ON THE ROLE OF SPACE DIMENSION $n = 2 + 2\sqrt{2}$ IN THE SEMILINEAR BREZIS-NIRENBERG EIGENVALUE PROBLEM [1]

Filippo Gazzola and Hans-Christoph Grunau

Received:

Abstract: We show that the number $n = 2 + 2\sqrt{2}$ plays a crucial role in the description of the branches of radially symmetric solutions of the equation $-\Delta u = \lambda u + |u|^{2^*-2}u$ in the unit ball.

AMS-Classification (1991): 35J65, 35J25

1 Nonresonant dimensions

Let Ω be an open bounded set in $\mathbb{R}^n$ ($n \geq 3$); we consider the problem of the existence of nontrivial solutions of the following equation

$$(1) \qquad \begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent and $\lambda > 0$. This equation has been widely studied in recent years but it still has several interesting open problems; in particular, a somehow surprising phenomenon is that the existence of nontrivial solutions of (1) depends not only on λ but on the couple (n, λ). Our starting point is the celebrated paper by Brezis-Nirenberg [7] where it is shown that:

- if $n = 3$, there exist constants $\lambda_1 > \lambda^*(\Omega) \geq \lambda^{**}(\Omega) > 0$ (presumably the same) such that (1) admits a positive solution if $\lambda \in (\lambda^*, \lambda_1)$ and not if $\lambda \in (0, \lambda^{**}]$; here λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet conditions.
- if $n \geq 4$, then (1) admits a positive solution if and only if $\lambda \in (0, \lambda_1)$.

It is well-known [17] that if $\Omega = B$ (the unit ball) positive solutions of (1) are radially symmetric; in this case, we have $\lambda^* = \lambda^{**} = \lambda_1/4$, see [7]. Subsequently, Capozzi-Fortunato-Palmieri [8] (see also [1, 16, 21]) considered the case $\lambda \geq \lambda_1$ and proved the following results:

- if $n = 4$, $\lambda > 0$ and $\lambda \notin \sigma(-\Delta)$ (the spectrum of $-\Delta$), then (1) admits a nontrivial solution.
- if $n \geq 5$, for all $\lambda > 0$ (1) admits a nontrivial solution.

[1] This work was supported by the Vigoni-programme of CRUI and DAAD

The result stated in [8] does not make the distinction between the two cases $n = 4$ and $n \geq 5$ but, as pointed out in [18, p.79], a careful analysis of the proof shows that it works only in the above stated situations. Up to now this problem is still open: it is not known if (1) admits nontrivial solutions when $n = 4$ and $\lambda \in \sigma(-\Delta)$; some partial (positive) results are found by Fortunato-Jannelli [13] in domains having some symmetries. Therefore, it seems natural to ask whether there exists indeed a difference between the dimensions $n = 4$ and $n \geq 5$ or if it is only a technical problem due to the particular proofs suggested in [8, 16, 21]: in agreement with [14], we name $n = 4$ *nonresonant dimension*.

These phenomena involving the space dimension also appear for more general operators as in the following polyharmonic problem

$$(2) \qquad \begin{cases} (-\Delta)^K u = \lambda u + |u|^{K_*-2}u & \text{in } \Omega \\ D^k u = 0 \quad \text{on } \partial\Omega & k = 0, ..., K-1 \end{cases}$$

where $K \in \mathbb{N}$ ($\mathbb{N}$ is the set of strictly positive integers), $\Omega \subset \mathbb{R}^n$ ($n \geq 2K+1$) is a bounded open set and $K_* = \frac{2n}{n-2K}$ is the critical Sobolev exponent; another way to generalize (1) is to consider the quasilinear problem

$$(3) \qquad \begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + |u|^{p^*-2}u & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $n > p > 1$ and $p^* = \frac{np}{n-p}$. A conjecture by Pucci-Serrin [20] states that the nonexistence result for radially symmetric solutions of (1) when $\Omega = B$ in dimension $n = 3$ "bifurcates" for (2) to the dimensions $n = 2K+1, ..., 4K-1$: Pucci-Serrin call these dimensions *critical*. This conjecture is "almost" completely proved [10, 18, 19, 20]. It is also known that the critical dimensions for the p-Laplacian are $n \in (p, p^2)$, see [11]. Recently, we have shown [15] that the critical dimensions are the dimensions for which a "linear" remainder term may be added in a Sobolev inequality with optimal constant for the spaces H^K_0; for a similar result in $W^{1,p}_0$, see [12]. On the other hand, it has been found independently in [14, 18] that the nonresonant dimensions for the polyharmonic operator are $n \in [4K, (2+2\sqrt{2})K]$, while for the p-Laplacian they are $n \in [p^2, \frac{p^2+p\sqrt{p^2+4}}{2}]$, see [2].

Let us come back to the simpler equation (1) for which the existence of nontrivial solutions seems to be different in the three cases $n = 3$, $n = 4$ and $n \geq 5$. Our purpose in this note is to study the nonresonant dimensions, to define them in a precise way and to give an interpretation of the limit value $n = 2 + 2\sqrt{2}$.

Let $\Omega = B$ be the unit ball in $\mathbb{R}^n$; we consider radially symmetric solutions of (1), namely solutions of the ODE problem

$$(4) \qquad u'' + \frac{n-1}{r}u' + \lambda u + |u|^{2^*-2}u = 0 \quad r \in (0,1)\,, \qquad u'(0) = u(1) = 0$$

for which n may be considered as a real parameter; for all $j \in \mathbb{N}$ let μ_j denote the eigenvalue of $-\Delta$ in B corresponding to a radial eigenfunction having $j-1$ nodes in $[0,1)$ and let u^j_λ be a solution of (4) having j nodes. Thanks to Atkinson-Brezis-Peletier [3] we know that if $n \in [4,6)$ then

$$(5) \qquad \|u^j_\lambda\|_\infty \to \infty \quad \Longrightarrow \quad \lambda \to \mu_j \qquad \forall j \in \mathbb{N}\,.$$

We are here interested in finding out if the convergence in (5) occurs from above or from below: it turns out that $n = 2 + 2\sqrt{2}$ plays a crucial role. More precisely, we prove

THEOREM 1: *For all* $j \in \mathbb{N}$ *let* u_λ^j *denote a solution of (4) having* j *nodes; then,*
(i) if $4 \le n \le 2 + 2\sqrt{2}$*, then* $\quad \|u_\lambda^1\|_\infty \to \infty \quad \Longrightarrow \quad \lambda \to \mu_1^+$
(ii) if $2 + 2\sqrt{2} < n < 6$*, then* $\quad \|u_\lambda^j\|_\infty \to \infty \quad \Longrightarrow \quad \lambda \to \mu_j^- \qquad \forall j \in \mathbb{N}.$

We can now define more rigorously nonresonant dimensions:

DEFINITION 2: *We say that a dimension* n *(*$n \in \mathbb{R}$*,* $n > 2$*) is a* **nonresonant dimension** *for* $-\Delta$ *if there exists* $m \in \mathbb{N}$ *such that* $\quad \|u_\lambda^m\|_\infty \to \infty \quad \Longrightarrow \quad \lambda \to \mu_m^+$*.*

Therefore, thanks to the results in [3, 4, 9], by Theorem 1 we have:

COROLLARY 3: *The nonresonant dimensions for* $-\Delta$ *are* $4 \le n \le 2 + 2\sqrt{2}$*.*

We conjecture that (4) admits no nontrivial solution when $\lambda = \lambda_1$ and $4 \le n \le 2 + 2\sqrt{2}$.

2 Proof of Theorem 1

As we are basing our analysis on the paper by Atkinson-Brezis-Peletier [3], we adopt their notation: in the sequel, the letter L after the reference number to an equation means that we refer to the reference number in [3].
By means of scaling and of Emden-Fowler inversion $y(t) := \lambda^{(2-n)/4} u\left((n-2)\,\lambda^{-1/2}\, t^{1/(2-n)}\right)$, (4) becomes

$$y'' + t^{-k}(1 + |y|^{2^*-2})y = 0 \quad t \in (0, \infty)\ , \qquad y(t) \to \gamma \text{ as } t \to \infty$$

where $k = 2\frac{n-1}{n-2}$ and the "shooting level" γ may be assumed to be positive. In [3] it is shown that y has infinitely many zeros $T_1(\gamma) > T_2(\gamma) > ...$, and that

$$(6) \qquad \text{if } 4 \le n < 6 \qquad \text{then } \lim_{\gamma \to \infty} T_1(\gamma) = \infty \qquad \text{and } \lim_{\gamma \to \infty} T_j(\gamma) = \tau_{j-1} \quad \forall j \ge 2\ .$$

Here $\tau_1 > \tau_2 > ...$ are the zeros of the function

$$\alpha(t) = (k-2)^{1/(k-2)} \Gamma\left(\frac{k-1}{k-2}\right) \sqrt{t} J_{1/(k-2)}\left(\frac{2}{(k-2)t^{(k-2)/2}}\right)$$

where $J_{1/(k-2)}$ is the first kind (regular) Bessel function of order $1/(k-2)$; note that α solves the equation $z'' + t^{-k} z = 0$. Define the function b as in (4.6)L, then (4.12)L entails

$$(7) \qquad b(\tau_1) = b(T_1) + \int_{\tau_1}^{T_1} \frac{|y(s)|^{2k-4} y(s) \alpha(s)}{s^k} ds\ .$$

By (4.6)L we get $b(\tau_1) = -y(\tau_1)\alpha'(\tau_1)$; hence, since $\alpha'(\tau_1) > 0$, we have for γ large by means of (6) the following implications for λ corresponding to u_λ^1

$$(8) \qquad b(\tau_1) > 0 \Rightarrow y(\tau_1) < 0 \Rightarrow \tau_1 > T_2 \Rightarrow \lambda > \mu_1 \qquad b(\tau_1) < 0 \Rightarrow ... \Rightarrow \lambda < \mu_1\ .$$

The case $n < 2 + 2\sqrt{2}$ $(k > 2 + \sqrt{2}/2)$. By (8), in this case we have to show that for γ large enough we have $b(\tau_1) > 0$. By (3.2)L and (3.4)L-(3.5)L we have for large γ

$$\left|\int_{\tau_1}^{T_1} \frac{|y(s)|^{2k-4}y(s)\alpha(s)}{s^k}ds\right| \le C_1 \left(\frac{T_1(\gamma)}{\gamma}\right)^{2k-3} \le \frac{C_2}{\gamma^{4k^2-16k+15}} \quad \text{if } k<3$$
$$\left|\int_{\tau_1}^{T_1} \frac{|y(s)|^{2}y(s)\alpha(s)}{s^3}ds\right| \le C_3 \left(\frac{T_1(\gamma)}{\gamma}\right)^{3} \le C_4\left(\frac{\log\gamma}{\gamma}\right)^3 \quad \text{if } k=3;$$

on the other hand, (4.10)L states that $b(T_1) > C_5\gamma^{-1}$ for large γ; since $4k^2 - 16k + 15 > 1$ (for $2 + \sqrt{2}/2 < k < 3$) and $\gamma^{-3}\log^3\gamma \ll \gamma^{-1}$ as $\gamma \to \infty$, these two asymptotics inserted into (7) yield $b(\tau_1) > 0$ for γ large enough.

The case $n = 2 + 2\sqrt{2}$ $(k = 2 + \sqrt{2}/2)$. By (4.10)L, we have $b(T_1(\gamma)) = (1+\frac{\sqrt{2}}{2})^{\sqrt{2}}\gamma^{-1} + o(\gamma^{-1})$ as $\gamma \to \infty$. First, using (3.2)L, (3.5)L and proceeding as in the case before, for any $\varepsilon > 0$ one finds $T_0 > \tau_1$ such that $T_0 < T_1$ for large γ and

$$\left|\int_{\tau_1}^{T_1} \frac{|y(s)|^{2k-4}y(s)\alpha(s)}{s^k}\,ds\right| \le \int_{\tau_1}^{T_0} \frac{|y(s)|^{2k-3}\alpha(s)}{s^k}\,ds + \frac{\varepsilon}{\gamma}\,.$$

Second, on the fixed interval $[\tau_1, T_0]$, (4.13a)L may be applied to obtain

$$\begin{aligned}
&\int_{\tau_1}^{T_0} \frac{|y(s)|^{2k-3}\alpha(s)}{s^k}\,ds \\
\le\ & \frac{1}{\gamma}\left\{\left(\left(1+\frac{\sqrt{2}}{2}\right)\frac{\Gamma\left(\sqrt{2}-1\right)\,\Gamma\left(\sqrt{2}+1\right)}{\Gamma\left(2\sqrt{2}\right)}\right)^{1+\sqrt{2}} \cdot \int_{\tau_1}^{\infty} \frac{\alpha(s)^{2+\sqrt{2}}}{s^{2+(\sqrt{2}/2)}}\,ds + o(1)\right\} \\
=\ & \left(1+\frac{\sqrt{2}}{2}\right)^{\sqrt{2}} \cdot \frac{1}{\gamma}\cdot\left\{\left(1+\frac{\sqrt{2}}{2}\right)\left(\frac{\Gamma\left(\sqrt{2}-1\right)\,\Gamma\left(\sqrt{2}+1\right)}{\Gamma\left(2\sqrt{2}\right)}\right)^{1+\sqrt{2}} \cdot \frac{1}{\sqrt{2}^{2\sqrt{2}-1}}\right. \\
&\left. \cdot\Gamma\left(\sqrt{2}+1\right)^{2+\sqrt{2}} \cdot \int_0^{j_{\sqrt{2}}} t^{-1} J_{\sqrt{2}}(t)^{2+\sqrt{2}}\,dt + o(1)\right\} \\
\le\ & \left(1+\frac{\sqrt{2}}{2}\right)^{\sqrt{2}} \cdot \frac{1}{\gamma} \cdot (0.500041\ldots + o(1)) \le 0.6\left(1+\frac{\sqrt{2}}{2}\right)^{\sqrt{2}} \cdot \frac{1}{\gamma}
\end{aligned}$$

for γ large enough; here $j_{\sqrt{2}}$ denotes the first positive zero of $J_{\sqrt{2}}$. In the last step the numerical calculation was carried out with help of MAPLE. By an appropriate choice of ε, collecting terms yields $b(\tau_1) > 0$ for γ large also in this marginal case.

The case $n > 2 + 2\sqrt{2}$ $(k < 2 + \sqrt{2}/2)$. In order to prove (ii) we need to show that $\tau_j < T_{j+1}$ for all $j \in \mathbb{N}$ and for all sufficiently large γ; to this end, by Sturm separation Theorem it suffices to prove that $\tau_1 < T_2$ for γ large enough. By contradiction, assume that $\tau_1 \ge T_2$: then $y(s) \le 0$ (i.e. $y(s) = -|y(s)|$) for all $s \in [\tau_1, T_1]$ and for large γ by (4.13a)L we have

$$-\int_{\tau_1}^{T_1} \frac{|y(s)|^{2k-3}\alpha(s)}{s^k}ds \le -\frac{C_1}{\gamma^{4k^2-16k+15}}\int_{\tau_1}^{2\tau_1} \frac{[\alpha(s)]^{2k-2}}{s^k}ds \le -\frac{C_2}{\gamma^{4k^2-16k+15}}\ :$$

since $4k^2 - 16k + 15 < 1$, by (4.10)L and (7) this yields $b(\tau_1) < 0$ for large γ, contradiction; by (8) this completes the proof.

References

1. A. Ambrosetti, M. Struwe, A note on the problem $-\Delta u = \lambda u + u|u|^{2^*-2}$, *Manusc. Math.* **54**, 373-379 (1986).
2. G. Arioli, F. Gazzola, Some results on p-Laplace equations with a critical growth term, *Diff. Int. Eq.* **11**, 311-326 (1998).
3. F.V. Atkinson, H. Brezis, L.A. Peletier, Nodal solutions of elliptic equations with critical Sobolev exponents, *J. Diff. Eq.* **85**, 151-171 (1990).
4. F.V. Atkinson, L.A. Peletier, Large solutions of elliptic equations involving critical exponents, *Asymptotic Anal.* **1**, 139-160 (1988).
7. H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36**, 437-477 (1983).
8. A. Capozzi, D. Fortunato, G. Palmieri, An existence result for nonlinear elliptic problems involving critical Sobolev exponent, *Ann. Inst. H. Poincaré, Anal. non linéaire* **2**, 463-470 (1985).
9. G. Cerami, S. Solimini, M. Struwe, Some existence results for superlinear elliptic boundary value problems involving critical exponents, *J. Funct. Anal.* **69**, 289-306 (1986).
10. D.E. Edmunds, D. Fortunato, E. Jannelli, Critical exponents, critical dimensions and the biharmonic operator, *Arch. Rat. Mech. Anal.* **112**, 269-289 (1990).
11. H. Egnell, Elliptic boundary value problems with singular coefficients and critical nonlinearities, *Indiana Univ. Math. J.* **38**, 235-251 (1989).
12. H. Egnell, F. Pacella, M. Tricarico, Some remarks on Sobolev inequalities, *Nonlin. Anal. TMA* **13**, 671-681 (1989).
13. D. Fortunato, E. Jannelli, Infinitely many solutions for some nonlinear elliptic problems in symmetrical domains, *Proc. Roy. Soc. Edinburgh* **A 105**, 205-213 (1987).
14. F. Gazzola, Critical growth problems for polyharmonic operators, *Proc. Roy. Soc. Edinburgh* **A 128**, 251-263 (1998).
15. F. Gazzola, H.-Ch. Grunau, Critical dimensions and higher order Sobolev inequalities with remainder terms, *Nonlin. Diff. Eq. Appl.*, to appear.
16. F. Gazzola, B. Ruf, Lower order perturbations of critical growth nonlinearities in semilinear elliptic equations, *Adv. Diff. Eq.* **4**, 555-572 (1997).
17. B. Gidas, W.M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68**, 209-243 (1979).
18. H.-Ch. Grunau, Polyharmonische Dirichletprobleme: Positivität, kritische Exponenten und kritische Dimensionen, Habilitationsschrift, Universität Bayreuth, 1996.
19. H.-Ch. Grunau, On a conjecture of P. Pucci and J. Serrin, *Analysis* **16**, 399-403 (1996).
20. P. Pucci, J. Serrin, Critical exponents and critical dimensions for polyharmonic operators, *J. Math. Pures Appl.* **69**, 55-83 (1990).
21. D. Zhang, On multiple solutions of $\Delta u + \lambda u + |u|^{4/(n-2)} u = 0$, *Nonlin. Anal. TMA* **13**, 353-372 (1989).

F. G. : Dipartimento di Scienze T.A. - via Cavour 84, I-15100 Alessandria, gazzola@unipmn.it
H.-Ch. G. : Fachgruppe Mathematik, Universität, D-95440 Bayreuth, grunau@uni-bayreuth.de