, L)$ with $\dot{\gamma}^1(\tilde{s}) = 0$ exists. Lemma 4.1 yields that $\dot{\gamma}^2(\tilde{s}) > 0$. Here we need to distinguish a few cases to demonstrate how the above mentioned arguments apply:

1. $K(s^*) > 0$: The arguments for Claim 2, case 1, show that $K(s_*) < 0$, since $\gamma(s_*) \in B_-$. Figure 10 gives a sketch of the situation. Now two subcases need to be considered (see Figure 11):

   (a) $\ddot{\gamma}^1(\tilde{s}) \neq 0$: As in Claim 2, case 1, $K(\tilde{s}) > 0$ and we find an $s_1 \in (\tilde{s}, L]$ with $K(s_1) = 0$. $K(s_*) < 0$ yields a zero of K between s_* and $\tilde{s}$. Hence we found three zeros of K, contradicting Claim 1.

   (b) $\ddot{\gamma}^1(\tilde{s}) = 0$: The proof of Claim 2, case 2, yields a $\delta > 0$ such that $\forall s \in (\tilde{s} - \delta, \tilde{s})$ we have $K(s) > 0$. $K(s_*) < 0$ again yields three zeros of K which contradicts Claim 1.

2. $K(s^*) = 0$: The proof of Claim 2, case 2, yields a $\delta > 0$ such that $\forall s \in (s^*, s^* + \delta]$ we have $K(s) < 0$. Hence equation (7.1) yields γ to be a strictly concave graph on $(s^*, s^* + \delta]$. Now we need to consider two subcases (see Figure 12):

   (a) $\ddot{\gamma}^1(\tilde{s}) \neq 0$: Since $\dot{\gamma}^1(s^* + \delta) > 0$ we can apply the arguments of case 1 of the proof of Claim 2. Hence $K(\tilde{s}) > 0$ and we find an $s_1 \in (\tilde{s}, L]$ with $K(s_1) = 0$. $K(s^* + \delta) < 0$ yields again at least three zeros of K contradicting Claim 1.

(b) $\ddot{\gamma}^1(\tilde{s}) = 0$: Since γ is a graph on $(s^*, \tilde{s})$ the arguments of case 2 of the proof of Claim 2 yield $K(\tilde{s}) = 0$ and the existence of a $\tilde{\delta} > 0$ such that for all $s \in [\tilde{s} - \tilde{\delta}, \tilde{s})$ we have $K(s) > 0$. In combination with $K(s^* + \delta) < 0$ we obtain at least three zeros of K, contradicting again Claim 1.

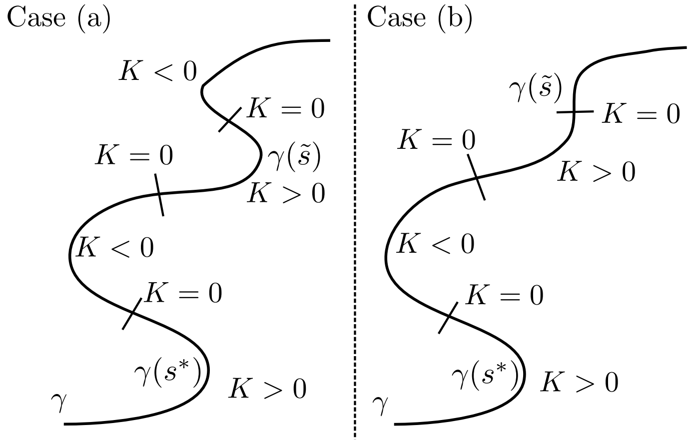

Figure 11: Uniqueness of s_* and s^*, case 1.

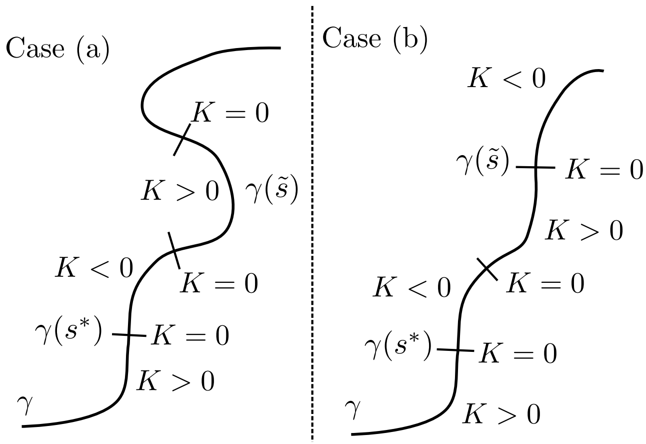

Figure 12: Uniqueness of $s_* = s^*$, case 2.

$\square$

Claim 3 also means we have at most two points $s^* \leq s_*$, such that $\dot{\gamma}^1(s^*_*) = 0$. For the sake of simplicity we set $s_* := s^*$, if we only have one such point. Claim 1 shows that K has at most two zeros. With this in mind we distinguish the following cases to prove our main result:

1. $K(L) < 0$:
   Claim 3 and Claim 1 show that $\forall s \in [s_*, L]$ we have $K(s) \leq 0$. Let

us take a closer look at the behaviour of γ on $[s_*, L]$. For this let us reparameterise γ locally around $s = L$ as a smooth graph $(\cdot, u(\cdot))$. The geodesic curvature $\kappa[u]$ then satisfies (7.1). Since $\forall s \in [s_*, L]$: $K(s) < 0$ the function u has to be strictly concave until $\gamma^1(s_*) =: x_0 < \gamma^1(L)$ with $\lim\limits_{x \searrow x_0} u'(x) = \infty$. By comparing this to Assumption 1.2 we obtain $\forall s \in [s_*, L]$: $\gamma^2(s) \leq p_{1,\alpha_+}(\gamma^1(s))$, because the geodesic curvature of p_{1,α_+} is zero. Then the intermediate value theorem and the boundary data give us an $\hat{s} \in (0, s_*)$ with $\gamma^2(\hat{s}) = p_{1,\alpha_+}(\gamma^1(\hat{s}))$.
To construct a suitable comparison curve for γ, let us define a family of catenoids $cat_s : [0, \infty) \to \mathbb{H}^2$ by $cat_s(0) = \gamma(s)$, $\dot{cat}_s(0) = \dot{\gamma}(s)$ and by cat_s being parameterised by hyperbolic arclength. For reasons of consistency cat_s is defined as a straight line parallel to the y-axis, whenever $\dot{\gamma}^1(s) = 0$. By the smallness condition 1.2 cat_0 cannot intersect p_{1,α_+}. The existence of $\hat{s}$ on the other hand gives us an $\bar{s} \in (0, \hat{s}]$, such that $cat_{\bar{s}}$ is tangent to p_{1,α_+}. Figure 13 explains the situation. Two different situations can now arise. To distinguish these subcases let us define $\bar{x} \in \mathbb{R}$ as the point, in which $p_{1,\alpha_+}(\bar{x})$ is tangent to $cat_{\bar{s}}$. The first case reflects the situation described by Figure 13.

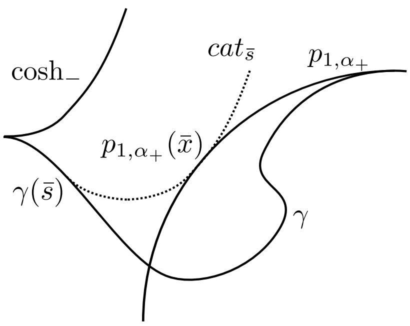

Figure 13: A comparison curve for γ.

(a) $\bar{x} \leq 1$:
For the following let $\tilde{p} : \mathbb{R} \to \mathbb{H}^2$ be the reparameterisation of p_{1,α_+} by hyperbolic arclength, satisfying $\tilde{p}^2(0) = p_{1,\alpha_+}(\bar{x})$. Let $\tilde{s} \in \mathbb{R}$ be given by $cat_{\bar{s}}(\tilde{s}) = \tilde{p}(0)$. Furthermore we need $\tilde{L} \geq 0$ defined by $\tilde{p}(\tilde{L}) = (1, \alpha_+)$. Now we can write down our comparison curve:

$$v(s) = \begin{cases} \gamma(s), & s \in [0, \bar{s}) \\ cat_{\bar{s}}(s - \bar{s}), & s \in [\bar{s}, \bar{s} + \tilde{s}) \\ \tilde{p}(s - (\bar{s} + \tilde{s})), & s \in [\bar{s} + \tilde{s}, \bar{s} + \tilde{s} + \tilde{L}]. \end{cases} \tag{7.2}$$

By construction $v \in C^{1,1}([0, \bar{s}+\tilde{s}+\tilde{L}], \mathbb{R}^2)$ and satisfies the boundary conditions. Let us compare v to $\gamma\big|_{[\bar{s},L]}$ in terms of the Willmore energy:

$$\begin{aligned} W_e(S(\gamma))\big|_{[\bar{s},L]} &\geq W_e(S(\gamma))\big|_{[s_*,L]} \\ &= \frac{\pi}{2} W_h(\gamma)\big|_{[s_*,L]} - 2\pi \left[\frac{\dot{\gamma}^2}{\sqrt{(\dot{\gamma}^1)^2 + (\dot{\gamma}^2)^2}} \right]_{s_*}^{L} \geq 2\pi. \end{aligned}$$

On the other hand Example 7.1 shows $W_e(S(v))\big|_{[\bar{s},\tilde{L}]} < 2\pi$. All in all we have

$$W_e(S(\gamma)) > W_e(S(v)).$$

Hence γ would not have been an energy minimiser, a contradiction.

(b) $\bar{x} > 1$:
This assumptions yields the existence of an $s_1 \in (0, \bar{s}]$ with $\gamma^2(s_1) > \alpha_+$. Additionally this point has to satisfy $\gamma^1(s_1) < \gamma^1(s_*)$, since we would otherwise find an $s \in (s_1, L)$ with $\dot{\gamma}^1(s) = 0$ and $\dot{\gamma}^2(s) < 0$. In combination with Lemma 4.1 this would be a contradiction. The perpendicularity condition with Σ then ensures $\gamma(s_1) \in \Sigma_-$. This in turn means $K(s_1) < 0$. Since this s_1 exists, we can find an $s_2 \in [0, s_1)$ with $K(s_2) > 0$. This is due to a concavity argument in combination with (7.1). Figure 14 gives a sketch of the situation. Claim 3 states

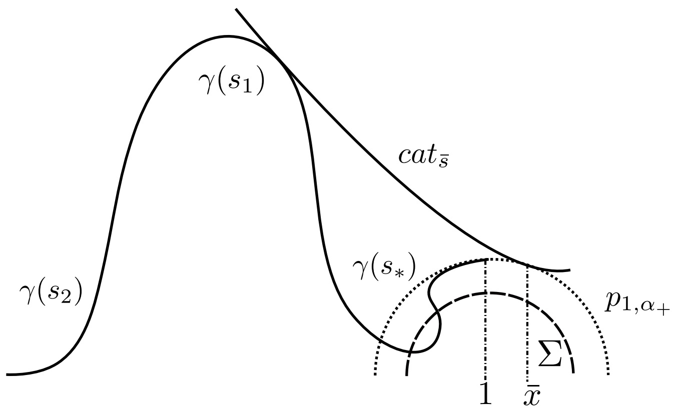

Figure 14: Behaviour of γ with $\bar{x} > 1$.

that K changes sign in s_0 from positive to negative. This gives us at least three zeros in K, which is a direct contradiction to Claim 1.

2. $K(L) > 0$:
The Claims 1 and 3 imply $K(0) > 0$ and the existence of exactly two zeros of K. In turn Claim 2 yields $\gamma(0), \gamma(L) \in \Sigma_+$. Let $(x_0, 0) \in \mathbb{R}^2$ be the center of Σ. Since every integral curve of J is an euclidean circle (see e.g. [18, Thm. 4.11 and Thm. A.3]) and J satisfies Example 6.6, every center of it is of the form $(x_0, y) \in \mathbb{R}^2$. We discuss in detail the case $x_0 \geq -1$. Otherwise we have $x_0 < -1 < 1$ and the same arguments as below can be applied around $\gamma(L)$. Let $s_0 > 0$ be the first zero of K. Now we like to show the existence of an $\tilde{s} \in [0, s_0)$ such that $K(\tilde{s})$ is a maximum. Lemma 6.4 would then give us $W_h(\gamma) > 3\pi > 8$, a contradiction.
Let us now assume such an $\tilde{s} \in [0, s_0)$ does not exist. We can extend γ as a solution of the differential equation (6.1) to $(-\infty, L]$ (see e.g. [18, Eq. (3.1)]. We can choose $s_1 < 0$ to be maximal with $K(s_1) = 0$. Then we can find an $\bar{s} \in (s_1, 0)$ with $K(\bar{s})$ being a maximum. Lemma 6.12 yields $\gamma(\bar{s}) \in B_+$. Let $I \subset \mathbb{H}^2$ be the integral curve of J starting at $\gamma(0) = (-1, \alpha_-)$. With $-1 \leq x_0$ and again Lemma 6.12 the derivative

$\dot\gamma(0) = (\alpha_-, 0)$ points strictly inward with respect to I. The intermediate value theorem gives us at least two intersecting points of $\gamma|_{(\bar s, s_0)}$ with I. This directly contradicts Lemma 6.11. Figure 15 describes the situation.

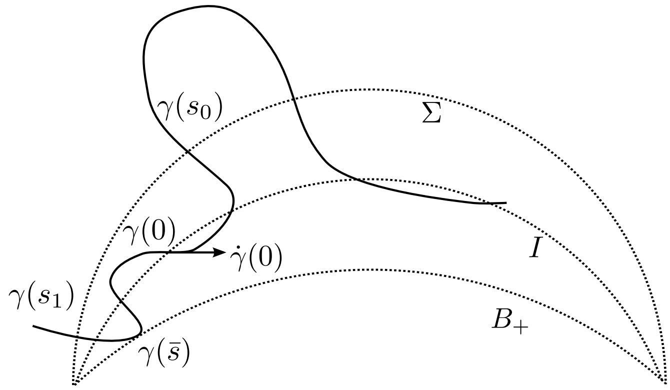

Figure 15: Existence of a maximum of K by contradiction.

Now we can switch to the other major case:

$$\dot\gamma^2(s^*) < 0.$$

With precisely the same arguments, we can give analogous statements to the Claims 2 and 3:

Claim 4. *From an euclidean viewpoint the geodesic Σ seperates $\mathbb{H}^2$ into an unbounded part Σ_- and a bounded part Σ_+. For all $s \in [0, L]$ we then have: $K(s) < 0$, iff $\gamma(s) \in \Sigma_-$. On the other hand $\gamma(s) \in \Sigma_+$, iff $K(s) > 0$.*
We also have $K(s^*) \leq 0$.

Claim 5. *There exists an $s_0 \in (0, L)$ with $K(s_0) = 0$ and in which the geodesic curvature changes sign from negative to positive. This point also satisfies $s^* \leq s_0$ and $\forall s \in (s^*, s_0)$ we have $\dot\gamma^1(s) \neq 0$.*
If $K(s^*) < 0$, there exists exactly one $s_* \in [s_0, L)$ with $\dot\gamma^1(s_*) = 0$ and $K(s_*) \geq 0$.
If on the other hand $K(s^*) = 0$ we do not have another point $s \in [0, L] \setminus \{s^*\}$ with $\dot\gamma^1(s) = 0$.

As in the first major case we identify without loss of generality $s_* := s^*$, if $K(s^*) = 0$. Again we have to distinguish cases by the sign of the geodesic curvature:

1. $K(0) < 0$: Here we need to consider two subcases:

   (a) $K(L) > 0$:
   We will show $\alpha_- > \alpha_+$, which would contradict our general assumption from the beginning of the proof. Claim 1 and Claim 5 yield exactly one zero $s_0 \in [s^*, s_*]$ of the geodesic curvature K. If we reparameterise γ as a smooth graph $(\cdot, u(\cdot))$ at $\gamma(0)$, this u would be strictly concave. This is due to $K(s) \leq 0$ for all $s \in [0, s^*]$ and equation (7.1). We then have $\gamma^2(s_0) < \gamma^2(0)$, since Lemma 7.3 yields

$\forall s \in [s^*, s_*]\ \dot\gamma^2(s) < 0$.
If we assume $\alpha_+ > \gamma^2(s_0)$, we would find an $\hat s \in (s_0, L)$ with $\dot\gamma^1(\hat s) = 0$ and $\dot\gamma^2(\hat s) > 0$, because $\gamma(L) \in \Sigma_+$ and $\dot\gamma^1(L) > 0$. Figure 16 explains the situation. Lemma 4.1 finally yields a contradiction.

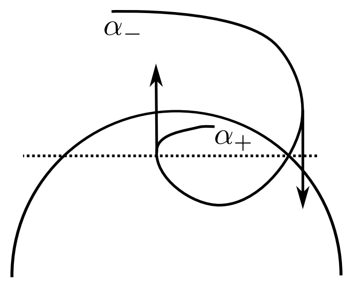

Figure 16: $W_h(\gamma) > 8$.

(b) $K(L) < 0$:
Claim 1 and Claim 5 yield exactly two zeros $s_0 < s_1$ of K with $s_0 \in [s^*, s_*]$ and $s_* < s_1$. Our aim is to construct a suitable comparison function to show that γ cannot be a minimiser for the Willmore energy. As in case $\dot\gamma(s^*) > 0$, subcase $K(L) < 0$, we define a family of catenoids $cat_s : \mathbb{R} \to \mathbb{H}^2$, $s \in [0, s^*)$, such that $cat_s(0)$ is tangent to $\gamma(s)$. Additionally we parameterise every catenoid by hyperbolic arclength.
At first we have to show that γ meets cat_0 only at $\gamma(0)$: As shown in case $\dot\gamma^2(s^*) < 0$, subcase $K(L) > 0$, we have for all $s \in (0, s_1]$ $\gamma^2(s) < \alpha_-$. Since $cat_0^2(t)$ is monotonically increasing for $t > 0$, $\gamma|_{(0,s_1]}$ and $cat_0(\mathbb{R})$ do not meet. On the other intervall $[s_1, L]$ the curvature satisfies $K \le 0$. Since p_{1,α_+} is a geodesic circle, the boundary datum $\dot\gamma^2(L) = 0$ yields for all $s \in [s_1, L]$ $p_{1,\alpha_+}(\gamma(s)) \ge \gamma^2(s)$. With Assumption 1.2 γ meets cat_0 only in $\gamma(0)$. Figure 17 gives a sketch of the situation.

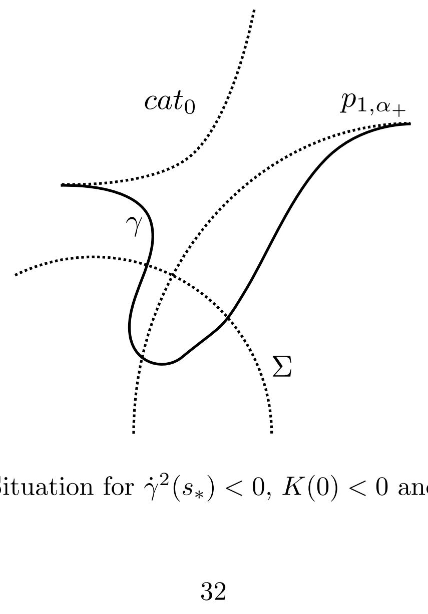

Figure 17: Situation for $\dot\gamma^2(s_*) < 0$, $K(0) < 0$ and $K(L) < 0$.

For $s \nearrow s^*$ the vertex of cat_s approaches zero. Then the boundary data for γ yield an $\bar{s} \in (0, s^*)$ and an $\tilde{s} \in (\bar{s}, L)$, such that $cat_{\bar{s}}$ touches γ at $\gamma(\tilde{s})$.
Before we can define a comparison curve, we have to show $\dot{\gamma}^1(\tilde{s}) > 0$. Since $\forall s \in [s^*, s_*]$ we have $\gamma^1(s) \leq \gamma^1(s^*)$ and $\gamma^2(s) \leq \gamma(s^*)$, the catenoid $cat_{\bar{s}}$ does not intersect $\gamma|_{[s^*, s_*]}$. Claim 5 on the other hand shows that γ can be reparameterised as a graph on $[0, L] \setminus [s^*, s_*]$. This yields $\dot{\gamma}^1(\tilde{s}) > 0$.
The resulting situation is sketched in Figure 18.

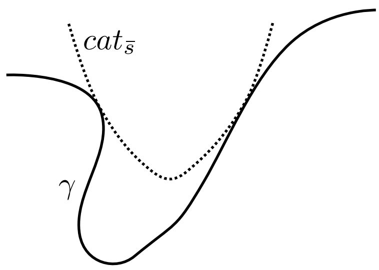

Figure 18: Inserting a catenoid.

Let us denote with $\tilde{L} > 0$ the point with $cat_{\bar{s}}(\tilde{L}) = \gamma(\tilde{s})$. We can now define the desired comparison curve:

$$v(s) = \begin{cases} \gamma(s), & s \in [0, \bar{s}) \\ cat_{\bar{s}}(s - \bar{s}), & s \in [\bar{s}, \bar{s} + \tilde{L}) \\ \gamma(s - (\bar{s} + \tilde{L}) + \tilde{s}), & s \in [\bar{s} + \tilde{L}, \bar{s} + \tilde{L} + L - \tilde{s}]. \end{cases} \tag{7.3}$$

Since catenoids are minimal surfaces their associated mean curvature vanishes. This means their Willmore energy is zero. This in turn yields $W_e(S(v)) < W_e(S(\gamma))$. Equality cannot arise, since it would mean that γ is a catenoid. This is not possible due to the boundary data.

2. $K(0) > 0$:
Claim 5 and Claim 1 yield exactly two zeros of K. This gives $K(L) > 0$. By Claim 4 the proof is from here on out completely analogous to $\dot{\gamma}(s^*) > 0$, subcase $K(L) > 0$.

□

A Estimates for the infimum of the Willmore energy

Here we provide an estimate, which shows that the elastic energy is not uniformely bounded for (α_-, α_+) in the class of graphs:

Lemma A.1. *If $\alpha_+ > \alpha_- + 1$, then for any $u \in C^{1,1}([-1, 1], (0, \infty))$ satisfying $u(-1) = \alpha_-$, $u(1) = \alpha_+$ and $u'(\pm 1) = 0$ we have*

$$W_h(u) \geq \frac{\alpha_+ - 1}{10}.$$

Proof. Equation (2.14) yields

$$\begin{aligned}
W_h(u) &= \int_{-1}^{1} \frac{u''(x)^2 u(x)}{(1+u'(x)^2)^{\frac{5}{2}}} + \frac{1}{u(x)\sqrt{1+u'(x)^2}}\,dx \\
&\geq \int_{-1}^{1} \frac{u''(x)^2 u(x)}{(1+u'(x)^2)^{\frac{5}{2}}}\,dx \\
&= \int_{-1}^{1} u(x)\sqrt{1+(u'(x))^2}\left(\frac{u''(x)}{(1+(u'(x))^2)^{\frac{3}{2}}}\right)^2 dx
\end{aligned}$$

We may now choose $x_1 \in (-1,1)$ such that $u'(x_1) \geq \frac{1}{2}$ and $u\big|_{[x_1,1]} \geq \alpha_+ - 1$. We conclude:

$$\begin{aligned}
W_h(u) &\geq \int_{-1}^{1} u(x)\sqrt{1+(u'(x))^2}\left(\frac{u''(x)}{(1+(u'(x))^2)^{\frac{3}{2}}}\right)^2 dx \\
&\geq \int_{x_1}^{1} u(x)\sqrt{1+(u'(x))^2}\left(\frac{u''(x)}{(1+(u'(x))^2)^{\frac{3}{2}}}\right)^2 dx \\
&\geq (\alpha_+ - 1)\int_{x_1}^{1} \left(\frac{u''(x)}{(1+(u'(x))^2)^{\frac{3}{2}}}\right)^2 dx \\
&\geq \frac{\alpha_+ - 1}{|1-x_1|}\left(\int_{x_1}^{1} \frac{u''(x)}{(1+(u'(x))^2)^{\frac{3}{2}}}\,dx\right)^2 \\
&= \frac{\alpha_+ - 1}{1-x_1}\left(\frac{u'(x_1)}{\sqrt{1+(u'(x_1))^2}}\right)^2 \geq \frac{\alpha_+ - 1}{2}\left(\frac{\frac{1}{2}}{\sqrt{1+\frac{1}{4}}}\right)^2 \geq \frac{\alpha_+ - 1}{10}.
\end{aligned}$$

□

If α_+ tends to ∞ the elastic energy of graphs will tend to ∞ as well.

In contrast to this we shall prove boundedness of the elastic energy in the class of curves. For this we need a lemma concerning circles first.

Lemma A.2 (see e.g. [18], Example 4.7)**.** *Let $r > 0$ and $M \in \mathbb{R}^2$ define an euclidean circle by*

$$C_{r,M}(t) := r(\sin t, \cos t) + (M^1, M^2).$$

If we further assume $C_{r,M}(\mathbb{R}) \cap \mathbb{H}^2 \neq \emptyset$ the geodesic curvature is here

$$\kappa[C_{r,M}] = -\frac{M^2}{r}.$$

In contrast to Lemma A.1 the following lemma shows that in the class of curves, the elastic energy is bounded for small α_- and big α_+:

Lemma A.3. *Let $(\alpha_-, \alpha_+) \in (0,\infty)^2$ satisfy*

$$\frac{\alpha_+}{\sqrt{2}+1} + \frac{\alpha_-}{\sqrt{2}-1} \geq 2 \text{ and } \alpha_+ > \alpha_-(\sqrt{2}+1)^4,$$

then

$$W^e_{\alpha_-,\alpha_+} < 2\pi^2.$$

Proof. Thanks to (2.11) we will show $W^h_{\alpha_-,\alpha_+} = \inf\{W_h(v) : v \in M_{\alpha_-,\alpha_+}\} < 4\pi$. Let us define two circles:

$$C_{\alpha_-} := C_{\frac{\alpha_-}{\sqrt{2}-1},\left(-1,\alpha_- + \frac{\alpha_-}{\sqrt{2}-1}\right)} \text{ and } C_{\alpha_+} := C_{\frac{\alpha_+}{\sqrt{2}+1},\left(1,\alpha_+ - \frac{\alpha_+}{\sqrt{2}+1}\right)}$$

Lemma A.2 yields

$$\kappa[C_{\alpha_\pm}] = -\sqrt{2}$$

and therefore the elastic energy of one half of a circle is with the help of maple$^{\mathrm{TM}}$

$$\int_0^\pi (\kappa[C_{\alpha_-}])^2(t)\,ds(t) = \int_0^\pi \frac{2\frac{\alpha_-}{\sqrt{2}-1}}{\frac{\alpha_-}{\sqrt{2}-1}\cos(t) + \alpha_- + \frac{\alpha_-}{\sqrt{2}-1}}\,dt = 2\pi.$$

Analogously we obtain to $\int_{-\pi}^0 (\kappa[C_{\alpha_+}])^2(t)\,ds(t) = 2\pi$.
Due to $\alpha_+ > \alpha_-(\sqrt{2}+1)^4$, C_{α_-} and C_{α_+} do not intersect, since this inequality is equivalent to

$$\alpha_+ - 2\frac{\alpha_+}{\sqrt{2}+1} > \alpha_- + 2\frac{\alpha_-}{\sqrt{2}-1}.$$

On the other hand the line $G_{\frac{\pi}{2}}$, which is parallel to the y-axis and starts at $C_{\alpha_-}(\frac{\pi}{2})$, intersects $C_{\alpha_+}\big|_{[-\pi,0]}$ because $\frac{\alpha_+}{\sqrt{2}+1} + \frac{\alpha_-}{\sqrt{2}-1} \geq 2$. We can now define a family of geodesics (upper half circles) G_t, which are tangent to C_{α_-} at $C_{\alpha_-}(t)$. Since G_0 does not intersect C_{α_+} we find a $t_0 \in (0, \frac{\pi}{2})$ such that G_{t_0} touches C_{α_+}. As in Lemma 7.4 we obtain a comparison curve v with

$$W_h(v) < 4\pi$$

Figure 19 sketches the situation. $\square$

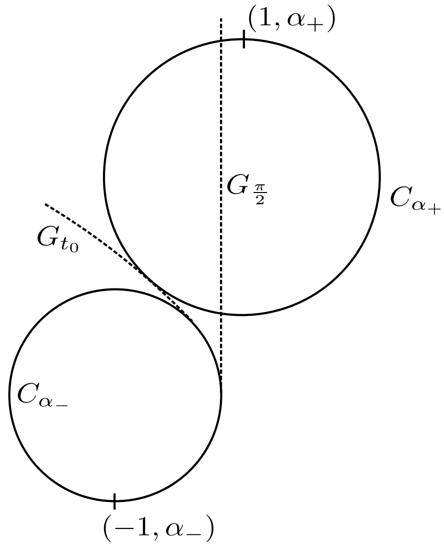

Figure 19: A comparison curve for small α_- and big α_+.

Remark A.4. The choice of the circles C_{α_-} and C_{α_+} in Lemma A.3 is not accidental. They are part of the Clifford torus and therefore minimal in the class of possible circles for the construction made above (see e.g. [26], Thm 4.1, or the proof of the Willmore conjecture [30]).

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Sascha Eichmann
Fakultät für Mathematik, Otto-von-Guericke-Universität, Postfach 4120, 39016 Magdeburg, Germany
E-mail: sascha.eichmann@ovgu.de

Hans-Christoph Grunau
Fakultät für Mathematik, Otto-von-Guericke-Universität, Postfach 4120, 39016 Magdeburg, Germany
E-mail: hans-christoph.grunau@ovgu.de