Stability and symmetry in the Navier problem for the one-dimensional Willmore equation

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Abstract

We consider the one-dimensional Willmore equation subject to Navier boundary conditions, i.e. the position and the curvature are prescribed on the boundary. In a previous work, explicit symmetric solutions to symmetric data have been constructed. Within a certain range of boundary curvatures one has precisely two symmetric solutions while for boundary curvatures outside the closure of this range there are none. The solutions are ordered; one is "small", the other "large". In the first part of this paper we address the stability problem and show that the small solution is (linearized) stable in the whole open range of admissible boundary curvatures, while the large one is unstable and has Morse index 1. A second goal is to investigate whether the small solution is minimal for the corresponding Willmore functional. It turns out that for a certain subrange of admissible boundary curvatures the small solution is the unique minimum, while for curvatures outside that range the minimum is not attained. As a by–product of our argument we show that for any admissible function there exists a symmetric function with smaller Willmore energy.

1 Introduction

Recently, Willmore surfaces (see[W]) and the related flow attracted quite some attraction, see e.g. [BK, KS1, KS2, KS3, MS, Sn, St], [DD] for numerical studies and [P, DKS] for elastic curves, which are the one-dimensional analoga. The mentioned work is concerned with closed surfaces and curves while only very few results concerning boundary value problems are available. Quite recently, Schätzle [Sch] considered Willmore surfaces with boundary, which are subject to the constraint to be submanifolds of $\mathbb{S}^n$ and which satisfy Dirichlet type boundary conditions.

In order to gain some more insight in general boundary conditions for the "free" Willmore equation, in [DG] we had a look at the one-dimensional case, where in some situations, almost explicit solutions can be found for suitable boundary value problems. For further background information and references, see [DG] and also [Nit]. In [DG], we were interested in Willmore graphs and studied among others the *Navier boundary value problem* with symmetric data $\alpha \in \mathbb{R}$ for the *one-dimensional Willmore equation*:

$$\begin{cases} \frac{1}{\sqrt{1+u'(x)^2}} \frac{d}{dx}\left(\frac{\kappa'(x)}{\sqrt{1+u'(x)^2}}\right) + \frac{1}{2}\kappa^3(x) = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0, \quad \kappa(0) = \kappa(1) = -\alpha. \end{cases} \tag{1}$$

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Here

$$\kappa(x) = \frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right) = \frac{u''(x)}{(1+u'(x)^2)^{3/2}} \tag{2}$$

denotes the curvature of the graph of u at the point $(x, u(x))$. Solutions of (1) are critical points of the modified one-dimensional Willmore functional

$$\tilde{W}_\alpha(u) = \int_{\mathrm{graph}(u)} \left(\kappa(x)^2 + 2\alpha\kappa(x)\right) \, ds(x) = \int_0^1 \left(\kappa(x)^2 + 2\alpha\kappa(x)\right) \sqrt{1+u'(x)^2}\, dx, \tag{3}$$

with $u \in H^2(0,1) \cap H_0^1(0,1)$. The boundary conditions $u(0) = u(1) = 0$ are formulated by working in the space H_0^1, while the curvature boundary conditions $\kappa(0) = \kappa(1) = -\alpha$ arise as natural boundary conditions since also the admissible testing functions only have to be in $H^2 \cap H_0^1$. By reflection it is sufficient to consider

$$\alpha \geq 0.$$

As for symmetric solutions of (1), in [DG], we proved the following result:

Proposition 1 ([DG, Theorem 1])**.** *There exists $\alpha_{\max} = 1.343799725\ldots$ such that for $0 < \alpha < \alpha_{\max}$, the Navier boundary value problem (1) has precisely two smooth (graph) solutions u in the class of smooth functions that are symmetric around $x = \frac{1}{2}$. If $\alpha = \alpha_{\max}$ one has precisely one such solution, for $\alpha = 0$ one only has the trivial solution and no such solutions exist for $\alpha > \alpha_{\max}$.*

Both solutions are positive and one of these solutions is larger than the other. The small solutions are ordered with respect to α while the large ones become smaller for increasing α, see Figure 1. For the bifurcation diagram, see Figure 2.

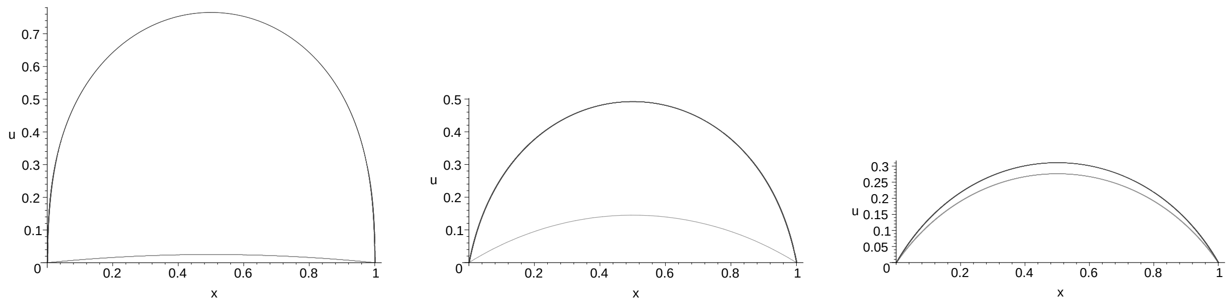

Figure 1: Solutions of the Navier boundary value problem (1) for $\alpha = 0.2$, $\alpha = 1$ and $\alpha = 1.34$ (left to right)

It is an obvious conjecture that for $0 \leq \alpha < \alpha_{\max}$ the small solutions are (linearized) stable. This property was left open in [DG], and to prove it is the first goal of this paper.

Theorem 1. *Assume that $0 \leq \alpha < \alpha_{\max}$, and that u is the symmetric* small solution *of the Navier boundary value problem (1). Then, this solution is linearized stable, i.e. the spectrum of the (self adjoint) linearization of (1) around u is contained in $(0, \infty)$.*

Observing that these linearizations are the second variation of the functional $\tilde{W}_\alpha$, this proves that the small solution is a local minimum of the functional $\tilde{W}_\alpha$ in $H^2 \cap H_0^1(0,1)$. Furthermore, we will show that on $0 < \alpha < \alpha_{\max}$, the large solutions are unstable. More precisely we know:

Theorem 2. *Assume that $0 < \alpha < \alpha_{\max}$, and that u is the symmetric* large solution *of the Navier boundary value problem (1). Then, this solution is unstable and has Morse index 1, i.e. one eigenvalue of the (self adjoint) linearization of (1) is negative while the remaining spectrum is contained in $(0, \infty)$.*

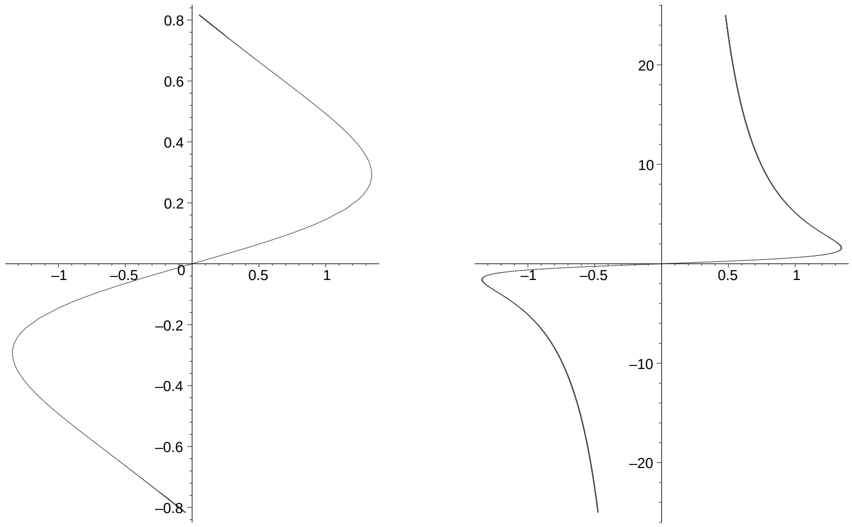

Figure 2: Bifurcation diagram for (1): The extremals value of the solution $u(1/2)$ (left) and of the derivative $u'(0)$ (right) plotted over α

We emphasize that no symmetry assumptions are made in the discussion of the linearizations of (1).

A further important question is whether the small solutions are not only a local but also a global minimum of the functional $\tilde{W}_\alpha$.

Theorem 3. *There exists* $\alpha^* = 1.132372323\ldots \in (0, \alpha_{\max})$ *such that for* $0 \le \alpha \le \alpha^*$ *the small solution* u *is the unique global minimum of the functional* $\tilde{W}_\alpha$ *in the class* $H^2 \cap H^1_0(0,1)$. *If* $\alpha^* < \alpha \le \alpha_{\max}$ *the infimum of* $\tilde{W}_\alpha$ *in* $H^2 \cap H^1_0(0,1)$ *is not attained and in that case*

$$\inf_{v \in H^2 \cap H^1_0(0,1)} \tilde{W}_\alpha(v) = \left(\int_{\mathbb{R}} \frac{1}{(1+\tau^2)^{5/4}} \, d\tau \right)^2 - 2\alpha\pi.$$

The main idea of proving Theorem 3 consists in reducing the minimization of $\tilde{W}_\alpha$ over $H^2 \cap H^1_0(0,1)$ to the minimization of a function of two variables. As a by–product of this approach we shall see that the infimum of the Willmore energy in $H^2 \cap H^1_0(0,1)$ coincides with the infimum in the subspace M of functions that are symmetric around $x = 1/2$, i.e for every function in $H^2 \cap H^1_0(0,1)$, there exists a symmetric function with the same or smaller Willmore energy. This is remarkable since we deal with a fourth order problem and the well–known symmetrization procedures do not apply.

Theorem 4. *Let* M *be the class of functions in* $H^2 \cap H^1_0(0,1)$, *which are symmetric around* $x = 1/2$. *Then we have*

$$\inf_{v \in H^2 \cap H^1_0(0,1)} \tilde{W}_\alpha(v) = \inf_{v \in M} \tilde{W}_\alpha(v).$$

2 Linearized stability

To prove Theorem 1 we describe in more detail how the symmetric solutions to (1) were obtained in [DG].

In what follows, the function

$$G:\mathbb{R}\to\left(-\frac{c_0}{2},\frac{c_0}{2}\right),\qquad G(s):=\int_0^s\frac{1}{(1+\tau^2)^{5/4}}\,d\tau, \tag{4}$$

$$c_0=\int_{\mathbb{R}}\frac{1}{(1+\tau^2)^{5/4}}\,d\tau=\mathcal{B}\left(\frac12,\frac34\right)=2.396280469\ldots,$$

plays a crucial role. It is straightforward to see that G is strictly increasing, bijective with $G'(s)>0$. So, also the inverse function

$$G^{-1}:\left(-\frac{c_0}{2},\frac{c_0}{2}\right)\to\mathbb{R} \tag{5}$$

is strictly increasing, bijective and smooth with $G^{-1}(0)=0$.

Lemma 1 ([DG, Lemma 4])**.** *Let $u\in C^4([0,1])$ be a function symmetric around $x=1/2$. Then u solves the Willmore equation in (1) iff there exists $c\in(-c_0,c_0)$ such that*

$$\forall x\in[0,1]:\qquad u'(x)=G^{-1}\left(\frac{c}{2}-cx\right). \tag{6}$$

For the curvature, one has that

$$\kappa(x)=-\frac{c}{\sqrt[4]{1+G^{-1}\left(\frac{c}{2}-cx\right)^2}}. \tag{7}$$

Moreover, if we additionally assume that $u(0)=u(1)=0$, then one has

$$u(x)=\frac{2}{c\sqrt[4]{1+G^{-1}\left(\frac{c}{2}-cx\right)^2}}-\frac{2}{c\sqrt[4]{1+G^{-1}\left(\frac{c}{2}\right)^2}}\quad(c\neq0). \tag{8}$$

In order to solve the Navier boundary value problem (1), in [DG], we had to study the function

$$h:(-c_0,c_0)\to\mathbb{R},\qquad h(c)=\frac{c}{\sqrt[4]{1+G^{-1}\left(\frac{c}{2}\right)^2}}. \tag{9}$$

The range of h is precisely the set of α, for which the Navier boundary value problem (1) has a smooth symmetric graph solution. The number of solutions c of the equation $\alpha=h(c)$ is the number of such solutions of the boundary value problem.

Lemma 2 ([DG, Lemma 6])**.** *We have $h>0$ in $(0,c_0)$, $h<0$ in $(-c_0,0)$, $\lim_{c\nearrow c_0}h(c)=\lim_{c\searrow -c_0}h(c)=0$. The function h is odd and has precisely one local maximum in $c_{\max}=1.840428142\ldots$ and one local minimum in $c_{\min}=-c_{\max}$. The corresponding value is $\alpha_{\max}=h(c_{\max})=1.343799725\ldots$.*

The small solutions correspond precisely to $c\in(0,c_{\max})$, the large ones to $c\in(c_{\max},c_0)$. Let us fix $c\in(0,c_0)$ with corresponding $\alpha=h(c)$ and solution u given by (8). First we have to calculate the linearization of (1) around u, i.e. the second variation of the modified Willmore functional $\tilde{W}_\alpha$ in u:

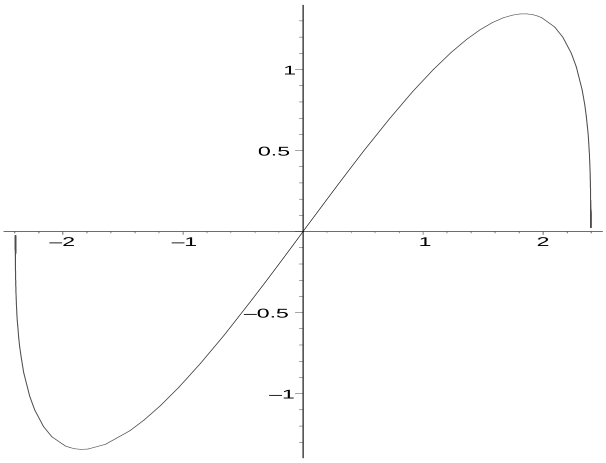

Figure 3: The function $c \mapsto h(c)$

Lemma 3. *We have*

$$\begin{aligned} D^2\tilde W_\alpha(u)(\varphi,\eta) &= 2\int_0^1 \frac{\varphi''(x)\eta''(x)}{(1+u'(x)^2)^{5/2}}\,dx + 5\int_0^1 \frac{1-u'(x)^2}{(1+u'(x)^2)^{3/2}}\kappa(x)^2\varphi'(x)\eta'(x)\,dx \\ &\quad +6\alpha\left[\frac{u'(x)\varphi'(x)\eta'(x)}{(1+u'(x)^2)^2}\right]_0^1, \quad \varphi,\eta\in H^2\cap H_0^1(0,1). \end{aligned}$$

Proof. According to [DG, Lemma 2 and Corollary 1], the first variation of $\tilde W_\alpha(u)$ is given by

$$\begin{aligned} D\tilde W_\alpha(u)(\varphi) &= 2\int_0^1 \frac{u''(x)\varphi''(x)}{(1+u'(x)^2)^{5/2}}\,dx - 5\int_0^1 \frac{u'(x)u''(x)^2\varphi'(x)}{(1+u'(x)^2)^{7/2}}\,dx \\ &\quad +2\alpha\left[\frac{\varphi'(x)}{1+u'(x)^2}\right]_0^1, \qquad \varphi\in H^2\cap H_0^1(0,1). \end{aligned}$$

In order to obtain the second derivative, we consider also $\eta\in H^2\cap H_0^1(0,1)$ and differentiate the previous expression with respect to this direction:

$$\begin{aligned} D^2\tilde W_\alpha(u)(\varphi,\eta) &= \frac{d}{dt}D\tilde W_\alpha(u+t\eta)(\varphi)|_{t=0} \\ &= 2\int_0^1 \frac{\varphi''(x)\eta''(x)}{(1+u'(x)^2)^{5/2}}\,dx - 10\int_0^1 \frac{u'(x)u''(x)\varphi''(x)\eta'(x)}{(1+u'(x)^2)^{7/2}}\,dx \\ &\quad -10\int_0^1 \frac{u'(x)u''(x)\varphi'(x)\eta''(x)}{(1+u'(x)^2)^{7/2}}\,dx - 5\int_0^1 \frac{u''(x)^2\varphi'(x)\eta'(x)}{(1+u'(x)^2)^{7/2}}\,dx \\ &\quad +35\int_0^1 \frac{u'(x)^2u''(x)^2\varphi'(x)\eta'(x)}{(1+u'(x)^2)^{9/2}}\,dx - 4\alpha\left[\frac{u'(x)\varphi'(x)\eta'(x)}{(1+u'(x)^2)^2}\right]_0^1 \\ &= 2\int_0^1 \frac{\varphi''(x)\eta''(x)}{(1+u'(x)^2)^{5/2}}\,dx - 5\int_0^1 \frac{\kappa(x)^2\varphi'(x)\eta'(x)}{\sqrt{1+u'(x)^2}}\,dx \\ &\quad -10\int_0^1 \kappa(x)\cdot\frac{u'(x)}{\sqrt{1+u'(x)^2}}\cdot\frac{1}{(1+u'(x)^2)^{3/2}}\cdot\frac{d}{dx}\big(\varphi'(x)\eta'(x)\big)\,dx \\ &\quad +35\int_0^1 \frac{u'(x)^2\kappa(x)^2\varphi'(x)\eta'(x)}{(1+u'(x)^2)^{3/2}}\,dx - 4\alpha\left[\frac{u'(x)\varphi'(x)\eta'(x)}{(1+u'(x)^2)^2}\right]_0^1. \end{aligned}$$

To proceed further we would like to integrate the third term by parts. Here we will exploit that u is a solution to (1). In particular, u is smooth and satisfies the Navier boundary data $\kappa(x) = -\alpha, x \in \{0,1\}$.

$$\begin{aligned} D^2\tilde{W}_\alpha(u)(\varphi,\eta) &= 2\int_0^1 \frac{\varphi''(x)\eta''(x)}{(1+u'(x)^2)^{5/2}}\,dx - 5\int_0^1 \frac{\kappa(x)^2\varphi'(x)\eta'(x)}{\sqrt{1+u'(x)^2}}\,dx \\ &\quad +35\int_0^1 \frac{u'(x)^2\kappa(x)^2\varphi'(x)\eta'(x)}{(1+u'(x)^2)^{3/2}}\,dx - 4\alpha\left[\frac{u'(x)\varphi'(x)\eta'(x)}{(1+u'(x)^2)^2}\right]_0^1 \\ &\quad -10\left[\kappa(x)\frac{u'(x)\varphi'(x)\eta'(x)}{(1+u'(x)^2)^2}\right]_0^1 + 10\int_0^1 \frac{\kappa'(x)u'(x)\varphi'(x)\eta'(x)}{(1+u'(x)^2)^2}\,dx \\ &\quad +10\int_0^1 \frac{\kappa(x)^2\varphi'(x)\eta'(x)}{(1+u'(x)^2)^{3/2}}\,dx - 30\int_0^1 \frac{\kappa(x)u'(x)^2u''(x)\varphi'(x)\eta'(x)}{(1+u'(x)^2)^3}\,dx \\ &= 2\int_0^1 \frac{\varphi''(x)\eta''(x)}{(1+u'(x)^2)^{5/2}}\,dx - 5\int_0^1 \frac{\kappa(x)^2\varphi'(x)\eta'(x)}{\sqrt{1+u'(x)^2}}\,dx \\ &\quad +5\int_0^1 \frac{u'(x)^2\kappa(x)^2\varphi'(x)\eta'(x)}{(1+u'(x)^2)^{3/2}}\,dx + 6\alpha\left[\frac{u'(x)\varphi'(x)\eta'(x)}{(1+u'(x)^2)^2}\right]_0^1 \\ &\quad +10\int_0^1 \frac{\kappa'(x)u'(x)\varphi'(x)\eta'(x)}{(1+u'(x)^2)^2}\,dx + 10\int_0^1 \frac{\kappa(x)^2\varphi'(x)\eta'(x)}{(1+u'(x)^2)^{3/2}}\,dx. \end{aligned}$$

We infer from (6) and (7) that

$$\forall x \in [0,1]: \qquad \kappa(x)\left(1+u'(x)^2\right)^{1/4} = -c$$

and hence

$$\forall x \in [0,1]: \quad \kappa'(x)\left(1+u'(x)^2\right)^{1/4} + \frac{1}{2}u'(x)\kappa(x)^2\left(1+u'(x)^2\right)^{3/4} = 0.$$

Consequently,

$$\begin{aligned} D^2\tilde{W}_\alpha(u)(\varphi,\eta) &= 2\int_0^1 \frac{\varphi''(x)\eta''(x)}{(1+u'(x)^2)^{5/2}}\,dx - 5\int_0^1 \frac{\kappa(x)^2\varphi'(x)\eta'(x)}{\sqrt{1+u'(x)^2}}\,dx \\ &\quad +6\alpha\left[\frac{u'(x)\varphi'(x)\eta'(x)}{(1+u'(x)^2)^2}\right]_0^1 + 10\int_0^1 \frac{\kappa(x)^2\varphi'(x)\eta'(x)}{(1+u'(x)^2)^{3/2}}\,dx. \end{aligned}$$

This proves our claim. ☐

Looking at η as a test function and plugging in the representation of u in terms of c according to Lemmas 1 and 2, the linearization of (1) around u reads as follows:

$$\begin{cases} \left(\frac{\varphi''(x)}{\left(1+G^{-1}\left(\frac{c}{2}-cx\right)^2\right)^{5/2}}\right)'' + \frac{5}{2}c^2\left(\frac{G^{-1}\left(\frac{c}{2}-cx\right)^2-1}{\left(1+G^{-1}\left(\frac{c}{2}-cx\right)^2\right)^2}\varphi'(x)\right)' = 0, \quad x \in (0,1), \\ \varphi(0) = \varphi(1) = 0, \\ \frac{\varphi''(0)}{\left(1+G^{-1}\left(\frac{c}{2}\right)^2\right)^{5/2}} + 3\frac{cG^{-1}\left(\frac{c}{2}\right)\varphi'(0)}{\left(1+G^{-1}\left(\frac{c}{2}\right)^2\right)^{9/4}} = 0, \quad \frac{\varphi''(1)}{\left(1+G^{-1}\left(\frac{c}{2}\right)^2\right)^{5/2}} - 3\frac{cG^{-1}\left(\frac{c}{2}\right)\varphi'(1)}{\left(1+G^{-1}\left(\frac{c}{2}\right)^2\right)^{9/4}} = 0. \end{cases} \tag{10}$$

For $c = 0$, the small solution of (1) is $u(x) \equiv 0$, and $D^2\tilde{W}_0(u)(\varphi,\varphi) = \int_0^1 \varphi''(x)^2\,dx$ is positive definite in $H^2 \cap H_0^1(0,1)$ with respect to the $L^2(0,1)$-norm. Since the eigenvalues of the linearization depend smoothly on u and u depends smoothly on c, $D^2\tilde{W}_\alpha(u)(\varphi,\varphi)$ remains positive definite for c increasing from 0 as long as (10) only has the trivial solution $\varphi(x)\equiv 0$.

We assume that (10) has a solution φ and put

$$\chi(x) := \varphi'(x).$$

Then, there exists a constant $A\in\mathbb{R}$ such that χ solves the second order differential equation

$$\left(\frac{\chi'(x)}{\left(1+G^{-1}\left(\frac{c}{2}-cx\right)^2\right)^{5/2}}\right)' + \frac{5}{2}c^2\left(\frac{G^{-1}\left(\frac{c}{2}-cx\right)^2-1}{\left(1+G^{-1}\left(\frac{c}{2}-cx\right)^2\right)^2}\chi\right) = c^2A.$$

We introduce more suitable variables:

$$\begin{aligned}
y &= G^{-1}\left(\frac{c}{2}-cx\right)\in\left[-G^{-1}\left(\frac{c}{2}\right),G^{-1}\left(\frac{c}{2}\right)\right], \quad x=\frac{1}{2}-\frac{G(y)}{c};\\
\psi(y) &:= \chi(x)=\chi\left(\frac{1}{2}-\frac{G(y)}{c}\right) \qquad \chi(x)=\psi\left(G^{-1}\left(\frac{c}{2}-cx\right)\right);\\
\chi'(x) &= -c\left(1+G^{-1}\left(\frac{c}{2}-cx\right)^2\right)^{5/4}\psi'\left(G^{-1}\left(\frac{c}{2}-cx\right)\right)
\end{aligned}$$

and conclude that ψ solves the following boundary value problem:

$$\begin{cases}\psi''(y)-\frac{5y}{2(1+y^2)}\psi'(y)+\frac{5(y^2-1)}{2(1+y^2)^2}\psi(y)=A, \quad y\in(-y_0,y_0),\\ \psi'(-y_0)+\frac{3y_0}{1+y_0^2}\psi(-y_0)=0, \qquad \psi'(y_0)-\frac{3y_0}{1+y_0^2}\psi(y_0)=0.\end{cases} \tag{11}$$

Here, we denote

$$y_0 := G^{-1}\left(\frac{c}{2}\right). \tag{12}$$

To simplify the boundary conditions we make a last change of variables and put

$$\Phi(y) := \frac{\psi(y)}{(1+y^2)^{3/2}}, \qquad y\in[-y_0,y_0] \tag{13}$$

and finally come up with considering the following boundary value problem:

$$\begin{cases}(1+y^2)^{3/2}\Phi''(y)+\frac{7}{2}y(1+y^2)^{1/2}\Phi'(y)+\left(y^2+\frac{1}{2}\right)(1+y^2)^{-1/2}\Phi(y)=A, \quad y\in(-y_0,y_0)\\ \Phi'(-y_0)=\Phi'(y_0)=0.\end{cases} \tag{14}$$

We recall the definition of $G(y):=\int_0^y \frac{1}{(1+\tau^2)^{5/4}}\,d\tau$ and put

$$\Phi_0(y) := -2\frac{1}{\sqrt{1+y^2}}, \qquad \Phi_1(y) := \frac{1}{\sqrt[4]{1+y^2}}, \qquad \Phi_2(y) := \frac{G(y)}{\sqrt[4]{1+y^2}}. \tag{15}$$

Then, one directly verifies that the general solution of the differential equation in (14) is given by

$$\Phi(y) := A\cdot\Phi_0(y)+\gamma_1\cdot\Phi_1(y)+\gamma_2\cdot\Phi_2(y) \tag{16}$$

with $\gamma_1,\gamma_2\in\mathbb{R}$. Since $A\cdot\Phi_0(y)+\gamma_1\cdot\Phi_1(y)$ is even and $\gamma_2\cdot\Phi_2(y)$ is odd, the boundary conditions in (14) are equivalent to

$$A\cdot\Phi_0'(y_0)+\gamma_1\cdot\Phi_1'(y_0)=0 \text{ and } \gamma_2\cdot\Phi_2'(y_0)=0 \tag{17}$$

in turn being equivalent to

$$\gamma_1 = \frac{4A}{\sqrt[4]{1+y_0^2}} \text{ and } \left(\gamma_2 = 0 \text{ or } \Phi_2'(y_0) = 0\right). \tag{18}$$

A beautiful coincidence between these solutions and the functions involved in the proof of Theorem 1 can be observed, namely

$$\Phi_2(y) = \frac{1}{2}h(2G(y)), \qquad \Phi_2'(y) = \frac{h'(2G(y))}{(1+y^2)^{5/4}}. \tag{19}$$

With the help of these observations we are now ready to conclude the

Lemma 4. *For $c \in [0, c_0) \setminus \{c_{\max}\}$, the boundary value problem (10) only has the trivial solution $\varphi(x) \equiv 0$. For $c = c_{\max}$, it has a one dimensional null space which is spanned by*

$$\varphi(x) = \frac{1}{c}\int_{G^{-1}(\frac{c}{2}-cx)}^{G^{-1}(\frac{c}{2})} G(\eta)\,d\eta.$$

If $c = c_{\max}, \alpha = \alpha_{\max}$, instabilities will occur first from the corresponding solution u in direction of this function φ, see Figure 4.

Proof. The case $c = 0$ is obvious and we consider only $c \in (0, c_0)$. We denote

$$\tilde{\Phi}_1(y) := \Phi_0(y) + \frac{4}{\sqrt[4]{1+y_0^2}} \cdot \Phi_1(y) = \frac{2}{\sqrt[4]{1+y^2} \cdot \sqrt[4]{1+y_0^2}}\left(2 - \frac{\sqrt[4]{1+y_0^2}}{\sqrt[4]{1+y^2}}\right).$$

According to (16), we have to study

$$\Phi(y) = A\tilde{\Phi}_1(y) + \gamma_2\Phi_2(y)$$

with some suitable $A, \gamma_2 \in \mathbb{R}$. Let φ be the corresponding solution of (10) which is obtained from Φ by tracing back the changes of variables and integrating χ. We want to show first that necessarily $A = 0$ for any $c \in [0, c_0)$.

$$\begin{aligned}
0 &= \varphi(1) - \varphi(0) = \int_0^1 \chi(x)\,dx = \int_0^1 \psi\left(G^{-1}\left(\frac{c}{2} - cx\right)\right)\,dx \\
&= \frac{1}{c}\int_{-G^{-1}(\frac{c}{2})}^{G^{-1}(\frac{c}{2})} \psi(y)(1+y^2)^{-5/4}\,dy = \frac{1}{c}\int_{-G^{-1}(\frac{c}{2})}^{G^{-1}(\frac{c}{2})} \Phi(y)(1+y^2)^{1/4}\,dy \\
&= \frac{A}{c}\int_{-G^{-1}(\frac{c}{2})}^{G^{-1}(\frac{c}{2})} \tilde{\Phi}_1(y)(1+y^2)^{1/4}\,dy + \frac{\gamma_2}{c}\int_{-G^{-1}(\frac{c}{2})}^{G^{-1}(\frac{c}{2})} \Phi_2(y)(1+y^2)^{1/4}\,dy \\
&= \frac{A}{c}\int_{-G^{-1}(\frac{c}{2})}^{G^{-1}(\frac{c}{2})} \tilde{\Phi}_1(y)(1+y^2)^{1/4}\,dy
\end{aligned}$$

since Φ_2 is odd. Hence we may conclude that

$$\begin{aligned}
0 &= \frac{A}{c}\int_{-G^{-1}(\frac{c}{2})}^{G^{-1}(\frac{c}{2})} \tilde{\Phi}_1(y)(1+y^2)^{1/4}\,dy \\
&= \frac{2A}{c}\int_{-G^{-1}(\frac{c}{2})}^{G^{-1}(\frac{c}{2})} \left(\frac{2}{\left(1+G^{-1}\left(\frac{c}{2}\right)^2\right)^{1/4}} - \frac{1}{(1+y^2)^{1/4}}\right) dy \\
&= \frac{4A}{c}F\left(G^{-1}\left(\frac{c}{2}\right)\right),
\end{aligned}$$

where F is defined by

$$F(\eta) := \frac{2\eta}{(1+\eta^2)^{1/4}} - \int_0^\eta \frac{1}{(1+s^2)^{1/4}}\, ds.$$

Since $F(0) = 0$ and

$$\begin{aligned} F'(\eta) &= \frac{2}{(1+\eta^2)^{1/4}} - \frac{\eta^2}{(1+\eta^2)^{5/4}} - \frac{1}{(1+\eta^2)^{1/4}} \\ &= \frac{1}{(1+\eta^2)^{1/4}} - \frac{\eta^2}{(1+\eta^2)^{5/4}} = \frac{1}{(1+\eta^2)^{5/4}} > 0, \end{aligned}$$

we have

$$F\left(G^{-1}\left(\frac{c}{2}\right)\right) > 0.$$

As a consequence, $A = 0$ and hence $\gamma_1 = 0$ by (18) and we are left with considering $\gamma_2\Phi_2$. We have that $h'(c) > 0$ for $c \in (0, c_{\max})$ and $h'(c) < 0$ for $c \in (c_{\max}, c_0)$. By making use of

$$\Phi_2'(y) = \frac{h'(2G(y))}{(1+y^2)^{5/4}},$$

and the boundary condition $\gamma_2\Phi_2'(G^{-1}(c/2)) = 0$, we conclude that $\gamma_2 = 0$, provided $c \in (0, c_0) \setminus \{c_{\max}\}$. If $c = c_{\max}$, then Φ_2 is a nontrivial solution of (14). For the corresponding nontrivial solution φ of (10), making use of the boundary conditions $\varphi(0) = \varphi(1) = 0$ we obtain that

$$\varphi(x) = \gamma_2 \frac{c_{\max}}{2} \int_0^x \left(1 + G^{-1}(\frac{c_{\max}}{2} - c_{\max}\xi)^2\right)^{5/4} (1-2\xi)\, d\xi = \frac{\gamma_2}{c_{\max}} \int_{G^{-1}(\frac{c_{\max}}{2} - c_{\max}x)}^{G^{-1}(\frac{c_{\max}}{2})} G(\eta)\, d\eta.$$

□

The *proof of Theorem 1* is now immediate. By the preceding lemma we have that on $[0, c_{\max})$, 0 is not an eigenvalue of (10). Since $D^2\tilde{W}_0(u)(\varphi,\varphi)$ is positive definite in $H^2 \cap H_0^1(0,1)$ with respect to the $L^2(0,1)$-norm, by continuity, the same holds true for $D^2\tilde{W}_\alpha(u)(\varphi,\varphi)$ for $c \in [0, c_{\max})$, which is the stated linearized stability of the corresponding small solutions of (1). □

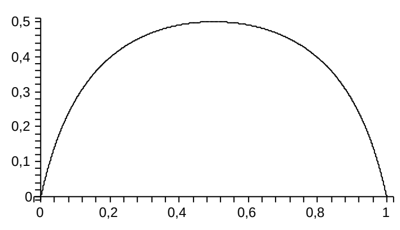

Figure 4: Profile of the unstable direction in $c = c_{\max}$

As an immediate consequence of Theorem 1 we obtain a global existence result for the geometric flow associated with (1), namely

$$V = -\kappa_{ss} - \frac{1}{2}\kappa^3 \quad \text{on } \Gamma(t).$$

Here, V denotes the upward normal velocity of the evolving graphs $\Gamma(t) = \{(x, v(x,t)) \,|\, x \in [0,1]\}$. The above evolution law then leads to the parabolic initial–boundary value problem (21) below. The principle of linearized stability as it was proved in great generality by Latushkin, Prüss and Schnaubelt [LPS, Proposition 16] can be applied to our situation and allows us to obtain global existence and asymptotic stability for initial data close to a small solution to (1).

Corollary 1. *Assume that* $c \in [0, c_{\max})$ *and let* $\alpha = h(c) = \frac{c}{\sqrt[4]{1+G^{-1}\left(\frac{c}{2}\right)^2}}$ *and*

$$u(x) = \frac{2}{c\sqrt[4]{1+G^{-1}\left(\frac{c}{2}-cx\right)^2}} - \frac{2}{c\sqrt[4]{1+G^{-1}\left(\frac{c}{2}\right)^2}} \tag{20}$$

be the corresponding small solution of (1). We fix some $p > 5$*. Then, there exist* $\delta, \rho, C > 0$ *such that for* $v_0 \in W^{4,p}(0,1)$ *with* $v_0(0) = v_0(1) = 0$, $\kappa_{v_0}(0) = \kappa_{v_0}(1) = -\alpha$ *and*

$$\|v_0 - u\|_{W^{4,p}(0,1)} \le \delta,$$

there exists a global solution $v \in L^p(0,\infty, W^{4,p}(0,1)) \cap W^{1,p}(0,\infty, L^p(0,1))$ *of the initial Navier boundary value problem*

$$\begin{cases} \frac{v_t(t,x)}{\sqrt{1+v_x(t,x)^2}} + \frac{1}{\sqrt{1+v_x(t,x)^2}}\frac{d}{dx}\left(\frac{\kappa_{v,x}(t,x)}{\sqrt{1+v_x(t,x)^2}}\right) + \frac{1}{2}\kappa_v^3(t,x) = 0, & (t,x) \in [0,\infty)\times[0,1], \\ v(t,0) = v(t,1) = 0, \quad \kappa_v(t,0) = \kappa_v(t,1) = -\alpha, & t \in [0,\infty), \\ v(0,x) = v_0(x), & x \in [0,1]. \end{cases} \tag{21}$$

One has exponential convergence towards the steady state u*:*

$$\|v(t,\,.\,) - u(\,.\,)\|_{W^{4,p}(0,1)} \le C\exp(-\rho t) \qquad (t \ge 1). \tag{22}$$

Remark 1. *With similar but simpler techniques and calculations one finds that the unique solution (cf. [DG, Theorem 2]) being symmetric around* $x = 1/2$ *of the Dirichlet problem*

$$\begin{cases} \frac{1}{\sqrt{1+u_x(x)^2}}\frac{d}{dx}\left(\frac{\kappa_x(x)}{\sqrt{1+u_x(x)^2}}\right) + \frac{1}{2}\kappa^3(t,x) = 0, & x \in [0,1], \\ u(0) = u(1) = 0, \quad u_x(0) = -u_x(1) = \beta, \end{cases} \tag{23}$$

$\beta \in \mathbb{R}$*, is (linearized) stable. Analogously, a global existence result follows for the following initial Dirichlet boundary value problem*

$$\begin{cases} \frac{v_t(t,x)}{\sqrt{1+v_x(t,x)^2}} + \frac{1}{\sqrt{1+v_x(t,x)^2}}\frac{d}{dx}\left(\frac{\kappa_{v,x}(t,x)}{\sqrt{1+v_x(t,x)^2}}\right) + \frac{1}{2}\kappa_v^3(t,x) = 0, & (t,x) \in [0,\infty)\times[0,1], \\ v(t,0) = v(t,1) = 0, \quad v_x(t,0) = -v_x(t,1) = \beta, & t \in [0,\infty), \\ v(0,x) = v_0(x), & x \in [0,1], \end{cases} \tag{24}$$

provided the initial datum v_0 *obeys the same boundary data and is sufficiently close to the stationary solution* u *of (23) with respect to the* $W^{4,p}$*-norm,* $(p > 5)$*.*

3 Morse index of the large solution

For $c \in (0, c_0)$ we consider as in (8)

$$u_c(x) = \frac{2}{c\sqrt[4]{1+G^{-1}\left(\frac{c}{2}-cx\right)^2}} - \frac{2}{c\sqrt[4]{1+G^{-1}\left(\frac{c}{2}\right)^2}}.$$

In order to prove Theorem 2 we have to show that exactly one eigenvalue of the quadratic form

$$\varphi \mapsto D^2\tilde{W}_\alpha(u_c)(\varphi,\varphi), \qquad \alpha = h(c)$$

passes through 0 when c passes through $c_{\max}$ and that for $c \in (c_{\max}, c_0)$, 0 is not an eigenvalue of $D^2\tilde{W}_\alpha(u_c)$, i.e. of (10). The latter was already done in Lemma 4. Moreover, its proof yields that is at most one eigenvalue, which crosses 0 in $c = c_{\max}$. It remains to show that for $c > c_{\max}$ and suitable $\varphi \in H^2 \cap H_0^1(0,1)$, one has that one has indeed $D^2\tilde{W}_\alpha(u_c)(\varphi,\varphi) < 0$. Making use of the same transformations and notations of Section 2 and restricting ourselves to symmetric φ we find:

$$\begin{aligned}
D^2\tilde{W}_\alpha(u_c)(\varphi,\varphi) &= 2\int_0^1 \frac{\chi'(x)^2}{\left(1+G^{-1}\left(\frac{c}{2}-cx\right)^2\right)^{5/2}}\,dx - 5c^2\int_0^1 \frac{G^{-1}\left(\frac{c}{2}-cx\right)^2-1}{\left(1+G^{-1}\left(\frac{c}{2}-cx\right)^2\right)^2}\chi(x)^2\,dx \\
&\quad -12h(c)\frac{G^{-1}\left(\frac{c}{2}\right)}{\left(1+G^{-1}\left(\frac{c}{2}\right)^2\right)^2}\chi(1)^2 \\
&= 2c\int_{-G^{-1}\left(\frac{c}{2}\right)}^{G^{-1}\left(\frac{c}{2}\right)} \frac{\psi'(y)^2}{(1+y^2)^{5/4}}\,dy - 5c\int_{-G^{-1}\left(\frac{c}{2}\right)}^{G^{-1}\left(\frac{c}{2}\right)} \frac{(y^2-1)}{(1+y^2)^{13/4}}\psi(y)^2\,dy \\
&\quad -12\frac{c\,G^{-1}\left(\frac{c}{2}\right)}{\left(1+G^{-1}\left(\frac{c}{2}\right)^2\right)^{9/4}}\psi\left(G^{-1}\left(\frac{c}{2}\right)\right)^2.
\end{aligned}$$

We choose

$$\psi_c(y) := (1+y^2)^{3/2}\Phi_2(y) = (1+y^2)^{5/4}G(y)$$

and obtain for the corresponding $\varphi_c \in H^2 \cap H_0^1(0,1)$:

$$\begin{aligned}
\frac{1}{4c}D^2\tilde{W}_\alpha(u_c)(\varphi_c,\varphi_c) &= \int_0^{G^{-1}\left(\frac{c}{2}\right)} \left(\left((1+y^2)^{-1/4}+\frac{5}{2}yG(y)\right)^2 - \frac{5}{2}(y^2-1)G(y)^2\right)\frac{dy}{(1+y^2)^{3/4}} \\
&\quad -\frac{3}{4}c^2G^{-1}\left(\frac{c}{2}\right)\left(1+G^{-1}\left(\frac{c}{2}\right)^2\right)^{1/4}. \qquad (25)
\end{aligned}$$

According to Theorem 1 we know that this expression is equal to 0 for $c = c_{\max}$. Writing $c = 2G(d)$ we see that the asymptotic behaviour of the right hand side is dominated by

$$\frac{c_0^2}{4}\left(\frac{25}{4}\cdot\frac{2}{3}-\frac{5}{2}\cdot\frac{2}{3}-3\right)d^{3/2} = -\frac{c_0^2}{8}d^{3/2} \to -\infty$$

for $d \to \infty$, i.e. $c \nearrow c_0$. This shows, together with Lemma 4 that $D^2\tilde{W}_\alpha(u_c)(\varphi_c,\varphi_c) < 0$ for $c \in (c_{\max}, c_0)$ and concludes the proof of Theorem 2.

The right hand side of (25) is plotted in Figure 5. Since $\varphi_c \to 0$ for $c \searrow 0$, the curve starts in $(0,0)$ although there, $D^2\tilde{W}_\alpha(u_0)$ is positive definite.

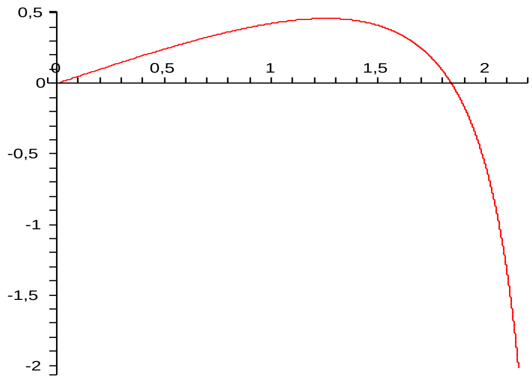

Figure 5: $\frac{1}{4c} D^2 \tilde{W}_\alpha(u_c)(\varphi_c, \varphi_c)$

4 Global minima and symmetry

The aim of this section is to examine whether the small solutions which were found to be local minima in Section 2 are also global minima for the functional $\tilde{W}_\alpha$. In what follows it will be convenient to write

$$\begin{aligned} \tilde{W}_\alpha(v) &= \int_0^1 \big(\kappa(x)^2 + 2\alpha\kappa(x)\big)\sqrt{1+v'(x)^2}\,dx, \\ &= \int_0^1 \kappa(x)^2\sqrt{1+v'(x)^2}\,dx + 2\alpha\left[\arctan(v'(x))\right]_0^1 =: W(v) + BC_\alpha(v). \end{aligned}$$

We remark that all quantities are geometric and so, invariant under rotation. Moreover, when stretching a curve by a factor k, W is multiplied by a factor $1/k$ while BC_α remains unchanged. We shall see that the task of minimizing $\tilde{W}_\alpha$ can be reduced to a minimization problem for a function of two variables. As a by–product of the analysis of this function we find that in order to determine $\inf_{v \in H^2 \cap H_0^1} \tilde{W}_\alpha(v)$ it is sufficient to minimize over all symmetric functions. The reduction to a two–dimensional problem is achieved in two steps. We begin by showing that it is enough to consider concave functions.

Lemma 5. *Suppose that $u \in H^2 \cap H_0^1(0,1)$ is not concave. Then there exists a concave function $v \in H^2 \cap H_0^1(0,1)$ with $\tilde{W}_\alpha(v) < \tilde{W}_\alpha(u)$.*

Proof. It is natural to think of v as the concave envelope of u, so that we are led to consider the following obstacle problem: find $v \in K$ such that

$$\forall \eta \in K \qquad \int_0^1 v'(\eta' - v') \geq 0, \tag{26}$$

where $K = \{\eta \in H_0^1(0,1) \,|\, \eta \geq u \text{ a.e. in } (0,1)\}$. It is shown in Chapter IV of [KS] that v can be obtained as the limit of a sequence $(v_\varepsilon)_{\varepsilon>0}$, where $v_\varepsilon \in H^2 \cap H_0^1(0,1)$ solves

$$-v_\varepsilon'' = (-u'')^+\,\vartheta_\varepsilon(v_\varepsilon - u) \quad \text{in } (0,1). \tag{27}$$

Here, $\vartheta_\varepsilon : \mathbb{R} \to \mathbb{R}$ satisfies

$$\vartheta_\varepsilon(t) = \begin{cases} 1, & t < 0 \\ 1 - \frac{t}{\varepsilon}, & 0 \leq t \leq \varepsilon \\ 0, & t > \varepsilon. \end{cases}$$

It follows from the analysis in [KS] that $v_\varepsilon \to v$ in $H^1(0,1)$, $v_\varepsilon'' \rightharpoonup v''$ in $L^2(0,1)$ as $\varepsilon \to 0$, so that $v \in H^2 \cap H_0^1(0,1)$ and $v'' \le 0$ a.e. in $(0,1)$; in particular v is concave. Denoting by $I = \{x \in [0,1] \,|\, v(x) = u(x)\}$ the coincidence set, we have that $v'' = 0$ a.e. in $[0,1]\setminus I$. Furthermore, using (27)

$$\begin{aligned} W(v) &= \int_0^1 \frac{|v''|^2}{(1+(v')^2)^{\frac{5}{2}}} = \int_I \frac{|v''|^2}{(1+(u')^2)^{\frac{5}{2}}} \le \liminf_{\varepsilon\to 0} \int_I \frac{|v_\varepsilon''|^2}{(1+(u')^2)^{\frac{5}{2}}} \\ &\le \int_I \frac{|(-u'')^+|^2}{(1+(u')^2)^{\frac{5}{2}}} \le \int_0^1 \frac{|(-u'')^+|^2}{(1+(u')^2)^{\frac{5}{2}}} \le \int_0^1 \frac{|u''|^2}{(1+(u')^2)^{\frac{5}{2}}} = W(u). \end{aligned}$$

If we had $W(v) = W(u)$, then the above argument would imply that $(-u'')^- = 0$ a.e. in $(0,1)$ and therefore $u'' \le 0$ a.e. in $(0,1)$ contradicting our assumption that u is not concave. Hence $W(u) < W(v)$; since $v \ge u$ we have that $u'(0) \le v'(0)$ and $u'(1) \ge v'(1)$ and therefore $\tilde{W}_\alpha(v) < \tilde{W}_\alpha(u)$. □

In what follows we shall make use of the prototype solution

$$U_0(x) = \frac{2}{c_0 \sqrt[4]{1 + G^{-1}\left(\frac{c_0}{2} - c_0 x\right)^2}}. \tag{28}$$

Formally, it is the large solution of the Navier boundary value problem (1) for $\alpha = 0$. However, one should observe that this solution is no longer smooth as a graph near $x = 0$ and $x = 1$, and for this reason, it was not included in Proposition 1.
Suppose that $0 \le x_0 < x_1 \le 1$ are two points with $x_1 - x_0 < 1$. Then $U_{0|[x_0,x_1]}$ can be written as a graph over the segment connecting $(x_0, U_0(x_0))$ and $(x_1, U_0(x_1))$. We denote by $u_{x_0,x_1} : [0,1] \to \mathbb{R}$ the strictly concave function which is obtained by translating, rotating and rescaling the above graph to the unit interval $[0,1]$. Note that $u_{x_0,x_1} \in H^2 \cap H_0^1(0,1)$. Our next lemma essentially reduces the minimization of $\tilde{W}_\alpha$ to a two–dimensional minimization problem.

Lemma 6. *Suppose that $u \in H^2 \cap H_0^1(0,1)\setminus\{0\}$ is concave. Then there exist $0 \le x_0 < x_1 \le 1$, $x_1 - x_0 < 1$ such that $v = u_{x_0,x_1}$ satisfies $BC_\alpha(u) = BC_\alpha(v)$ and either $W(v) \le W(u), u'(0) = v'(0)$ or $W(v) < W(u), u'(0) \ne v'(0)$.*

Proof. Let us denote by β_ℓ and β_r the boundary angles of graph(u) on the left and on the right respectively. Since u is assumed to be concave and nontrivial we have $\beta_\ell, \beta_r \in (0, \frac{\pi}{2})$. Consider

$$\mathcal{K} := \text{graph}(U_0) \cup \{(0,y) : \ y \le 0\} \cup \{(1,y) : \ y \le 0\}.$$

This is no longer neither a graph nor a solution of the Willmore equation. However, it is a regular H^2-curve, locally an H^2-graph over the x- or the y-axis respectively and it has minimal Willmore energy c_0^2 among all concave curves connecting any point from $\{(0,y) : \ y \le 0\}$ with any point from $\{(1,y) : \ y \le 0\}$ with tangential directions $(0,1)$ and $(0,-1)$ respectively. This minimality follows similarly as in [DG, end of Section 5].
Claim: There exist two points $P = (x_P, y_P), Q = (x_Q, y_Q) \in \mathcal{K}, P \ne Q$ such that the segment $[P,Q]$ intersects $\mathcal{K}$ under the angles β_ℓ at P and β_r at Q.
To see this, we start with the point $(x_1, y_1) = (1,0)$ and the orthogonal straight line through this point. This line intersects the left part of $\mathcal{K}$ in (x_0, y_0) under a right angle. Now we move the point (x_1, y_1) and the corresponding orthogonal straight line counterclockwise. The corresponding (x_0, y_0) finally moves down, the intersection angle (at least finally) decreases and becomes arbitrarily small. In particular, the left angle β_ℓ is attained. Now we keep this angle fixed and move the point (x_0, y_0) clockwise. We consider (x_1, y_1) on the right part of $\mathcal{K}$ as intersection point with the straight line

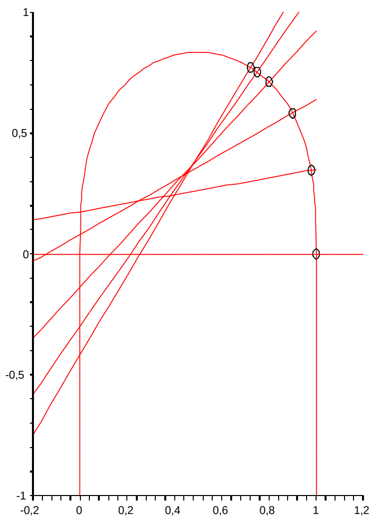

Figure 6: Left angle β_ℓ, right angle $\pi/2$

building the angle β_ℓ with $\mathcal{K}$ in (x_0, y_0). At the beginning this right angle is $\pi/2$ while it becomes arbitrarily small when (x_0, y_0) moves clockwise. In particular, β_r is attained as angle on the right and the claim is proved.
In view of the above–mentioned minimality property of $\mathcal{K}$, $\mathcal{K}'$ enjoys a similar minimality among those arcs with boundary angels β_ℓ, β_r. We infer that

$$W(\mathcal{K}') \leq \frac{1}{|P-Q|} W(u), \tag{29}$$

where $\mathcal{K}'$ denotes the subarc of $\mathcal{K}$ between P and Q. Observing that by construction y_P and y_Q cannot both be negative we may distinguish two cases:
Case 1: $y_P \geq 0$ and $y_Q \geq 0$. Setting $x_0 = x_P, x_1 = x_Q$ we have $x_1 - x_0 < 1$ since $\beta_\ell, \beta_r \in (0, \frac{\pi}{2})$. The function $v = u_{x_0,x_1}$ then satisfies

$$W(v) = |P-Q| W(\mathcal{K}') \leq W(u)$$

as well as $v'(0) = u'(0)$ and $v'(1) = u'(1)$.
Case 2: Either $y_P < 0$ or $y_Q < 0$. If $y_P < 0$, then $y_Q > 0$ since $\beta_r < \frac{\pi}{2}$ and we let $x_0 = 0, x_1 = x_Q$ as well as $v = u_{x_0,x_1}$. Denoting by $L(x_0, x_1)$ the length of the segment connecting $(x_0, U_0(x_0))$ and $(x_1, U_0(x_1))$ we have

$$W(v) = L(x_0, x_1) W(\mathcal{K}') \leq \frac{L(x_0, x_1)}{|P-Q|} W(u) < W(u)$$

since $u \neq 0$ and by construction any point on graph(U_0) is strictly closer to $(0,0)$ that to any other point on $\{(0, y) \,|\, y < 0\}$. A similar argument applies if $y_Q < 0$. Finally note that while $BC_\alpha(u) = BC_\alpha(v)$ we have $u'(0) \neq v'(0)$ in this case. $\square$

We deduce from Lemma 5 and Lemma 6 that when determining $\inf_{v \in H^2 \cap H_0^1(0,1)} \tilde{W}_\alpha(v)$ it is sufficient to calculate the Willmore energy for functions $v = u_{x_0,x_1}$ with $0 \leq x_0 < x_1 \leq 1$ and $x_1 - x_0 < 1$. The integrand for W on $[x_0, x_1]$ is c_0^2, so the integral is $c_0^2 \cdot (x_1 - x_0)$. The length of the base line is $\left((x_1-x_0)^2+(U_0(x_1)-U_0(x_0))^2\right)^{1/2}$. As for BC_α, we have $2\alpha(\arctan(U_0'(x_1))-\arctan(U_0'(x_0)))$. After rotation and rescaling we come up with:

$$\begin{aligned}\tilde{W}_\alpha(u_{x_0,x_1}) &= c_0^2\cdot(x_1-x_0)\left((x_1-x_0)^2+(U_0(x_1)-U_0(x_0))^2\right)^{1/2}\\ &\quad +2\alpha(\arctan(U_0'(x_1))-\arctan(U_0'(x_0)))\\ &= c_0^2\cdot(x_1-x_0)\\ &\quad \left((x_1-x_0)^2+\frac{4}{c_0^2}\left(\frac{1}{\sqrt[4]{1+G^{-1}(c_0/2-c_0x_1)^2}}-\frac{1}{\sqrt[4]{1+G^{-1}(c_0/2-c_0x_0)^2}}\right)^2\right)^{1/2}\\ &\quad +2\alpha(\arctan(G^{-1}(c_0/2-c_0x_1))-\arctan(G^{-1}(c_0/2-c_0x_0))).\end{aligned}$$

We now introduce the new variables

$$d_0 := G^{-1}(c_0/2-c_0x_0),\qquad d_1 := -G^{-1}(c_0/2-c_0x_1),\qquad d_1>-d_0 \tag{30}$$

so that

$$x_0=\frac{1}{2}-\frac{1}{c_0}G(d_0),\qquad x_1=\frac{1}{2}+\frac{1}{c_0}G(d_1).$$

Defining

$$\hat{W}_\alpha(d_0,d_1):=\tilde{W}_\alpha(u_{x_0,x_1})$$

we end up with

$$\begin{aligned}\hat{W}_\alpha(d_0,d_1) &= (G(d_0)+G(d_1))\left((G(d_0)+G(d_1))^2+4((1+d_0^2)^{-1/4}-(1+d_1^2)^{-1/4})^2\right)^{1/2}\\ &\quad -2\alpha(\arctan(d_0)+\arctan(d_1)).\end{aligned}$$

The following result summarizes what we have achieved so far.

Theorem 5. *Let $\alpha\geq 0$. Then*

$$\inf_{v\in H^2\cap H_0^1(0,1)}\tilde{W}_\alpha(v)=\inf_{(d_0,d_1)\in\mathbb{R}^2,d_1\geq -d_0}\hat{W}_\alpha(d_0,d_1).$$

It remains to discuss the two-dimensional function $\hat{W}_\alpha(d_0,d_1)$, $(d_1\geq -d_0)$. Here, the key step is proving positivity for the following expression:

Lemma 7. *For $d_1>-d_0$ we have that*

$$(G(d_0)+G(d_1))\cdot\left(\frac{d_0}{\sqrt[4]{1+d_0^2}}+\frac{d_1}{\sqrt[4]{1+d_1^2}}\right)-(G(d_0)+G(d_1))^2-2\left(\frac{1}{\sqrt[4]{1+d_0^2}}-\frac{1}{\sqrt[4]{1+d_1^2}}\right)^2>0.$$

Proof. By the fundamental theorem of calculus and since G is odd we have:

$$\begin{aligned}G(d_0)+G(d_1) &= G(d_0)-G(-d_1)=\int_{-d_1}^{d_0}\frac{1}{(1+\tau^2)^{5/4}}\,d\tau,\\ \frac{d_0}{\sqrt[4]{1+d_0^2}}+\frac{d_1}{\sqrt[4]{1+d_1^2}} &= \left[\frac{\tau}{(1+\tau^2)^{1/4}}\right]_{-d_1}^{d_0}=\int_{-d_1}^{d_0}\frac{1+\frac{1}{2}\tau^2}{(1+\tau^2)^{5/4}}\,d\tau,\\ \frac{1}{\sqrt[4]{1+d_0^2}}-\frac{1}{\sqrt[4]{1+d_1^2}} &= \left[\frac{1}{(1+\tau^2)^{1/4}}\right]_{-d_1}^{d_0}=-\frac{1}{2}\int_{-d_1}^{d_0}\frac{\tau}{(1+\tau^2)^{5/4}}\,d\tau.\end{aligned}$$

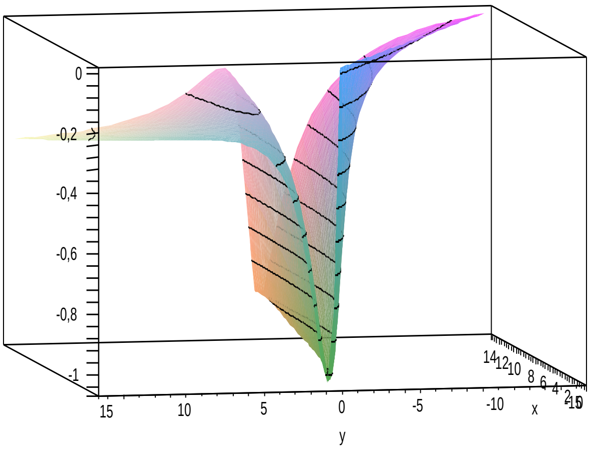

Figure 7: Cross section of the graph of $\hat{W}_1$ along the axis $d_0 = d_1$

One may observe that $d_1 > -d_0$ is equivalent to $-d_1 < d_0$. The first two terms in the expression under consideration combine as follows:

$$\begin{aligned}&(G(d_0)+G(d_1))\cdot\left(\frac{d_0}{\sqrt[4]{1+d_0^2}}+\frac{d_1}{\sqrt[4]{1+d_1^2}}\right)-(G(d_0)+G(d_1))^2\\ =\;&\frac{1}{2}\left(\int_{-d_1}^{d_0}\frac{1}{(1+\tau^2)^{5/4}}\,d\tau\right)\cdot\left(\int_{-d_1}^{d_0}\frac{\tau^2}{(1+\tau^2)^{5/4}}\,d\tau\right).\end{aligned}$$

We now apply the Cauchy-Schwarz inequality and make use of $\tau\mapsto\frac{1}{(1+\tau^2)^{5/8}}$ and $\tau\mapsto\frac{\tau}{(1+\tau^2)^{5/8}}$ being linearly independent:

$$\begin{aligned}2\left(\frac{1}{\sqrt[4]{1+d_0^2}}-\frac{1}{\sqrt[4]{1+d_1^2}}\right)^2 &= \frac{1}{2}\left(\int_{-d_1}^{d_0}\frac{\tau}{(1+\tau^2)^{5/4}}\,d\tau\right)^2\\ &< \frac{1}{2}\left(\int_{-d_1}^{d_0}\frac{1}{(1+\tau^2)^{5/4}}\,d\tau\right)\cdot\left(\int_{-d_1}^{d_0}\frac{\tau^2}{(1+\tau^2)^{5/4}}\,d\tau\right)\\ &= (G(d_0)+G(d_1))\cdot\left(\frac{d_0}{\sqrt[4]{1+d_0^2}}+\frac{d_1}{\sqrt[4]{1+d_1^2}}\right)-(G(d_0)+G(d_1))^2,\end{aligned}$$

thereby proving the claim. □

Next we show that in the open interior of the domain of definition of the two dimensional energy function $\hat{W}_\alpha$, critical points may occur at most on the diagonal, i.e. on symmetric graphs in the original context.

Lemma 8. *Let* $\alpha \geq 0$ *and assume that*

$$\begin{aligned}(d_0,d_1)\;&\mapsto\;\hat{W}_\alpha(d_0,d_1)\\ &=\;(G(d_0)+G(d_1))\cdot\left((G(d_0)+G(d_1))^2+4\left((1+d_0^2)^{-1/4}-(1+d_1^2)^{-1/4}\right)^2\right)^{1/2}\\ &\quad-2\alpha\left(\arctan(d_0)+\arctan(d_1)\right)\end{aligned}$$

has a critical point (d_0, d_1) *with* $d_1 > -d_0$*. Then*

$$d_0 = d_1.$$

Proof. In a critical point of $\hat{W}_\alpha$, we have that

$$\begin{aligned}
0 &= \frac{\partial}{\partial d_0}\hat{W}_\alpha(d_0, d_1) \\
&= \frac{1}{2}\left(G(d_0)+G(d_1)\right)\cdot\left((G(d_0)+G(d_1))^2 + 4\left((1+d_0^2)^{-1/4} - (1+d_1^2)^{-1/4}\right)^2\right)^{-1/2} \\
&\quad \cdot\left(2(G(d_0)+G(d_1))(1+d_0^2)^{-5/4} - 4d_0(1+d_0^2)^{-5/4}\left((1+d_0^2)^{-1/4} - (1+d_1^2)^{-1/4}\right)\right) \\
&\quad +(1+d_0^2)^{-5/4}\left((G(d_0)+G(d_1))^2 + 4\left((1+d_0^2)^{-1/4} - (1+d_1^2)^{-1/4}\right)^2\right)^{1/2} \\
&\quad -2\alpha\frac{1}{1+d_0^2};
\end{aligned}$$

$$\begin{aligned}
0 &= \frac{\partial}{\partial d_1}\hat{W}_\alpha(d_0, d_1) \\
&= \frac{1}{2}\left(G(d_0)+G(d_1)\right)\cdot\left((G(d_0)+G(d_1))^2 + 4\left((1+d_0^2)^{-1/4} - (1+d_1^2)^{-1/4}\right)^2\right)^{-1/2} \\
&\quad \cdot\left(2(G(d_0)+G(d_1))(1+d_1^2)^{-5/4} + 4d_1(1+d_1^2)^{-5/4}\left((1+d_0^2)^{-1/4} - (1+d_1^2)^{-1/4}\right)\right) \\
&\quad +(1+d_1^2)^{-5/4}\left((G(d_0)+G(d_1))^2 + 4\left((1+d_0^2)^{-1/4} - (1+d_1^2)^{-1/4}\right)^2\right)^{1/2} \\
&\quad -2\alpha\frac{1}{1+d_1^2}.
\end{aligned} \tag{31}$$

Equivalently:

$$\begin{aligned}
0 &= (G(d_0)+G(d_1))\cdot\left((G(d_0)+G(d_1))^2 + 4\left((1+d_0^2)^{-1/4} - (1+d_1^2)^{-1/4}\right)^2\right)^{-1/2} \\
&\quad \cdot\left((G(d_0)+G(d_1))(1+d_0^2)^{-1/4} - 2d_0(1+d_0^2)^{-1/4}\left((1+d_0^2)^{-1/4} - (1+d_1^2)^{-1/4}\right)\right) \\
&\quad +(1+d_0^2)^{-1/4}\left((G(d_0)+G(d_1))^2 + 4\left((1+d_0^2)^{-1/4} - (1+d_1^2)^{-1/4}\right)^2\right)^{1/2} - 2\alpha; \\
0 &= (G(d_0)+G(d_1))\cdot\left((G(d_0)+G(d_1))^2 + 4\left((1+d_0^2)^{-1/4} - (1+d_1^2)^{-1/4}\right)^2\right)^{-1/2} \\
&\quad \cdot\left((G(d_0)+G(d_1))(1+d_1^2)^{-1/4} + 2d_1(1+d_1^2)^{-1/4}\left((1+d_0^2)^{-1/4} - (1+d_1^2)^{-1/4}\right)\right) \\
&\quad +(1+d_1^2)^{-1/4}\left((G(d_0)+G(d_1))^2 + 4\left((1+d_0^2)^{-1/4} - (1+d_1^2)^{-1/4}\right)^2\right)^{1/2} - 2\alpha.
\end{aligned}$$

Subtracting both equations yields

$$\begin{aligned} 0 &= \Big((1+d_0^2)^{-1/4}-(1+d_1^2)^{-1/4}\Big) \\ &\quad\cdot\left((G(d_0)+G(d_1))^2+4\left((1+d_0^2)^{-1/4}-(1+d_1^2)^{-1/4}\right)^2\right)^{-1/2} \\ &\quad\cdot\Bigg\{(G(d_0)+G(d_1))^2-2(G(d_0)+G(d_1))\left(\frac{d_0}{\sqrt[4]{1+d_0^2}}+\frac{d_1}{\sqrt[4]{1+d_0^2}}\right) \\ &\qquad+\left((G(d_0)+G(d_1))^2+4\left((1+d_0^2)^{-1/4}-(1+d_1^2)^{-1/4}\right)^2\right)\Bigg\}. \end{aligned}$$

By Lemma 7, the curly bracket is strictly negative, since we assume that $d_1 > -d_0$. We conclude that

$$0 = \Big((1+d_0^2)^{-1/4}-(1+d_1^2)^{-1/4}\Big),$$

which yields that $d_0 = d_1$. □

We are now in position to solve the two–dimensional minimization problem.

Proposition 2. *Let $0 < \alpha \le \alpha_{\max}$. There exists $\alpha^* = 1.132372323\ldots \in (0, \alpha_{\max})$ such that*

$$\inf_{(d_0,d_1)\in\mathbb{R}^2, d_1\ge -d_0} \hat{W}_\alpha(d_0,d_1) = \begin{cases} \hat{W}_\alpha(G^{-1}(\frac{c}{2}), G^{-1}(\frac{c}{2})), & 0<\alpha\le\alpha^* \\ c_0^2-2\alpha\pi, & \alpha^*<\alpha\le\alpha_{\max}, \end{cases}$$

where $c \in (0, c_{\max})$ solves $h(c) = \alpha$. In the first case $d_0 = d_1 = G^{-1}(\frac{c}{2})$ is the only point for which the minimum is attained, while it is not attained for $\alpha^* < \alpha \le \alpha_{\max}$.

Proof. In view of Lemma 8 and the symmetry of $\hat{W}_\alpha$,

$$\inf_{(d_0,d_1)\in\mathbb{R}^2, d_1\ge -d_0} \hat{W}_\alpha(d_0,d_1)$$

is the minimum between

$$\inf_{d\in(0,\infty)} \hat{W}_\alpha(d,d), \tag{32}$$

$$\inf_{d\in\mathbb{R}} \hat{W}_\alpha(d,-d) = 0, \tag{33}$$

and

$$\inf_{d\in\mathbb{R}} \hat{W}_\alpha(d,\infty). \tag{34}$$

Since

$$\hat{W}_\alpha(d,d) = 4G(d)^2 - 4\alpha\arctan(d)$$

is certainly negative for $d > 0$ close to 0, we see that $\inf_{d\in(0,\infty)} \hat{W}_\alpha(d,d) < 0$, so we need not consider (33). As for (34) we have

$$\hat{W}_\alpha(d,\infty) = \left(G(d)+\frac{c_0}{2}\right)\cdot\left((G(d)+\frac{c_0}{2})^2+4(1+d^2)^{-1/2}\right)^{1/2} - 2\alpha\left(\arctan(d)+\frac{\pi}{2}\right).$$

It is sufficient to discuss local mimima, since $\hat{W}_\alpha(\infty,\infty)$ is already covered by (32) and $\hat{W}_\alpha(-\infty,\infty) = 0$ by (33). Passing to the $c = 2G(d)$-variable, we see that $\hat{W}_\alpha$ attains its minimum on $\{(d_0,d_1) : d_0 \in [-\infty,\infty], d_1 \in [-d_0,\infty]\}$. For fixed $d_0 \in \mathbb{R}$, we infer from (31) that for d_1 large enough, $\frac{\partial \hat{W}_\alpha}{\partial d_1} > 0$. This follows since the slowest term $4d_1(1+d_1^2)^{-5/4}(1+d_0^2)^{-1/4}$ decays of order $-3/2$ and has a positive coefficient. Hence, the minimum is not attained on $\mathbb{R} \times \{\infty\}$, but either in (∞, ∞) or in the interior of our domain. This proves that

$$\inf_{(d_0,d_1)\in\mathbb{R}^2, d_1 \geq -d_0} \hat{W}_\alpha(d_0, d_1) = \inf_{d\in(0,\infty)} \hat{W}_\alpha(d, d). \tag{35}$$

It remains to evaluate the right hand side of (35). Let

$$\phi(d) := \hat{W}_\alpha(d, d) = 4G(d)^2 - 4\alpha \arctan(d).$$

We have

$$\phi'(d) = \frac{8G(d)}{(1+d^2)^{\frac{5}{4}}} - \frac{4\alpha}{1+d^2} = \frac{4}{1+d^2}\big(h(2G(d)) - \alpha\big),$$

with h defined in (9). Thus, $\phi'(d) = 0$ if and only if $d = G^{-1}(\frac{c}{2})$, where c is one of the solutions of $h(c) = \alpha$. Only the solution $c \in (0, c_{\max})$ is a local minimum so that

$$\inf_{d\in(0,\infty)} \phi(d) = \min\big(c^2 - 4\alpha \arctan(G^{-1}(\frac{c}{2})), c_0^2 - 2\alpha\pi\big)$$

taking into account that $\phi(0) = 0$ and $\phi(d) < 0$ for small $d > 0$. In order to calculate the last minimum we introduce the following auxiliary function $f : [0, c_{\max}] \to \mathbb{R}$,

$$f(c) := c_0^2 - 2h(c)\pi - c^2 + 4h(c)\arctan G^{-1}(\frac{c}{2}).$$

We find that $f(0) = c_0^2 > 0$, $f(c_{\max}) = -0.6674542140\ldots < 0$ and a short calculation shows that

$$f'(c) = \big(4\arctan G^{-1}(\frac{c}{2}) - 2\pi\big)h'(c) < 0, \quad c \in (0, c_{\max})$$

so that f has a unique zero

$$c^* = 1.274998908\ldots \in [0, c_{\max}] \text{ with } \alpha^* := h(c^*) = 1.132372323\ldots.$$

This proves the formula for $\inf_{(d_0,d_1)\in\mathbb{R}^2,d_1\geq -d_0} \hat{W}_\alpha(d_0, d_1)$. The uniqueness of the minimum for $0 \leq \alpha \leq \alpha^*$ follows from Lemma 2. $\square$

We are now in position to prove Theorem 3 and Theorem 4. The second result is an immediate consequence of (35) and Theorem 5. As for Theorem 3 we focus on the case $0 < \alpha \leq \alpha^*$. Let $c \in (0, c_{\max})$ be the unique solution of $h(c) = \alpha$ with corresponding small solution u_c. Clearly,

$$\tilde{W}_\alpha(u_c) = \hat{W}_\alpha(G^{-1}(\frac{c}{2}), G^{-1}(\frac{c}{2})) = \inf_{(d_0,d_1)\in\mathbb{R}^2, d_1\geq -d_0} \hat{W}_\alpha(d_0, d_1) = \inf_{v\in H^2\cap H_0^1(0,1)} \tilde{W}_\alpha(v)$$

by Proposition 2 and Lemma 6. It remains to show that u_c is the only function in $H^2 \cap H_0^1(0, 1)$ for which the minimum is attained. Suppose that $u \in H^2 \cap H_0^1(0, 1)$ satisfies $\tilde{W}_\alpha(u) = \inf_{v\in H^2\cap H_0^1(0,1)} \tilde{W}_\alpha(v)$. In view of Lemma 5 u is necessarily concave. Let $v = u_{x_0,x_1} \in H^2 \cap H_0^1(0, 1)$ be the function appearing in Lemma 6 with d_0, d_1 given by (30). Using the minimality of u, Proposition 2 and Lemma 6 we obtain

$$\tilde{W}_\alpha(u) \leq \tilde{W}_\alpha(u_c) = \hat{W}_\alpha(G^{-1}(\frac{c}{2}), G^{-1}(\frac{c}{2})) \leq \hat{W}_\alpha(d_0, d_1) = \tilde{W}_\alpha(v) \leq \tilde{W}_\alpha(u).$$

This implies that $\hat{W}_\alpha(G^{-1}(\frac{c}{2}), G^{-1}(\frac{c}{2})) = \hat{W}_\alpha(d_0, d_1)$ and hence by Proposition 2 that $d_0 = d_1 = G^{-1}(\frac{c}{2})$ so that $v = u_c$. In particular we infer with the help of Lemma 6 that

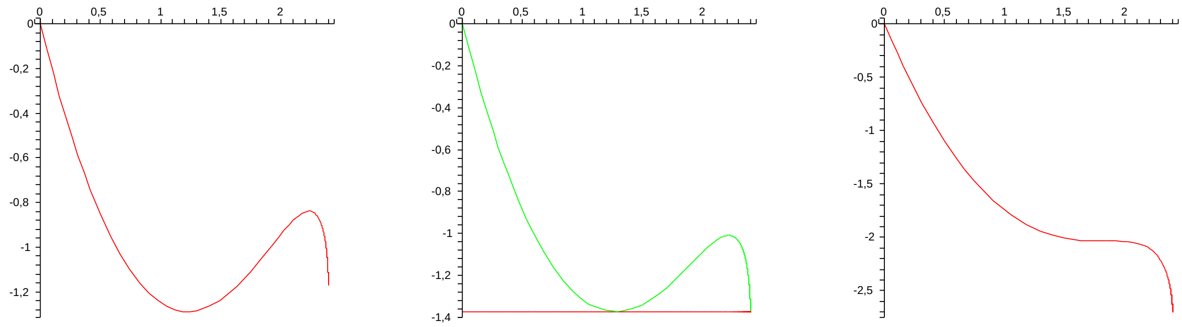

Figure 8: Graphs of the function $c \mapsto \tilde{W}_\alpha(u_c)$ for $\alpha = 1.1$, $\alpha = \alpha^*$ and $\alpha = 1.34$ (left to right)

$u'(0) = v'(0) = u_c'(0)$ and $u'(1) = v'(1) = u_c'(1)$. As a consequence we have $BC_\alpha(u) = BC_\alpha(u_c)$ and therefore $W(u) = W(u_c)$. However, in view of Theorem 2 in [DG] u_c is the *unique* minimum of W in the class $M_\beta = \{w \in H^2 \cap H_0^1(0,1) \,|\, w'(0) = -w'(1) = \beta\}$ ($\beta = u_c'(0)$) so that we must have $u = u_c$. This completes the proof of Theorem 3.

For selected values of α, Fig. 8 shows plots of the function $c \mapsto \tilde{W}_\alpha(u_c)$ on the interval $[0, c_0)$.

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