Rotationally symmetric classical solutions to the Dirichlet problem for Willmore surfaces

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Abstract

We consider the Willmore equation with Dirichlet boundary conditions for a surface of revolution obtained by rotating the graph of a positive smooth even function. We show existence of a regular solution by minimisation. Instead of minimising the Willmore functional we reformulate the problem in the hyperbolic half plane and we minimise the corresponding "hyperbolic Willmore functional".

1 Introduction

Recently, the Willmore functional and the associate L^2–gradient flow, the so–called Willmore flow, have attracted a lot of attention. Given a smooth immersed surface $f : M \to \mathbb{R}^3$, the Willmore functional is defined by

$$W(f) := \int_{f(M)} H^2 dA,$$

where $H = (\kappa_1 + \kappa_2)/2$ denotes the mean curvature of $f(M)$. Apart from being of geometric interest, the functional W is a model for the elastic energy of thin shells or biological membranes. Furthermore, it is used in image processing for problems of surface restoration and image inpainting. In these applications one is usually concerned with minima, or more generally with critical points of the Willmore functional. It is well–known that the corresponding surface Γ has to satisfy the Willmore equation

$$\Delta_g H + 2H(H^2 - K) = 0 \qquad \text{on } \Gamma, \tag{1}$$

where Δ_g denotes the Laplace–Beltrami operator on Γ and K its Gauss curvature with respect to the induced metric g. A particular difficulty arises from the fact that Δ_g depends on the unknown surface so that the equation is highly nonlinear. Moreover, it is of fourth order where many of the established techniques do not apply. A solution of (1) is called a Willmore surface. Existence of closed Willmore surfaces of prescribed genus has been proved by Simon [Sn] and Bauer & Kuwert [BK]. Recently, Rivière [R] proved a far reaching regularity result. Also, local and global existence results for the Willmore flow of closed surfaces are available, see e.g. [KS1, KS2, KS3, St]. On the other hand, Mayer and Simonett [MS] gave a numerical example providing evidence that the Willmore flow may develop singularities in finite time. An analytic proof for occurence of a singularity in finite or infinite time for particular initial data was given by Blatt [Bl]. The Willmore flow for one dimensional closed curves was studied by [DKS, P].

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If one is interested in surfaces with boundaries, then appropriate boundary conditions have to be added to (1). Since this equation is of fourth order one requires two sets of conditions and a discussion of possible choices can be found in [Nit] along with corresponding existence results. These results, however, are based on perturbation arguments and hence require severe smallness conditions on the data, which are by no means explicit. Thus the question arises whether it is possible to specify more general conditions on the boundary data that will guarantee the existence of a solution to (1). Such a task seems to be quite difficult since the problem is highly nonlinear and in addition lacks a maximum principle. Quite recently, Schätzle [Sch] proved an important general result concerning existence of branched Willmore immersions in $\mathbb{S}^n$ with boundary which satisfy Dirichlet boundary conditions. Assuming the boundary data to obey some explicit geometrically motivated smallness condition these immersions can even be shown to be embedded. By working in $\mathbb{S}^n$, some compactness problems could be overcome; on the other hand, when pulling pack these immersions to $\mathbb{R}^n$ it cannot be excluded that they contain the point ∞. Moreover, in general, the existence of branch points cannot be ruled out, and due to the generality of the approach, it seems to us that no topological information about the solutions can be extracted from the existence proof. We think that it is quite interesting to identify situations where it is possible to work with a-priori-bounded minimising sequences or where solutions with additional properties like e.g. being a graph or enjoying certain symmetry properties can be found. In order to outline possible directions of further research and to see, which kind of phenomena and results concerning compact embedded solutions in $\mathbb{R}^3$ of boundary value problems for the Willmore equation might be expected, we investigate boundary value problems for (1) in a specific symmetric situation. More precisely, we look at surfaces of revolution, which are obtained by rotating a graph over the $x = x_1$-axis in $\mathbb{R}^3$ around the x_1-axis. These are described by a sufficiently smooth function

$$u : [-1,1] \to (0,\infty)$$

and are parametrised as follows:

$$(x,\varphi) \mapsto f(x,\varphi) = (x, u(x)\cos\varphi, u(x)\sin\varphi), \quad x \in [-1,1], \quad \varphi \in [0,2\pi].$$

Numerical experiments concerning such kind of Willmore surfaces were performed by Fröhlich [F]. In the present article we consider the Willmore problem under Dirichlet boundary conditions, where the height $u(\pm 1) = \alpha > 0$ and a horizontal angle $u'(\pm 1) = 0$ are prescribed at the boundary:

Theorem 1. *For every $\alpha > 0$, there exists a smooth function $u \in C^\infty([-1,1],(0,\infty))$ such that the corresponding surface of revolution solves the Dirichlet problem for the Willmore equation*

$$\begin{cases} \Delta_g H + 2H(H^2 - K) = 0 & \text{in } (-1,1), \\ u(\pm 1) = \alpha, \qquad u'(\pm 1) = 0. \end{cases} \tag{2}$$

This solution u is even and has the following additional properties:

$$\forall x \in [0,1]: \quad 0 \le x + u(x)u'(x), \qquad u'(x) \le 0.$$

$$\forall x \in [-1,1]: \qquad \alpha \le u(x) \le \alpha + 1, \qquad |u'(x)| \le \frac{1}{\alpha}.$$

When comparing this result with the situation for minimal surfaces of revolution one may be surprised that existence holds true even for $\alpha \searrow 0$.

We solve (2) by minimising the Willmore functional in the class of surfaces of revolution, which are given by even functions $u : [-1,1] \to (0,\infty)$. In the following section we reformulate this problem in the hyperbolic half plane. In Section 3, taking advantage of using geodesic arcs in the hyperbolic half plane and refined energy reducing constructions, we show that one may pass to suitable minimising sequences satisfying quite strong a-priori-estimates. The latter ensure the required compactness. Further interesting properties of minimising sequences and the minimal Willmore energy as e.g. monotonicity in α are also proved in Section 3. In developing these techniques we benefit from previous works on related one-dimensional problems [DG1], [DG2].

Langer and Singer [LS1] gave explicit expressions for the curvature of elastic curves in the hyperbolic half plane in terms of the arclength of the unknown curve. However, there does not seem to be a direct way to use these results for the question being studied in the present article. Moreover, we think that the constructions made below in order to improve the properties of minimising sequences are of independent interest and *explain* to a good extent the shape of solutions.

2 Geometric background

2.1 Geometric quantities for surfaces of revolution

The calculations below are based on the formulas given in [BLe]. Let

$$u : [-1, 1] \to (0, \infty)$$

be a sufficiently smooth function. We consider the surface generated by the graph of u, the parametrisation of which is given by

$$(x, \varphi) \mapsto f(x, \varphi) = (x, u(x) \cos\varphi, u(x) \sin\varphi).$$

Here, we consider $x = x_1$ as first and $\varphi = x_2$ as second parameter. First and second fundamental form and the interior normal on the surface of revolution are given as follows:

$$\begin{aligned}
(g_{ij}) &= \begin{pmatrix} 1 + u'(x)^2 & 0 \\ 0 & u(x)^2 \end{pmatrix}, \qquad g = u(x)^2 \left(1 + u'(x)^2\right) \\
(L_{ij}) &= \frac{1}{\sqrt{1 + u'(x)^2}} \begin{pmatrix} -u''(x) & 0 \\ 0 & u(x) \end{pmatrix} \\
\nu(x, \varphi) &= \frac{1}{\sqrt{1 + u'(x)^2}} \left(u'(x), -\cos\varphi, -\sin\varphi\right).
\end{aligned}$$

We use the sign convention that the mean curvature H is positive if the surface is mean convex and negative if it is mean concave with respect to the interior normal ν. The mean curvature and Gauss curvature are then given respectively by

$$\begin{aligned}
H &= -\frac{u''(x)}{2\left(1 + u'(x)^2\right)^{3/2}} + \frac{1}{2u(x)\sqrt{1 + u'(x)^2}} = \frac{1}{2u(x)u'(x)} \left(\frac{u(x)}{\sqrt{1 + u'(x)^2}}\right)', \\
K &= -\frac{u''(x)}{u(x)\left(1 + u'(x)^2\right)^2}.
\end{aligned} \tag{3}$$

The Laplace-Beltrami operator on the surface of revolution acts on smooth functions h as follows

$$\begin{aligned}
\Delta_g h &= \frac{1}{\sqrt{g}} \sum_{i,j=1}^{2} \partial_i \left(\sqrt{g} g^{ij} \partial_j h\right) \\
&= \frac{1}{u(x)\sqrt{1 + u'(x)^2}} \left(\partial_x \left(\frac{u(x)}{\sqrt{1 + u'(x)^2}} \, \partial_x h\right) + \partial_\varphi \left(\frac{\sqrt{1 + u'(x)^2}}{u(x)} \, \partial_\varphi h\right)\right),
\end{aligned}$$

where g^{ij} are the entries of the inverse of $(g_{ij})_{i,j}$. The terms in the Willmore equation (1) for a surface of revolution are then

$$\begin{aligned}
\Delta_g H &= \frac{1}{u(x)\sqrt{1+u'(x)^2}}\,\partial_x\left(\frac{u(x)}{\sqrt{1+u'(x)^2}}\,\partial_x\left(\frac{1}{2u(x)\sqrt{1+u'(x)^2}}-\frac{u''(x)}{2\,(1+u'(x)^2)^{3/2}}\right)\right),\\
2H(H^2-K) &= \frac{1}{4\,(1+u'(x)^2)^{3/2}}\left(\frac{1}{u(x)}-\frac{u''(x)}{1+u'(x)^2}\right)\left(\frac{1}{u(x)}+\frac{u''(x)}{1+u'(x)^2}\right)^2.
\end{aligned}$$

So, for surfaces Γ of revolution as described above, the Willmore functional reads as follows

$$W(\Gamma)=\int_\Gamma H^2\,dS=\frac{\pi}{2}\int_{-1}^{1}\left(\frac{1}{u(x)\sqrt{1+u'(x)^2}}-\frac{u''(x)}{(1+u'(x)^2)^{3/2}}\right)^2 u(x)\sqrt{1+u'(x)^2}\,dx. \tag{4}$$

2.2 Surfaces of revolution as elastic curves in the hyperbolic half plane

The following formulae and calculations are mainly based on [LS2]. We will recall a different and for our purposes more suitable interpretation and reformulation of the Willmore functional.

The hyperbolic half plane $\mathbb{R}^2_+ := \{(x,y) : y>0\}$ is equipped with the metric

$$ds^2=\frac{1}{y^2}(dx^2+dy^2).$$

Geodesics are circular arcs centered on the x-axis and lines parallel to the y-axis; the first will play a crucial role in choosing suitable minimising sequences for the modified Willmore functional.

Let $s\mapsto\gamma(s)=(\gamma_1(s),\gamma_2(s))$, where we do not raise the indices, be a curve in $\mathbb{R}^2_+$ parametrised with respect to its arclength, i.e.

$$1\equiv\frac{\gamma_1'(s)^2+\gamma_2'(s)^2}{\gamma_2(s)^2}.$$

Then, its curvature is given by

$$\kappa(s)=-\frac{\gamma_2(s)^2}{\gamma_2'(s)}\frac{d}{ds}\left(\frac{\gamma_1'(s)}{\gamma_2(s)^2}\right)=\frac{\gamma_2(s)^2}{\gamma_1'(s)}\left(\frac{1}{\gamma_2(s)}+\frac{d}{ds}\left(\frac{\gamma_2'(s)}{\gamma_2(s)^2}\right)\right). \tag{5}$$

We think that this is the most frequently used sign convention. However, our arguments would not be affected by choosing the opposite sign. For graphs $[-1,1]\ni x\mapsto(x,u(x))\in\mathbb{R}^2_+$, formula (5) yields

$$\kappa(x)=-\frac{u(x)^2}{u'(x)}\frac{d}{dx}\left(\frac{1}{u(x)\sqrt{1+u'(x)^2}}\right)=\frac{u(x)u''(x)}{(1+u'(x)^2)^{3/2}}+\frac{1}{\sqrt{1+u'(x)^2}}. \tag{6}$$

Concerning the Willmore energy (in this metric) we find:

$$\begin{aligned}
\hat{W}(u) &:= \int_{-1}^{1}\kappa(x)^2\,ds(x)=\int_{-1}^{1}\kappa(x)^2\frac{\sqrt{1+u'(x)^2}}{u(x)}\,dx\\
&= \int_{-1}^{1}\left(\frac{u''(x)}{(1+u'(x)^2)^{3/2}}-\frac{1}{u(x)\sqrt{1+u'(x)^2}}\right)^2 u(x)\sqrt{1+u'(x)^2}\,dx\\
&\quad +4\int_{-1}^{1}\frac{u''(x)}{(1+u'(x)^2)^{3/2}}\,dx\\
&= \frac{2}{\pi}\int_\Gamma H^2\,dS-\frac{2}{\pi}\int_\Gamma K\,dS=\frac{2}{\pi}\int_\Gamma H^2\,dS+4\left[\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right]_{-1}^{1},
\end{aligned}$$

with H and K as given in (3). This means that

$$W(\Gamma) = \frac{\pi}{2}\hat{W}(u) - 2\pi \left[\frac{u'(x)}{\sqrt{1+u'(x)^2}} \right]_{-1}^{1},$$

where $W(\Gamma)$ is defined in (4) and Γ is the surface of revolution generated by u. In our situation where we assume Dirichlet data

$$u(\pm 1) = \alpha, \qquad u'(\pm 1) = 0,$$

we even have

$$W(\Gamma) = \frac{\pi}{2}\hat{W}(u). \tag{7}$$

In proving Theorem 1, we benefit a lot from considering $\hat{W}$ instead of W. We do not only take technical advantage from this point of view, but we think that it is geometrically more suitable as the constructions in Section 3 will make clear.

Concerning the Euler-Lagrange equation for critical points of the "hyperbolic Willmore functional" $\hat{W}$ one has:

Lemma 1. *Assume that $u \in C^4([-1,1])$ is such that for all $\varphi \in C_0^\infty(-1,1)$ one has that $0 = \frac{d}{dt}\hat{W}(u + t\varphi)|_{t=0}$. Then u satisfies the following Euler-Lagrange equation:*

$$\frac{u(x)}{\sqrt{1+u'(x)^2}} \frac{d}{dx}\left(\frac{u(x)}{\sqrt{1+u'(x)^2}}\kappa'(x) \right) - \kappa(x) + \frac{1}{2}\kappa(x)^3 = 0, \qquad x \in (-1,1), \tag{8}$$

with κ as defined in (6).

This observation was formulated in [LS1, LS2] and goes back to U. Pinkall and R. Bryant, P. Griffiths [BG]. For the reader's convenience and because it will be used in the proof of regularity, we present the proof of Lemma 1 in Appendix A.

3 Minimisation of the Willmore functional

For $\alpha \in (0, \infty)$ we denote

$$N_\alpha := \{u \in C^{1,1}([-1,1]), u \text{ is even and positive}, u(1) = \alpha, u'(1) = 0\}, \tag{9}$$

and

$$M_\alpha := \inf\{\hat{W}(u) : u \in N_\alpha\}. \tag{10}$$

In this section, we will show that M_α is attained: i.e there exists $u_\alpha \in N_\alpha$, which is even in $C^\infty([-1,1])$, such that $\hat{W}(u_\alpha) = M_\alpha$.

According to (7) we have for all $u \in N_\alpha$

$$W(\Gamma) = \frac{\pi}{2}\int_{-1}^{1} \kappa(x)^2 \, ds(x) = \frac{\pi}{2}\hat{W}(u),$$

with Γ the surface of revolution generated by the graph u. Hence, the surface of revolution generated by the graph of u_α is a minimizer of the Willmore functional in the class of surfaces of revolution generated by the graph of functions in N_α. The corresponding Willmore equation is the following Dirichlet problem

$$\begin{cases} \Delta_g H + 2H^3 - 2HK = 0 & \text{in } (-1,1), \\ u(\pm 1) = \alpha, \qquad u'(\pm 1) = 0. \end{cases} \tag{11}$$

By minimising the functional $\hat{W}$ on N_α we construct a symmetric solution to (11).

Remark 1. *We will use the following rescaling property. If u is a positive function in $C^{1,1}([-r,r])$, for some $r>0$, then the function $v \in C^{1,1}([-1,1])$ defined by $v(x)=\frac{1}{r}u(rx)$ is such that*

$$\hat{W}(v)=\int_{-r}^{r}\kappa^2[u]ds[u].$$

Here and in the following $\kappa[u]$ denotes the curvature of the graph of u in the hyperbolic half plane (defined in (6)*) and $ds[u]$ denotes the corresponding line element.*

3.1 Upper bound for M_α

Lemma 2. *Let M_α be defined as in* (10)*. Then*

$$M_\alpha \le 8\int_0^{\arctan(1/(2\alpha))}\frac{d\varphi}{2-\cos\varphi}\le\frac{16}{9}\sqrt{3}\pi.$$

In particular,

$$\lim_{\alpha\to\infty}M_\alpha=0.$$

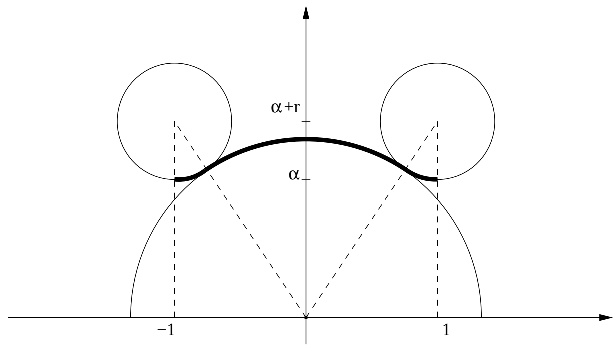

Figure 1: Comparison functions

Proof. Let $r>0$ to be suitably chosen. On the circle centered at $(1,\alpha+r)$ with radius r we consider the shortest arc starting at P, point of intersection between the circle and the segment from $(0,0)$ to $(1,\alpha+r)$, and ending in $(1,\alpha)$. This arc has opening angle $\arctan(1/(\alpha+r))$. Then we extend the curve in a $C^{1,1}$-way by considering on the geodesic circle centered at the origin and going through the point P the arc that starts at the intersection point between the circle and the y-axis and ends in P. Notice that the geodesic arc touches the original arc tangentially. Then we extend the curve on $[-1,0]$ by symmetry. This yields a curve $u_{\alpha,r}$ in N_α with equation

$$u_{\alpha,r}(x):=\begin{cases}\left((\sqrt{1+(\alpha+r)^2}-r)^2-|x|^2\right)^{\frac{1}{2}}, & \text{if } 0\le|x|<1-\frac{r}{\sqrt{1+(\alpha+r)^2}},\\ \alpha+r-\sqrt{r^2-(|x|-1)^2}, & \text{if } 1-\frac{r}{\sqrt{1+(\alpha+r)^2}}\le|x|\le 1.\end{cases}$$

The part of the curve given by the geodesic circular arc does not contribute to the hyperbolic Willmore energy. The graph of the other circular arcs has hyperbolic curvature

$$\kappa=\frac{\alpha+r}{r},$$

and line element

$$ds=\frac{r}{\alpha+r(1-\cos\varphi)}d\varphi.$$

Then $u_{\alpha,r}$ has Willmore energy:

$$\hat{W}(u_{\alpha,r}) = 2\frac{(\alpha+r)^2}{r}\int_0^{\arctan(1/(\alpha+r))}\frac{1}{\alpha+r(1-\cos\varphi)}\,d\varphi,$$

and the claim follows choosing $r=\alpha$. □

3.2 Monotonicity of the optimal Willmore energy

We show that $M_{\alpha'} \le M_\alpha$ for $\alpha' > \alpha$.

Lemma 3. *Fix $a>0$. Assume that $u \in C^{1,1}([-a,a])$ has only finitely many critical points and it is positive and symmetric with $u'(a)=0$ and such that $u'(x)\le 0$ for all $x\in[0,a]$. Then, for each $\rho\in(0,a]$, there exists a positive symmetric function $u_\rho \in C^{1,1}([-\rho,\rho])$ such that $u_\rho(\rho)=u(a)$, $u'_\rho(\rho)=0$, u_ρ has at most as many critical points as u and*

$$\forall x\in[0,\rho]:\ u'_\rho(x)\le 0 \ \ \textit{as well as} \ \ \int_{-\rho}^{\rho}\kappa[u_\rho]^2\,ds[u_\rho] \le \int_{-a}^{a}\kappa[u]^2\,ds[u].$$

In particular if $a=1$

$$\int_{-\rho}^{\rho}\kappa[u_\rho]^2\,ds[u_\rho]\le \hat{W}(u).$$

Proof. Let $r\in(0,a)$ be a parameter. The normal to the graph of u in $(r,u(r))$ has direction $(-u'(r),1)$. The straight line generated by the normal intersects the x-axis left of r, since u is decreasing. We take this intersection point $(c(r),0)$ as center for a geodesic circular arc, where the radius is chosen such that the arc is tangential to the graph of u in $(r,u(r))$ (i.e. the radius is given by the distance between $(c(r),0)$ and $(r,u(r))$). We build a new symmetric function with smaller curvature integral as follows. On $[c(r),r]$ we take this geodesic arc, which has horizontal tangent in $c(r)$, while on $[r,a]$ we take u. By construction, this function is $C^{1,1}([c(r),a])$ and decreasing. We shift it such that $c(r)$ is moved to 0, and extend this to an even function, which is again $C^{1,1}$, now on a suitable interval $[-\ell(r),\ell(r)]$. This function has the same boundary values as u, at most as many critical points as u and, by construction, a smaller curvature integral. This construction yields the claim since $r\mapsto\ell(r)$ is continuous and $\lim_{r\searrow 0}\ell(r)=a$, $\lim_{r\nearrow a}\ell(r)=0$. □

Lemma 4. *Fix $a>0$. Assume that $u\in C^{1,1}([-a,a])$ has only finitely many critical points and it is symmetric, positive with $u'(a)=0$ and such that $u'(x)\ge 0$ for all $x\in[0,a]$. Then there exists a positive symmetric function $v\in C^{1,1}([-a,a])$ with $v(a)=u(a)$, $v'(a)=0$, v has at most as many critical points as u and*

$$\forall x\in[0,a]:\ v'(x)\le 0 \ \ \textit{as well as} \ \ \int_{-a}^{a}\kappa[v]^2\,ds[v]\le\int_{-a}^{a}\kappa[u]^2\,ds[u].$$

In particular if $a=1$, $\hat{W}(v)\le\hat{W}(u)$.

Proof. We may assume that $u(0)<u(a)$. We consider

$$\tilde{u}(x) := \begin{cases} u(x+a), & \text{if } x\in[-a,0] \\ u(x-a), & \text{if } x\in[0,a]. \end{cases}$$

We apply the procedure of Lemma 3 to $\tilde{u}$ and find for all $\rho\in(0,a]$ a symmetric positive function $\tilde{u}_\rho\in C^{1,1}([-\rho,\rho])$ with lower Willmore energy, at most as many critical point as $\tilde{u}$ and such that $\tilde{u}_\rho(\rho)=\tilde{u}(a)=u(0)$, $\tilde{u}'_\rho(\rho)=0$ and $\tilde{u}'_\rho(x)\le 0$ for all $x\in[0,\rho]$. Let $\rho_0\in(0,a]$ be such that $\tilde{u}(a)=u(0)=\frac{\rho_0}{a}u(a)$. Then, by rescaling (Remark 1), the function $v(x)=\frac{a}{\rho_0}\tilde{u}_{\rho_0}(\frac{\rho_0}{a}x)$ defined on $[-a,a]$ is the desired decreasing function with smaller Willmore energy. □

Lemma 5. *Fix $a > 0$. Assume that $u \in C^{1,1}([-a,a])$ is a symmetric, positive function having only finitely many critical points and satisfying $u'(a) = 0$. Then, for each $\rho \in (0,a]$, there exists a symmetric positive function $u_\rho \in C^{1,1}([-\rho,\rho])$ with $u_\rho'(\rho) = 0$ and $u_\rho(\rho) = u(a)$ with at most as many critical points as u such that*

$$\int_{-\rho}^{\rho} \kappa[u_\rho]^2 \, ds[u_\rho] \leq \int_{-a}^{a} \kappa[u]^2 \, ds[u].$$

If $u'(x) < 0$ for x close to a, the same may be achieved for $u_\rho'(x)$ for x close to ρ. In particular if $a = 1$

$$\int_{-\rho}^{\rho} \kappa[u_\rho]^2 \, ds[u_\rho] \leq \hat{W}(u).$$

Proof. We may assume that u is not a constant. Let $x_0 > 0$ be such that $[-x_0, x_0]$ is the smallest possible symmetric interval with $u'(x_0) = 0$. In $[0, x_0]$ the derivative of u has a fixed sign. If $u'(x) \geq 0$ in $[0, x_0]$ then by Lemma 4 there is a positive symmetric function $v \in C^{1,1}([-x_0, x_0])$ with lower Willmore energy such that $v(x_0) = u(x_0)$, $v'(x_0) = 0$ and $v'(x) \leq 0$ in $[0, x_0]$. Hence we may assume that $u'(x) \leq 0$ in $[0, x_0]$. By Lemma 3 for all $r \in (0, x_0]$ there exists a positive symmetric function $v_r \in C^{1,1}([-r, r])$ such that $v_r(r) = u(x_0)$ and $v_r'(r) = 0$ and $v_r'(x) \leq 0$ in $[0, r]$. Hence the function

$$u_r(x) := \begin{cases} u(x + x_0 - r), & \text{if } \ r < x \leq a + r - x_0, \\ v_r(x), & \text{if } \ -r \leq x \leq r, \\ u(x - x_0 + r), & \text{if } \ -a - r + x_0 < x \leq -r, \end{cases}$$

is in $C^{1,1}([-a - r + x_0, a + r - x_0])$, is symmetric , $u_r'(a + r - x_0) = 0$, $u_r(a + r - x_0) = u(a)$ and

$$\int_{-(a+r-x_0)}^{a+r-x_0} \kappa[u_r]^2 \, ds[u_r] \leq \int_{-a}^{a} \kappa[u]^2 \, ds[u].$$

With this construction the claim is proved for $\rho \geq a - x_0$.

For $\rho < a - x_0$ we start from the function just constructed obtained at the limit for r going to zero. That is $v(x) = u(x + x_0)$ for $x \in [0, a - x_0]$ and extended by symmetry on $[-a + x_0, 0]$. This function is in $C^{1,1}([-a + x_0, a - x_0])$, positive and symmetric. We can repeat the same construction just done. We continuously decrease the interval of definition and, at the same time, the curvature integral. Since we have only finitely many critical points and at each iteration step we do not increase the number of critical points, this procedure is well defined and terminates after finitely many iterations.

If $u' < 0$ close to a the same may be achieved for u_ρ' since in the construction we do not change the function near the end-points of the interval of definition. □

Corollary 1. *Fix $a > 0$ and $\alpha > 0$. For each positive symmetric $u \in C^{1,1}([-a,a])$ having only finitely many critical points and satisfying*

$$u(\pm a) = \alpha, \qquad u'(\pm a) = 0$$

and for each $\beta \geq \alpha$, we find a symmetric $v \in C^{1,1}([-a,a])$ having at most as many critical points as u, satisfying

$$v(\pm a) = \beta, \qquad v'(\pm a) = 0$$

and

$$\int_{-a}^{a} \kappa[v]^2 \, ds[v] \leq \int_{-a}^{a} \kappa[u]^2 \, ds[u].$$

If $u'(x) < 0$ for x close to a, the same may be achieved for v'. In particular if $a = 1$, $\hat{W}(v) \leq \hat{W}(u)$.

Proof. By Lemma 5 for each $\rho \in (0,a]$, there exists a symmetric positive function $u_\rho \in C^{1,1}([-\rho,\rho])$ having at most as many critical points as u with $u_\rho'(\rho)=0$ and $u_\rho(\rho)=u(a)=\alpha$ such that

$$\int_{-\rho}^{\rho} \kappa[u_\rho]^2\, ds[u_\rho] \le \int_{-a}^{a} \kappa[u]^2\, ds[u].$$

Choosing ρ_0 such that $\frac{a}{\rho_0}\alpha = \beta$ the function $v(x) = \frac{a}{\rho_0} u_{\rho_0}(\frac{\rho_0}{a}x)$ for $x \in [-a,a]$ yields the claim. □

Theorem 2. *Let M_α for $\alpha \in \mathbb{R}^+$ be as defined in* (10)*. Then for $0<\alpha<\hat\alpha$ we have that*

$$M_{\hat\alpha} \le M_\alpha.$$

Proof. Since the polynomials are dense in H^2, a minimising sequence for M_α may be chosen (in N_α), which consists of symmetric positive polynomials. Corollary 1 yields the claim. □

3.3 Properties of minimising sequences

The first main step consists in finding a procedure which does not increase the Willmore energy but allows to restrict to functions v in N_α (defined in (9)) such that $v'(x) \le 0$ for all $x \in [0,1]$. Here, the techniques developed in subsection 3.2 are used essentially.

Theorem 3. *Let N_α be as defined in* (9)*. For each $u \in N_\alpha$ having only finitely many critical points, we find $v \in N_\alpha$ having at most as many critical points as u, satisfying*

$$v'(x) \le 0 \text{ for all } x \in [0,1] \quad \text{and} \quad \hat W(v) \le \hat W(u).$$

Proof. If u does not have the claimed property then there exist $x_0, x_1 \in [0,1]$, $x_0 < x_1$, with $u'(x)>0$ in (x_0,x_1), $u'(x_0)=u'(x_1)=0$ and $u'(x)\le 0$ in $[x_1,1]$. Using that $u(x_0)<u(x_1)$, we construct a positive symmetric function $v_1 \in C^{1,1}([-x_1,x_1])$ such that v_1 has at most as many critical points as $u|_{[-x_1,x_1]}$, $v_1'(x)\le 0$ in $[\tilde x_0, x_1]$, where $\tilde x_0$ is the largest critical point of v_1 below x_1. Moreover

$$v_1'(x_1)=0, \quad v_1(x_1)=u(x_1), \quad \int_{-x_1}^{x_1} \kappa[v_1]^2 ds[v_1] \le \int_{-x_1}^{x_1} \kappa[u]^2 ds[u]. \tag{12}$$

The claim will then follow by finitely many iterations proceeding from the boundary points towards the central point 0.

We consider $u|_{[-x_0,x_0]}$ and apply Corollary 1 with $\beta = u(x_1)$. If $x_0 = 0$ one simply skips this first step. There exists a symmetric positive function $w_1 \in C^{1,1}([-x_0,x_0])$ with $w_1(x_0)=u(x_1)$, $w_1'(x_0)=0$, having no more critical points than $u|_{[-x_0,x_0]}$ and satisfying

$$\int_{-x_0}^{x_0} \kappa[w_1]^2 ds[w_1] \le \int_{-x_0}^{x_0} \kappa[u]^2 ds[u].$$

We define on $[-x_1,x_1]$

$$\tilde v_1(x) := \begin{cases} u(x+x_1+x_0), & \text{if } x \in [-x_1,-x_0],\\ w_1(x), & \text{if } x \in [-x_0,x_0],\\ u(x-x_1-x_0), & \text{if } x \in [x_0,x_1]. \end{cases}$$

Certainly, $\tilde v_1 \in C^{1,1}([-x_1,x_1])$ is positive, symmetric and it does not have more critical points than $u|_{[-x_1,x_1]}$. Moreover, $\tilde v_1'(x)\le 0$ for $x \in [x_0,x_1]$ and

$$\int_{-x_1}^{x_1} \kappa[\tilde v_1]^2 ds[\tilde v_1] \le \int_{-x_1}^{x_1} \kappa[u]^2 ds[u], \quad \tilde v_1(x_1) = u(x_0), \quad \tilde v_1'(x_1)=0.$$

Corollary 1 now yields a positive symmetric function $v_1 \in C^{1,1}([-x_1, x_1])$, having no more critical points than $u|_{[-x_1,x_1]}$ and satisfying (12), with $v_1'(x) \le 0$ in $[\tilde{x}_0, x_1]$, where $\tilde{x}_0$ is the largest critical point of v_1 below x_1. The last property is verified first close to x_1; it holds on the whole interval since no further critical points arise. $\square$

Moreover, in choosing a minimising sequence for M_α we may restrict to functions in N_α satisfying

$$\forall x \in [0,1]: \qquad 0 \le x + v(x)v'(x). \tag{13}$$

For $x = 0$ and $x = 1$, this inequality is trivially satisfied. If for some $x_0 \in (0,1)$ we have that $0 = x_0 + v(x_0)v'(x_0)$, then the normal in $(x_0, v(x_0))$ to the graph of v goes through the origin. Hence, with the same construction as in Lemma 2 we could substitute over $[-x_0, x_0]$ the original graph by a geodesic circular arc lowering the Willmore energy. Observe that this procedure, applied to a positive symmetric $C^{1,1}$-function with $v'(x) \le 0$ for all $x \in [0,1]$ preserves all these properties.

Combining (13) with Theorem 3 we may restrict ourselves to minimising sequences $(v_k)_k$ for M_α (defined in (10)) having the following properties:

$$v_k \in C^{1,1}([-1,1]) \text{ are positive, symmetric and s.t. } \forall x \in [0,1]: \ 0 \le x + v_k(x)v_k'(x),\ v_k'(x) \le 0. \tag{14}$$

This implies immediately the following a-priori-estimates for this suitably chosen minimising sequence:

$$\forall x \in [-1,1]: \qquad \alpha \le v_k(x) \le \sqrt{\alpha^2 + 1 - x^2} \le \alpha + 1 \qquad |v_k'(x)| \le \frac{|x|}{\alpha}. \tag{15}$$

3.4 Attainment of the minimal Willmore energy

We are now able to state and to prove a more precise result than the main existence result Theorem 1 from the introduction:

Theorem 4. *For each $\alpha > 0$, there exists a positive symmetric function $u \in H^2(-1,1) \cap C^{1,1/2}([-1,1])$ satisfying*

$$u(\pm 1) = \alpha, \qquad u'(\pm 1) = 0,$$

such that

$$\hat{W}(u) = M_\alpha \overset{def}{=} \inf\{\hat{W}(v) : v \in C^{1,1}([-1,1]), v \textit{ is even, } v(\pm 1) = \alpha, v'(\pm 1) = 0\}.$$

This minimum is a weak solution to the Dirichlet problem (11) satisfying

$$\forall x \in [0,1]: \quad 0 \le x + u(x)u'(x), \qquad u'(x) \le 0. \tag{16}$$

$$\forall x \in [-1,1]: \qquad \alpha \le u(x) \le \sqrt{\alpha^2 + 1 - x^2} \le \alpha + 1 \qquad |u'(x)| \le \frac{|x|}{\alpha}. \tag{17}$$

Moreover, u is a classical solution, i.e. $u \in C^\infty([-1,1])$.

Proof. **Step1. Existence and quantitative properties of a minimiser.**
Let $(v_k)_k \subset N_\alpha$ be a minimizing sequence for M_α satisfying $(14-15)$. By the uniform bounds in (15) we find

$$\begin{aligned} \hat{W}(v_k) &= \int_{-1}^{1} \frac{v_k''(x)^2 v_k(x)}{(1+v_k'(x)^2)^{5/2}}\,dx + \int_{-1}^{1} \frac{1}{v_k(x)\sqrt{1+v_k'(x)^2}}\,dx \\ &\ge \frac{\alpha}{\left(1+\frac{1}{\alpha^2}\right)^{5/2}} \int_{-1}^{1} v_k''(x)^2\,dx + 2\frac{1}{(\alpha+1)\sqrt{1+\frac{1}{\alpha^2}}}. \end{aligned}$$

This shows uniform boundedness of $(v_k)_k$ in $H^2(-1,1)$. After passing to a subsequence, we find a positive symmetric function $u \in H^2(-1,1)$ such that

$$v_k \rightharpoonup u \text{ in } H^2(-1,1), \quad v_k \to u \in C^1([-1,1]),$$

and satisfying $(16-17)$. Since

$$\begin{aligned} M_\alpha + o(1) &= \hat{W}(v_k) = \int_{-1}^1 \frac{v_k''(x)^2 u(x)}{(1+u'(x)^2)^{5/2}}\, dx + \int_{-1}^1 \frac{1}{u(x)\sqrt{1+u'(x)^2}}\, dx + o(1) \\ &\geq \int_{-1}^1 \frac{u''(x)^2 u(x)}{(1+u'(x)^2)^{5/2}}\, dx + \int_{-1}^1 \frac{1}{u(x)\sqrt{1+u'(x)^2}}\, dx + o(1), \end{aligned}$$

it follows that u minimises $\hat{W}$ in the class of all positive symmetric $H^2(-1,1)$-functions v, satisfying $v(\pm 1) = \alpha$, $v'(\pm 1) = 0$. So, u weakly solves (11) and hence, also (8) in the sense of (18) below (see also (19)).

Step 2. Regularity of the minimiser.
From the calculations in the Appendix A, concerning the derivation of the Euler-Lagrange equation, we see that for any *even* $\varphi \in C^2([-1,1])$ with $\varphi(1) = 0$, $\varphi'(1) = 0$ one has that

$$\begin{aligned} -2\int_{-1}^1 \kappa \frac{1}{1+u'^2} \varphi''\, dx &= \int_{-1}^1 \kappa^2 \frac{\sqrt{1+u'^2}}{u^2} \varphi\, dx - 5\int_{-1}^1 \kappa^2 \frac{u'}{u\sqrt{1+u'^2}} \varphi'\, dx \\ &\quad -2\int_{-1}^1 \kappa \frac{1}{u^2} \varphi\, dx + 4\int_{-1}^1 \kappa \frac{u'}{u(1+u'^2)} \varphi'\, dx. \end{aligned} \tag{18}$$

First, we observe that (18) is still true for any $\varphi \in C^2([-1,1])$ with $\varphi(\pm 1) = 0$ and $\varphi'(\pm 1) = 0$. This follows by decomposition of φ in its even and odd part and using that they satisfy the same boundary conditions and that integrals over odd functions vanish. We take for arbitrary $\eta \in C_0^\infty(-1,1)$

$$\varphi(x) := \int_{-1}^x \int_{-1}^y \eta(s)\, ds dy - \beta(x+1)^2 - \gamma(x+1)^3,$$

where

$$\begin{aligned} \beta &= -\frac{1}{2}\int_{-1}^1 \eta(s)\, ds + \frac{3}{4}\int_{-1}^1 \int_{-1}^y \eta(s)\, ds dy \\ \gamma &= \frac{1}{4}\int_{-1}^1 \eta(s)\, ds - \frac{1}{4}\int_{-1}^1 \int_{-1}^y \eta(s)\, ds dy \end{aligned}$$

are chosen such that $\varphi(\pm 1) = 0$ and $\varphi'(\pm 1) = 0$. Since $\hat{W}(u)$ is finite, u obeys (17) and since

$$\beta, \gamma, \|\varphi\|_{C^1} \leq C\|\eta\|_{L^1},$$

we can conclude from (18) that for each $\eta \in C_0^\infty(-1,1)$,

$$\left| \int_{-1}^1 \kappa \frac{1}{1+u'^2} \eta\, dx \right| \leq C(u)\|\eta\|_{L^1}.$$

By the bounds on u in (17), the inequality above shows that κ is bounded and so,

$$u \in W^{2,\infty}(-1,1).$$

Next, for arbitrary $\eta \in C_0^\infty(-1,1)$ we choose

$$\varphi(x) = \int_{-1}^{x} \eta(s)\,ds - \frac{3}{4}\left(\int_{-1}^{1} \eta(s)\,ds\right)(x+1)^2 + \frac{1}{4}\left(\int_{-1}^{1} \eta(s)\,ds\right)(x+1)^3$$

so that

$$\varphi(\pm 1) = 0, \qquad \varphi'(\pm 1) = 0, \qquad \|\varphi\|_{C^0} \le C\|\eta\|_{L^1}, \qquad \|\varphi'\|_{L^1} \le C\|\eta\|_{L^1}.$$

Since we already know that κ is bounded, we conclude from (18) that for each $\eta \in C_0^\infty(-1,1)$,

$$\left|\int_{-1}^{1} \kappa \frac{1}{1+u'^2}\eta'(x)\,dx\right| \le C(u)\|\eta\|_{L^1}.$$

This proves that

$$\kappa\frac{1}{1+u'^2} \in W^{1,\infty}(-1,1), \quad \kappa \in W^{1,\infty}([-1,1]) = C^{0,1}([-1,1]), \quad u \in W^{3,\infty}([-1,1]) = C^{2,1}([-1,1]).$$

Finally, rewriting (8) as follows

$$\frac{d}{dx}\left(\frac{u(x)}{\sqrt{1+u'(x)^2}}\kappa'(x)\right) = \frac{\sqrt{1+u'(x)^2}}{u(x)}\left(\kappa(x) - \frac{1}{2}\kappa(x)^3\right) \text{ in } (-1,1),$$

we get an equation for κ with $W^{1,\infty}$-coefficients and right hand side. Hence, $\kappa \in W^{3,\infty}([-1,1]) = C^{2,1}([-1,1])$, $u \in C^{4,1}([-1,1])$ and finally, by straightforward bootstrapping, $u \in C^\infty([-1,1])$. $\square$

A Proof of Lemma 1

In order to calculate the Euler-Lagrange equation for the functional $\hat{W}$, we observe first that for arbitrary $\varphi \in C_0^\infty(-1,1)$:

$$\begin{aligned}
\frac{d}{dt}\kappa[u+t\varphi]|_{t=0} &= -\frac{d}{dt}\left\{\frac{(u+t\varphi)^2}{u'+t\varphi'}\frac{d}{dx}\left(\frac{1}{(u+t\varphi)\sqrt{1+(u'+t\varphi')^2}}\right)\right\}\Bigg|_{t=0}\\
&= -2\frac{u\varphi}{u'}\frac{d}{dx}\left(\frac{1}{u\sqrt{1+u'^2}}\right) + \frac{u^2\varphi'}{u'^2}\frac{d}{dx}\left(\frac{1}{u\sqrt{1+u'^2}}\right)\\
&\quad + \frac{u^2}{u'}\frac{d}{dx}\left(\frac{\varphi}{u^2\sqrt{1+u'^2}}\right) + \frac{u^2}{u'}\frac{d}{dx}\left(\frac{u'\varphi'}{u(1+u'^2)^{3/2}}\right)
\end{aligned}$$

and writing it in terms of κ

$$\begin{aligned}
\frac{d}{dt}\kappa[u+t\varphi]|_{t=0} &= 2\frac{\varphi}{u}\kappa - \frac{\varphi'}{u'}\kappa - \frac{\varphi}{u}\kappa + \frac{u}{u'\sqrt{1+u'^2}}\left(\frac{\varphi}{u}\right)'\\
&\quad - \frac{u'\varphi'}{1+u'^2}\kappa + \frac{u}{u'\sqrt{1+u'^2}}\left(\frac{u'\varphi'}{1+u'^2}\right)'\\
&= \frac{\varphi}{u}\kappa - \frac{\varphi'}{u'}\kappa - \frac{u'\varphi'}{1+u'^2}\kappa + \frac{\varphi'}{u'\sqrt{1+u'^2}} - \frac{\varphi}{u\sqrt{1+u'^2}}\\
&\quad + \frac{u}{u'\sqrt{1+u'^2}}\left(\frac{\varphi''u'}{1+u'^2} + \frac{\varphi'u''}{1+u'^2} - 2\frac{\varphi'u'^2u''}{(1+u'^2)^2}\right).
\end{aligned}$$

As for the last large bracket we have

$$
\begin{aligned}
&\left(\frac{\varphi'' u'}{1+u'^2}+\frac{\varphi' u''}{1+u'^2}-2\frac{\varphi' u'^2 u''}{(1+u'^2)^2}\right)=\frac{\varphi'' u'}{1+u'^2}-\frac{\varphi' u''}{1+u'^2}+2\frac{\varphi' u''}{(1+u'^2)^2}=\\
&=\quad \frac{\varphi'' u'}{1+u'^2}+\varphi'\sqrt{1+u'^2}\left(-\frac{\kappa}{u}+\frac{1}{u\sqrt{1+u'^2}}\right)-\frac{2\varphi'}{\sqrt{1+u'^2}}\left(-\frac{\kappa}{u}+\frac{1}{u\sqrt{1+u'^2}}\right)\\
&=\quad \frac{\varphi'' u'}{1+u'^2}-\frac{\kappa\varphi'}{u}\sqrt{1+u'^2}+\frac{\varphi'}{u}+\frac{2\kappa\varphi'}{u\sqrt{1+u'^2}}-\frac{2\varphi'}{u(1+u'^2)}
\end{aligned}
$$

so that

$$
\frac{d}{dt}\kappa[u+t\varphi]|_{t=0}=\frac{\varphi\kappa}{u}-3\frac{u'\varphi'\kappa}{1+u'^2}-\frac{\varphi}{u\sqrt{1+u'^2}}+\frac{2u'\varphi'+u\varphi''}{(1+u'^2)^{3/2}}.
$$

So, if $u\in C^4([-1,1])$ is such that for all $\varphi\in C_0^\infty(-1,1)$ one has that $0=\frac{d}{dt}\hat{W}(u+t\varphi)|_{t=0}$ we have:

$$
\begin{aligned}
0 \quad &= \quad \frac{d}{dt}\hat{W}(u+t\varphi)|_{t=0}=\frac{d}{dt}\int_{-1}^{1}\kappa[u+t\varphi]^2\frac{\sqrt{1+(u'+t\varphi')^2}}{u+t\varphi}\,dx\Big|_{t=0}\\
&= \quad \int_{-1}^{1}2\kappa\frac{\sqrt{1+u'^2}}{u}\left(\frac{\varphi\kappa}{u}-3\frac{u'\varphi'\kappa}{1+u'^2}-\frac{\varphi}{u\sqrt{1+u'^2}}+\frac{2u'\varphi'+u\varphi''}{(1+u'^2)^{3/2}}\right)dx\\
&\quad +\int_{-1}^{1}\kappa^2\left(\frac{u'\varphi'}{u\sqrt{1+u'^2}}-\frac{\varphi\sqrt{1+u'^2}}{u^2}\right)dx\\
&= \quad \int_{-1}^{1}\kappa^2\frac{\sqrt{1+u'^2}}{u^2}\varphi\,dx-5\int_{-1}^{1}\kappa^2\frac{u'}{u\sqrt{1+u'^2}}\varphi'\,dx-2\int_{-1}^{1}\kappa\frac{1}{u^2}\varphi\,dx\\
&\quad +4\int_{-1}^{1}\kappa\frac{u'}{u(1+u'^2)}\varphi'\,dx+2\int_{-1}^{1}\kappa\frac{1}{1+u'^2}\varphi''\,dx=\dots
\end{aligned}
\tag{19}
$$

integrating by parts first in the last integral and then in the second one

$$
\begin{aligned}
\dots \quad &= \quad \int_{-1}^{1}\kappa^2\frac{\sqrt{1+u'^2}}{u^2}\varphi\,dx-\int_{-1}^{1}\kappa^2\frac{u'}{u\sqrt{1+u'^2}}\varphi'\,dx-2\int_{-1}^{1}\kappa\frac{1}{u^2}\varphi\,dx-2\int_{-1}^{1}\kappa'\frac{1}{1+u'^2}\varphi'\,dx\\
&= \quad \int_{-1}^{1}\kappa^2\frac{\sqrt{1+u'^2}}{u^2}\varphi\,dx+\int_{-1}^{1}\kappa^2u'\left(\frac{1}{u\sqrt{1+u'^2}}\right)'\varphi\,dx+2\int_{-1}^{1}\kappa\kappa'\frac{u'}{u\sqrt{1+u'^2}}\varphi\,dx\\
&\quad +\int_{-1}^{1}\kappa^2\frac{u''}{u\sqrt{1+u'^2}}\varphi\,dx-2\int_{-1}^{1}\kappa\frac{1}{u^2}\varphi\,dx-2\int_{-1}^{1}\kappa'\frac{1}{1+u'^2}\varphi'\,dx\\
&= \quad \int_{-1}^{1}\kappa^2\varphi\left(\frac{\sqrt{1+u'^2}}{u^2}-\frac{u'^2}{u^2\sqrt{1+u'^2}}-\frac{u'^2u''}{u(1+u'^2)^{3/2}}+\frac{u''}{u\sqrt{1+u'^2}}\right)dx\\
&\quad -2\int_{-1}^{1}\kappa\frac{1}{u^2}\varphi\,dx+2\int_{-1}^{1}\kappa\kappa'\frac{u'}{u\sqrt{1+u'^2}}\varphi\,dx-2\int_{-1}^{1}\frac{u}{\sqrt{1+u'^2}}\kappa'\frac{1}{u\sqrt{1+u'^2}}\varphi'\,dx=\dots
\end{aligned}
$$

and, finally, integrating by parts in the last integral

$$
\dots \quad = \quad \int_{-1}^{1}\kappa^3\frac{1}{u^2}\varphi\,dx-2\int_{-1}^{1}\kappa\frac{1}{u^2}\varphi\,dx+2\int_{-1}^{1}\frac{u}{\sqrt{1+u'^2}}\frac{d}{dx}\left(\frac{u}{\sqrt{1+u'^2}}\kappa'\right)\frac{1}{u^2}\varphi\,dx.
$$

□

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